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ON COMPACTNESS AND COMPLETENESS OF CONFORMAL METRICS IN R^N*

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Abstract. For a smooth function $K(x)$ in $\mathbf{R}^n$, we consider the problem of finding a metric g conformal to the flat metric $|dx|^2$ such that $K(x)$ is the scalar curvature of g. Let $g = u^{\frac{4}{n-2}}|dx|^2$. Then when $n \geq 3$ the problem is equivalent to finding a positive smooth solution of the equation

$$\Delta u + K(x)u^{\frac{n+2}{n-2}} = 0 \text{ in } \mathbf{R}^n. \tag{0.1}$$

A solution u of (0.1) is called of slow decay if the conformal metric g can not be realized as a smooth metric on S^n. In this paper, under some conditions on K, we prove that if u is a solution of slow decay, then the conformal metric $g = u^{\frac{4}{n-2}}|dx|^2$ is a complete metric in $\mathbf{R}^n$ which has bounded curvatures. As a corollary of the result, we show that under the same conditions on K as above, the Kazdan-Warner identity holds for a solution u if and only if the conformal metric $g = u^{\frac{4}{n-2}}|dx|^2$ can be realized as a smooth metric on S^n. A compactness theorem is also proved.

1. Introduction. Let (M, g_0) be a Riemannian manifold of dimension n with $n \geq 3$. Given a smooth function $K(x)$ defined on M, one would like to find a metric g conformal to g_0 such that K is the scalar curvature of the new metric g. In the last several years, there have been considerable works devoted to studying this problem of prescribing scalar curvature. However, most works were only concerned with the case when M is a compact manifold; pariticularly, the standard n-dimensional unit sphere S^n. In this paper, we want to consider the case when (M, g_0) is the n-dimensional Euclidean space $\mathbf{R}^n$. When (M, g_0) is the Euclidean space and let $g = u^{\frac{4}{n-2}}g_0$ for some positive function u, then the question above is equivalent to finding positive smooth solutions of

$$\Delta u + K(x)u^{\frac{n+2}{n-2}} = 0 \quad \text{in } \mathbf{R}^n \tag{1.1}$$

after an appropriate scaling, where Δ is the Laplace operator of $\mathbf{R}^n$. *In the paper, we always assume the limit $K(\infty) = \lim_{|x| \longrightarrow +\infty} K(x)$ exists and is positive.* We will discuss the case $K(\infty) = 0$ in another paper.

From the viewpoint of geometry, it is natural to ask a solution u of (1.1) such that the conformal metric $g = u^{\frac{4}{n-2}}g_0$ is a complete metric in $\mathbf{R}^n$. Following conventional notations, *a solution of (1.1) is called of slow decay if the conformal metric $u^{\frac{4}{n-2}}g_0$ can not be realized as a smooth metric on S^n.* From the viewpoint of PDE, many basic questions about solutions of equation (1.1) remain to be investigated. Three of them are listed below.

Question 1. Is every solution of equation (1.1) bounded in $\mathbf{R}^n$?

Question 2. Suppose that u is a solution of slow decay. Is the conformal metric $g = u^{\frac{4}{n-2}}g_0$ a complete metric in $\mathbf{R}^n$?

Question 3. If the conformal metric g is complete in $\mathbf{R}^n$, does g always have bounded curvatures in $\mathbf{R}^n$?

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The answer of these questions was implicitly contained in the work of Caffarell-Gidas-Spruck [CGS], when the curvature function $K(x)$ is identically a positive constant for $|x|$ large. In [CGS], they proved that if u is a solution of slow decay, then there exists an entire singular solution u_0 of

$$\Delta u_0(x) + K(\infty)u_0^{\frac{n+2}{n-2}} = 0 \ \text{ in } \ \mathbf{R}^n\backslash\{0\} \tag{1.2}$$

such that

$$u(x) = u_0(x)(1 + o(1)) \tag{1.3}$$

as $|x| \longrightarrow +\infty$. Since singular solutions of (1.2) can be completely classified, an immediate consequence of (1.3) is that there exist two positive constants c_1 and c_2 such that

$$c_1|x|^{-\frac{n-2}{2}} \le u(x) \le c_2|x|^{-\frac{n-2}{2}} \tag{1.4}$$

holds for $|x|$ large. Thus, the completeness of the conformal metric g follows immediately and it is not difficult to show that g has bounded curvatures.

Another insteresting implication of (1.3) is the followings. A result of Kazdan-Warner states that if $g = u^{\frac{4}{n-2}}|dx|^2$ can be realized as a smooth metric on S^n, then the identity

$$\int_{\mathbf{R}^n} (x \cdot \nabla K(x))u^{\frac{2n}{n-2}}(x)dx = 0 \tag{1.5}$$

always holds. An immediate consequence of (1.3) says that when $K(x)$ is a positive constant for $|x|$ large, then the Kazdan-Warner identity holds for a solution u is sufficient and necessary for the conformal metric $g = u^{\frac{4}{n-2}}|dx|^2$ to be realized as a smooth metric on S^n. Therefore, it should be an interesting question to ask whether these results can be extended to functions K which are not constant near ∞.

Actually, the work of [CGS] was extended in [CLn3] to a more general class of curvature functions K. To state the result in [CLn3], we suppose that K satisfies

$$\begin{cases} K(\infty) = \lim_{|x|\longrightarrow\infty} K(x) > 0 \ \text{ and} \\ c_1 \le |\nabla K(x)||x|^{l+1} \le c_2 \ \text{ for } |x| \text{ large and for some } l > 0. \end{cases} \tag{1.6}$$

In [CLn3], we proved that if (1.6) is satisfied with $l \ge \dfrac{n-2}{2}$, then (1.3) holds for any solution u of slow decay. In particular, we answer Question1 through 3 affirmatively. On the other hand, when $l < \dfrac{n-2}{2}$ we have constructed a solution u such that the conformal metric g is a complete metric, but, with unbounded curvatures. In this example, it is easy to see that the curvature function $K(|x|)$ has a local maximum at ∞. In this paper, we want to consider the case when $K(x)$ has its local minimum at ∞. In this case, the condition $l \ge \dfrac{n-2}{2}$ will be removed. Instead, we need the following assumption:

(1.7) There exists a constant $c_0 > 1$ such that $K(y) \le K(x)$ whenever $|y| \ge c_0|x|$ for $|x|$ large.

THEOREM 1.1. *Assume K satisfies both (1.6) and (1.7). Then there exist $c > 0$ and $R > 0$ such that for any solution u of equation (1.1),*

$$u(x) \le c\,|x|^{-\frac{n-2}{2}} \tag{1.8}$$

holds for $|x| \ge R$. In addition, if we assume $x \cdot \nabla K(x) \le 0$ for large $|x|$ and $u(x)$ is a solution of slow decay, then there exists a constant $c_1 = c_1(u) > 0$ such that

$$u(x) \ge c_1 |x|^{-\frac{n-2}{2}} \tag{1.9}$$

holds for $|x|$ large. Therefore the conformal metric $g = u^{\frac{4}{n-2}}|dx|^2$ is complete in $\mathbf{R}^n$

We have to emphasize the constant c in (1.8) is independent of u, an improvement of our previous result in [CLn1] even under the condition $l \ge \dfrac{n-2}{2}$. Inequality (1.8) is very important because it enables us to apply the Harnack inequality to the solution u for $|x| \ge R$. As in [CLn1], inequality (1.8) will be proved through the application of the method of moving planes. The method of moving planes was first invented by A.D. Alexandrov, and has been developed further to study the radial symmetry of solutions of elliptic equations by Serrin [S], Gidas-Ni-Nirenberg [GNN], Caffarelli-Gidas-Spruck [CGS] and Chen-Li [CL]. Here, we use a modified version which was developed in [CLn1].

It is easy to see that by the Kazdan-Warner identity, if $K(x) \not\equiv$constant is nonincreasing along any ray issuing from the origin, then any solution of (1.1) must be of slow decay. Hence, we have the following consequence of Theorem 1.1.

COROLLARY 1.2. *Suppose $K \not\equiv$ a positive constant satisfies (1.6),(1.7) and $x \cdot \nabla K(x) \le 0$ for $x \in \mathbf{R}^n$. Then for any solution $u(x)$, $u(x)|x|^{\frac{n-2}{2}}$ is bounded by two positive constants for $|x|$ large. Furthermore, the conformal metric $g = u^{\frac{4}{n-2}} g_0$ is complete and of bounded curvature on $\mathbf{R}^n$.*

Corollary 1.2 answers a question which was arised in [DN]. In [DN], Ding-Ni proved the existence of infinitely many solutions of (1.1) when K is almost symmetry and nonincreasing along any ray from the origin. But, they did not establish the completeness of their solutions. Thus, from Corollary 1.2, the existence of complete metrics is obtained for such class of K considered in [DN].

It is easy to see the upper bound (1.8) implies that $|x||\nabla K(x)|u^{\frac{2n}{n-2}} \in L^1(\mathbf{R}^n)$ if $|\nabla K(x)|$ decays like $|x|^{-l}$ at ∞ for $l > 0$.

COROLLARY 1.3. *Suppose that K satisfies (1.6), (1.7) and $x \cdot \nabla K(x) \le 0$ for $|x|$ large. Then the Kazdan-Warner identity (1.5) holds if and only if the conformal metric $g = u^{\frac{4}{n-2}}|dx|^2$ can be realized as a smooth metric on S^n. In the case when g is complete in $\mathbf{R}^n$, the quantity*

$$P = \int_{\mathbf{R}^n} (x \cdot \nabla K(x)) u^{\frac{2n}{n-2}} < 0. \tag{1.10}$$

The quantity P defined above is called *the Pohozaev number* because it comes from the Pohozaev identity. It is quite intersting to note that a negative upper bound of the Pohozaev number can guarantee the compactness of conformal metrics. To see it, we assume that K has only a finite number of critical points $\{P_1, \dots, P_N\}$ in $\mathbf{R}^n$. At each critical point P_i, K satisfies the nondegenerate conditions :

$$|\nabla K(x)| \sim |x - P_i|^{\beta_i - 1} \tag{1.11}$$

in a neighborhood of P_i for some constant $\beta_i > 1$, where two functions $f(x) \sim g(x)$ means that the ratio $f(x)/g(x)$ is bounded by two positive constants.

THEOREM 1.4. *Suppose the assumptions of Corollary 1.3 hold and K has a finite number of critical points $\{P_1, \ldots, P_N\}$. At each critical point P_i of K, assume (1.11) holds for some $\beta_i > \frac{n-2}{2}$. Let $\varepsilon_0 > 0$ be fixed. Then there exists a constant $c = c(\varepsilon_0)$ such that for any $x \in \mathbf{R}^n$, the inequality*

$$u(x) \leq c\,(1+|x|)^{-\frac{n-2}{2}} \tag{1.12}$$

holds for $x \in \mathbf{R}^n$ and for any positive solution u satisfying

$$\int_{\mathbf{R}^n} (x \cdot \nabla K(x)) u^{\frac{2n}{n-2}}(x)dx \leq -\varepsilon_0. \tag{1.13}$$

We remark that for the case of two dimension, the conformal metric of a presbcribed Gaussian curvature might be neither a complete metric on $\mathbf{R}^2$, nor a smooth metric on S^2. Thus, the conclusions of both Theorem 1.1 and Corollary 1.2 on the completeness of conformal metrics do not hold in two dimension.

The paper is organized as follows. In Section 2, the first part of Theorem 1.1 is proved via the method of moving planes. The inequality (1.9) and Theorem 1.4 will be proved in Section 3.

2. The method of moving planes. To prove Theorem 1.1, we will apply a modified version of the well-known reflection method, as developed in [CLn1]. Following conventional notations, we let for any $\lambda < 0$, $T_\lambda = \{x = (x_1, \ldots, x_n) \mid x_1 = \lambda\}$, $\Sigma_\lambda = \{x \mid x_1 > \lambda\}$ and x^λ denote the reflection point of x with respect to T_λ. Let u be a positive C^2 solution of

$$\Delta u + K(x)u^{\frac{n+2}{n-2}} = 0 \quad \text{in } \mathbf{R}^n \backslash \Lambda, \tag{2.1}$$

where Λ is a finite set of singular points. Assume $\Lambda \subset \Sigma_{\lambda_0}$ for some $\lambda_0 < 0$. Let $w_\lambda(x) = u(x) - u(x^\lambda)$ for $x \in \Sigma_\lambda \backslash \Lambda$ and $\lambda \leq \lambda_0$. Then w_λ satisfies

$$\Delta w_\lambda + b_\lambda w_\lambda = Q_\lambda(x) \quad \text{in } \Sigma_\lambda \backslash \Lambda, \tag{2.2}$$

where

$$b_\lambda(x) = K(x)\frac{u(x)^{\frac{n+2}{n-2}} - u(x^\lambda)^{\frac{n+2}{n-2}}}{u(x) - u(x^\lambda)} \quad \text{and}$$
$$Q_\lambda(x) = (K(x^\lambda) - K(x))u^{\frac{n+2}{n-2}}(x^\lambda).$$

Suppose $u(x) = O(|x|^{2-n})$ at ∞. Then for $\lambda \leq \lambda_0$, $b_\lambda(y)$ satisfies

$$0 \leq b_\lambda(x) \leq C|x|^{-4} \tag{2.3}$$

for large $|x|$ and a positive constant $C > 0$.

To apply the method of moving planes, we want to construct a family of comparison functions $h_\lambda \in C^1(\overline{\Sigma}_\lambda)$ satisfying the following conditions.

$$\begin{cases} \Delta h_\lambda(x) \geq Q_\lambda(x) & \text{in } \Sigma_\lambda \\ h_\lambda(x) > 0 \quad \text{in } \Sigma_\lambda \text{ and } h_\lambda(x) = 0 \text{ on } T_\lambda, \end{cases} \tag{2.4}$$

(2.5) $h_{\lambda_1}(x) < w_{\lambda_1}(x)$ holds for $x \in \sum_{\lambda_1} \backslash \Lambda$ and for some $\lambda_1 < \lambda_0$.

(2.6) $h_\lambda(x) = O(|x|^{-\tau})$ at ∞ for some $\tau > 0$. Both h_λ and ∇h_λ are continuous in λ, x variables.

LEMMA 2.1 *Suppose $u(x)$ satisfies (2.1) and $u(x) = O(|x|^{2-n})$ at ∞. Assume there exists a family of functions h_λ satisfying (2.4), (2.5) and (2.6). Then $w_\lambda(x) > 0$ in Σ_λ for $\lambda_1 \le \lambda \le \lambda_0$.*

We refer the reader to [CLn1] for the proof of Lemma 2.1 (See Lemma 2.1 in [CLn1].) To apply Lemma 2.1, we need the following lemma about the Green function $G^\lambda(x,\eta)$ of Δ on Σ_λ with zero boundary value. The Green function has the form of

$$G^\lambda(x,\eta) = c_n\left(\frac{1}{|\eta - x|^{n-2}} - \frac{1}{|\eta - x^\lambda|^{n-2}}\right) \tag{2.7}$$

for $x,\eta \in \bar{\Sigma}_\lambda$, where c_n is a positive constant depending on n only.

LEMMA 2.2. *Let $\lambda < 0$. Then the followings hold.*

(i) For $|x| \le \frac{|\lambda|}{2}, G^\lambda(x,0) \ge c_1|x|^{-n+2}$.

(ii) For $|x| \ge \frac{|\lambda|}{2}, G^\lambda(x,0) \ge c_1|\lambda|(x_1 - \lambda)|x|^{-n}$.

(iii) For $x,\eta \in \Sigma_\lambda$, we have

$$G^\lambda(x,\eta) \le c_2 \min(|x-\eta|^{2-n}, |x-\eta|^{1-n}(x_1 - \lambda), |\eta - x|^{-n}(x_1-\lambda)(\eta_1 - \lambda))$$

where c_1 and c_2 are two positive constants depending on n only.

Now we are in the position to prove Theorem 1.1.

Proof of Theorem 1.1. The upper bound. Without loss of generality, K is assumed to satisfy

$$0 < c_1 \le |\nabla K(x)||x|^{l+1} \le c_2 \quad \text{for } |x| \ge 1. \tag{2.8}$$

Let u be a positive smooth solution of equation (1.1), and let $u^*(x) = |x|^{2-n}u(\frac{x}{|x|^2})$ be the Kelvin transformation of u. Then u^* satisfies

$$\Delta u^*(x) + K^*(x){u^*}^{\frac{n+2}{n-2}}(x) = 0 \quad \text{in } \mathbf{R}^n\backslash\{0\}, \tag{2.9}$$

where $K^*(x) = K(\frac{x}{|x|^2})$. Inequality (1.8) is equivalent to the following

$$u^*(x) \le C|x|^{\frac{2-n}{2}} \quad \text{for } |x| \le 1/2. \tag{2.10}$$

Suppose that (2.10) does not hold. By applying a blowing-up argument due to R. Schoen [P], there exist a sequence of solutions u_i^* and a sequence of local maximum points x_i^* of u_i^* such that

$$u_i^*(x_i^*)|x_i^*|^{\frac{n-2}{2}} \longrightarrow +\infty \tag{2.11}$$

and the rescaled function

$$v_i^*(y) = M_i^{-1}u_i^*(x_i^* + M_i^{\frac{-2}{n-2}}y) \tag{2.12}$$

converges to $U_0^*(y)$ in $C^2_{loc}(\mathbf{R}^n)$, where $M_i = u_i^*(x_i^*)$ and $U_0^*(y)$ is the positive solution of

$$\begin{cases} \Delta U_0^* + K(\infty) U_0^{*\frac{n+2}{n-2}} = 0 \quad \text{in } \mathbf{R}^n, \\ U_0^*(0) = 1 = \max_{\mathbf{R}^n} U_0^*(y). \end{cases} \tag{2.13}$$

For a proof of the statement above, we refer the readers to [P] and [CLn1]. Since K^* has no critical points in $0 < |x| < 1$. Hence by Corollary 1.4 in [CLn1], we have $x_i^* \longrightarrow 0$ as $i \longrightarrow +\infty$.

Obviously, v_i^* satisfies

$$\Delta v_i^* + K_i^*(y) v_i^{*\frac{n+2}{n-2}}(y) = 0 \quad \text{in } \mathbf{R}^n \backslash \{-M_i^{\frac{2}{n-2}} x_i^*\}, \tag{2.14}$$

where $K_i^*(y) = K^*(x_i^* + M_i^{-\frac{2}{n-2}} y)$ and $\lim_{i \longrightarrow +\infty} M_i^{\frac{2}{n-2}} |x_i^*| = +\infty$. Without loss of generality, we may assume

$$e_1 = (1, 0, \ldots, 0) = \lim_{i \longrightarrow +\infty} |\nabla K^*(x_i^*)|^{-1} \nabla K^*(x_i^*). \tag{2.15}$$

For any $\delta > 0$, set

$$v_i^\delta(y) = |y|^{2-n} \left|\frac{y}{|y|^2} - \delta e_1\right|^{2-n} v_i^*(I_\delta(y)), \tag{2.16}$$

and

$$I_\delta(y) = \left(\frac{y}{|y|^2} - \delta e_1\right) \left|\frac{y}{|y|^2} - \delta e_1\right|^{-2}. \tag{2.17}$$

Obviously, $I_\delta(y)$ is a composition of two inversions and v_i^δ is a composition of two Kelvin transformations. Since equation (2.14) is invariant under the Kelvin transformation, $v_i^\delta(y)$ satisfies

$$\Delta v_i^\delta + K_\delta(y)(v_i^\delta)^{\frac{n+2}{n-2}} = 0 \quad \text{in } \mathbf{R}^n \backslash \{\xi_i\},$$

where $K_\delta(y) = K^*(x_i^* + M_i^{-\frac{2}{n-2}} I_\delta(y))$ and ξ_i satisfies $I_\delta(\xi_i) = -M_i^{\frac{2}{n-2}} x_i^*$. It is not difficult to see

$$\lim_{i \longrightarrow \infty} \xi_i = \frac{e_1}{\delta}. \tag{2.18}$$

Let $U_\delta(y) = |y|^{2-n} \left|\frac{y}{|y|^2} - \delta e_1\right|^{2-n} U_0^*(I_\delta(y))$. By a straightforward calculation, $U_\delta(y)$ has a nondegenerate maximum point e_δ such that $e_\delta \longrightarrow 0$ as $\delta \longrightarrow 0$. Hence, v_i^δ has a local maximum at y_i with $\lim_{i \longrightarrow +\infty} y_i = e_\delta$. In the followings, δ_0 will be chosen sufficiently small so that y_i is contained in the strip $\{y = (y_1, \ldots, y_n) \mid -1/2 \le y_1 \le 1/2\}$ for all i and $\delta \le \delta_0$.

Let $w_\lambda(y) = v_i^\delta(y) - v_i^\delta(y^\lambda)$. (For simplicity of notations, indices i and δ are dropped.) Then $w_\lambda(y)$ satisfies

$$\Delta w_\lambda + b_\lambda(y) w_\lambda(y) = Q_\lambda(y) \quad \text{in } \Sigma_\lambda \backslash \{\xi_i\}, \tag{2.19}$$

where $b_\lambda(y) w_\lambda(y) = K_\delta(y)\{v_i^\delta(y)^{\frac{n+2}{n-2}} - v_i^\delta(y^\lambda)^{\frac{n+2}{n-2}}\}$ and

$$Q_\lambda(y) = [K_\delta(y^\lambda) - K_\delta(y)](v_i^\delta(y^\lambda))^{\frac{n+2}{n-2}}.$$

First, we claim that there exist constants $\lambda_0 < 0$ and $\delta_0 > 0$ such that for all $0 < \delta \leq \delta_0$, we have

$$w_{\lambda_0}(y) \geq c_1(1+|y|)^{-n}(y_1 - \lambda_0) \tag{2.20}$$

for $y \in \Sigma_{\lambda_0} \backslash \{\xi_i\}$, where c_1 is a positive constant independent of δ and i.

The proof of (2.20) goes similarly as the proof of Lemma 3.1 in [CLn1]. However, the lower bound of u is not assumed here. For the sake of completeness, we present the proof here.

Since $v_i^*(y)$ converges to $U_0^*(y)$ in $C_{loc}^2(\mathbf{R}^n)$, for any $\varepsilon > 0$ there exists a sequence of $R_i \longrightarrow +\infty$ such that $|v_i^*(y) - U_0^*(y)| \leq \varepsilon R_i^{2-n}$ for $|y| \leq R_i$. Let $B = \{y \mid ||y - \frac{e_1}{\delta}| \leq 1\}$. Since v_i^δ is superharmonic in $B \backslash \{\xi_i\}$, by the maximum principle,

$$v_i^\delta(y) \geq \inf_{\partial B} v_i^\delta \quad \text{for } y \in B. \tag{2.21}$$

(e.g., see Lemma 2.1 in [CLn4]). Since the right-hand side of (2.21) tends to $\inf_B U_\delta(y)$ as $i \longrightarrow +\infty$ and $U_\delta(y) > (1 + c_1\delta)U_\delta(z^\lambda)$ for $\lambda \leq -2$ and for $y, z \in B$ where c_1 is a constant depending on n only, we have by (2.21),

$$w_\lambda(y) \geq \frac{1}{2}c_1\delta^{n-1} \quad \text{for } y \in B \tag{2.22}$$

By a direct computation, we have

$$I_\delta(y) + \frac{e_1}{\delta} = \delta^{-2}|y - \frac{e_1}{\delta}|^{-2}(y' + (\frac{1}{\delta} - y_1)e_1), \tag{2.23}$$

and by (2.23),

$$|I_\delta(y) - I_\delta(y^\lambda)| \leq 2\delta^{-2}[1 + (2|\lambda| + \frac{2}{\delta})|y - \frac{e}{\delta}|^{-1}]|y^\lambda - \frac{e}{\delta}|^{-2}(y_1 - \lambda) \tag{2.24}$$

for $y_1 > \lambda$, where $y' = (0, y_2, \ldots, y_n)$. By the uniform convergence of v_i^*, we have

$$\begin{aligned} \delta^{1-n}|\bigtriangledown v_i^*(z)| &= (n-2)\delta^{2-n}|v_i^*(z)|(1 + o(1) + O(\delta)) \\ &= c_n(1 + o(1) + O(\delta)) \end{aligned} \tag{2.25}$$

holds for $|z + \frac{e_1}{\delta}| \leq 1$. Applying (2.23) through (2.25), we have for $|y - \frac{e_1}{\delta}| \geq \frac{1}{\delta^2}$,

$$\begin{aligned} v_i^\delta(y) - v_i^\delta(y^\lambda) &= \delta^{2-n}\{(|y - \frac{e_1}{\delta}|^{2-n} - |y^\lambda - \frac{e_1}{\delta}|^{2-n})v_i^*(I_\delta(y)) \\ &\quad + [v_i^*(I_\delta(y)) - v_i^*(I_\delta(y^\lambda))]|y^\lambda - \frac{e_1}{\delta}|^{2-n}\} \\ &\geq c_n|y^\lambda - \frac{e_1}{\delta}|^{-n}(y_1 - \lambda), \end{aligned}$$

provided that $|\lambda|$ is sufficiently large, δ is small and i is large. Thus, let $|\lambda_0|$ be large enough so that (2.20) holds for $|y - \frac{e_1}{\delta}| \geq \frac{1}{\delta^2}$.

For $1 \leq |y - \frac{e_1}{\delta}| \leq \delta^{-2}$, (2.20) follows immediately from the C^2 convergence of v_i^δ to U_δ as $i \longrightarrow +\infty$. Therefore, the proof of (2.20) is complete.

Let

$$B_i = \{y \mid |y - \frac{e_1}{\delta}| \le \frac{1}{4c_0\delta^2} M_i^{-\frac{2}{n-2}} |x_i^*|^{-1}\}, \tag{2.26}$$

where c_0 is the positive constant in (1.7). Since $|I_\delta(y)| = |y|(\delta|y - \frac{e_1}{\delta}|)^{-1}$, we have

$$|I_\delta(y)| \ge 3c_0 M_i^{\frac{2}{n-2}} |x_i^*| \tag{2.27}$$

for $y \in B_i$ and large i. Since $|I_\delta(y^\lambda)| \le \frac{3}{\delta}$ for $y \in B_i$ and $\lambda_0 \le \lambda \le \frac{1}{2}$, we have

$$\begin{aligned} |x_i^* + M_i^{-\frac{2}{n-2}} I_\delta(y^\lambda)| &\le |x_i^*| + M_i^{-\frac{2}{n-2}} |I_\delta(y^\lambda)| \\ &\le M_i^{-\frac{2}{n-2}} (M_i^{\frac{2}{n-2}} |x_i^*| + \frac{3}{\delta}) \\ &\le c_0^{-1} |x_i^* + M_i^{-\frac{2}{n-2}} I_\delta(y)| \end{aligned}$$

for large i. Thus, by (1.7),

$$K_\delta(y) \ge K_\delta(y^\lambda) \tag{2.28}$$

for $y \in B_i$.

Let

$$Q_\lambda^+(y) = \begin{cases} Q_\lambda(y) & \text{if } y \notin B_i, \\ 0 & \text{if } y \in B_i \end{cases} \tag{2.29}$$

and

$$h_\lambda(y) = -\int_{\Sigma_\lambda} G^\lambda(y,\eta) Q_\lambda^+(\eta) d\eta, \tag{2.30}$$

for $y \in \Sigma_\lambda$ where $G^\lambda(y,\eta)$ is the Green function (2.7). Clearly, $h_\lambda(y) \in C^1(\overline{\Sigma}_\lambda)$. By (2.28), we have $\Delta h_\lambda(y) \ge Q_\lambda(y)$ for $y \in \Sigma_\lambda$. In order to apply Lemma 2.1, it suffices for us to prove

$$h_\lambda(y) > 0 \quad \text{in } \Sigma_\lambda \tag{2.31}$$

for $\lambda_0 \le \lambda \le 1/2$, and

$$h_{\lambda_0}(y) < w_{\lambda_0}(y) \ \text{ for } y \in \Sigma_{\lambda_0}. \tag{2.32}$$

To estimate $h_\lambda(y)$, we follow the computations in [CLn1]. For the details of computations, we refer the readers to [CLn1]. Note that by (2.15), we have

$$K_\delta(y) - K_\delta(y^\lambda) \ge M_i^{-\frac{2}{n-2}} |x_i^*|^{l-1}((y_1 - \lambda) - o(1)|y^\lambda|) \tag{2.33}$$

for $|y| \le R = \frac{\epsilon_0}{\delta}$, where $\delta << \varepsilon_0 << 1$ and $o(1)$ denotes a positive constant which tends to 0 as $i \longrightarrow +\infty$.

For $y \notin B_i$, we have

$$|K_\delta(y) - K_\delta(y^\lambda)| \le c_1\, M_i^{-\frac{2}{n-2}} |x_i^*|^{l-1}(|I_\delta(y)| + |I_\delta(y^\lambda)|). \tag{2.34}$$

Thus, by applying the same computations as in Step 2 of the proof of Theorem 1.1 in [CLn1], we have for small δ,

$$h_\lambda(y) \geq c_2 \ (\log R) M_i^{-\frac{2}{n-2}} |x_i^*|^{l-1} (y_1 - \lambda)(1+|y|)^{-n} \tag{2.35}$$

for $y \in \Sigma_\lambda$ and $\lambda_0 \leq \lambda \leq \frac{1}{2}$. By the same computation, we have the upper bound also, i.e.,

$$h_\lambda(y) \leq c_3 \ (\delta) M_i^{-\frac{2}{n-2}} |x_i^*|^{l-1} (y_1 - \lambda)(1+|y|)^{-n} \tag{2.36}$$

where $c_3(\delta)$ is a positive constant independent of i. Hence, if δ is chosen sufficiently small such that both (2.20) and (2.35) hold, then for large i, we have $h_\lambda(y) > 0$ for $y \in \Sigma_\lambda$ and $\lambda_0 \leq \lambda \leq 1/2$, and $h_{\lambda_0}(y) \leq w_{\lambda_0}(y)$ for $y \in \Sigma_{\lambda_0}$. Thus, conditions (2.4) through (2.6) are proved. Applying Lemma 2.1, we have $w_\lambda(y) > 0$ for $y \in \Sigma_\lambda$ and $\lambda \leq 1/2$, which yields a contradiction to the fact that $v_i^\delta(y)$ has a local maximum at y_i which is contained in the strip $\{y \mid -1/2 \leq y_1 \leq 1/2\}$. Therefore, the proof of (1.8) is finished. □

3. Lower bound and compactness theorem.

Proof of Theorem 1.1 The lower bound. To prove the lower bound (1.9), we note that (1.1) can be written as $\Delta u(x) + c(x)u = 0$, where $c(x) = K(x)u^{\frac{4}{n-2}} = O(|x|^{-2})$ by (1.8). Thus, the Harnack inequality and the gradient estimates imply that there exists a constant c such that

$$\max_{|x|=r} u(x) \leq c \min_{|x|=r} u(x) \tag{3.1}$$

and

$$|\nabla u(x)| \leq c|x|^{-1}u(x) \tag{3.2}$$

hold for large $|x|$. By the Pohozaev identity, we have for $r > s$,

$$\int_{B_r \setminus B_s} x \cdot \nabla K(x) u^{\frac{2n}{n-2}}(x) dx = P(r;u) - P(s;u), \tag{3.3}$$

where

$$P(r;u) = \int_{|x|=r} \Big(\frac{n-2}{2} u(x) \frac{\partial u}{\partial \nu} - \frac{|x|}{2} |\nabla u|^2 + |x| |\frac{\partial u}{\partial \nu}|^2 + \frac{n-2}{2n} K(x)|x| u^{\frac{2n}{n-2}}\Big) d\sigma. \tag{3.4}$$

By (1.8), we have $(x \cdot \nabla K(x)) u^{\frac{2n}{n-2}}(x) \in L^1(\mathbf{R}^n)$. Hence $P = \lim_{r \longrightarrow +\infty} P(r;u)$ always exists and by the assumption on K, we have

$$P \leq P(r;u) \tag{3.5}$$

for large r. Let $w(t) = \bar{u}(r) r^{\frac{n-2}{2}}$ with $t = \log r$, where $\bar{u}(r) = \fint_{|x|=r} u d\sigma$ is the average of u over the sphere $|x| = r$. Assume that (1.9) fails, i.e., $\underline{\lim}_{t \longrightarrow +\infty} w(t) = 0$. Since $g = u^{\frac{4}{n-2}} |dx|^2$ can not be realized as a smooth metric on S^n, we have

$\overline{\lim}_{t\longrightarrow+\infty}w(t) > 0$. Thus, there exists a sequence of $t_i \longrightarrow +\infty$ such that $w'(t_i) = 0$ and $\lim_{i\longrightarrow+\infty} w(t_i) = 0$. By using (3.1) and (3.2), we have

$$\begin{aligned} P(r_i; u) &= \sigma_{n-1}\{\frac{1}{2}w'^2(t_i) - \frac{1}{2}(\frac{n-2}{2})^2 w^2(t_i) + \frac{n-2}{2n}\bar{K}(t_i)w^{\frac{2n}{n-2}}(t_i) \\ &\quad +[w'(t_i)^2 + w^2(t_i)]o(1)\} \\ &= -(1+o(1))\frac{\sigma_{n-1}}{2}(\frac{n-2}{2})^2 w^2(t_i) < 0, \end{aligned}$$

where $t_i = \log r_i$, $\bar{K}(t) = ⨍_{\partial B_r} K$ and σ_n is the volume of S^{n-1}. But by (3.5), we have

$$0 = \lim_{i\longrightarrow+\infty} P(r_i; u) = P \leq P(r_i; u) < 0$$

which obviously yields a contradiction. Therefore (1.9) is proved. □

Proof of Corollary 1.3. Suppose that u is a solution of slow decay. By (3.3), we have

$$\int_{\mathbf{R}^n} (x \cdot \nabla K(x))u^{\frac{2n}{n-2}}(x)dx = \lim_{r\longrightarrow+\infty} P(r; u). \tag{3.6}$$

For any sequence $r_i \longrightarrow +\infty$, we let $u_i(x) = r_i^{\frac{n-2}{2}} u(r_i x)$. Then u_i satisfies

$$\Delta u_i + K(r_i x)u_i^{\frac{n+2}{n-2}} = 0 \text{ in } \mathbf{R}^n.$$

By (1.8) and (1.9), we have

$$c_1|x|^{-\frac{n-2}{2}} \leq u_i(x) \leq c_2|x|^{-\frac{n-2}{2}} \tag{3.7}$$

for $x \in \mathbf{R}^n\backslash\{0\}$. By elliptic estimates, there exists a subsequence of u_i (still denoted by u_i) such that u_i converges in $C^2_{loc}(\mathbf{R}^n\backslash\{0\})$ to v, where v is a singular solution of

$$\Delta v + K(\infty)v^{\frac{n+2}{n-2}} = 0 \text{ in } \mathbf{R}^n\backslash\{0\}.$$

By a simple scaling, we hve

$$\lim_{i\longrightarrow+\infty} P(r_i; u_i) = P(1; v).$$

By the result of Caffarelli-Gidas-Spruck, $v(x)$ is radially symmetric with respect to the origin. Thus, $P(r; v)$ is always a negative constant independent of r. Hence, the proof of Corollary 1.3 is finished. □

Proof of Theorem 1.4. Let R_0 be large such that for any solution u of (1.1), one has

$$u(x) \leq c_0|x|^{-\frac{n-2}{2}}$$

for $|x| \geq R_0$. Assume also that all critical points of K are contained in $\{x \mid |x| < R_0\}$. Now suppose that there exists a sequence of solutions u_i of (1.1) such that $M_i = \max_{|x|\leq R_0} u_i \longrightarrow +\infty$ and

$$\int_{\mathbf{R}^n} (x \cdot \nabla K(x))u_i^{\frac{2n}{n-2}}(x)dx \leq -\varepsilon_0.$$

Let $m_i = \inf_{|x|\le R_0} u_i$. Then by Theorem 1.3 of [CLn2] (also see Lemma 3.3 in [CLn2]), we have $m_i \longrightarrow 0$. Moreover, u_i satisfies

$$u_i(x) \le c_1|x - P_i|^{-\frac{n-2}{2}}$$

for $|x - P_j| \le \delta_0$ with some small δ_0. By the Harnack inequality, we have

$$u_i(x) \le c_2\ m_i$$

for $|x| \le R_0$ and $|x - P_j| \ge \delta_0$ with $1 \le j \le N$, where $c_2 = c_2(R_0, \delta_0)$ is independent of i. Let $\beta_0 = \min_{1\le j\le N}\{\beta_j\} > 1$. Then

$$\begin{aligned}\varepsilon_0 &\le |\int_{\mathbf{R}^n}(x\cdot\nabla K(x))u_i^{\frac{2n}{n-2}}(x)dx| \\ &\le c_3[\int_{|x|\ge R_0}|x|^{-n-l}dx + \sum_{j=1}^N\int_{B(P_j,\delta_0)}|x-P_j|^{\beta_j-1+n}dx + m_i^{\frac{2n}{n-2}}R_0^n] \\ &\le c_4(R_0^{-l} + \delta_0^{\beta_0-1} + (c_2m_i)^{\frac{2n}{n-2}}R_0^n),\end{aligned}$$

which yields a contradiction when $i \longrightarrow +\infty$ if both R_0^{-1} and δ_0 are chosen sufficiently small. □

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