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GLOBAL EXISTENCE OF SOLUTIONS OF THE YANG–MILLS EQUATIONS ON GLOBALLY HYPERBOLIC FOUR DIMENSIONAL LORENTZIAN MANIFOLDS*

PIOTR T. CHRUŚCIEL† AND JALAL SHATAH‡

Abstract. We prove global solvability of the Cauchy problem for the Yang–Mills equations on smooth globally hyperbolic four dimensional Lorentzian manifolds.

1. Introduction. In a classical paper [9], Eardley and Moncrief have shown that solutions of the Yang–Mills equations on Minkowski space–time (with a compact gauge group) do not develop singularities in finite time, provided the initial data are sufficiently regular. This result has been generalized[1] to the Einstein cylinder $\mathbb{R} \times \mathbb{S}^3$ in [6, 8, 4], and to anti-de-Sitter space–time in [1], making use of the conformal invariance of the Yang–Mills equations. It is of interest to enquire whether or not this result remains true for general globally hyperbolic Lorentzian manifolds; such a result in dimensions $1+1$ and $2+1$ has been established, for quite a large class of manifolds, via direct energy methods, in [3]. It is the purpose of this paper to show that global existence holds true in $3+1$ dimensions: we show global solvability of the Cauchy problem for the Yang–Mills equations on any globally hyperbolic Lorentzian manifold[2] (M, g).

It is convenient to start with a description of the geometric context. Recall that a Lorentzian manifold is globally hyperbolic if it admits a Cauchy surface Σ, i.e. a hypersurface $\Sigma \subset M$ that is intersected precisely once by every inextendible causal curve. For such manifolds there exists a smooth time function t on M such that $\Sigma = \{t = 0\}$, with each level set Σ_t of t being a Cauchy surface [11, 16]. Moreover, flowing along the gradient of t one obtains a diffeomorphism between M and $\mathbb{R} \times \Sigma$.

Let G be any connected Lie group with Lie algebra $\mathfrak{g}$. Throughout we shall assume that 1) $\mathfrak{g}$ is compact, that is, $\mathfrak{g}$ admits an Ad–invariant positive scalar product $k(\cdot, \cdot)$ (that will be the case if, *e.g.*, G is compact). 2) $\mathfrak{g}$ has a faithful representation as a subalgebra of the algebra of matrices over some finite dimensional vector space V, so that the bracket operation corresponds to the commutator of matrices, with the adjoint representation of G acting on $\mathfrak{g}$ as an appropriate product of matrices. Let P be a G–principal bundle over M and let P^{Σ} be the pull–back of P to a Cauchy surface Σ Choose any smooth connection on P, let X denote the horizontal lift of the gradient of t to P, we can identify P with $\mathbb{R} \times P^{\Sigma}$ by flowing along the integral curves of X. This leads to the commutative diagram:

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†Département de Mathématiques, Faculté des Sciences, Parc de Grandmont, F37200 Tours, France (Chrusciel@Univ-Tours.Fr). Supported in part by a Polish Research Council grant number 2 P301 050 07.
‡Courant Institute, New York University, 251 Mercer Street, New York NY 10012, USA (shatah@cims.nyu.edu). Supported in part by N.S.F. DMS 9401558.

[1] Some further improvements of the Eardley–Moncrief theorem can be found in [12, 14].
[2] We use the signature $-+++$, and we assume that both the manifold and the metric are smooth. M is moreover assumed to be paracompact, Hausdorff, and time oriented.

$$\begin{array}{ccc} P & \longleftrightarrow & \mathbb{R} \times P^\Sigma \\ \downarrow & & \downarrow \\ M & \longleftrightarrow & \mathbb{R} \times \Sigma \end{array}$$

Given a trivialisation of P over a coordinate patch $\mathcal{U}$ of M, a connection A on P can be expressed as a $\mathfrak{g}$-valued one form on $\mathcal{U}$,

$$A = A_\mu(x)\, dx^\mu, \qquad A_\mu(x) \in \mathfrak{g}\,,$$

and the curvature F as the $\mathfrak{g}$-valued two form given by

$$F_{\mu\nu} = \partial_\mu A_v - \partial_\nu A_\mu + [A_\mu, A_\nu]\,.$$

A connection A is a Yang-Mills connection if it satisfies

$$\mathcal{D}_\mu F^{\mu\nu} = F^{\mu\nu}{}_{;\mu} + [A_\mu, F^{\mu\nu}] = 0\,. \tag{1.1}$$

Here and everywhere the symbol $\mathcal{D}$ denotes a space–time and gauge covariant derivative, while ∇_μ or the addition of a sub- or superscript $";\mu"$ denotes a space–time covariant derivative.

The Cauchy problem for the Yang–Mills equations on M consists in prescribing on P^Σ a connection field A^Σ together with an Ad–equivariant $\mathfrak{g}$–valued one form E. (Loosely speaking, E represents the time–derivative of the connection form on Σ.) Moreover A^Σ and E are assumed to satisfy the Yang–Mills constraint[3] equation,

$$\mathcal{D}_i E^i = 0. \tag{1.2}$$

We shall say that a connection field is of differentiability class $H^k_{\rm loc}(\Sigma)$ if there exists a covering of Σ by coordinate balls $\mathcal{U}_\alpha$ together with trivialisations of $P^\Sigma|_{U_\alpha}$ such that in local coordinates the components A^Σ_i of A^Σ are in $H^k_{\rm loc}(\mathcal{U}_\alpha)$. A similar definition is used for E. A connection A on P will be called a Yang–Mills development of $(A^\Sigma$,$E)$ if A is a solution of the Yang–Mills equations (1.1), such that the pull–back of A to P^Σ coincides with A^Σ, and such that the pull–back of $n \vee F$ to P^Σ coincides with E. (We also impose some differentiability conditions on A, which are made precise in Section 4.) Here n is the horizontal lift to P of the field of future directed unit normals to Σ, and $\vee$ denotes contraction of a vector with a form. In this paper we prove the following:

THEOREM 1.1. *Let P be a principal bundle over a smooth globally hyperbolic four dimensional Lorentzian manifold (M, g) with structure group G, whose Lie algebra $\mathfrak{g}$ is compact. Let Σ be a Cauchy surface in M and let P^Σ be the pull-back of P to Σ. Let $A^\Sigma \in H^{k+1}_{\rm loc}(\Sigma)$ be a connection on P^Σ and let $E \in H^k_{\rm loc}(\Sigma)$ be an Ad-equivariant $\mathfrak{g}$-valued one form on P^Σ, $k \geq 2$, satisfying the constraint (1.2). Then there exists a unique Yang–Mills development A of $(A^\Sigma$,$E)$ on P. (If the initial data are smooth then the Yang–Mills development is smooth.)*

To establish theorem 1.1 we adapt several of the Eardley–Moncrief ideas to the curved space–time setting. Let us highlight some elements of our proof:

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[3]Some results concerning existence of solutions of the constraint equation (1.2) can be found in [4, 2].

- One of the key ingredients of the Eardley–Moncrief proof is the use of the Cronström gauge. We also use this gauge, and this carries over to curved space–time without introducing any difficulties.
- Eardley and Moncrief's strategy is to establish an L^∞ *a priori* estimate on the Yang–Mills curvature field F, using the spherical means representation of solutions of the wave equation. We follow a similar strategy, replacing the spherical means integral by a representation formula of Friedlander [10]. There are two differences between those formulae and the Minkowski space–time ones: First, there are more terms to deal with, because of a solid–cone integral in a generic curved space–time which is absent in Minkowski space–time. This introduces some tediousness to the proof but does not lead to any difficulties because the solid–cone integral terms that arise can be estimated in a rather straightforward way. Next, the formulae of Friedlander are valid only in causal domains (as defined in [10]), while the spherical means are valid in the whole of Minkowski space–time. To handle that issue, roughly speaking we first obtain an $\|F\|_{H^2_{\mathrm{loc}}}$ *a priori* estimate in causal domains, and use a globalisation argument essentially due to Choquet–Bruhat and Geroch [5] to obtain global existence on any globally hyperbolic manifold. This is here that the condition of global hyperbolicity of the space–time enters in our argument.
- It turns out that the arguments used by Eardley–Moncrief to control all the terms that arise in the light cone integral carry over to the curved space–time case, except for the term $[A^\alpha{}_{;\alpha}, F_{\mu\nu}]$. To take care of that term we have to use a completely different argument, requiring simultaneous control of $\|F\|_{L^\infty_{\mathrm{loc}}}$ and of a H^2_{loc} semi–norm of F. The observation that this term can be handled in this way is the main new idea of this paper.
- We note, finally, that a slight difficulty is introduced by the constraint part of the Yang–Mills equations. While there are several ways to handle this problem, we use here the observation of Kapitanskii and Goganov [12] that one can obtain Eardley–Moncrief type estimates for appropriately modified Yang–Mills equations, in which the constraint part of the equations is only partially satisfied.

This paper is organized as follows: In Section 2 we review Friedlander's representation formulae. In Section 3 we derive various integral *a priori* estimates. including the mixed $\|F\|_{L^\infty_{\mathrm{loc}}}$ and H^2_{loc} estimates. The proof of Theorem 1.1 is given in Section 4.

NOTATIONS. Given a time oriented four dimensional Lorentzian manifold (M, g), we denote local coordinates by $x = (x^0, x^1, x^2, x^3) = (x^0, \mathrm{x})$, $r = |\mathrm{x}|$, and the volume element by $\mu = \sqrt{|g|}dx$. Given a hypersurface $S = \{x;\ \ f(x) = 0\}$, μ_f will denote the Leray form associated with S, i.e. $\mu = df \wedge \mu_f$. We use the following notations for functions and tensors:

$X(f) = X^\alpha \partial_\alpha f$ the derivative of f in the direction of a vector X,
$\nabla f = \{g^{\mu\nu}\partial_\nu f\}$ the gradient of a function f,
$\nabla_\alpha T^r_s = T^r_{s;\alpha}$ the covariant derivative of a tensor field T of type $\{r, s\}$,
$\nabla_X T^r_s = X^\alpha T^r_{s;\alpha}$ the derivative of T in the direction of a vector X,
$\langle T, S\rangle =$ the inner product of T and S with respect to the metric g.

We will assume that M is globally hyperbolic, which implies that M is foliated by space-like hypersurfaces Σ_t, which are the level surfaces of a smooth time function t, and denote by $\underline{t} = \frac{dt}{\sqrt{-\langle dt, dt\rangle}}$. On TM, the tangent space of M, we introduce basis vectors $\{\hat{t}, \hat{z}_1, \hat{z}_2, \hat{z}_3\}$ where $\hat{t}$ is a unit vector in the direction of $-\nabla t$, and

$$\langle \hat{t}, \hat{t}\rangle = -1, \qquad \langle \hat{t}, \hat{z}_i\rangle = 0, \qquad \langle \hat{z}_i, \hat{z}_j\rangle = \delta_{ij}\,.$$

For any vector field X, let

$$|X|^2 = |\langle \hat{t}, X\rangle|^2 + \sum |\langle \hat{z}_i, X\rangle|^2\,, \tag{1.3}$$

and for a tensor T, $|T|$ is defined in a similar fashion.

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2. The wave equation on M. In this section we will state some basic results about the wave equation on a four dimensional Lorentzian manifold (M, g). We review the explicit formula for the fundamental solution of the wave equation on M of [10]. All of our statements and results are restricted to a geodesically convex neighbourhood $\Omega \subset M$. The derivation of the fundamental solution formula can be found in Friedlander [10].

On a geodesically convex neighbourhood Ω we denote by
$\gamma(p,q)$ = square of the geodesic distance between $p, q \in \Omega$,
$C^-(p)$ = past null semicone with vertex $p \in \Omega$,
$J^-(p)$ = the closure of the causal past of p,
$D(t_0) = \{q \in J^-(p);\ \ t(q) = t_0 < t(p)\}$ a cross section of $J^-(p)$,
$K(t_1, t_2) = \{q \in J^-(p);\ \ t_1 \le t(q) \le t_2 < t(p)\}$ a truncated cone,
$M(t_1, t_2) = \{q \in C^-(p);\ \ t_1 \le t(q) \le t_2 < t(p)\}$ the mantle of the truncated cone $K(t_1, t_2)$,
with similar definitions for $C^+(p), J^+(p)$.

DEFINITION 2.1. *Given a point $p \in M$ define null basis vectors on the tangent space T_pM, as basis vectors $\{\hat{\ell}, \hat{m}, \hat{e}_1, \hat{e}_2\}$, that satisfy*

$$\begin{aligned} &\langle \hat{\ell}, \hat{\ell}\rangle = \langle \hat{m}, \hat{m}\rangle = \langle \hat{\ell}, \hat{e}_i\rangle = \langle \hat{m}, \hat{e}_i\rangle = 0\ , \\ &\langle \hat{\ell}, \hat{m}\rangle = 2 \quad \text{and} \quad \langle \hat{e}_i, \hat{e}_j\rangle = \delta_{ij}\,. \end{aligned} \tag{2.1}$$

The forms dual to $\{\hat{\ell}, \hat{m}, \hat{e}_1, \hat{e}_2\}$ are denoted by $\{\underline{\ell}, \underline{m}, \underline{e_1}, \underline{e_2}\}$. We will always work with the following null basis on $C^-(p) \setminus p$. For any $q \in C^-(p) \setminus p$, let $\hat{n}$ denote the unit vector orthogonal to $\hat{t}$ such that

$$\nabla\gamma = a\hat{t} + b\hat{n}\,.$$

Here the gradient of γ is taken with respect to the q variable, p being held fixed. Define $\hat{\ell} = \hat{t} - \hat{n}$, and $\hat{m} = -(\hat{t} + \hat{n})$; then for any orthonormal basis $\{\hat{n}, \hat{e}_1, \hat{e}_2\}$ of the tangent space $T_q\Sigma_{t(q)}$, we obtain a null basis $\{\hat{\ell}, \hat{m}, \hat{e}_1, \hat{e}_2\}$.

The representation formula. The fundamental solution of the wave equation $G^-(p,q)$,

$$\Box G^- := -\nabla^\alpha \partial_\alpha\, G^- = \delta(p)\,,$$

whose support is contained in $J^-(p)$, can be computed inside a causal domain Ω_0:

$$G^-(p,q) = U(p,q)\delta_-(\gamma) + V^-(p,q)\ ,$$

where, in a coordinate chart ($\pi(p) = x$ and $\pi(q) = y$),

$$U(x,y) = \frac{|\det \gamma_{x^i y^j}|^{\frac{1}{2}}}{8\pi|\det(g(x))\det(g(y))|^{\frac{1}{4}}},$$
$$\delta_-(\gamma) = \lim_{\epsilon\to 0} \delta_-(\gamma - \epsilon).$$

Here $\delta_-(\gamma-\epsilon)$ is the Dirac measure supported on the lower branch of the hyperboloid $\gamma = \epsilon$, and, following [10, p. 146], we say that a connected open set Ω is a causal domain when it is globally hyperbolic and contained in a geodesically convex set. V^- is a solution to the characteristic problem

$$\Box V^- = 0 \qquad \text{in} \quad J^-(p),$$
$$V^- \big|_{C^-(p)} = V_0\big|_{C^-(p)} := -\frac{1}{4}U(x,y)\int_0^1 \frac{\Box U}{U}\, ds\,\big|_{C^-(p)},$$

where the integral is over a geodesic $c(s)$ joining $p = c(0)$ to $q = c(1)$. In a small causal domain Ω_0 the above problem has a solution $V^-(p,q)$ such that $\operatorname{supp} V^- \subset \Delta^-$, $V^- \in C^\infty(\Delta^-)$, where

$$\Delta^- = \{(p,q): \quad q \in J^-(p)\}$$

is the closure of one component of $\{(p,q);\ \gamma(p,q) > 0\}$. The function $q \to V^-(p,q)$ is supported in $J^-(p)$ and is C^∞ at the boundary of its support.

Using this fundamental solution we can find a representation for solutions of the Cauchy problem,

$$\begin{aligned} \Box u &= f, \\ u|_D &= u_0, \\ \nabla u|_D &= u_1, \end{aligned} \tag{2.2}$$

where $D \subset \Sigma_a \cap \Omega_0$ is a space-like hypersurface, and $X(u_0) = \langle X, u_1\rangle$ for any vector field X tangent to D.

For $p \in \Omega_0$ let $K(p)$ be the cone of vertex p and whose base lies on D, that is, $K(p) = J^-(p) \cap J^+(D)$, where $J^+(D)$ denotes the set of points which lie to the causal future of D. Let $C(p) = C^-(p) \cap K(p)$ be the mantle of this cone, and let $S = \Sigma_a \cap C^-(p)$ be the topological boundary of $C(p)$. Then u is given by

$$u(p) = u_h(p) + \int_{C(p)} U(p,q)f(q)\mu_\gamma + \int_{K(p)} V^-(p,q)f(q)\mu, \tag{2.3}$$

where $\mu_\gamma =$ the Leray form associated with $C(p)$, *i.e.*, $\mu = d\gamma \wedge \mu_\gamma$, and $u_h(p)$ is the solution of the homogenous wave equation with the same initial data. To express $u_h(p)$ explicitly in terms of the initial data we need to define the following. Let $C^+(S)$ be the null hypersurface whose normal on S is given by $\hat{m}$, and S_ε the intersection of $C^+(S)$ and the set $\{q; \gamma(p,q) = \varepsilon\}$. Denote by μ_S and μ_{S_ε} the volume elements induced by the the metric g on the surface S and S_ε respectively. On S define the functions ρ and θ

$$\rho = \langle \hat{m}, -\nabla\gamma\rangle/2,$$
$$\theta\mu_S = \rho\,\frac{d}{d\varepsilon}\mu_{S_\varepsilon}\bigg|_{\varepsilon=0},$$

then

$$u_h(p) = \int_S [\frac{1}{\rho}U[\langle \hat{m}, \nabla u\rangle + \theta u)] + V^- u]\, dS + \int_{D(a)} (V^- t^a \nabla_a u - u t^a \nabla_a V^-)\mu_a\, .$$

If we parametrize the cone $C(p)$ by $(\zeta, \omega) \in \mathbb{R}^+ \times \mathbb{S}^2$, the solution u can be expressed in the following way. Let $\hat{\xi}$ be a unit vector orthogonal to $\hat{t}(p)$, parameterized by $\omega \in \mathbb{S}^2$. For any past directed null vector $\hat{v}$ of the form $\hat{v} = -\hat{t} + \hat{\xi}$, and $\zeta \in \mathbb{R}^+$ let $q = \exp_p(\zeta\hat{v})$. Since we are in a geodesically convex set, the $\exp_p$ map is a diffeomorphism from $C(p) \setminus p$ into $(0, \zeta_0(\omega)] \times \mathbb{S}^2$, where $\exp_p(\zeta_0(\omega)\hat{v}) \in S$. In these coordinates

$$\begin{aligned} U\mu_\gamma &= \frac{\kappa(\zeta,\omega)}{\zeta}\,\mu_\ell = \tilde{\kappa}(\zeta,\omega)\zeta\, d\zeta \wedge d\omega\, ,\\ \mu_S &= \chi(\zeta_0(\omega),\omega)\zeta_0(\omega)^2\, d\omega\, ,\\ \rho &= \tilde{\kappa}(\zeta,\omega)\zeta\, ,\\ \theta &= \frac{\tilde{\theta}(\zeta,\omega)}{\zeta}\, , \end{aligned}$$

where κ, $\tilde{\kappa}$, χ, $\tilde{\chi}$, and $\tilde{\theta}$ are smooth positive functions, and where μ_ℓ denotes the Leray form on $C^-(p) \setminus p$. Equation (2.3) can be written as

$$\begin{aligned} u(p) &= u_h(p) + \int_{C(p)} f(\zeta,\omega)\frac{\kappa(\zeta,\omega)}{\zeta}\,\mu_\ell + \int_{K(p)} V^- f\mu\, ,\\ u_h(p) &= \int_S \left(\frac{1}{\zeta}\tilde{U}[\langle \hat{m}, \nabla u\rangle + \frac{1}{\zeta}\tilde{\theta}u] + V^- u\right)\mu_S + \int_{D(a)} (V^- t^a\nabla_a u - u t^a \nabla_a V^-)\mu_a\, , \end{aligned}$$

(2.4)
where $\tilde{U}$ is a smooth function.

Energy estimates. Given a space-like hypersurface $D \subset \Sigma_a$, local energy estimates for solutions of the Cauchy problem can be derived in a region

$$\Delta(a,b) = \{q \in \text{Domain of dependence of}\, D;\ a \le t(q) \le b\}\, .$$

For any vector field V, using the notation introduced earlier, we define energy norms on $D(t)$ and on $M(t_1, t_2)$ as

$$\begin{aligned} ||V(t)||^2_{H^1(D(t))} &= ||\nabla_{\hat{t}}V||^2_{L^2(D(t))} + \sum ||\nabla_{\hat{z}_i}V||^2_{L^2(D(t))} + ||V||_{L^2(D(t))}\, ,\\ ||V_{tan}||^2_{L^2(M(t_1,t_2))} &= ||\langle \hat{\ell}, V\rangle||^2_{L^2(M(t_1,t_2))} + \sum ||\langle \hat{e}_i, V\rangle||^2_{L^2(M(t_1,t_2))}\, ,\\ ||\nabla_{tan}V||^2_{L^2(M(t_1,t_2))} &= ||\nabla_{\hat{\ell}}V||^2_{L^2(M(t_1,t_2))} + \sum ||\nabla_{\hat{e}_i}V||^2_{L^2(M(t_1,t_2))}\, ,\\ ||V||_{L^{\infty,2}(K(t_1,t_2))} &= \sup_{t\in[t_1,t_2]} ||V(t)||_{L^2(D(t))}\, , \end{aligned}$$

where L^p and H^k denote the standard Lebesgue and Sobolev spaces on $D(t)$ and $M(t_1,t_2)$, with measure μ_t and μ_ℓ respectively. In a similar manner we define $||T||_{H^1}$, $||T_{tan}||_{L^2}$, and $||\nabla_{tan}T||_{L^2}$, for any tensor T.

To derive energy estimates for equation (2.2), multiply the equation by $X(u)$ to obtain

$$\nabla^\alpha[X^\beta(u_{;\alpha}u_{;\beta} - \frac{1}{2}g_{\alpha\beta}u_{;\nu}u^{;\nu})] - X^{\alpha;\beta}[(u_{;\alpha}u_{;\beta} - \frac{1}{2}g_{\alpha\beta}u_{;\nu}u^{;\nu})] = fX(u)\, .$$

The above equation can be expressed as

$$\nabla^\alpha(X^\beta T_{\alpha\beta}) - X^{\alpha;\beta}T_{\alpha\beta} = fX(u)\,, \tag{2.5}$$

where the energy–momentum tensor $T_{\alpha\beta}$ is defined as

$$T_{\alpha\beta} = \partial_\alpha u \partial_\beta u - \frac{1}{2} g_{\alpha\beta}(g^{\mu\nu}\partial_\mu u \partial_\nu u)\ . \tag{2.6}$$

For any $p \in \Delta(a,b)$ integrate equation (2.5) on the truncated cone $K(a,t) \subset K(p)$ to obtain

$$\begin{aligned} &\int_{D(t)} T(X,\hat{t})\mu_t + \int_{M(a,t)} T(X,\hat{\ell})\mu_\ell \\ &= \int_{D(a)} T(X,\hat{t})\mu_a + \int_{K(a,t)} (X^{\alpha;\beta}T_{\alpha\beta} + fX(u))\mu\,. \end{aligned} \tag{2.7}$$

Substituting $X = \hat{t}$, the terms in equation (2.8) can be written as

$$\begin{aligned} E(u,D(t)) &:= +\int_{D(t)} t^\alpha t^\beta T_{\alpha\beta}\mu_t = \int_{D(t)} \left[\frac{1}{2}|\hat{t}(u)|^2 + \frac{1}{2}\sum|\hat{z}_i(u)|^2\right]\mu_t\,, \\ \text{flux }(u,M(a,t)) &:= \int_{M(a,t)} \ell^\alpha t^\beta T_{\alpha\beta}\mu_\ell = \frac{1}{2}\int_{M(a,t)} \ell^\alpha\left(\ell^\beta + m^\beta\right)T_{\alpha\beta}\mu_\ell\,, \\ &= \frac{1}{2}\int_{M(a,t)} \left[|\hat{\ell}(u)|^2 + \sum|\hat{e}_i(u)|^2\right]\mu_\ell\,, \end{aligned}$$

where we used the decomposition of ∇u

$$\begin{aligned} \nabla u &= \hat{t}(u)\hat{t} + \sum \hat{z}_i(u)\hat{z}_i \qquad \text{on } D(t)\,, \\ \nabla u &= \frac{1}{2}\hat{m}(u)\hat{\ell} + \frac{1}{2}\hat{\ell}(u)\hat{m} + \sum \hat{e}_i(u)\hat{e}_i \quad \text{on } M(a,t)\,. \end{aligned}$$

Thus the conservation of energy can be writtten as

$$E(u,D(t)) + \text{flux }(u,M(a,t)) = E(u,D(a)) + \int_{K(a,t)} \left[t^{\alpha;\beta}T_{\alpha\beta} + f\hat{t}(u)\right]\mu\,. \tag{2.8}$$

We have

$$\int_{K(a,t)} t^{\alpha;\beta}T_{\alpha\beta}\ \mu \le C\int_a^t E(u,D(s))ds\ ,$$

so that (2.8) together with Gronwall's lemma implies

$$||\nabla u(t)||_{L^2(D(t))} + ||\nabla_{tan}u||_{L^2(M(a,t))} \le C||\nabla u||_{L^2(D(a))} + \int_a^t ||f(s)||_{L^2(D(s))}ds\,.$$

Here $||\nabla u(t)||_{L^2(D(t))}$ is defined as $\sqrt{E(u,D(t))}$. On $K(p) = \{q \in J^-(p);\ a \le t(q) \le t(p)\}$ the above equation can be written as

$$||\nabla u||_{L^{\infty,2}(K(p))} + ||\nabla_{tan}u||_{L^2(C(p))} \le C\left(||\nabla u(a)||_{L^2(D)} + |t(p)-a|||f||_{L^{\infty,2}(K(p))}\right)\,. \tag{2.9}$$

We can derive similar estimates on $\Delta(a,b)$

$$||\nabla u(t)||_{L^2(\Delta(t))} + ||\nabla_{tan} u||_{L^2(\Lambda(a,t))} \le C||\nabla u(a)||_{L^2(\Delta(a))} + \int_a^t ||f(s)||_{L^2(\Delta(s))} ds\,,$$

where

$$\Delta(\tau) = \{q \in \Delta(a,b);\ t(q) = \tau\}\,,$$
$$\Lambda(a,\tau) = \{q \in \text{null boundary of}\,\Delta(a,b);\ a \le t(q) \le \tau\}\,.$$

The estimate on $\Delta(a,b)$ combined with the estimate on $C(p)$ implies

$$||\nabla u||_{L^{\infty,2}(\Delta(a,b))} + \sup_{p\in\Delta(a,b)} ||\nabla_{tan} u||_{L^2(C(p))} \le C\left(||\nabla u(a)||_{L^2(D)} + |b-a|||f||_{L^{\infty,2}(\Delta(a,b))}\right). \tag{2.10}$$

where $||\nabla u||_{L^{\infty,2}(\Delta(a,b))} := \sup_{a\le t\le b} ||\nabla u(t)||_{L^2(\Delta(t))}$.

3. The Yang-Mills Equations. Recall that a connection A on P, a G–principal bundle over M, can be expressed in a coordinate patch as

$$A = A_\mu(x)\, dx^\mu, \qquad A_\mu(x) \in \mathfrak{g}\ ,$$

and the curvature F as

$$F_{\mu\nu} = \partial_\mu A_v - \partial_\nu A_\mu + [A_\mu, A_\nu]\,. \tag{3.1}$$

We shall assume for simplicity that the elements of $\mathfrak{g}$ are matrices, and that the adjoint action Ad of G on $\mathfrak{g}$ corresponds to an appropriate multiplication by matrices. Then the gauge transformations are given by a smooth map $\mathcal{G} \in G$

$$A'_\mu = \mathcal{G}^{-1} A_\mu \mathcal{G} + \mathcal{G}^{-1}\partial\mathcal{G}\,,$$

and gauge covariant derivative by $\mathcal{D}_\mu \ = \nabla_\mu \ + [A_\mu,\ ]$. This implies

$$\begin{aligned} F'_{\mu\nu} &= \mathcal{G}^{-1} F_{\mu\nu}\mathcal{G}\,,\\ \mathcal{D}'_\alpha F'_{\mu\nu} &= \mathcal{G}^{-1}\mathcal{D}_\alpha F_{\mu\nu}\mathcal{G}\,. \end{aligned} \tag{3.2}$$

The Yang-Mills equations and the Bianchi identity are given by

$$\mathcal{D}_\mu F^{\mu\nu} = F^{\mu\nu}{}_{;\mu} + [A_\mu, F^{\mu\nu}] = 0\,, \tag{3.3a}$$
$$\mathcal{D}_\alpha F_{\mu\nu} + \mathcal{D}_\mu F_{\nu\alpha} + \mathcal{D}_\nu F_{\alpha\mu} = 0\ . \tag{3.3b}$$

From this first order system, we can derive a covariant wave equation for F by differentiating (3.3b) with respect to $\mathcal{D}_\alpha$ and using (3.3a) and the commutation relation of covariant derivatives

$$\mathcal{D}_\alpha\mathcal{D}_\beta F_{\mu\nu} = \mathcal{D}_\beta\mathcal{D}_\alpha F_{\mu\nu} - R^\gamma{}_{\mu\alpha\beta}F_{\gamma\nu} - R^\gamma{}_{\nu\alpha\beta}F_{\mu\gamma} + [F_{\alpha\beta}, F_{\mu\nu}], \tag{3.4}$$

where R is the curvature tensor, to obtain

$$-\Box F_{\mu\nu} + 2[F_\mu{}^\beta, F_{\beta\nu}] - 2R_{\alpha\mu\nu\beta}F^{\alpha\beta} - R_{\mu\alpha}F^\alpha{}_\nu - R_{\nu\alpha}F_\mu{}^\alpha = 0\,, \tag{3.5}$$

where the covariant wave operator is given by

(3.6)
$$-\Box F_{\mu\nu} := \mathcal{D}^\alpha \mathcal{D}_\alpha F_{\mu\nu} = g^{\alpha\beta} F_{\mu\nu;\alpha\beta} + [A^\alpha, F_{\mu\nu}]_{;\alpha} + [A^\alpha, F_{\mu\nu;\alpha}] + [A^\alpha, [A_\alpha, F_{\mu\nu}]] .$$

In a manner similar to the wave equation, we can derive estimates for solutions of the Cauchy problem for the Yang-Mills equations, in a geodesically convex neighbourhood, with data given on a space-like hypersurface $D \subset \Sigma_a$. As before, define $\Delta(a, b)$ and $K(p)$,

$$\Delta(a, b) = \{q \in \text{Domain of dependence of} D;\ a \leq t(q) \leq b\} ,$$
$$K(p) = \{q \in J^-(p);\ a \leq t(q) \leq t(p)\} ,$$

with $\delta = b-a > 0$ small; and denote by $\Lambda(a, b)$ and $C(p)$ the null boundaries of $\Delta(a, b)$ and $K(p)$ respectively. In what follows, we derive a priori estimates for solutions of the Yang-Mills equations in the region $\Delta(a, b)$.

L^2 estimates. As for the wave equation, we can define an energy-momentum tensor for the Yang-Mills equations,

$$T_{\alpha\beta} := F_{\alpha\mu} \cdot F_\beta{}^\mu - \frac{1}{4} g_{\alpha\beta} F_{\mu\nu} \cdot F^{\mu\nu}$$

where $A \cdot B = k(A, B)$, a positive definite Ad–invariant metric on $\mathfrak{g}$. This tensor satisfies $\nabla^\beta T_{\alpha\beta} = 0$, which implies an a priori L^2 bound on F. To derive this bound integrate $t^\alpha T_{\alpha\beta}$ on a truncated cone $K(a, t) \subset K(p)$, to obtain

$$\int_{D(t)} T(\hat{t}, \hat{t})\mu_t + \int_{M(a,t)} T(\hat{t}, \hat{\ell})\mu_\ell = \int_{D(a)} T(\hat{t}, \hat{t})\mu_a + \int_{K(a,t)} \hat{t}^{\alpha;\beta} T_{\alpha\beta}\mu .$$

We note that

$$\int_{D(t)} t^\alpha t^\beta T_{\alpha\beta}\mu_t = \int_{D(t)} \left[\frac{1}{2} F_{\hat{z}_i \hat{t}} \cdot \sum F_{\hat{z}_i \hat{t}} + \frac{1}{4} \sum F_{\hat{z}_i \hat{z}_j} \cdot F_{\hat{z}_i \hat{z}_j}\right] \mu_t \approx \| F(t) \|^2_{L^2(D(t))} ,$$

$$\int_{M(a,t)} \ell^\alpha t^\beta T_{\alpha\beta}\mu_\ell = \int_{M(a,t)} \left[\frac{1}{2} \sum F_{\hat{e}_i \hat{\ell}} \cdot F_{\hat{e}_i \hat{\ell}} + \frac{1}{4} \sum F_{\hat{e}_i \hat{e}_j} \cdot F_{\hat{e}_i \hat{e}_j} + \frac{1}{8} F_{\hat{\ell}\hat{m}} \cdot F_{\hat{\ell}\hat{m}}\right] \mu_\ell$$
$$\approx ||F_{tan}||^2_{L^2(M(a,t))} .$$

It follows that we have the a priori estimate

$$||F(t)||_{L^2(D(t))} + ||F_{tan}||_{L^2(M(a,t))} \leq C||F(a)||_{L^2(D(a))} .$$

If we denote the square root of energy on D by

$$E_0 = ||F(a)||_{L^2(D)} ,$$

the above energy estimate can be written as

$$||F||_{L^{\infty,2}(K(p))} + ||F_{tan}||_{L^2(C(p))} \leq CE_0 . \tag{3.7}$$

Similarly we derive the following estimate on $\Delta(a, b)$

$$||F||_{L^{\infty,2}(\Delta(a,b))} + \sup_{p \in \Delta(a,b)} ||F_{tan}||_{L^2(C(p))} \leq CE_0 . \tag{3.8}$$

H^1 estimates. Derivative estimates for F can be derived from equation (3.5) in a manner similar to that of the wave equation: Let $\hat{t}, \hat{z}_i$ be an ON basis as described at the end of the introduction, and let $h^{\alpha\beta}$ be the Riemannian metric on M defined by $h^{\alpha\beta} = \hat{t}^\alpha \hat{t}^\beta + \sum_{i=1}^3 \hat{z}_i^\alpha \hat{z}_i^\beta$. Consider the new "energy–momentum tensor" $T^{\alpha\beta}$ defined as

$$T^{\alpha\beta} = h^{\mu\nu} h^{\rho\sigma} \{k(\mathcal{D}^\alpha F_{\mu\rho}, \mathcal{D}^\beta F_{\nu\sigma}) - \frac{1}{2} k(\mathcal{D}^\lambda F_{\mu\rho}, \mathcal{D}_\lambda F_{\nu\sigma}) g^{\alpha\beta}\} \ . \tag{3.9}$$

We have

$$\nabla_\alpha (T^{\alpha\beta} \hat{t}_\beta) = \nabla_\alpha (h^{\mu\nu} h^{\rho\sigma} \hat{t}_\beta) \{k(\mathcal{D}^\alpha F_{\mu\rho}, \mathcal{D}^\beta F_{\nu\sigma}) - \frac{1}{2} k(\mathcal{D}^\lambda F_{\mu\rho}, \mathcal{D}^\lambda F_{\nu\sigma}) g^{\alpha\beta}\} + \\ h^{\mu\nu} h^{\rho\sigma} \hat{t}_\beta \{k(\mathcal{D}_\alpha \mathcal{D}^\alpha F_{\mu\rho}, \mathcal{D}^\beta F_{\nu\sigma}) + k(\mathcal{D}^\alpha F_{\mu\rho}, (\mathcal{D}_\alpha \mathcal{D}^\beta - \mathcal{D}^\beta \mathcal{D}_\alpha) F_{\nu\sigma})\} \ .$$

Integrating this equation over $K(a,t)$ and using (3.4) and (3.5) to get rid of second derivatives of F we obtain

$$\| \mathcal{D}F(t) \|^2_{L^2(D(t))} + \| \mathcal{D}_{tan} F \|^2_{L^2(M(a,t))} \\ \leq \| \mathcal{D}F(a) \|^2_{L^2(D(a))} + C \int_{K(a,t)} |\mathcal{D}F| (|F|^2 + |F| + |\mathcal{D}F|) \, ,$$

where $\mathcal{D}F$ denotes the gauge covariant gradient of F, and

$$|\mathcal{D}_{tan} F| = |\mathcal{D}_{\hat{\ell}} F| + \sum |\mathcal{D}_{\hat{e}_i} F| \, .$$

From Gronwall's lemma it then follows

$$||\mathcal{D}F(t)||_{L^2(D(t))} + ||\mathcal{D}_{tan} F||_{L^2(M(a,t))} \\ \leq c||\mathcal{D}F(a)||_{L^2(D(a))} + C \int_a^t (||F(s)||^2_{L^4(D(s))} + ||F(s)||_{L^2(D(s))}) ds \, . \tag{3.10}$$

This implies

$$||\mathcal{D}F(t)||_{L^2(D(t))} + ||\mathcal{D}_{tan} F||_{L^2(M(a,t))} \\ \leq C \left(||\mathcal{D}F(a)||_{L^2(D(a))} + ||F(a)||_{L^2(D(a))} + |t-a| \sup_{q \in K(p)} |F(q)| \right) \, ,$$

where C depends on the energy $E_0 = ||F(a)||_{L^2(D)}$. The above equation is equivalent to

$$||DF||_{L^{\infty,2}(K(p))} + ||\mathcal{D}_{tan} F||_{L^2(C(p))} \leq C \left(||DF(a)||_{L^2(D(a))} + \delta ||F||_{L^\infty(K(p))} \right) \, . \tag{3.11}$$

Here, as before, $\delta = |b-a|$. Repeating the same argument on $\Delta(a,b)$ we obtain

$$||\mathcal{D}F(t)||_{L^2(\Delta(t))} + ||\mathcal{D}_{tan} F||_{L^2(\Lambda(a,t))} \leq C_1 ||DF(a)||_{L^2(D)} + C_2 |t-a| \sup_{q \in \Delta(a,b)} |F(q)| \, ,$$

where $\Delta(\tau) = \{q \in \Delta; \ t(q) = \tau\}$. Combining the above with equation (3.11) gives

$$||\mathcal{D}F||_{L^{\infty,2}(\Delta(a,b))} + \sup_{p \in \Delta(a,b)} ||\mathcal{D}_{tan} F||_{L^2(C(p))} \leq C \left(||\mathcal{D}F(a)||_{L^2(D)} + \delta ||F||_{L^\infty(\Delta(a,b))} \right) \, . \tag{3.12}$$

Cronström gauge estimates. To obtain bounds on higher energy norms of F, we see from equation (3.12) that we need a pointwise estimate on F. This will be done, as in [9], by using the Cronström gauge,

$$\langle d\gamma(p,q),\ A(q)\rangle = 0 \qquad A(p) = 0\,.$$

This choice of gauge has the advantage of allowing us to express A in terms of F in a simple manner. Using coordinates that are normal and Minkowskian at p the Cronström gauge is written as

$$\gamma^{;\alpha}(x)A_\alpha(x) = 2x^\alpha A_\alpha(x) = 0, \qquad A_\alpha(0) = 0.$$

From equation (3.1) we obtain

$$\begin{aligned} x^\alpha F_{\alpha\beta}(x) &= x^\alpha(\partial_\alpha A_\beta - \partial_\beta A_\alpha) \\ &= x^\alpha \partial_\alpha A_\beta + A_\beta. \end{aligned}$$

The above equation has the solution

$$A_\beta(x) = \int_0^1 x^\alpha F_{\alpha\beta}(sx)s\,ds\,. \tag{3.13}$$

Estimates of A on $C(p)$. On $C(p)$, A can be estimated in terms of F in the following manner. As before, parameterize $C(p)\backslash p$ by $(\zeta,\omega) \in \mathbb{R}^+ \times \mathbb{S}^2$ where any $q \in C^-(p)\backslash p$ can be represented by $q = \exp_p(-\zeta\hat{\ell})$. From (3.13) we obtain

$$\left|\frac{A_\alpha(x)}{\zeta}\right| \le \int_0^1 |F_{\alpha\hat{\ell}}(sx)|s\,ds \le ||F||_{L^\infty(K(p))}\,. \tag{3.14}$$

By integrating the above equation on $C(p)$, we obtain

$$\left|\left|\frac{A_\alpha}{\zeta}\right|\right|_{L^2(C(p))} \le C\int_0^1 s^{-3/2}||F_{\alpha\hat{\ell}}||_{L^2(C(p))}s\,ds \le C||F_{tan}||_{L^2(C(p))} \le CE_0\,. \tag{3.15}$$

By integrating equation (3.14) over $S = D(a)\cap C(p)$, we have

$$\int_S |A_\alpha|\,\mu_S \le C\zeta_1\int_{C(p)}\frac{1}{\zeta}|F_{\alpha\hat{\ell}}|\mu_\ell \le C\zeta_1^{3/2}||F_{tan}||_{L^2(C(p))} \le C\zeta_1^{3/2}E_0\,, \tag{3.16}$$

where $\zeta_1 = \max\{\zeta(q);\ q \in C(p)\}$.

Estimates on $D(t)$. The following estimates are obtained from equation (3.13) and (3.14) in a straightforward manner.

$$||A(t)||_{L^2(D(t))} \le C\zeta_1||F||_{L^{\infty,2}(K(p))} \le C\zeta_1E_0\,, \tag{3.17}$$

$$||\nabla A(t)||_{L^2(D(t))} \le C\left(\zeta_1||\mathcal{D}F||_{L^{\infty,2}(K(p))} + \zeta_1^2||F||_{L^\infty(K(p))}E_0 + E_0\right)\,, \tag{3.18}$$

where in the above inequality we used equation (3.14) to substitute $\mathcal{D}F$ for ∇F.

Pointwise estimates on F. Since F satisfies equation (3.5), we have from the tensorial equivalent of equation (2.3) (*cf.* [10], Section 5.5, together with the arguments of the proof of Theorem 5.3.3 there) the following representation for F

$$F_{\mu\nu}(p) = \int_{C(p)} H\frac{\kappa(\zeta,\omega)}{\zeta}\,\mu_\ell + \int_{K(p)} HV^-\mu + F^{lin}_{\mu\nu}(p)\,, \tag{3.19}$$

where $F^{lin}_{\mu\nu}$ is a solution of the homogeneous covariant wave equation (3.6), and where

$$\begin{aligned} H = {} & 2[F_\mu{}^\beta, F_{\beta\nu}] - 2R_{\alpha\mu\nu\beta}F^{\alpha\beta} - R_{\mu\alpha}F_\nu{}^\alpha + R_{\nu\alpha}F_\mu{}^\alpha \\ & +[A^\alpha, F_{\mu\nu}]_{;\alpha} + [A^\alpha, F_{\mu\nu;\alpha}] + [A^\alpha,[A_\alpha, F_{\mu\nu}]]\,. \end{aligned}$$

Here κ and V^- are smooth tensor fields; for notational convenience we have suppressed indices on H, κ and V^-. If A and F are given in the Cronström gauge, so that A is tangential to $C(p)$, we can integrate by parts on the term $[A^\alpha, F_{\mu\nu}]_{;\alpha}$ to rewrite equation (3.19) as

(3.20)

$$F_{\mu\nu}(p) = \int_{C(p)} \frac{\tilde{H}\kappa(\zeta,\omega)}{\zeta}\,\mu_\ell + \int_{K(p)} HV^-\mu + F^{lin}_{\mu\nu}(p) + \int_S \frac{\kappa}{\zeta} m^\alpha\,[A_\alpha, F_{\mu\nu}]\,\mu_S\,.$$

where $S = C(p)\cap D(a)$, and

$$\begin{aligned} \tilde{H} = {} & 2[F_\mu{}^\beta, F_{\beta\nu}] - 2R_{\alpha\mu\nu\beta}F^{\alpha\beta} - R_{\mu\alpha}F_\nu{}^\alpha + R_{\nu\alpha}F_\mu{}^\alpha \\ & -\kappa_{;\alpha}[A^\alpha, F_{\mu\nu}] + [A^\alpha, F_{\mu\nu;\alpha}] + [A^\alpha,[A_\alpha, F_{\mu\nu}]]\,. \end{aligned}$$

The integral on the surface of the cone in equation (3.20) can be bounded using equation (3.13),

$$\begin{aligned} \left|\left|\frac{1}{\zeta}[A^\alpha, F_{\mu\nu}]\right|\right|_{L^1(C(p))} &\le C\left|\left|\frac{A^\alpha}{\zeta}\right|\right|_{L^1(C(p))} ||F||_{L^\infty(C(p))} \le C\sqrt{\delta}\,||F||_{L^\infty(K(p))}\,, \\ \left|\left|\left[\frac{1}{\zeta}A^\alpha, F_{\mu\nu;\alpha}\right]\right|\right|_{L^1(C(p))} &\le C||\mathcal{D}_{tan}F||_{L^2(C(p))}\,, \\ \left|\left|\frac{1}{\zeta}\,[A_\alpha,[A^\alpha, F_{\mu\nu}]]\right|\right|_{L^1(C(p))} &\le C\delta||F||_{L^\infty(K(p))}\,, \\ \left|\left|\frac{1}{\zeta}[F_\mu{}^\beta, F_{\beta\nu}]\right|\right|_{L^1(C(p))} &\le C||F||_{L^\infty(K(p))}\left|\left|\frac{F}{\zeta}\right|\right|_{L^1(C(p))} \le C\sqrt{\delta}\,||F||_{L^\infty(K(p))}\,, \end{aligned}$$

where in the last inequality we used the observation of Eardley and Moncrief [9], that $[F_\mu{}^\beta, F_{\beta\nu}]$ consists of tangential terms only (i.e. $[F_{\hat{\ell}\hat{e_i}}, F_{\hat{\ell}\hat{e_i}}]$ and $[F_{\hat{e_i}\hat{e_j}}, F_{\hat{e_i}\hat{e_j}}]$). Similarly the integral over the solid cone can be estimated using (3.17) and (3.18) to obtain

$$\begin{aligned} \left|\int_{K(p)} V^-H\mu\right| \le {} & C(||A||_{L^2(K(p))}||\mathcal{D}F||_{L^2(K(p))} \\ & +||A||_{L^\infty(K(p))}||A||_{L^2(K(p))}||F||_{L^2(K(p))} + \delta E_0) \\ \le {} & C\left(\delta||\mathcal{D}F||_{L^{\infty,2}K(p)} + \delta^2||F||_{L^\infty(K(p))} + \delta E_0\right)\,. \end{aligned}$$

Since the curvature is bounded on bounded sets, then the first two terms in equation (3.20) can be bounded by a constant times

$$E_0 + ||\mathcal{D}F||_{L^{\infty,2}(K(p))} + ||\mathcal{D}_{tan}F||_{L^2(C(p))} + \sqrt{\delta}\; ||F||_{L^\infty(K(p))}\,. \tag{3.21}$$

The remaining terms, $F^{lin}_{\mu\nu}(p)$ together with the last term from eq. (3.20), consist of the following expressions

$$\int_{D(a)} (V^- t^\alpha \nabla_\alpha F_{\mu\nu} - F_{\mu\nu}V^-)\mu_a \tag{3.22a}$$
$$= \int_{D(a)} (V^- t^\alpha \mathcal{D}_\alpha F_{\mu\nu} - F_{\mu\nu}V^- - V^-[A^\alpha, F_{\mu\nu}])\mu_a\,,$$

$$\int_S \frac{1}{\zeta}\tilde{U}\left(m^\alpha \nabla_\alpha F_{\mu\nu} + \frac{1}{\zeta}\tilde{\theta}F_{\mu\nu}\right)\mu_S \tag{3.22b}$$
$$= \int_S \frac{1}{\zeta}\tilde{U}\left(m^\alpha \mathcal{D}_\alpha F_{\mu\nu} + \frac{1}{\zeta}\tilde{\theta}F_{\mu\nu}\right)\mu_S - \int_S \frac{1}{\zeta}\tilde{U} m^\alpha [A_\alpha, F_{\mu\nu}]\,\mu_S\,,$$

$$\int_S V^- F_{\mu\nu}\,\mu_S\,, \tag{3.22c}$$

$$\int_S \frac{\kappa}{\zeta} m^\alpha\,[A_\alpha, F_{\mu\nu}]\,\mu_S\,. \tag{3.22d}$$

Here $\tilde{U}$ and $\tilde{\theta}$ are smooth tensor fields (with indices suppressed). The terms in (3.22b) and (3.22c) can be bounded by a constant by

$$||\mathcal{D}F(a)||_{L^2(D(a))} + ||F(a)||_{L^2(D(a))}\,.$$

By the divergence theorem, the terms in (3.22c) and (3.22d) can be bounded by a constant times

$$\frac{1}{r_1}\int_{D(a)} (|\nabla \mathcal{D}F(a)| + \frac{1}{r}(|\nabla F(a)| + |\mathcal{D}F(a)|) \tag{3.23}$$
$$+ \frac{1}{r^2}|F(a)|)\mu_a + \frac{1}{r_1}||F||_{L^\infty(K(p))}\int_S |A| d\mu_S\,,$$

where in normal coordinates centered at p, $r = |\mathrm{x}|$, $r_1 = \min\{\zeta(q);\; q \in S\}$ and $r_2 = \max\{\zeta(q);\; q \in S\}$. The ratio $\frac{r_2}{r_1}$ is bounded by a constant depending on p.

The first term in equation (3.24) can be bounded by a constant times

$$\frac{1}{r_1}\int_{D(a)} |\nabla \mathcal{D}F(a)|\mu_a \le \frac{1}{r_1}\int_{D(a)} (|\mathcal{D}\mathcal{D}F(a)| + C|A||\mathcal{D}F|)\mu_a \le CE_2(a)\,,$$

where we have used $\int_{D(a)} \mu_a \le Cr_1^3$, and where we have defined

$$E_2(\tau) = ||\mathcal{D}\mathcal{D}F(t)||_{L^2(\Delta(\tau))} + ||\mathcal{D}F(t)||_{L^2(\Delta(\tau))} + ||F(t)||_{L^2(\Delta(\tau))}\,. \tag{3.24}$$

The second term in equation (3.24) can be bounded by a constant times

$$\begin{aligned}
&\frac{1}{r_1}\int_{D(a)}\frac{1}{r}(|\nabla F|+|\mathcal{D}F|)\mu_a \\
&\leq \frac{1}{r_1}\int_{D(a)}\frac{1}{r}|\mathcal{D}F(a)|\mu_a + CE_0 \\
&\leq \frac{1}{r_1}\int_{D(a)}(|\nabla\mathcal{D}F(a)| + C|\mathcal{D}F(a)|)\mu_a + CE_0 \\
&\leq C\left(\frac{1}{r_1}\int_{D(a)}(|\mathcal{D}\mathcal{D}F(a)| + |A\mathcal{D}F(a)| + |\mathcal{D}F(a)|)\mu_a + E_0\right) \\
&\leq CE_2(a)\,.
\end{aligned}$$

The third term in equation (3.24) can be bounded by a constant times

$$\frac{1}{r_1}\int_{D(a)}\frac{1}{r^2}|F(a)|\mu_a \leq \frac{1}{r_1}\int_{D(a)}\frac{1}{r}|\mathcal{D}F(a)|\mu_a + CE_0 \leq CE_2(a)\,.$$

From equation (3.16) the last term in equation (3.24) can be bounded by

$$C(r_2)^{\frac{1}{2}}E_0||F||_{L^\infty(K(p))}\,.$$

Equations (3.21) and (3.24) imply

$$|F(p)| \leq C\left(E_2(a) + ||\mathcal{D}_{tan}F||_{L^2(C(p))} + \delta^{1/2}||F||_{L^\infty(K(p))}\right)\,.$$

Since all of the terms in the above estimate are gauge invariant, by (3.2) the above inequality remains valid in any gauge. By taking sup over $\Delta(a,b)$ we obtain

$$||F||_{L^\infty(\Delta(a,b))} \leq C\left(E_2(a) + \sup_{p\in\Delta(a,b)}||\mathcal{D}_{tan}F||_{L^2(C(p))} + \delta^{1/2}||F||_{L^\infty(\Delta(a,b))}\right)\,.$$

By choosing δ small, depending on the energy E_0 and a only, we obtain from the above equation

$$||F||_{L^\infty(\Delta(a,b))} \leq C\left(E_2(a) + \sup_{p\in\Delta(a,b)}||\mathcal{D}_{tan}F||_{L^2(C(p))}\right)\,. \tag{3.25}$$

H^2 bound on F. From the energy inequality (3.12) and inequality (3.25) we have

$$\begin{aligned}
&||\mathcal{D}F||_{L^{\infty,2}(\Delta(a,b))} + \sup_{p\in\Delta(a,b)}||\mathcal{D}_{tan}F||_{L^2(C(p))} \\
&\leq C\left(||\mathcal{D}F(a)||_{L^2(D(a))} + \delta(E_2(a) + ||\mathcal{D}_{tan}F||_{L^2(C(p))})\right)\,.
\end{aligned}$$

By taking δ small enough, depending on the energy E_0 and a only, we obtain

$$||\mathcal{D}F||_{L^{\infty,2}(\Delta(a,b))} + \sup_{p\in\Delta(a,b)}||\mathcal{D}_{tan}F||_{L^2(C(p))} \leq CE_2(a)\,. \tag{3.26}$$

Taking a space–time and gauge covariant derivative of equation (3.5) we obtain

$$\begin{aligned}\mathcal{D}^\alpha\mathcal{D}_\alpha\mathcal{D}_\sigma F_{\mu\nu} &= -\mathcal{D}_\sigma\left\{2[F_\mu{}^\beta, F_{\beta\nu}] - 2R_{\alpha\mu\nu\beta}F^{\alpha\beta} - R_{\mu\alpha}F^\alpha{}_\nu - R_{\nu\alpha}F_\mu{}^\alpha\right\} \\ &\quad +[\mathcal{D}^\alpha\mathcal{D}_\alpha, \mathcal{D}_\sigma]F_{\mu\nu}\,. \end{aligned} \tag{3.27}$$

The right–hand–side of this equation can be estimated as

$$C(|F| + |\mathcal{D}F| + |F||\mathcal{D}F|)\ . \tag{3.28}$$

We can now repeat the argument leading to (3.10) with the energy–momentum tensor (3.9) replaced by

$$T^{\alpha\beta} = h^{\psi\phi}h^{\mu\nu}h^{\rho\sigma}\{k(\mathcal{D}^\alpha\mathcal{D}_\psi F_{\mu\rho}, \mathcal{D}^\beta\mathcal{D}_\phi F_{\nu\sigma}) - \frac{1}{2}k(\mathcal{D}^\lambda\mathcal{D}_\psi F_{\mu\rho}, \mathcal{D}_\lambda\mathcal{D}_\phi F_{\nu\sigma})g^{\alpha\beta}\}\ , \tag{3.29}$$

to obtain

$$E_2(t) \le C\left(E_2(a) + \delta||\mathcal{D}F||_{L^{\infty,2}(\Delta(a,b))}\left(1 + ||F||_{L^\infty(\Delta(a,b))}\right)\right)\ .$$

From equations (3.25)–(3.26) we conclude

$$E_2(t) \le C(a, E_2(a))\,,$$

for all $a \le t \le b$. Since the size of the cone δ depends only on a and E_0, which is bounded in terms of the intial data we conclude that for any point $p \in M$ and compact hypersurface $\mathcal{S}(p) \subset \{q \in M;\ t(q) = t(p)\ \}$

$$||\mathcal{D}\mathcal{D}F||_{L^2(\mathcal{S}(p))} + ||\mathcal{D}F||_{L^2(\mathcal{S}(p))} \le C(\mathcal{S}(p))||F^{ini}||_{H^2}\,, \tag{3.30}$$

where F^{ini} corresponds to the initial data for F.

4. Global existence. Let $\mathcal{O}$ be a coordinate patch on Σ_t, using the flow of ∇t on M we can extend the coordinates x^i on $\mathcal{O}$ to coordinates $(t, x^i) = (x^0, x^i)$ on $\mathbb{R}\times\mathcal{O}$. In this coordinate system the metric takes the form

$$g_{\mu\nu}dx^\mu dx^\nu = -\alpha^2dt^2 + g_{ij}dx^idx^j\ , \tag{4.1}$$

with some function $\alpha > 0$. In all our considerations below we shall only use coordinate systems of this type. While this is clearly not necessary for our analysis, it simplifies some of the calculations involved.

Let $\Omega \subset M$ be an open set. We shall consider connections on $P|\Omega$ with the following differentiability properties: We shall assume that Ω can be covered by coordinate patches Ω_i with coordinates $(t, x^i) \in I \times \mathcal{U}_i$ together with local trivialisations of P over Ω_i, such that the coordinate components A_μ of the connection are in $C(I; H^{k+1}(\mathcal{U}_i)) \cap C^1(I; H^k(\mathcal{U}_i))$, for some $k \ge 2$.

Consider a trivialisation of P over a coordinate patch $\mathcal{U}$ of the form $I \times \mathcal{O}$, where I is a time interval, such that the hypersurfaces Σ_t are given by the equation $x^0 = t$. For the purposes of this section it will be convenient to impose the *temporal gauge* condition

$$A_0 = 0\ , \tag{4.2}$$

and to define

$$G_{\mu\nu} = F_{\mu\nu} - \partial_\mu A_\nu + \partial_\nu A_\mu - [A_\mu, A_\nu] \ , \tag{4.3a}$$

$$B^\lambda = \epsilon^{\lambda\alpha\beta\gamma}\mathcal{D}_\alpha F_{\beta\gamma} \ , \tag{4.3b}$$

$$C^\lambda = \mathcal{D}_\alpha F^{\alpha\lambda} \ . \tag{4.3c}$$

In the temporal gauge, the F_{0i} component of eq. (3.1) reduces to

$$\partial_t A_i = F_{0i}. \tag{4.4}$$

To establish local in time existence of solutions of the Cauchy problem we shall be solving for A_i, F_{ij}, and F_{0i} using equation (4.4) together with the equations obtained by setting $B^i = C^i = 0$:

$$\mathcal{D}_t F_{ij} = \mathcal{D}_j F_{i0} - \mathcal{D}_i F_{j0} \ , \tag{4.5}$$

$$\mathcal{D}_t F^{0i} = \mathcal{D}_j F^{ji} \ . \tag{4.6}$$

Let us note, that when eqs. (4.4)–(4.6) hold throughout $\mathcal{U}$, and when $G_{\mu\nu}$ defined in eq. (4.3a) vanishes at one value of t, then $G_{\mu\nu}$ will vanish throughout $\mathcal{U}$ (and consequently we shall have $B^\alpha = 0$ throughout $\mathcal{U}$). Indeed, the time derivative of the right–hand–side of eq. (4.3a) with $\mu\nu = ij$ vanishes when eqs. (4.4)–(4.5) hold. It then follows that $F_{\mu\nu}$ is the curvature field of the connection A_μ. The vanishing of B^α is then simply the Bianchi identity. It is important to note that this will be true regardless of whether or not the Yang–Mills constraint equation $C^0 = 0$ holds.

Let us start with a result that involves Lorentzian metrics on $M = \mathbb{R}^{3+1}$. In that case every principal bundle is trivial so that both the connection and its curvature can be considered as fields defined on M. We have the following (which, again, holds irrespective of the vanishing of C^0):

PROPOSITION 4.1. *Let $g_{\mu\nu}$ be a smooth Lorentzian metric on $\mathbb{R}^{3+1}$ such that the level sets of $t = x^0$ are Cauchy surfaces. Consider two $\mathfrak{g}$–valued one-forms A_i^D and E_i defined on $\mathbb{R}^3 \approx \{t = 0\} \subset \mathbb{R}^{3+1}$, with $(A_i^D, E_i) \in H^{k+1}_{\rm loc}(\mathbb{R}^3) \times H^k_{\rm loc}(\mathbb{R}^3)$. Then*

1. *Suppose that $k \geq 4$. There exists a $\mathfrak{g}$–valued one form A_μ defined on $\mathbb{R}^{3+1}$ such that the field $F_{\mu\nu}$ defined by eq. (4.3a) with[4] $G_{\mu\nu} = 0$ is the unique solution of equations (4.4)–(4.6), satisfying $F_{0i}|_{t=0} = E_i$, $A_i|_{t=0} = A_i^D$, and $A_0 = 0$.*
2. *Suppose that $k \geq 2$ and that $\mathcal{D}_j E^j = 0$ on a neighbourhood of a coordinate ball $D \equiv B(R) \subset \{t = 0\}$. Assume moreover that the interior of the future domain of dependence int$\Delta^+(D)$ of D has the property that for all τ its sections int$\Delta^+(D) \cap \{t = \tau\}$ are diffeomorphic (as manifolds with boundary) to coordinate balls, when not empty. Then there exists a $\mathfrak{g}$–valued one form A_μ defined on a neighbourhood of $\Delta^+(D)$ and satisfying the Yang–Mills equations in int$\Delta^+(D)$.*

Proof. The proof of the first assertion is a straightforward consequence of the *a priori* estimates of Section 3, we shall give some details for completeness[5]. Let $B(R)$ be a coordinate ball of radius R in $\{t = 0\}$, multiplying the initial data (A_i^D, E_i) by a cut–off function we obtain initial data $(A_i^{D,R}, E_i^R)$ which coincide with (A_i^D, E_i) on $B(R)$ and which vanish outside $B(2R)$. Modifying the metric in an appropriate way

[4]Missing condition inserted.

[5]The idea of allowing a source term $c_{\mu\nu}$ in eq. (4.12) below is due to [12].

we also obtain a metric $g^R_{\mu\nu}$ which coincides with $g_{\mu\nu}$ in the domain of dependence $\Delta(B(R))$ of $B(R)$ and is the standard flat Minkowski metric outside of $I \times B(2R)$, where $I \subset \mathbb{R}$ is a compact time interval.

Now the system of equations (4.4)–(4.6) is symmetric hyperbolic, which is easily checked as follows: Introduce $E_i = F_{i0}$, $H_i = \epsilon_i{}^{jk}F_{jk}/2$, where $\epsilon_{ijk} = \sqrt{\det g_{ij}}\partial_i \vee \partial_j \vee \partial_k \vee dx^1 \wedge dx^2 \wedge dx^3$. Eqs. (4.4)–(4.6) can be rewritten as

$$g^{ij}\partial_t A_j = l.o. \ , \tag{4.7}$$

$$\alpha^{-2} g^{ij}\partial_t E_j = \epsilon^{ijk}\partial_j H_k + l.o. \ , \tag{4.8}$$

$$g^{ij}\partial_t H_j = -\epsilon^{ijk}\partial_j E_k + l.o. \ , \tag{4.9}$$

where α is the "lapse" function appearing in (4.1), and "*l.o.*" denotes terms which do not involve derivatives of A_i, E_i and H_i. Set $u = (A_i, E_i, H_i)$ and

$$L^0 = \begin{pmatrix} g^{ij} & 0 & 0 \\ 0 & \alpha^{-2}g^{ij} & 0 \\ 0 & 0 & g^{ij} \end{pmatrix} .$$

It follows that (4.7)–(4.9) is of the form

$$L^0(t,x)\partial_t u + L^i(t,x)\partial_i u = M(t,x,u) \ , \tag{4.10}$$

with symmetric matrices L^μ (*cf. e.g.* [15, p. 199]).

Since equations (4.4)–(4.6) form a semilinear symmetric hyperbolic system, we have from standard energy estimates that the modified Cauchy problem has a local solution in $C([-T_1, T_2], H^k(\mathbb{R}^n))$, provided $k > \frac{n}{2}$. (See, for example[6], Volume III, Theorem 1.2, p. 362 and Proposition 2.1, page 370 of [17].)

Consider the field C^λ defined by eq. (4.3c). We have from equation (4.6) $C^i = 0$, and from the identity $\mathcal{D}_\alpha C^\alpha = 0$ we have

$$\partial_t(\sqrt{-\det g}C^0) = 0 \Rightarrow C^\lambda = \delta^\lambda_0 a, \qquad a = C^0(t=0)/\sqrt{-\det g} \ . \tag{4.11}$$

This shows that a is uniformly bounded on $\mathbb{R}^{3+1}$. The modified curvature tensor F^R of A^R satisfies the equation

$$-\Box F_{\mu\nu} + 2[F_\mu{}^\beta, F_{\beta\nu}] - 2R_{\alpha\mu\nu\beta}F^{\alpha\beta} - R_{\mu\alpha}F^\alpha{}_\nu - R_{\nu\alpha}F_\mu{}^\alpha = c_{\mu\nu} \ , \tag{4.12}$$

where $c_{\mu\nu} \equiv \mathcal{D}_\mu C_\nu - \mathcal{D}_\nu C_\mu$ is globally controlled in the L^∞ norm (all the C^k norms if the initial data are smooth). Equation (4.12) differs from eq. (3.5) of Section 3 only by the $c_{\mu\nu}$ term. Moreover, under the gauge transformation $A_\mu \to \mathcal{G}^{-1}\partial_\mu\mathcal{G} + \mathcal{G}^{-1}A_\mu\mathcal{G}$, $F_{\mu\nu} \to \mathcal{G}^{-1}F_{\mu\nu}\mathcal{G}$, eq. (4.12) will be transformed to the same equation with $c_{\mu\nu}$ replaced by $\mathcal{G}^{-1}c_{\mu\nu}\mathcal{G}$. The *a priori* estimates derived in Section 3 carry over to the above equation when $k \geq 4$ (and, hence, $c_{\mu\nu} \in L^\infty_{\rm loc}$) and thus imply global existence of solutions for the modified problem. Point 1 follows now from the fact that the solutions of equations (4.4)–(4.6) on $\Delta(B(R))$ are uniquely determined by their initial data on $B(R)$, so that for $R_1 > R$ we have $(F^{R_1}|_{\Delta(B(R))}, A^{R_1}|_{\Delta(B(R))}) = (F^R|_{\Delta(B(R))}, A^R|_{\Delta(B(R))})$. One can therefore patch together all the (F^R, A^R) 's to obtain a globally defined solution on $\mathbb{R}^{3+1}$.

[6] Since equations (4.4)–(4.6) are semilinear it is easy to see, by a simple modification of the arguments given in [17], that the differentiability threshold $k > n/2 + 1$ imposed in [17] can be lowered, in our setting, to $k > n/2$.

To prove point 2, note that to obtain the H^2 estimates of Section 3, one has to be able to perform a gauge transformation to the Cronström gauge. This requires some sufficient degree of differentiability of the fields, which might not be satisfied for $k = 2$. Further, the estimates used in the proof of point 1 fail to hold globally with $k = 2$, because of the potential lack of uniform boundedness of the $c_{\mu\nu}$ term. To overcome those problems, consider a sequence of smooth initial data $(A_i^{D,R,n}, \hat{E}_i^{R,n})$ supported in the ball $B(2R)$ which converge to $(A_i^{D,R}, E_i^R)$ in $H^{k+1}(\mathbb{R}^3) \times H^k(\mathbb{R}^3)$. By point 1 we have a corresponding sequence of solutions of eqs. (4.4)–(4.6) defined on $\mathbb{R}^{3+1}$. Because the H^2 estimates of Section 3 in $\Delta(B(R))$, as adapted to include a supplementary $c_{\mu\nu}$ term in eq. (3.5), involve only the $H^2(B(R))$ norm of $F_{\mu\nu}(t = 0)$ and the $L^\infty(B(R))$ norm of $c_{\mu\nu}$, they will continue to hold in $\Delta(B(R))$ when passing to the limit if we show that we can choose an approximating sequence for which the norms $\|c_{\mu\nu}\|_{L^\infty(B(R))} \le 2\|\mathcal{D}_i E^i\|_{C^1(B(R))}$ have a bound independent of n. To obtain such a sequence, let the $\mathfrak{g}$–algebra valued field $\phi^{R,n}$ be obtained as a solution of the equation

$$\mathcal{D}^i(\mathcal{D}_i\phi^{R,n} + \hat{E}_i^{R,n}) = \psi_{1/n} * \mathcal{D}^i E_i^R \ . \tag{4.13}$$

Here $\psi_{1/n} * f$ denotes the convolution of a function f with a Friedrichs mollifier. As discussed in [2, Section 5], eq. (4.13) has a unique solution in an appropriate weighted Sobolev space. Note that $\psi_{1/n} * \mathcal{D}^i E_i^R$ converges to $\mathcal{D}^i E_i^R = 0$ on $B(R)$ in the $C^1(B(R))$ norm, since E_i coincides with E_i^R in a neighbourhood of $B(R)$. Moreover both $\mathcal{D}^i \hat{E}_i^{R,n}$ and $\psi_{1/n} * \mathcal{D}^i E_i^R$ converge to $\mathcal{D}^i E_i^R$ on $\mathbb{R}^3$ in the $H^{k-1}(\mathbb{R}^3)$ norm. It follows that $\phi^{R,n}$ converges to zero in $H^{k+1}(B(2R))$. Setting $E_i^{R,n} = \mathcal{D}_i\phi^{R,n} + \hat{E}_i^{R,n}$, the sequence $(A_i^{D,R,n}, E_i^{R,n})$ will have all the desired properties. To finish the proof of point 2, note that from eq. (4.11) one has that $C^\mu = 0$ on $\mathbb{R}\times D$. Since $\Delta(D) \subset \mathbb{R}\times D$ the result follows. □

Proof of Theorem 1.1. Consider the collection $\mathcal{P}$ of open subsets Ω of M with the following properties:

1. Ω is globally hyperbolic with Cauchy surface Σ;
2. $P|_\Omega$ carries a connection which is a Yang–Mills development of the initial data (A^Σ, E);
3. Ω is covered by coordinate patches Ω_i with coordinates $(t, x^i) \in I\times\mathcal{U}_i$ together with local trivialisations of P over Ω_i, such that the coordinate components A_μ of the connection are in $C(I; H^{k+1}(\mathcal{U}_i)) \cap C^1(I; H^k(\mathcal{U}_i))$, with $A_0 = 0$.

By Proposition 4.1 together with a standard patching procedure $\mathcal{P}$ is not empty. $\mathcal{P}$ is directed by inclusion, and by the Kuratowski–Zorn lemma there exists a maximal element $\widetilde{\Omega}$ in $\mathcal{P}$. Suppose that $\partial\widetilde{\Omega}$ is not empty, by global hyperbolicity of M and of $\widetilde{\Omega}$, changing time orientation if necessary, there exists a point $p \in \partial\widetilde{\Omega}$ with the property that $J^-(p) \cap \partial\widetilde{\Omega} = \{p\}$ (*cf. e.g.* [5, 7]; here we follow the standard notation [13] in which $J^-(p)$ consists of the set of points causally related to p, and lying to the past of p). Choose ϵ small enough so that the set $K = J^-(p) \cap \Sigma_{t(p)-\epsilon}$ is covered by a single coordinate system. Since $\partial\widetilde{\Omega}$ is closed we can find a cover of K by open sets $\mathcal{O}_p$, $p \in K$, such that $\partial\widetilde{\Omega} \cap \overline{\mathcal{O}}_p = \emptyset$. (Here $\overline{\mathcal{O}}$ denotes the closure of the set $\mathcal{O}$.) By compactness of K a finite number of the $\mathcal{O}_p$'s can be chosen, $p = p_i$, $i = 1, ..., N$. Set $\widehat{\mathcal{O}} = \cup_{i=1}^N \mathcal{O}_{p_i}$. Without loss of generality we can assume that the domain of dependence $\Delta(\widehat{\mathcal{O}})$ of $\widehat{\mathcal{O}}$ is conditionally compact and lies within a single coordinate patch. It should be clear that p lies in the interior of $\Delta(\widehat{\mathcal{O}})$. Let q be any point in the interior of $\Delta(\widehat{\mathcal{O}}) \cap J^+(p)$, where $J^+(p)$ is the set of points in the causal future of p. Let $\mathcal{O}$ be the interior (relative to the topology of $\Sigma_{t(p)-\epsilon}$) of $J^-(q) \cap \Sigma_{t(p)-\epsilon}$. We have $\mathcal{O} \subset \widehat{\mathcal{O}}$, hence $\overline{\mathcal{O}} \cap \partial\widetilde{\Omega} = \emptyset$. Now the interior of the future domain of dependence $\Delta^+(\mathcal{O})$ of $\mathcal{O}$ coincides with the interior of $J^-(q) \cap J^+(\Sigma_{t(p)-\epsilon})$, so that (decreasing ϵ if necessary) $\Delta^+(\mathcal{O}) \cap \Sigma_t$ is diffeomorphic to a three–dimensional coordinate ball for $t \in (t(p)-\epsilon, t(q))$. We can extend the metric $g_{\mu\nu}|_{\Delta(\mathcal{O})}$ from $\Delta(\mathcal{O})$ to a smooth globally hyperbolic metric defined on $\mathbb{R}^{3+1}$ in any way. Similarly we can extend the Yang–Mills initial data induced on $\mathcal{O}$ by the Yang–Mills field on $\widetilde{\Omega}$ in any way. By point 2 of Proposition 4.1 there exists a global solution $\widehat{A}$ of equations (4.4)–(4.6) on $\mathbb{R}^{3+1}$. By uniqueness of solutions of equations (4.4)–(4.6) in domains of dependence, the connection $\widehat{A}$ coincides with A on $\Delta(\mathcal{O}) \cap \widetilde{\Omega}$. Therefore one can extend the connection A from $P|_{\widetilde{\Omega}}$ to a connection field on $\Delta(\mathcal{O}) \cup \widetilde{\Omega}$, the extended connection being a Yang–Mills development of the initial data (A^Σ, E). This, however, contradicts maximality of $\widetilde{\Omega}$ so that $\widetilde{\Omega} = M$, and Theorem 1.1 is proved. □

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