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THE HARNACK ESTIMATE FOR THE RICCI FLOW ON A SURFACE – REVISITED*

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1. The Harnack estimate for the Ricci flow on a surface of positive curvature occurs in [H]. This was generalized to the case with some negative curvature by Ben Chow [C]. However, he introduced the global potential function φ solving

$$\Delta\varphi = R - r,$$

where R is the scalar curvature and r is the mean scalar curvature. Since this does not lend itself well to three dimensions, we rederive a Harnack estimate for surfaces with some negative curvature using only local quantities. The idea of use square root was used by the second author in [Y1] and [Y2] (note that an error of [Y1] is corrected in this paper).

THEOREM. 1.1. *For any constants K and L we can find (positive) constants A, B, C and D with the following property. If we have any solution to the Ricci flow on a compact surface such that at the initial time $t = 0$ we have*

$$R \geq 1 - K$$

and

$$\frac{1}{R+K}\frac{\partial R}{\partial t} - \frac{1}{(R+K)^2}|\nabla R|^2 \geq -L,$$

then for all $t \geq 0$ we have

$$\frac{1}{R+K}\frac{\partial R}{\partial t} - \frac{1}{(R+K)^2}|\nabla R|^2 + F\left(\frac{|\nabla R|^2}{(R+K)^2}, R+K\right) \geq 0,$$

where

$$F(X,Y) = A + \sqrt{2B(X+Y)+C} + D\log Y.$$

Proof. First note if $\sqrt{C} \geq L$ the inequality holds at $t = 0$, we only need to show it is preserved by the maximum principle. To simplify the notation we let

$$\Box = \frac{\partial}{\partial t} - \Delta$$

and

$$L = \log(R+K).$$

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Note $R+K \geq 1$ so $L \geq 0$. We compute

$$\begin{aligned}
\Box R &= R^2, \\
\Box L &= |\nabla L|^2 + e^L - 2K + K^2 e^{-L}, \\
\Box e^L &= e^{2L} - 2Ke^L + K^2, \\
\Box |\nabla L|^2 &= 2\nabla L \cdot \nabla |\nabla L|^2 - 2\left|\nabla^2 L\right|^2 + 2e^L |\nabla L|^2 - 2K^2 e^{-L} |\nabla L|^2, \\
\Box \Delta L &= 2\nabla L \cdot \nabla \Delta L + 2\left|\nabla^2 L\right|^2 + \left(2e^L - K - K^2 e^{-L}\right) \Delta L \\
&\quad + \left(2e^L - K + K^2 e^{-L}\right) |\nabla L|^2 .
\end{aligned}$$

We define the Harnack expression:

$$H = \Delta L + e^L.$$

Then we have

$$\begin{aligned}
\Box H &= 2\nabla L \cdot \nabla H + 2\left|\nabla^2 L\right|^2 + \left(2e^L - K - K^2 e^{-L}\right) \Delta L \\
&\quad + \left(-K + K^2 e^{-L}\right) |\nabla L|^2 + e^{2L} - 2Ke^L + K^2.
\end{aligned}$$

Let $X = |\nabla L|^2$ and $Y = e^L$ (note $Y \geq 1$). Then we have

$$\begin{aligned}
\Box X &= 2\nabla L \cdot \nabla X - 2\left|\nabla^2 L\right|^2 + 2XY - 2K^2 X/Y, \\
\Box Y &= 2\nabla L \cdot \nabla Y - 2XY + Y^2 - 2KY + K^2, \\
\Box H &= 2\nabla L \cdot \nabla H + 2\left|\nabla^2 L\right|^2 + \left(2Y - K - K^2/Y\right) \Delta L \\
&\quad + \left(-K + K^2/Y\right) X + Y^2 - 2KY + K^2.
\end{aligned}$$

Let $F = F(X,Y)$ so $\nabla F = F_X \nabla X + F_Y \nabla Y$. Then

$$\Box F = F_X \Box X + F_Y \Box Y - F_{XX} |\nabla X|^2 - 2F_{XY} \nabla X \cdot \nabla Y - F_{YY} |\nabla Y|^2 .$$

Assume F is *concave*, i.e.

$$\nabla^2 F = \begin{pmatrix} F_{XX} & F_{XY} \\ F_{XY} & F_{YY} \end{pmatrix} \leq 0.$$

Then

$$\Box F \geq F_X \Box X + F_Y \Box Y$$

and we get

$$\begin{aligned}
\Box F &\geq 2\nabla L \cdot \nabla F + F_X \left[-2\left|\nabla^2 L\right|^2 + 2e^L |\nabla L|^2 - 2K^2 e^{-L} |\nabla L|^2\right] \\
&\quad + F_Y \left[-2e^L |\nabla L|^2 + e^{2L} - 2Ke^L + K^2\right] \\
&= 2\nabla L \cdot \nabla F + F_X \left[-2\left|\nabla^2 L\right|^2 + 2XY - 2K^2 X/Y\right] \\
&\quad + F_Y \left[-2XY + Y^2 - 2KY + K^2\right].
\end{aligned}$$

Introduce the compensated Harnack expression

$$\tilde{H} = H + F = \Delta L + e^L + F\left(|\nabla L|^2, e^L\right),$$

where $X = |\nabla L|^2$ and $Y = e^L$. We compute

$$\begin{aligned}\Box\tilde{H} \geq\, & 2\nabla L \cdot \nabla\tilde{H} + 2\left|\nabla^2 L\right|^2 + 2Y\Delta L + Y^2 - K(\Delta L + X + 2Y) \\ & + K^2(-\Delta L/Y + X/Y + 1) + F_X\left[-2\left|\nabla^2 L\right|^2 + 2XY - 2K^2X/Y\right] \\ & + F_Y\left[-2XY + Y^2 - 2KY + K^2\right].\end{aligned}$$

Thus we have

$$\begin{aligned}\Box\tilde{H} \geq\, & 2\nabla L \cdot \nabla\tilde{H} + 2(1 - F_X)\left|\nabla^2 L\right|^2 + 2Y\Delta L + 2XY(F_X - F_Y) \\ & + Y^2(1 + F_Y) - K\left[\Delta L + X + 2Y(1 + F_Y)\right] \\ & + K^2\left[-\Delta L/Y + (1 - 2F_X)X/Y + (1 + F_Y)\right].\end{aligned}$$

Use

$$2\left|\nabla^2 L\right|^2 \geq (\Delta L)^2$$

and assume $F_X \leq 1$ (which happens if $B \leq \sqrt{C}$).

At $\tilde{H} = \min$, we have

$$\nabla\tilde{H} = 0$$

and at $\tilde{H} = 0$ we have

$$\Delta L = -(Y + F).$$

To keep $\tilde{H} \geq 0$ by the maximum principle we need

$$\begin{aligned}& (1 - F_X)(Y + F)^2 - 2Y(Y + F) + 2XY(F_X - F_Y) + Y^2(1 + F_Y) \\ & - K\left[X + Y + 2YF_Y - F\right] + K^2\left[2 + F_Y + F/Y + (1 - 2F_X)X/Y\right] \\ & \geq 0.\end{aligned}$$

If in addition $F_Y \geq 0$ and $F_X \leq 1/2$ (which happens if $B \leq \sqrt{C}/2$) we only need

$$\begin{aligned}& (1 - F_X)F^2 - 2YFF_X + \left(Y^2 - 2XY\right)(F_Y - F_X) \\ & \geq K\left[X + Y + 2YF_Y - F\right].\end{aligned}$$

Look for $F(X, Y)$ in the form

$$F = A + \sqrt{2B(X + Y) + C} + D\log Y.$$

For ease write

$$\begin{aligned}Q &= 2B(X + Y) + C, \\ F &= A + \sqrt{Q} + D\log Y.\end{aligned}$$

Compute

$$F_X = \frac{B}{\sqrt{Q}},\ F_Y = \frac{B}{\sqrt{Q}} + \frac{D}{Y},\ F_Y - F_X = \frac{D}{Y}.$$

We have

$$\begin{aligned}
&(1-F_X)F^2 - 2YFF_X + \left(Y^2-2XY\right)(F_Y-F_X)\\
&= \left(1-\frac{B}{\sqrt{Q}}\right)\left(A+\sqrt{Q}+D\log Y\right)^2 - 2Y\left(A+\sqrt{Q}+D\log Y\right)\frac{B}{\sqrt{Q}}\\
&\quad +Y(Y-2X)\frac{D}{Y}\\
&= 2(B-D)X + DY + (C-2AB) + (A-B)\sqrt{Q} + D\log Y\left(\sqrt{Q}-2B\right)\\
&\quad +(A+D\log Y)\left(\sqrt{Q}-\frac{2BY}{\sqrt{Q}}\right) + \left(1-\frac{B}{\sqrt{Q}}\right)(A+D\log Y)^2\\
&\geq 2(B-D)X + DY,
\end{aligned}$$

provided $C \geq 2AB$, $A \geq B$, $C \geq 4B^2$ (so $\sqrt{Q} \geq 2B$), and using $\sqrt{Q} \geq 2BY/\sqrt{Q}$.

On the other hand if $B \leq \sqrt{C}/2$ then

$$2YF_Y = \frac{2BY}{\sqrt{Q}} + 2D \leq A + \sqrt{Q} \leq F$$

provided $A \geq 2D$ since $2BY \leq Q$. Thus we now only need

$$2(B-D)X + DY \geq K(X+Y),$$

which happens if

$$D \geq K \text{ and } 2(B-D) \geq K.$$

To summarize, we need

$$D \geq K,$$

$$B \geq D + \frac{1}{2}K,$$

$$A \geq B \text{ and } A \geq 2D,$$

$$C \geq L^2,\ \ C \geq 4B^2 \text{ and } C \geq 2AB,$$

all of which is easily done by choosing them in this order. This completes the proof. □

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