ASIAN J. MATH.
Vol. 1, No. 2, pp. 330–417, June 1997
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009

K–THEORY FOR TRIANGULATED CATEGORIES I(A): HOMOLOGICAL FUNCTORS *

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0. Introduction. We should perhaps begin by reminding the reader briefly of Quillen's Q–construction on exact categories.

DEFINITION 0.1. *Let $\mathcal{E}$ be an exact category. The category $Q(\mathcal{E})$ is defined as follows.*

0.1.1. *The objects of $Q(\mathcal{E})$ are the objects of $\mathcal{E}$.*

0.1.2. *The morphisms $X \bullet\!\!\rightarrow X'$ in $Q(\mathcal{E})$ between $X, X' \in Ob(Q(\mathcal{E})) = Ob(\mathcal{E})$ are isomorphism classes of diagrams of morphisms in $\mathcal{E}$*

$$\begin{array}{ccccc} X & & & & X' \\ & \searrow & & \swarrow & \\ & & Y & & \end{array}$$

where the morphism $X \longrightarrow Y$ is an admissible mono, while $X' \longrightarrow Y$ is an admissible epi. Perhaps a more classical way to say this is that X is a subquotient of X'.

0.1.3. *Composition is defined by composing subquotients;*

$$\begin{array}{ccccccccccc} X & & & & X' & & X' & & & & X'' \\ & \searrow & & \swarrow & & \textit{and} & & \searrow & & \swarrow & \\ & & Y & & & & & & Y' & & \end{array}$$

compose to give

$$\begin{array}{ccccccccc} X & & & & X' & & & & X'' \\ & \searrow & & \swarrow & & \searrow & & \swarrow & \\ & & Y & & PO & & Y' & & \\ & & & \searrow & & \swarrow & & & \\ & & & & Z & & & & \end{array}$$

where the square marked PO is a pushout square.

The category $Q(\mathcal{E})$ can be realised to give a space, which we freely confuse with the category. The Quillen K–theory of the exact category $\mathcal{E}$ was defined, in [9], to be the homotopy of the loop space of $Q(\mathcal{E})$. That is,

$$K_i(\mathcal{E}) = \Pi_{i+1}\left[Q(\mathcal{E})\right].$$

Quillen proved many nice functoriality properties for his K–theory, and the one most relevant to this article is the resolution theorem. The resolution theorem asserts the following.

THEOREM 0.2. *Let $F : \mathcal{E} \longrightarrow \mathcal{F}$ be a fully faithful, exact inclusion of exact categories. Suppose further that every object $y \in \mathcal{F}$ admits a resolution*

$$0 \longrightarrow x_n \longrightarrow x_{n-1} \longrightarrow \cdots \longrightarrow x_1 \longrightarrow x_0 \longrightarrow y \longrightarrow 0,$$

with all the x_i's in $\mathcal{E}$. Then the natural map

$$Q(\mathcal{E}) \longrightarrow Q(\mathcal{F})$$

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is a homotopy equivalence.
The hypothesis in Quillen's resolution theorem, namely that every object of $\mathcal{F}$ have a resolution by objects of $\mathcal{E}$, is one of the standard conditions under which the induced map of derived categories

$$D^b(\mathcal{E}) \longrightarrow D^b(\mathcal{F})$$

is an equivalence. The theorem therefore strongly suggests that Quillen's K-theory should depend not on the exact category $\mathcal{E}$, but only on its derived category $D^b(\mathcal{E})$. To make this even more plausible, there is a theorem of Waldhausen which improves on Quillen's resolution.

THEOREM 0.3. *Let $F : \mathcal{E} \longrightarrow \mathcal{F}$ be any exact functor of exact categories. Suppose the induced map*

$$D^b(\mathcal{E}) \longrightarrow D^b(\mathcal{F})$$

is an equivalence of categories. Then the natural map

$$Q(\mathcal{E}) \longrightarrow Q(\mathcal{F})$$

is a homotopy equivalence.
In other words, Waldhausen improved Quillen's theorem to any exact functor F inducing an equivalence of derived categories, not only to one induced by a somewhat special fully faithful inclusion. See [13].

It is natural enough to ask whether one can define K-theory directly for the derived category. This is the main problem addressed by the present series of articles. Our first attempt to answer this question was [3]. Let me therefore discuss first [3]. In [3], Giffen and the author constructed, for every triangulated category $\mathcal{T}$, a simplicial set $\mathcal{Q}(\mathcal{T})$. For $\mathcal{T} = D^b(\mathcal{E})$, $\mathcal{E}$ an exact category, we claimed the following theorem.

MAIN "THEOREM" OF [3]. *The natural inclusion $\mathcal{E} \to D^b(\mathcal{E})$ extends to a simplicial map $Q(\mathcal{E}) \to \mathcal{Q}(D^b(\mathcal{E}))$. This map induces a homotopy equivalence provided no object of $\mathcal{E}$ is a proper direct summand of itself.*

Since then we have discovered a gap in the argument; we no longer know whether the main "theorem" of [3] is true. The problem seems to be that in [3] we studied the wrong Q-construction.

The most simple-minded Q-construction for a triangulated category $\mathcal{T}$ would be the category $Q(\mathcal{T})$, defined as follows:

(1) The objects of $Q(\mathcal{T})$ are the objects of $\mathcal{T}$.

(2) The morphisms $X \bullet\!\!\rightarrow X'$ in $Q(\mathcal{T})$ between $X, X' \in Ob(Q(\mathcal{T})) = Ob(\mathcal{T})$ are isomorphism classes of diagrams of morphisms in $\mathcal{T}$

$$\begin{array}{ccccc} X & & & & X' \\ & \searrow & & \swarrow & \\ & & Y & & \end{array}$$

(3) Composition is defined by

$$\begin{array}{ccccc} X & & & & X' \\ & \searrow & & \swarrow & \\ & & Y & & \end{array} \quad \text{and} \quad \begin{array}{ccccc} X' & & & & X'' \\ & \searrow & & \swarrow & \\ & & Y' & & \end{array}$$

compose to give

$$
\begin{array}{ccccccccc}
X & & & & X' & & & & X'' \\
& \searrow & & \swarrow & & \searrow & & \swarrow & \\
& & Y & & \text{MV} & & Y' & & \\
& & & \searrow & & \swarrow & & & \\
& & & & Z & & & &
\end{array}
$$

where the square marked $M - V$ is a Mayer-Vietoris square (i.e., there exists a morphism $Z \to \Sigma X'$ in $\mathcal{T}$ making $X' \to Y \oplus Y' \to Z \to \Sigma X'$ into a triangle).

The geometric realization of $Q(\mathcal{T})$ is a space, and there is a natural map $Q(\mathcal{E}) \to Q(D^b(\mathcal{E}))$. This map is clearly a Π_1-isomorphism, but it is hard to prove any more than that.

We could try to study instead a more rigid simplicial set. Let us begin with the K-theory of exact categories, where these constructions are well-known. We define a bisimplicial set

$$
\boxed{\begin{array}{l} \twoheaduparrow \\ \mathcal{E}\ \rightarrowtail \end{array}}
$$

as follows. A (p, q)-simplex is a commutative diagram

$$
\begin{array}{ccccc}
X_{p0} & \rightarrowtail & \cdots & \rightarrowtail & X_{pq} \\
\twoheaduparrow & & & & \twoheaduparrow \\
\vdots & & & & \vdots \\
\twoheaduparrow & & & & \twoheaduparrow \\
X_{00} & \rightarrowtail & \cdots & \rightarrowtail & X_{0q}
\end{array}
$$

where all squares are bicartesian. For readers unfamiliar with this simplicial set, let us observe that in this diagram, every object is a subquotient of X_{0q}. In fact, if $p = q$ the terms X_{ij} with $i + j = p$, that is the diagonal terms of the square, form a sequence of subquotients of X_{0q}. All the other information in the diagram amounts to choices, unique up to canonical isomorphism, of objects in $\mathcal{A}$ isomorphic to the intermediate subobjects and quotients. Thus we easily identify a map from the diagonal realisation of the simplicial set $\boxed{\begin{array}{l} \twoheaduparrow \\ \mathcal{E}\ \rightarrowtail \end{array}}$ to Quillen's Q-construction. In the notation above, the map takes the simplex

$$\begin{array}{ccccc} X_{p0} & \rightarrowtail & \cdots & \rightarrowtail & X_{pp} \\ \uparrow & & & & \uparrow \\ \vdots & & & & \vdots \\ \uparrow & & & & \uparrow \\ X_{00} & \rightarrowtail & \cdots & \rightarrowtail & X_{0p} \end{array}$$

to the composable morphisms in $Q(\mathcal{E})$

$$\begin{array}{ccccccccc} X_{p0} & \longrightarrow & X_{p1} & & & & & & \\ & & \uparrow & & & & & & \\ & & X_{p-1,1} & \longrightarrow & X_{p-1,2} & & & & \\ & & & & \uparrow & & & & \\ & & & & X_{p-2,2} & \longrightarrow & & & \\ & & & & & & \ddots & & \\ & & & & & & \uparrow & & \\ & & & & & & X_{1,p-1} & \longrightarrow & X_{1p} \\ & & & & & & & & \uparrow \\ & & & & & & & & X_{0p} \end{array}$$

It is well–known that this map is a homotopy equivalence; see for instance Proposition 1.4 on page 1178 of [6].

We could imitate this construction with triangulated categories. Let $\mathcal{T}$ be a triangulated category. We could define the bisimplicial set

$$\boxed{\begin{array}{c} \;\uparrow \\ \mathcal{T}\,\lfloor\!\longrightarrow \end{array}}$$

whose (p,q)-simplices are diagrams of $M-V$ squares

$$\begin{array}{ccccc} X_{p0} & \longrightarrow & \cdots & \longrightarrow & X_{pq} \\ \uparrow & & & & \uparrow \\ \vdots & & & & \vdots \\ \uparrow & & & & \uparrow \\ X_{00} & \longrightarrow & \cdots & \longrightarrow & X_{0q} \end{array}$$

As in the case of exact categories, there is a natural simplicial map

$$\boxed{\begin{array}{c} \;\uparrow \\ \mathcal{T}\,\lfloor\!\longrightarrow \end{array}} \longrightarrow Q(\mathcal{T})$$

which takes the diagonal realisation of the bisimplicial set to the simplicial set. Concretely, the map takes the simplex

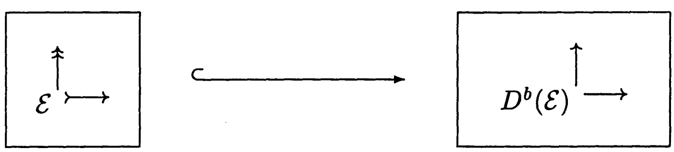
$$\begin{array}{ccccc} X_{p0} & \longrightarrow & \cdots & \longrightarrow & X_{pp} \\ \uparrow & & & & \uparrow \\ \vdots & & & & \vdots \\ \uparrow & & & & \uparrow \\ X_{00} & \longrightarrow & \cdots & \longrightarrow & X_{0p} \end{array}$$

to the chain of composable morphisms

$$\begin{array}{ccccccccc} X_{p0} & \longrightarrow & X_{p1} & & & & & & \\ & & \uparrow & & & & & & \\ & & X_{p-1,1} & \longrightarrow & X_{p-1,2} & & & & \\ & & & & \uparrow & & & & \\ & & & & X_{p-2,2} & \longrightarrow & & & \\ & & & & & \ddots & & & \\ & & & & & & \uparrow & & \\ & & & & & & X_{1,p-1} & \longrightarrow & X_{1p} \\ & & & & & & & & \uparrow \\ & & & & & & & & X_{0p} \end{array}$$

I have no idea whether this map is a homotopy equivalence. The main "theorem" of [3] was stated for the rigidified simplicial sets. It asserted, very precisely:

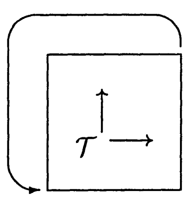
MAIN "THEOREM" OF [3]. *The natural inclusion $\mathcal{E} \to D^b(\mathcal{E})$ extends to a bisimplicial map*



This map induces a homotopy equivalence provided no object of $\mathcal{E}$ is a proper direct summand of itself.

I repeat, we have not been able to decide the truth of the "theorem." However, one can obtain something if one changes the simplicial sets even more.

DEFINITION 0.4. *Let $\mathcal{T}$ be a small triangulated category. Let the simplicial set*



be defined by having for its (p, q)-simplices the diagrams in $\mathcal{T}$

$$\begin{array}{ccccc} X_{p0} & \longrightarrow & \cdots & \longrightarrow & X_{pq} \\ \uparrow & & & & \uparrow \\ \vdots & & & & \vdots \\ \uparrow & & & & \uparrow \\ X_{00} & \longrightarrow & \cdots & \longrightarrow & X_{0q} \end{array}$$

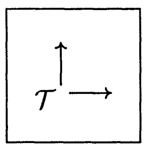
together with a coherent differential $d : X_{pq} \to \Sigma X_{00}$. *A coherent differential* $d : X_{pq} \to \Sigma X_{00}$ *is a map such that for any* $0 \leq i \leq i' \leq p$, $\quad 0 \leq j \leq j' \leq q$,

$$X_{ij} \to X_{i'j} \oplus X_{ij'} \to X_{i'j'} \xrightarrow{\phi} \Sigma X_{ij}$$

is a triangle, where $\phi : X_{i'j'} \to \Sigma X_{ij}$ *is the composite*

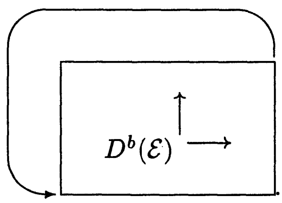
$$X_{i'j'} \to X_{pq} \xrightarrow{d} \Sigma X_{00} \to \Sigma X_{ij}.$$

The reader should note that for all standard triangulated categories, a simplex in



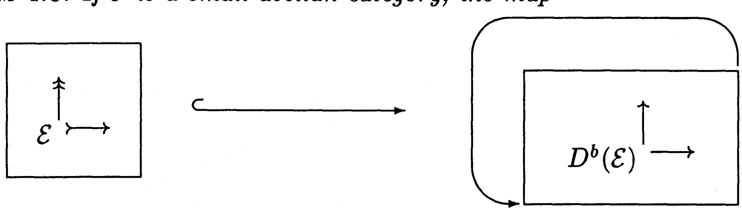
admits at least one coherent differential; however, I do not know whether this is a consequence of the axioms of triangulated categories.

There is still a map from Quillen's K–theory of the exact category $\mathcal{E}$, to the diagonal realization of

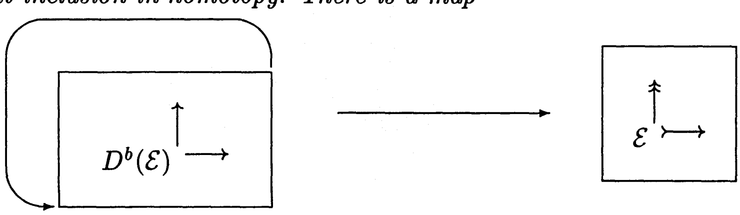


In this article we will prove

THEOREM 4.8. *If* $\mathcal{E}$ *is a small abelian category, the map*



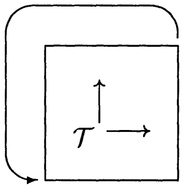
is a split inclusion in homotopy. There is a map



which is homotopy left inverse to it.

If we are willing to change some more the definition of the simplicial set we work with, we do better.

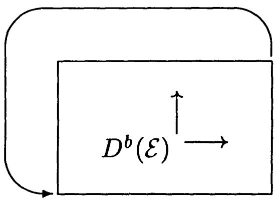
DEFINITION 0.5. *Suppose $\mathcal{T}$ is a small triangulated category. Let S be some non-empty class of biWaldhausen complicial categories in the sense of [10], whose homotopy categories are all $\mathcal{T}$. Define*



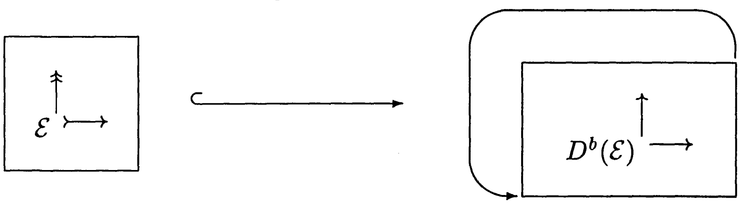
to be as in Definition 0.4, except that now we furthermore assume that any simplex has at least one lifting to a diagram of bicartesian squares in at least one model $s \in S$ for the triangulated category $\mathcal{T}$.

With this new definition of the simplicial set, we will prove:

THEOREM 7.1. *Let $\mathcal{E}$ be a small abelian category, and let*

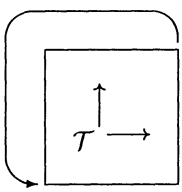


be as in Definition 0.5. Then the map



induces a homotopy equivalence.

REMARK 0.6. The particular model for the simplicial set



given in Definition 0.5 has several unpleasant properties. First, it is not functorial in the triangulated category $\mathcal{T}$. Given a triangulated functor, it is very unclear whether it takes a diagram admitting a lifting to a model to another diagram with a lifting.

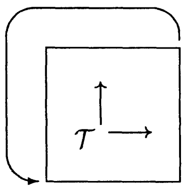
Note that the class S of permissible model categories is only a class, not a set. Typically, we would like to take S to be the class of all models, and even if $\mathcal{T}$ is small, the author sees no reason that the models for it should form a set. If we take S to be the collection of all models, it is perfectly clear that the construction depends only on the triangulated category $\mathcal{T}$, albeit non–functorially.

The most unpleasant property of the space defined in Definition 0.5 is that it is NOT an H–space. Given two simplices, they have in general liftings to different models. The direct sum will probably have no lifting at all. This is actually a serious shortcoming.

Theorem 7.1 tells us, among other things, that two abelian categories with isomorphic derived categories have isomorphic K–theory spaces. But with the construction of Definition 0.5, the isomorphism is not clearly an H–map. In particular, we have no reason to conclude that it is an infinite loop map of infinite loop spaces.

This problem is relatively easily rectified.

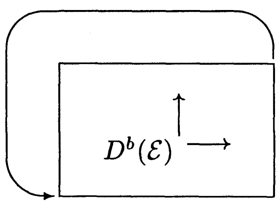
DEFINITION 0.7. *Suppose $\mathcal{T}$ is a small triangulated category. Let S be some non–empty class of biWaldhausen complicial categories in the sense of [10], whose homotopy categories are all $\mathcal{T}$. Define*



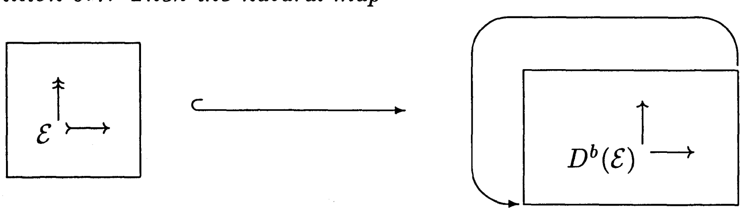
to be as in Definition 0.4, except that now we furthermore assume that any simplex has A DIRECT SUM DECOMPOSITION WHERE EACH SUMMAND has at least one lifting to a diagram of bicartesian squares in at least one model $s \in S$ for the triangulated category $\mathcal{T}$.

With this new definition of the simplicial set, we will also prove:

THEOREM 7.1. *Let $\mathcal{E}$ be a small abelian category, and let*



be as in Definition 0.7. Then the natural map



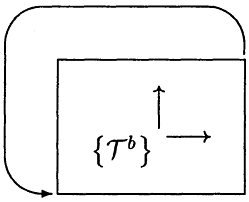
induces a homotopy equivalence.

REMARK 0.8. With this definition, it is clear we have an H–map of H–spaces; in fact, one sees easily that it is an infinite loop map of infinite loop spaces.

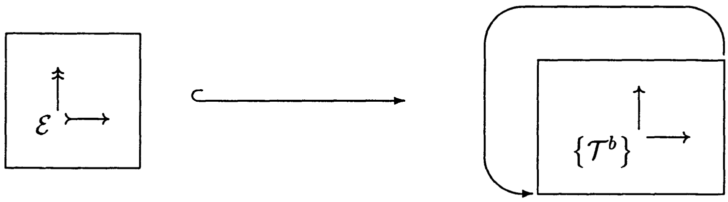
A stronger theorem is actually true.

STRONG THEOREM 7.1. *Let $\mathcal{T}$ be a small triangulated category with a t–structure. Assume $\mathcal{T}$ admits at least one biWaldhausen complicial model. Let S be a non–empty class of such models, as in Definitions 0.5 or 0.7. Let $\mathcal{E}$ be the heart of the t–structure.*

Let $\mathcal{T}^b$ be the bounded part of $\mathcal{T}$. If the t–structure is non–degenerate, $\mathcal{T}^b = \mathcal{T}$. Define the simplicial set



to be either the simplicial set of Definition 0.5 or the simplicial set of Definition 0.7. In either case, the natural map



induces a homotopy equivalence.

In the fifth part of this sequence, we will prove the Strong Theorem 7.1. But the proof is sufficiently complicated and requires setting up so much notation, that the author felt justified in giving first somewhat simpler proofs of weaker theorems.

It is unpleasant that, in the statement of the theorems, we need to assume the existence of models for $\mathcal{T}$, and that the simplicial set is defined in terms of some such class of models. We could try to prove a better statement. We begin with some definitions.

DEFINITION 0.9. *Let $\mathcal{T}$ be a triangulated category. A* candidate triangle *in $\mathcal{T}$ is a sequence*

$$x \xrightarrow{u} y \xrightarrow{v} z \xrightarrow{w} \Sigma x$$

in which the three composites

$$vu, \qquad wv, \qquad \{\Sigma u\}w$$

all vanish.

DEFINITION 0.10. *A candidate triangle is called* contractible *if it is isomorphic to a sum of three trivial candidate triangles*

$$\begin{array}{ccccccc} x & \xrightarrow{1} & x & \longrightarrow & 0 & \longrightarrow & \Sigma x \\ 0 & \longrightarrow & y & \xrightarrow{1} & y & \longrightarrow & 0 \\ \Sigma^{-1}z & \longrightarrow & 0 & \longrightarrow & z & \xrightarrow{1} & z \end{array}$$

Having defined candidate triangles, and defined which of them we view as contractible, it is now time to define morphisms between them. A morphism of candidate triangles is a commutative diagram

$$\begin{array}{ccccccc} x & \xrightarrow{u} & y & \xrightarrow{v} & z & \xrightarrow{w} & \Sigma x \\ {\scriptstyle f}\downarrow & & {\scriptstyle g}\downarrow & & {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle \Sigma f} \\ x' & \xrightarrow{u'} & y' & \xrightarrow{v'} & z' & \xrightarrow{w'} & \Sigma x' \end{array} .$$

With this definition, the collection of candidate triangles forms a category. We denote this category $CT(\mathcal{T})$. Given a morphism of candidate triangles as above, its mapping cone is the diagram

$$y \oplus x' \xrightarrow{\begin{pmatrix} -v & 0 \\ g & u' \end{pmatrix}} z \oplus y' \xrightarrow{\begin{pmatrix} -w & 0 \\ h & v' \end{pmatrix}} \Sigma x \oplus z' \xrightarrow{\begin{pmatrix} -\Sigma u & 0 \\ \Sigma f & w' \end{pmatrix}} \Sigma y \oplus \Sigma x'.$$

The mapping cone construction takes a morphism in $CT(\mathcal{T})$ to an object of $CT(\mathcal{T})$.

DEFINITION 0.11. *The subcategory $ST(\mathcal{T}) \subset CT(\mathcal{T})$ is the smallest full subcategory which*

0.11.1. *Contains all the distingushed triangles.*

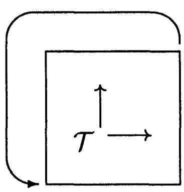
0.11.2. *Contains the mapping cone on any map of its objects.*

0.11.3. *If C is a candidate triangle in $ST(\mathcal{T})$, and C is isomorphic to a direct sum $C' \oplus C''$ with C' contractible, then $C'' \in ST(\mathcal{T})$.*

The objects of $ST(\mathcal{T})$ will be referred to as semi–triangles.

Now we define a simplicial set

DEFINITION 0.12. *Let $\mathcal{T}$ be a triangulated category. The simplicial set*



is a modified version of that given in Definition 0.4. As in Definition 0.4, the (p,q)–simplices are diagrams in $\mathcal{T}$

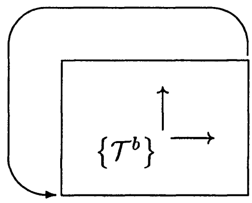
$$\begin{array}{ccccc} X_{p0} & \longrightarrow & \cdots & \longrightarrow & X_{pq} \\ \uparrow & & & & \uparrow \\ \vdots & & & & \vdots \\ \uparrow & & & & \uparrow \\ X_{00} & \longrightarrow & \cdots & \longrightarrow & X_{0q} \end{array}$$

together with a coherent differential $d : X_{pq} \to \Sigma X_{00}$. Here, a coherent differential $d : X_{pq} \to \Sigma X_{00}$ is a map such that for any $0 \le i \le i' \le p$, $\qquad 0 \le j \le j' \le q$,

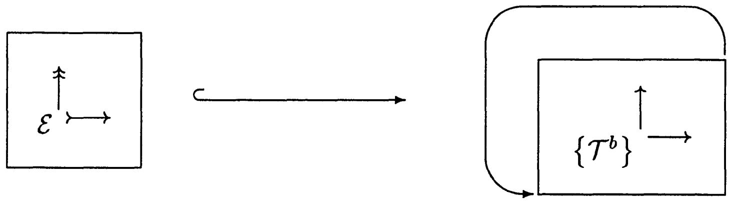
$$X_{ij} \to X_{i'j} \oplus X_{ij'} \to X_{i'j'} \xrightarrow{\phi} \Sigma X_{ij}$$

is a semi–triangle; in Definition 0.4 we insisted on it being a distinguished triangle.

CONJECTURE 0.13. *Let $\mathcal{T}$ be a small triangulated category with a t-structure. Let $\mathcal{E}$ be the heart of the t-structure. Let $\mathcal{T}^b$ be the bounded part of $\mathcal{T}$. Define the simplicial set*



as in Definition 0.12. Then the natural map



induces a homotopy equivalence.

REMARK 0.14. Let me explain the status of this conjecture. Four or five years ago, I believed I knew how to prove it. I gave talks claiming the result, and promised to write a careful account, developing the theory of semi–triangles and indicating how the proof of Conjecture 0.13 can be obtained by modifying the arguments in the existing articles.

I never wrote the article. Four years ago, there appeared to be little probability the work will ever be published, and there seemed little point in going to great effort. Since then, I have not really thought about the problem. I have largely forgotten what argument I had. All I have left are somewhat sketchy notes which I wrote for myself. In an Appendix, I include these notes. I no longer remember the subtle points of the argument. The notes are reproduced purely for the benefit of future workers. See Appendix A.

It seems relevant to note that the results of this article fly in the face of expert opinion at the time it was written. The experts had given up for hopeless the attempt to define a sensible K–theory for the derived category. In [5] the authors venture the opinion:

> "It seems that the derived category is too coarse even to recover from it the group K_1."

In a letter to me, Waldhausen offers the same opinion:

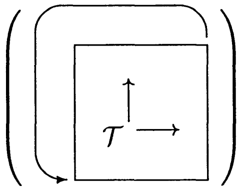
> "From my own experience I feel rather pessimistic about the prospects of constructing a K–theory of triangulated categories with the aim of, say, reconstructing the K–theory of a ring from the underlying derived category."

It is relevant to note the date on Waldhausen's letter. It was sent to me on October 4, 1988. In other words, Waldhausen wrote his letter after receiving my preprint, in response to a note that I enclosed with the preprint. He was not merely venturing an opinion on what might be true. He was telling me that he did not believe my result. Waldhausen was not alone; a number of referees said essentially the same in referees' reports. It is fair to say that it took two years before anyone read this article beyond the introduction.

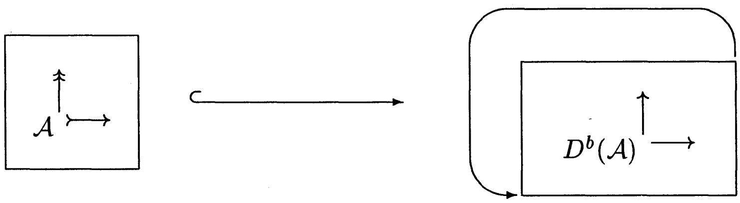
Unlike many others, Waldhausen was very kind to me. He invited me to visit him in Bielefeld and encouraged my work, even if he only half took it seriously. I would like to use this opportunity to thank him. I have not always been as grateful as I should have been.

Now it is natural to define the K-groups

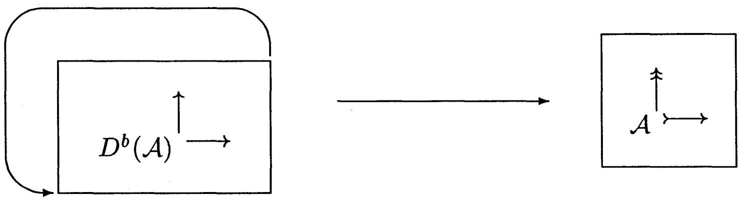
$$K_i(\mathcal{T}) \quad \stackrel{def}{\equiv} \quad \Pi_{i+1}$$



and Theorem 7.1 says that, for an abelian category $\mathcal{A}$, there is a natural isomorphism $K_i(\mathcal{A}) \cong K_i(D^b(\mathcal{A}))$. This isomorphism begins its life as a natural map



which can be proved to be a homotopy equivalence by a series of fairly involved arguments. It is natural to ask whether the inverse has a simple simplicial description. Is there a relatively straightforward simplicial map



giving the inverse? What makes this seem plausible is that such a map exists on the level of Π_1; there is a natural map

$$p : K_0(D^b(\mathcal{A})) \to K_0(\mathcal{A}),$$

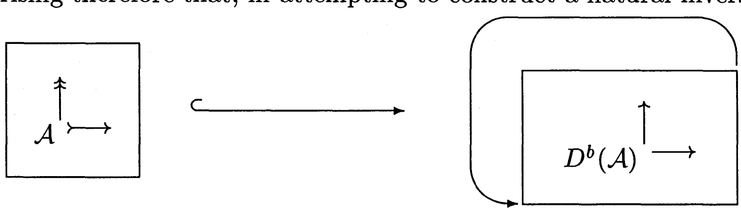
inverse to the map

$$K_0(I) : K_0(\mathcal{A}) \to K_0(D^b(\mathcal{A}))$$

induced by the inclusion $I : \mathcal{A} \hookrightarrow D^b(\mathcal{A})$. The map p takes an object X of $D^b(\mathcal{A})$ (which may be viewed as an element X in the abelian group $K_0(D^b(\mathcal{A}))$ and sends it to

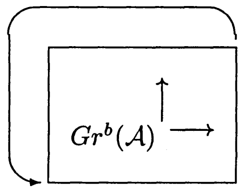
$$p(X) = \sum_{i \in \mathbb{Z}} (-1)^i H^i(X).$$

It is not surprising therefore that, in attempting to construct a natural inverse to



we will fall back on the homology functor $H : D^b(\mathcal{A}) \to \mathcal{A}$.

The form our theorem will take is quite general. Starting with any abelian category $\mathcal{A}$, one can construct a rather curious simplicial set, which we will denote



It is now time to define it; for more detail, the reader is referred to Construction 4.7.

DEFINITION 0.15. *Let $\mathcal{A}$ be an abelian category. Let $Gr^b(\mathcal{A})$ be the category of bounded graded objects in $\mathcal{A}$. That is, an object $X \in Gr^b(\mathcal{A})$ is a functor*

$$X : \mathbb{Z} \longrightarrow \mathcal{A}$$

where $\mathbb{Z}$ is the dicrete category of all integers. The boundedness asserts that X vanishes on all but finitely many $n \in \mathbb{Z}$. The morphisms is $Gr^b(\mathcal{A})$ are natural transformations.

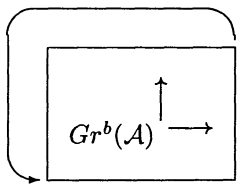
REMARK 0.16. The category $\mathbb{Z}$ admits a translation functor,

$$n \mapsto n+1.$$

Denote this functor by $\Sigma : \mathbb{Z} \longrightarrow \mathbb{Z}$. Then Σ induces a translation functor on $Gr^b(\mathcal{A})$, which we also denote by Σ. By definition,

$$\Sigma X = X \circ \Sigma.$$

DEFINITION 0.17. *Let $\mathcal{A}$ be an abelian category. The simplicial set*



is defined as follows. A (p,q)–simplex is a diagram in $Gr^b(\mathcal{A})$

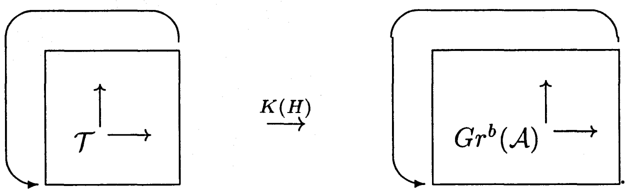
$$\begin{array}{ccccc}
X_{p0} & \longrightarrow & \cdots & \longrightarrow & X_{pq} \\
\uparrow & & & & \uparrow \\
\vdots & & & & \vdots \\
\uparrow & & & & \uparrow \\
X_{00} & \longrightarrow & \cdots & \longrightarrow & X_{0q}
\end{array}$$

together with a coherent differential $d : X_{pq} \to \Sigma X_{00}$. Here, a coherent differential $d : X_{pq} \to \Sigma X_{00}$ is a map such that for any $0 \le i \le i' \le p$, $0 \le j \le j' \le q$,

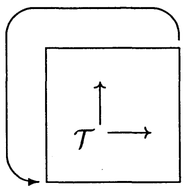
$$X_{ij} \to X_{i'j} \oplus X_{ij'} \to X_{i'j'} \xrightarrow{\phi} \Sigma X_{ij}$$

delivers a long exact sequence in $\mathcal{A}$.

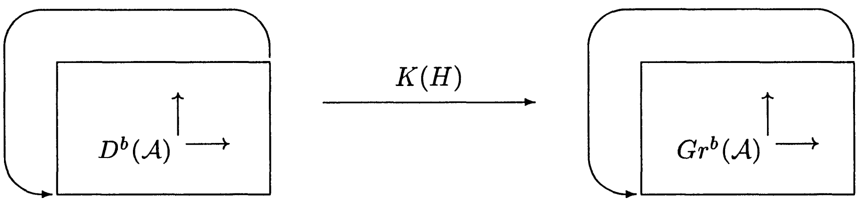
It is essentially trivial from the construction that, given any triangulated category $\mathcal{T}$ and any bounded homological functor $H : \mathcal{T} \to \mathcal{A}$, there is an induced simplicial map



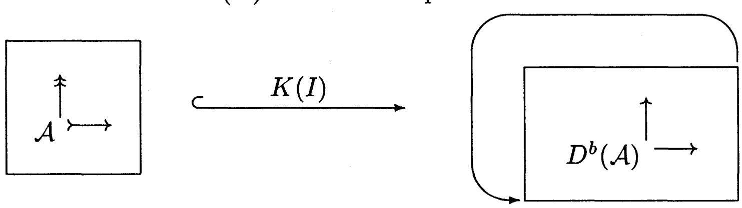
We remind the reader that a homological functor $H : \mathcal{T} \to \mathcal{A}$ is called *bounded* if for all X objects of $\mathcal{T}$, $H(\Sigma^n X) = 0$ if $n \ll 0$ or $n \gg 0$. We note also that the simplicial map $K(H)$ exists for any of the many definitions we have seen for



The construction of the map $K(H)$ is insensitive to the choice of model for triangulated K-theory. In particular, the ordinary homology functor $H : D^b(\mathcal{A}) \to \mathcal{A}$ induces a map

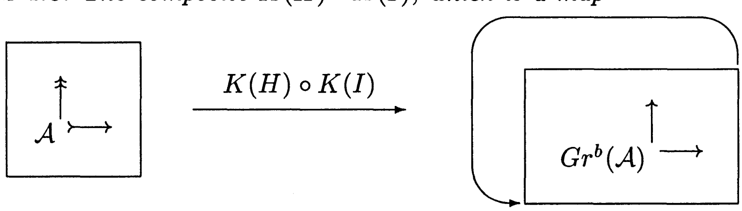


But the inclusion of $I : \mathcal{A} \hookrightarrow D^b(\mathcal{A})$ induces a map



We will prove:

THEOREM 4.8. *The composite $K(H) \circ K(I)$, which is a map*



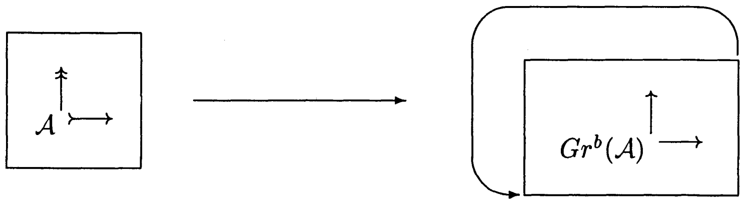
induces a homotopy equivalence.

An immediate corollary is

THEOREM 4.1. *A homological functor $H : \mathcal{T} \to \mathcal{A}$ induces a map $K_i(H) : K_i(\mathcal{T}) \to K_i(\mathcal{A})$, with strong naturality properties which are discussed more fully in Section 4.*

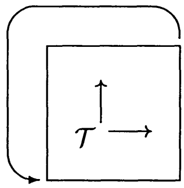
It turns out that Theorem 4.8 remains true for exact categories. That is

STRONG THEOREM 4.8. *Let $\mathcal{A}$ be any exact category. The natural map*



induces a homotopy equivalence.

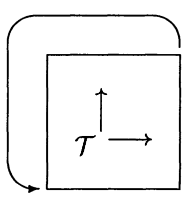
Now we have seen two theorems, Theorem 4.8 and 7.1. Theorem 7.1 came in many versions, depending on the definition of the simplicial set



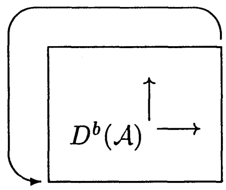
The next point we wish to make is that, in some sense that the author does not fully understand, all these theorems are really the same. More precisely, their proofs are almost all the same. In *K–theory for triangulated categories I, II* and *III,* I tried to stress this. The articles are therefore informal in style. Because we are proving more than one theorem at a time, the theorems are rarely precisely stated. The statements of the theorems, as well as large parts of the proof, are left as exercises to the reader. I try to explain the central idea of how the proof might go, without entering into the grubby detail of just which simplicial set we are dealing with.

Although I still believe that *K–theory for triangulated categories I, II* and *III* focus on the important questions, it is a matter of record that the expository style did not inspire confidence in the correctness of the results. Under duress, the author has also written two more formal articles, *K–theory for triangulated categories* $3\frac{1}{2}$ and $3\frac{3}{4}$. In those articles, the theorems are carefully stated and the proofs complete and detailed. The reader might want to start with one of the more formal articles. The author nevertheless believes that the informal articles *K–theory for triangulated categories I, II* and *III* contain more interesting mathematics. The rest of the introduction will focus on them. When I speak of "this article", I will be referring to parts I, II and III. Parts 3.5 and 3.75 amount to a very detailed account of some, but not all, of the results that may be found is parts I, II and III.

As I said, all the theorems have essentially the same proof. But it turns out that Theorem 4.8 is somewhat less technical to prove than Theorem 7.1, perhaps because it really does not involve triangulated categories. In some sense Theorem 4.8 is also the simplest and clearest result of the article. To prove Theorem 7.1, one needs to make some modifications to Definition 0.4 of the simplicial set



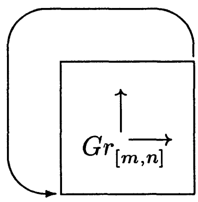
and/or make some restrictive hypotheses on the triangulated category $\mathcal{T}$. Theorem 4.8, on the other hand, is free of technical baggage. It says that with any reasonable modification of the definitions, the bisimplicial set



contains $Q(\mathcal{A})$ as a retract (up to homotopy). Since it is hard to imagine what the other factor could possibly be, Theorem 7.1 is not so surprising. The surprising thing is all the technical baggage one must carry to modify a proof of Theorem 4.8 to a proof of Theorem 7.1.

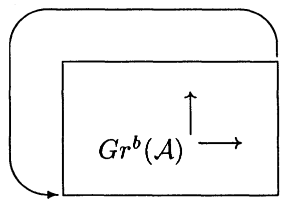
My colleague Nick Kuhn suggested it would be a good idea to sketch an outline of the proof in the introduction. Because the proof of Theorem 4.8 is less technical, we will sketch it. Here we go.

The objects of $Gr^b(\mathcal{A})$ come with a natural notion of length. A graded object $X \in Gr^b(\mathcal{A})$ is said to be supported on the interval $[m,n]$ if X vanishes on integers outside the interval. We denote by $Gr_{[m,n]}$ the full subcategory of $Gr^b(\mathcal{A})$, whose objects are supported in the interval $[m,n]$. It is possible to construct a K–theory for every $Gr_{[m,n]}$. Our notation for it will be



.

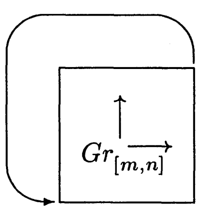
By definition, this is a simplicial subset of the larger



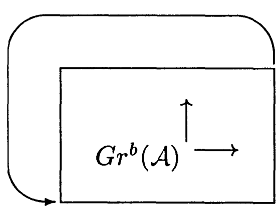
. The simplex

$$\begin{array}{ccccc} X_{p0} & \longrightarrow & \cdots & \longrightarrow & X_{pq} \\ \uparrow & & & & \uparrow \\ \vdots & & & & \vdots \\ \uparrow & & & & \uparrow \\ X_{00} & \longrightarrow & \cdots & \longrightarrow & X_{0q} \end{array}$$

lies in

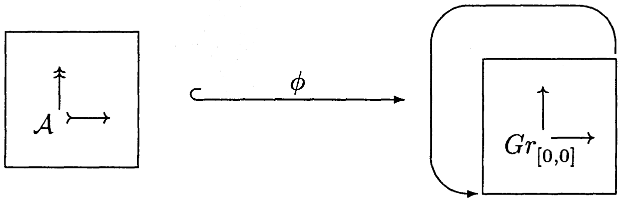


$\subset$

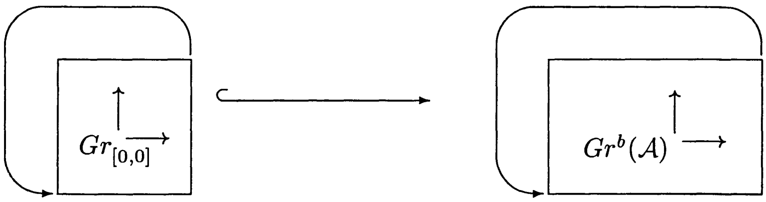


if and only if all the X_{ij} lie in $Gr_{[m,n]}$.

There is a natural inclusion

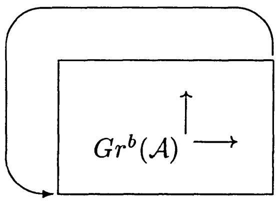


and it turns out to be very easy to show that ϕ is a homotopy equivalence; see Theorem 3.7. To complete the proof of Theorem 4.8 it suffices therefore to prove that the inclusion

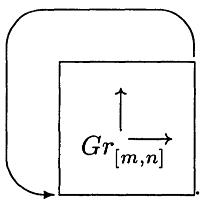


also induces a homotopy equivalence.

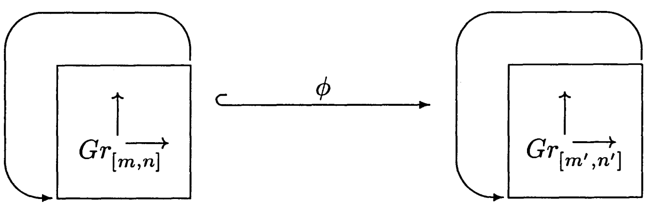
Given integers $m' \leq m \leq n \leq n'$, there are inclusions $Gr_{[m,n]} \hookrightarrow G_{[m',n']}$. The category $Gr^b(\mathcal{A})$ is the (directed) union of all these inclusions, and it follows easily from the definitions that the simplicial set



is the directed union of all the simplicial subsets

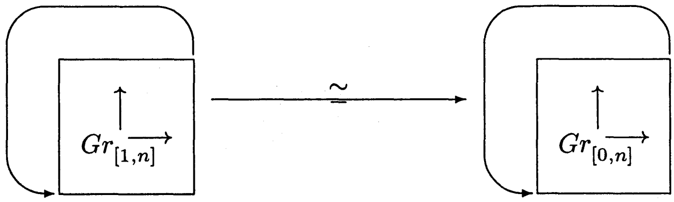


It therefore suffices to prove that, for any integers $m' \leq m \leq n \leq n'$, the inclusions

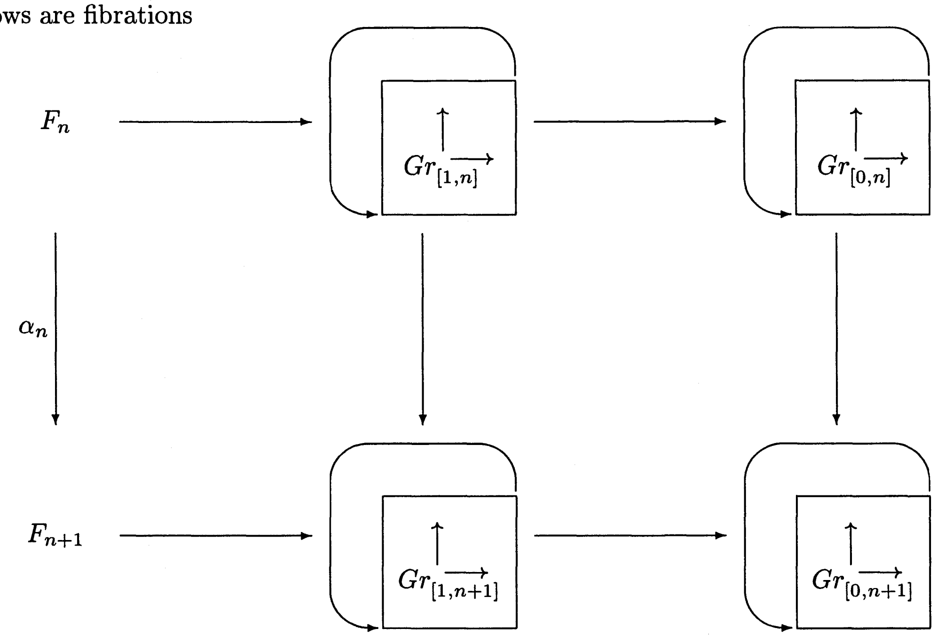


all induce homotopy equivalences. It clearly also suffices to consider the case where only one of the pairs of integers $\{m', m\}$ and $\{n, n'\}$ is actually a pair (that is, either $n = n'$ or $m' = m$). We also lose no generality by assuming that the pair of distinct ones differ by only 1. The two possibilities being dual to each other, we may assume $m = m' + 1$, and $n' = n$. Finally, by translating the complex, we reduce to the case $m' = 0$.

So we have to prove that the natural inclusion induces a homotopy equivalence



The strategy of the proof is the following. There is a commutative diagram, whose rows are fibrations



We will prove two lemmas:

LEMMA 1. (=Lemma 8.8) $\alpha_n : F_n \to F_{n+1}$ *induces a homotopy equivalence.*
LEMMA 2. (=Lemma 8.10) $\alpha_n : F_n \to F_{n+1}$ *is null homotopic.*

Concretely, we will find two simplicial models for the homotopy fiber F_n; one of these lends itself to the proof of Lemma 1, whereas the other is more useful for considering Lemma 2. The surprising fact is that Lemma 1 is true even for the construction without the differentials. This will allow us to say something about that construction.

REMARK 0.18. The proof we just sketched works only if $\mathcal{A}$ is an abelian category. The proof for exact categories is more delicate.

This summarises the proof we will find in part I. Parts II and III will deal with stronger versions of the theorem, and their proofs are more difficult to briefly outline.

When writing any article, the author must keep in mind the potential audience. For any article, the audience will divide into three groups, listed in order of probable size.

Group 1: The people who want a rough idea of the contents of the article, and at the very most a sketch of the proofs in an easy special case.

Group 2: The people who want to check the result, because they might consider using it in their own work.

Group 3: The people reading the article because they might work on the problem themselves.

In general, an author writing an article addresses mostly Groups 1 and 2, because they greatly outnumber any possible Group 3 audience. In particular, the referee will undoubtedly belong to Group 1 or Group 2, and if the article is to be published the author has to keep the referee happy. This is the way the present author has written most of his other work.

But nevertheless the author felt this result was different enough to justify a departure from the accepted norm. The article is most definitely written with a Group 3 audience in mind. Since it is impossible to keep everyone happy, Group 1 and Group 2 readers will have to have some patience. The author feels obliged to explain first why he thought that he should write the article for that particular audience, and then explain how, without an enormous effort, a Group 1 or Group 2 reader can still find the article worthwhile.

The first point that should be made clear is that the author does not view this article as the definitive last work on the subject—quite the opposite. The subject is embryonic. Until this article, no expert seriously thought it possible to define K–theory in terms of the derived category.

My contribution in this article is only a small first step. It shows that there is a K–theory one can define for a triangulated category, and this K–theory is sensible enough to agree with Quillen's, in the special case of the derived category of an abelian category. But so far the author has been unsuccessful in his attempts to prove functoriality properties for the new K–theory, even though there is reason to believe that it will have far nicer functorial properties than Waldhausen's K–theory of model categories. Part of the problem may be that we still do not understand triangulated categories well enough, and part may be that the homotopy theory that goes into the new arguments is really a quantum leap harder than in the classical arguments of Quillen and Waldhausen.

In any case, the author firmly believes that the subject is open to a great deal of future progress. In fifty to one hundred years from now, people will undoubtedly look back at this article and laugh at the clumsy arguments and bad notation. But in between, someone will have to think hard about the problem, and prove better theorems than I have here. What I tried to do in this article is to help the future someones working on this subject. I have written down essentially everything I know, providing motivation whenever possible, and giving on occasion more than one proof of one result.

There is no getting around the fact that this article is hard to read. We want to prove that some map is a homotopy equivalence. At some point, the proof has to degenerate into a sequence of intermediate maps and spaces, and we will have to prove the maps homotopy equivalences. Given the simplicial way in which the spaces are defined, we will at some point find ourselves constructing a long string of simplicial homotopies.

In this article, the going starts getting rough in Section 7. Sections 1-6 are "soft"; we begin with preliminaries. Sections 1 and 2 were lifted straight out of [3], and they are mostly a review of triangulated categories for the non-expert. Sections 3, 4, and 5 are mostly the definitions of the various simplicial sets we will study, with their very elementary properties discussed in some detail. The purpose of Sections 3, 4, and 5 is to familiarize the reader with the notation, and the general style of argument. If the reader will quickly leaf through the article, he will see lots of strange-looking little diagrams of squares, triangles, and arrows in bizarre configurations. These diagrams are a shorthand notation for the simplicial sets that come up in the proof.

People have attacked my notation a great deal. Let me say a few words in my own defence.

By training, I am neither K-theorist nor homotopy theorist. I came to Virginia in September 1987, and during the fall Giffen gave a number of talks on the foundations of K-theory, leading up to his delooping construction. I found all this very mysterious, and in January 1988, I asked Giffen if there was any way to construct K-theory directly from the derived category. Giffen became excited, and we started working together. Until January 1988, I had never read Quillen's paper on K-theory, and was blissfully ignorant of the geometric realization of a simplicial nerve of a category.

In the following months I worked very hard, and this article was finished and ready to send off to the referee by late September of 1988. In other words, this article was written by somebody who had known nothing about the subject nine months prior to finishing it.

I have no doubt that an expert could present the results better than me. Undoubtedly, experts can and will find a notation better adapted for writing down the myriad simplicial homotopies that make up the core of the proof. But there are two observations one should nevertheless make. Because there are so few homotopies to work with, one is forced to write strings of spaces and maps which are much longer than is typical of a proof by Quillen or Waldhausen. One needs to have a notation flexible enough to be able to provide symbols for all of the spaces and maps that occur, but simple enough to be readable. This is quite a tall order. The notation I adopt may be less than perfect, but a perfect notation may prove hard to find, even for experts.

The second observation is that it is much too early in the development of the theory to be concerned about the optimal presentation of the results. Before one sets out to develop the perfect notation for the exposition of this theory, there should be some more theory to expose.

I promised that I would tell the Group 1 and Group 2 audiences how to read this article, to get at least the flavor of the theory without spending an enormous effort. In this I have tried to offer the reader some help by dividing up the article into parts. This part, *K-theory for triangulated categories I,* is intended primarily for a Group 1 audience. It sets up the notation, and then gives the simplest proof of the weakest useful version of Theorem 7.1. Thus, after the introductory material of Sections 1–6, Sections 7 and 8 will be devoted to the proof of Theorem 4.8. A Group 1 reader (a reader who only wants the idea of the proof, seeing at most one simple special case discussed in detail) is advised to read *K-theory for triangulated categories I* and then quit.

The next part, *K-theory for triangulated categories II,* is for the exclusive use of Group 3 readers—readers interested in working on the subject. When someone writes a long and very difficult paper, he has an obligation to tell the reader why the proof must be so complicated. *K-theory for triangulated categories II* was written to achieve just exactly that. Precisely, we study at some length the construction without the differentials. There is good reason to believe that this construction is for the birds; it is therefore very instructive to see how close the proof comes to showing that its homotopy agrees with the homotopy of the construction with the differentials. The two simplicial sets are difficult to distinguish. The arguments in the proof that tell them apart are remarkably subtle. This is a warning to the reader. An argument that is too simple will probably fail to distinguish the construction with the differentials from the construction without, and should therefore be viewed as suspect.

The third part, *K–theory for triangulated categories III,* was written with Group 2 in mind. The reader may well want to carefully check the result, without being troubled with subtleties about similar constructions that fail, and what they signify regarding the delicacy of the proof. Such a reader should omit *K–theory for triangulated categories II* and go directly to *K–theory for triangulated categories III.* This part has the best version of Theorem 7.1, and the proof is careful and complete, but without any unnecessary frills.

In order to keep the cross–references in the articles from getting out of hand, we will write Theorem II.3.4 for Theorem 3.4 of the second part of the series. When the reference and the theorem lie within the same part, we will allow ourselves to drop the Roman numerals. Thus Theorem 3.4 stands for Theorem 3.4 in whatever part you are currently reading.

I sincerely hope that someone finds a simpler proof, involving fewer intermediate spaces and maps. In the revised version of the article I have tried to explain how I arrived at the particular sequence of spaces considered in this article. Starting from Section 7, at the beginning of each section there is a "Motivation" introduction, explaining how the proof should have worked and what difficulties I ran into. In the section itself is a modified version of the argument given in the Motivation, often very substantially modified, which has the virtue that it is a rigorous proof. In some sense my reasons for arriving at the proof I give are entirely irrelevant to the proof. But several people have complained that the article was nearly unreadable, and the author felt that the Motivation paragraphs might make the construction more transparent. The original exposition was very terse, probably partly as a reaction to [3], which had been all motivation and no proof.

Now I come to the acknowledgements. First and foremost I must acknowledge the help I received from Giffen. Sections 1 and 2 were lifted straight out of [3], as was the idea to prove the theorem by induction. But my debt to Giffen goes far beyond that. It is fair to say that everything I know about classical K–theory I learned from Giffen.

I should perhaps explain why the collaboration with Giffen fell apart. When we discovered the error in our joint article [3], we naturally wanted to fix it. But we had very different ideas about how to proceed. Giffen felt the correct proof should follow method A, while I believed, equally strongly, that only method B could work. To be fair to Giffen, he had the backing of Thomason. Even long after I proved my theorem, and after he had checked it carefully and believed the result, Thomason kept telling me that a good proof of the theorem would proceed along the lines of method A of Giffen. I continue to be convinced that method A has no chance of working. In fact, I wrote *K–theory for triangulated categories II* largely to show that. When Thomason would propose a major simplification of the proof, I would point out that such a simplification would prove the stronger theorem for the construction without the differentials, as in part II. Needless to say, part II was Thomason's least favorite part of the article. But it is relevant to note that, three or four years after Giffen and I had these disagreements, Thomason was telling me exactly what Giffen had said four years earlier. Maybe they are both right and I am just a stubborn fool. I do not know if Giffen is still working on trying to push his ideas through. If Thomason were still alive, such an effort would undoubtedly have his blessing.

In the introduction, there were several allusions to the history of the article, which is very unusual. It seems only fair to give a review of this history. In this review, I try to stick to the facts and offer very little commentary. The names of individuals are omitted, except when I feel they have acted commendably. I mention by name only Thomason, Deligne, Faltings and Franke.

Since I will be offering very little commentary, I should say now what was commendable about the people mentioned above. Each went to some considerable effort to check a difficult result. It was a result they did not start out believing. This shows an admirable intellectual honesty. Having said this, I will now stick to the narrative, without offering my judgements.

I have submitted different versions of the result to many places. Only four of the places treated the article at all seriously. These were, in chronological order,

Publications Math. IHES
Journal K–Theory
Annals of Math.
Inventiones Math.

The version Giffen and I submitted to *Publications Math. IHES* in April 1988 contained a serious gap; this has been extensively discussed in the introduction. The article was rejected. When I fixed the gap, I resubmitted, in September 1988. The article was now much longer, and *Publications Math. IHES* did not want to handle it. It was suggested I submit to *Journal K–Theory.*

I submitted different manuscripts to *Journal K–Theory* three times. The first time the work was rejected on a technical point, within four weeks. The next two times Thomason was the referee, and each time he rejected the article because he believed he had found a major error. The first time Thomason rejected the article, the editor was very kind and encouraged me to submit an improved manuscript. The second time he was less kind. In his rejection email, which I have unfortunately not kept, he told me that now the scientific merit of my work had been decided; I was obviously a charlatan, and under no circumstances should I ever submit the article back to *Journal K–Theory.*

Thomason was more willing to be open minded. He invited me to Paris in May 1992, for an interrogation. He was convinced there had to be an error, and was determined to find it. He did not spare himself, and he certainly was not about to spare me. We met in his office for many hours every day, going into the minutest detail of the proof of the theorem he wanted to see. After a week and a half he pronounced himself satisfied.

By then, of course, it was impossible to resubmit to *Journal K–Theory.* The next place I submitted it was *Annals of Mathematics.* To be completely accurate, I did not even wait until after meeting Thomason to submit the articles to the *Annals.* I submitted the result on May 9, 1992. The papers were rejected almost immediately, on May 13, 1992. In his rejection letter Deligne tells me that the result is interesting and "if true, worth publishing in the *Annals of Mathematics.*" He goes on to say that he has read "a little more than the introduction", and based on what he read, it is already obvious that the article does not contain a proof of the theorem I claim.

On June 3, 1992 I resubmitted the then current versions. By then, I had Thomason to back up my assertion that I knew how to prove a theorem. There were three parts, *K–theory for triangulated categories* 1, 2 and 3. On November 10, 1992 Deligne sent me a rejection for the articles. He refereed the articles himself, and did not believe the result. In his own words:

> The most important reason [for the rejection] is that I have strong doubts about the correctness both of the proof and of the statements you make.

The second major reason Deligne gives for the rejection is that, as written, the articles make it nearly impossible for anyone to check the results.

Since in a footnote Deligne mentioned Theorem 4.8 on pp. 63–64 as one of the statements he doubts, I wrote *K–theory for triangulated categories* 3.5 to give a detailed, checkable proof. I submitted this manuscript to the *Annals of Mathematics* on May 28, 1993. The article was rejected December 6, 1993, because the referee thought he had found a major mistake, on page 35. The rejection letter quotes the referee: "the main result is certainly interesting, and if correct deserves publication."

It turns out there was no mistake, only a misunderstanding of what the proof said. I sent to the *Annals* a detailed explanation of why there was no mistake on page 35, to be forwarded to the referee. I never received an answer. In March 1994, Deligne was kind enough to listen to me present the result orally. On August 30, 1994 I sent to the *Annals of Mathematics* a revised version of *K–theory for triangulated categories* 3.5. The order in which the lemmas are presented was permuted, and some subtle points were elaborated, for example the one where the referee thought he had found a mistake. On September 21, 1994 I submitted also *K–theory for triangulated categories* 3.75, the proof presented to Thomason.

In September 1994, the *Annals of Mathematics* asked Thomason to be the referee. I happen to know this, because Thomason sent me a copy of his email, declining Deligne's request to referee either article. In his refusal he says that, having gone through the argument carefully orally, he is unwilling to check through the written version. He goes on to describe the main theorem of *K–theory for triangulated categories* 3.75 as "very striking", "at the borderline of statements for which there are well–known counterexamples". Thomason concludes his email by telling Deligne that, in his opinion, the result of *K–theory for triangulated categories* 3.75 (the result Thomason checked orally) is better than the theorem of *K–theory for triangulated categories* 3.5 (the result Deligne checked orally).

The *Annals of Mathematics* rejected both articles on February 5, 1996, almost a year and a half later. In the rejection letter they mention that the papers went to a referee (presumably someone they contacted after Thomason declined the job). The referee sent a preliminary report, expressing doubts "that the organisation of the manuscript is reasonably economical". The rejection letter goes on to say that, despite contacting the referee "numerous times", the *Annals of Mathematics* was "unable to receive a formal report".

This left the *Annals of Mathematics* with no choice but to reject the articles. They promise they will keep pressing the referee for a "more constructive report", and suggest I try to publish the result elsewhere.

Let us suppose the *Annals of Mathematics* got in touch with the current referee sometime in October or November 1994. It is now July 1997, more than two and a half years later. I am still waiting for the "more constructive report", and from time to time I send Deligne an email to remind him of the fact.

The next journal to seriously consider the articles was *Inventiones Mathematicae.* On February 28, 1996 I sent to Faltings *K–theory for triangulated categories* 3.75. Faltings gave it to Jens Franke to referee. Franke refereed the article and found it correct. But on November 25, 1996 Faltings rejected the article. In the rejection letter he tells me that the good news is that the referee believes the result, the bad news is that the backlog in *Inventiones* is so long that he cannot at the moment accept an article as long as mine. He concludes by saying he felt it was important that someone should check the result carefully, and that he hopes the endorsement will be useful.

This completes the account of the four journals that took the results seriously. I have also sent versions of this result to other journals, which rejected them essentially out-of-hand. Aside from the journals already mentioned, I have sent articles to

Journal of Pure and Applied Algebra
Ann. Sci. Ecole Normale Superieure
Acta Mathematica
Memoirs AMS
Memoires Soc. Math. France
Lecture Notes in Math.
American J. Math.

often submitting different articles to the same journal on more than one occasion. The only common feature to these journals is that they sometimes publish very long articles. The pattern I have observed is that whereas prestigious journals like the *Annals of Mathematics* or *Inventiones Mathematicae* take the results seriously, less prestigious ones reject the work very quickly. For example, *Journal of Pure and Applied Algebra* took only three days. If you wonder how this is possible, it was all done very efficiently, by email.

I should make precise what I mean by rejecting the articles "out of hand". Each of the seven journals mentioned above rejected the articles in a matter of a few weeks. The only time I received a referee's report was the third time I submitted a manuscript to *American J. Math.* This unique referee's report was very brief, and clearly based only on a reading of the introduction.

A recent example is the *Lecture Notes in Mathematics.* The rejection letter was very kind. It explains to me in detail that the *Lecture Notes* faces financial pressures. Their sales have dropped, forcing them to raise prices. They were forced to reduce their output from ≥ 60 volumes in 1989 to ≤ 30 in 1991. If they publish esoteric articles like mine, with a very limited readership, then many libraries will cancel their subscriptions.

The rejection letter suggests I try to break up the result into smaller articles. But I have tried; there is a small article, *Loop spaces for the Q-construction,* which proves a modest result in 35 pages. I have not been able to publish it anywhere. The best I could get was a promise from van der Geer that, if my longer articles get published somewhere, then he is willing to reconsider *Loop spaces for the Q-construction* for *Compositio.*

One of the consequences of this tortuous history has been that there were results I never carefully checked. The articles looked for a long time like they would never be published. There seemed little point in writing yet more. So although I had some ideas about proving better theorems, the motivation to check these carefully was missing. For the last four years, I have not thought about the problems. In the meantime I have forgotten what I once may have known. All I now have are some sketchy notes.

In an appendix to part I (see Appendix A), we will talk about semi-triangles. In appendices to part III, we will include some even sketchier ideas; the appendices should not be viewed as theorems the author claims to have carefully checked.

I am very grateful to Chai, Dold, Mumford and more recently Faltings for their help and encouragement. Without them I would have given up years ago. I am also grateful to Yau for agreeing to accept the series of articles for the *Asian Journal of Mathematics.*

1. Triangulated categories and exact subcategories.

DEFINITION 1.1. *An additive category $\mathcal{T}$ is called a triangulated category if it comes equipped with:*

1.1.1. *An automorphism $\Sigma : \mathcal{T} \to \mathcal{T}$ called the suspension functor;*

1.1.2. *A collection of "triangles"; i.e. sextuples (X, Y, Z, u, v, w) where X, Y, and Z are objects of $\mathcal{T}$, u, v, w are morphisms $u : X \to Y$, $v : Y \to Z$, and $w : Z \to \Sigma X$.*

A morphism of triangles is a triple $(f; g; h)$ of morphisms in $\mathcal{T}$, rendering commutative the diagram

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & \Sigma X \\ {\scriptstyle f}\downarrow & & {\scriptstyle g}\downarrow & & {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle \Sigma f} \\ X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z' & \xrightarrow{w'} & \Sigma X' \end{array} .$$

These data must further satisfy the following compatibilities.

1.1.3. **[TR1]:** *Every sextuple isomorphic to a triangle is a triangle. Every morphism $u : X \to Y$ can be embedded in a triangle (X, Y, Z, u, v, w). The sextuple $(X, X, 0, 1_X, 0, 0)$ is a triangle.*

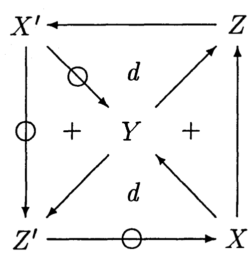
1.1.4. **[TR2]:** *(X, Y, Z, u, v, w) is a triangle if and only if $(Y, Z, \Sigma X, v, w, -\Sigma u)$ is.*

1.1.5. **[TR3]:** *Given two triangles (X, Y, Z, u, v, w) and (X', Y', Z', u', v', w') and a commutative square*

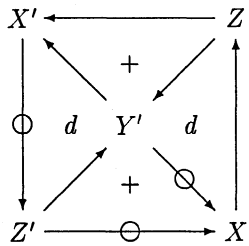
$$\begin{array}{ccc} X & \xrightarrow{u} & Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ X' & \xrightarrow{u'} & Y' \end{array}$$

there exists a map $h : Z \to Z'$ which completes the above to a morphism of triangles.

1.1.6. **[TR4] (The octahedral axiom):** *Given the following diagram*



There exists a way to construct a diagram



1.1.7. **Notation**. The arrows labeled with a circle are of degree one: a map

$$X \xrightarrow{\ \ominus\ } Y$$

means a morphism $X \to \Sigma Y$. The triangles marked + are commutative, while the ones marked d are distinguished; they are the "triangles" in the sense of Definition 1.1.2. The symbols + and d are used as in [1]. The notation for morphisms of degree one is different from [1], but is more in line with the conventions of this article, where we feel free to introduce curious–looking arrows to a stand for the various possible classes of morphisms we are led to consider.

1.1.8. **Hidden Hypothesis** Furthermore, the two composites

$$Y \to Z \to Y' \quad \text{and} \quad Y \to Z \to Y'$$

agree, as do the two composites

$$Y' \to \Sigma X \to \Sigma Y \quad \text{and} \quad Y' \to X' \to \Sigma Y$$

It should perhaps be noted that axiom [TR4] has an equivalent formulation, which is perhaps more in keeping with the ideas of this article. Given a morphism of triangles, one can construct its "mapping cone": i.e. given a morphism

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & \Sigma X \\ {\scriptstyle f}\downarrow & & {\scriptstyle g}\downarrow & & {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle \Sigma f} \\ X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z' & \xrightarrow{w'} & \Sigma X' \end{array}$$

there is a chain complex (*)

$$Y \oplus X' \xrightarrow{\begin{pmatrix} -v & 0 \\ g & u' \end{pmatrix}} Z \oplus Y' \xrightarrow{\begin{pmatrix} -w & 0 \\ h & v' \end{pmatrix}} \Sigma X \oplus Z' \xrightarrow{\begin{pmatrix} -\Sigma u & 0 \\ \Sigma f & w' \end{pmatrix}} \Sigma Y \oplus \Sigma X'$$

which may or may not be a triangle. [TR4] is equivalent to [TR4'].

1.1.9. **[TR4']**: *Given a commutative diagram*

$$\begin{array}{ccc} X & \xrightarrow{u} & Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ X' & \xrightarrow{u'} & Y' \end{array}$$

as in 1.1.5 = [TR3], then the map h, whose existence is guaranteed by [TR3], may be chosen so that (*) is a triangle.

The proof that $[TR1]+[TR2]+[TR3]+[TR4]$ imply $[TR4']$ is Theorem 1.8 in [7]. The proof that $[TR1]+[TR2]+[TR3]+[TR4']$ imply $[TR4]$ is Lemma 5.1 in [8].

The theory of triangulated categories has been expounded at length elsewhere. We refer the reader to [4], [1] and [11] for details. We only wish to remind the reader of three examples.

EXAMPLE 1.2. The *Spanier-Whitehead category* (also called the stable homotopy category). The objects are desuspensions of finite CW-complexes (i.e. $\Sigma^{-m}X$, where X is a finite CW complex). By definition

$$Hom_{\mathcal{T}}(X,Y) = \lim_{n\to\infty}[\Sigma^n X, \Sigma^n Y]$$

where $[X,Y]$ is the set of homotopy classes of maps $X \to Y$. The suspension functor Σ is obvious. Triangles are by definition sequences isomorphic to

$$\Sigma^{-2k}(X) \xrightarrow{f} \Sigma^{-2k}(Y) \to \Sigma^{-2k}(C_f) \to \Sigma^{-2k+1}X$$

where C_f is the mapping cone of the map $f: X \to Y$.

The axioms are easy to verify; the octahedral axiom is the statement that, given maps $f: X \to Y$ and $g: Y \to Z$, then ΣC_{gf} is the cone of the natural map $C_g \to \Sigma C_f$.

EXAMPLE 1.3. The category $\mathcal{K}(\mathcal{A})$. Let $\mathcal{A}$ be an additive category. The objects of $\mathcal{K}(\mathcal{A})$ are chain complexes of objects of $\mathcal{A}$, the morphisms in $\mathcal{K}(\mathcal{A})$ are the homotopy equivalence classes of chain maps. $\Sigma: \mathcal{K}(\mathcal{A}) \to \mathcal{K}(\mathcal{A})$ is left translation of a complex, with the sign of the differential switched. For any map $X \to Y$, the sequence $X \to Y \to C(X \to Y) \to \Sigma X$ is a triangle, where $C(X \to Y)$ stands for the mapping cone on $X \to Y$. The triangles are all sextuples isomorphic to ones obtained as above.

EXAMPLE 1.4. The category $D(\mathcal{A})$. If $\mathcal{A}$ is abelian, we may invert the quasi-isomorphisms in $\mathcal{K}(\mathcal{A})$, i.e. the maps inducing isomorphisms in homology. More generally, if $\mathcal{A}$ is an exact category, we may invert maps $X \to Y$ provided the cone $C(X \to Y)$ is acyclic. The category obtained from $K(\mathcal{A})$ by formally inverting the quasi–isomorphisms is called the derived category of $\mathcal{A}$, and denoted $D(\mathcal{A})$.

DEFINITION 1.5. *Let $\mathcal{T}$ be a triangulated category, $\mathcal{A}$ an abelian category. A* homological functor *$H : \mathcal{T} \to \mathcal{A}$ is a functor which takes triangles to long exact sequences: i.e. if (X,Y,Z,u,v,w) is a triangle, then*

$$H(X) \to H(Y) \to H(Z)$$

is exact. More generally, one could define the concept of homological functors $H : \mathcal{T} \to \mathcal{A}$ even when $\mathcal{A}$ is only an exact category.

EXAMPLE 1.6. If $X \in Ob(\mathcal{T})$, then $Hom_{\mathcal{T}}(X,-)$ is a homological functor, where $\mathcal{A}$ is the category of abelian groups.

EXAMPLE 1.7. If $\mathcal{T} = \mathcal{K}(\mathcal{A})$ or $D(\mathcal{A})$, $\mathcal{A}$ abelian, then $H(X) = H^0(X)$ defines a homological functor.

DEFINITION 1.8. *A full subcategory $\mathcal{S} \subset \mathcal{T}$ is called* exact *if whenever (X,Y,Z,u,v,w) is a triangle, $X \in \mathcal{S}$ and $Z \in \mathcal{S} \Rightarrow Y \in \mathcal{S}$.*

REMARK 1.9. In the original version of this paper, I referred to such subcategories as thick. Thomason strongly objected, and I have agreed that another term might be better.

The terminology in the literature is actually quite bad. If $\mathcal{A}$ is an abelian category, it is customary to refer to a number of types of full subcategories.

1.9.1. *Abelian subcategories. These are subcategories which are closed with respect to the formation of kernels and cokernels.*

1.9.2. *Thick subcategories. These are the categories closed with respect to extensions.*

1.9.3. *Serre subcategories. These are the abelian categories closed with respect to the formation of extensions and subquotients.*

In K–theory, the most important are the thick subcategories. Quillen's formalism of exact categories is an axiomatic description of categories capable of being embedded in abelian categories as thick subcategories. Exact categories provide a useful framework for defining K–theory.

Naturally, we want to generalize all this to the triangulated setting. Let $\mathcal{T}$ be a triangulated category, $\mathcal{S}$ a full subcategory. Then

1.9.1.' $\mathcal{S}$ is called triangulated if it is closed with respect to the formation of triangles; the mapping cone on any morphism in $\mathcal{S}$ is in $\mathcal{S}$.

It would be natural to go on and define

1.9.2.' $\mathcal{S}$ is thick if it is closed under extensions, as in Definition 1.8.

1.9.3.' $\mathcal{S}$ is a Serre subcategory if it is triangulated, and any direct summand of an object in $\mathcal{S}$ is in $\mathcal{S}$.

The reason 1.9.3' is a natural definition is that such $\mathcal{S}'s$ are precisely kernels of triangulated functors. Given a triangulated functor $F : \mathcal{T} \to \mathcal{T}'$, i.e. a functor of triangulated categories commuting with the suspension and taking triangles to triangles, define ker(F) to be the full subcategory of $\mathcal{T}$ whose objects X satisfy $F(X) \cong 0$. Then ker(F) satisfies the hypothesis of 1.9.3'. Given any $\mathcal{S}$ satisfying the hypothesis of 1.9.3', one can form a triangulated category $\mathcal{T}/\mathcal{S}$ and a triangulated functor $F : \mathcal{T} \to \mathcal{T}/\mathcal{S}$, whose kernel is precisely $\mathcal{S}$. This is closely parallel to the construction of the quotient of an abelian category by a Serre subcategory.

The unfortunate terminological fact is that categories satisfying 1.9.3' are called épaisse subcategories of $\mathcal{T}$, not Serre subcategories. With my abysmal knowledge of French I did not realize that épaisse is the French for thick, and I went ahead and made 1.9.2' a definition. I have to thank Thomason for some basic linguistic instruction. Evidently, the people who wrote in English about abelian categories translated Gabriel's *sous catégorie épaisse* to *thick subcategory*. But the same translation was not made by the people writing about triangulated categories in English.

It would be interesting to have a description of exact subcategories of a triangulated category $\mathcal{T}$ which, like Quillen's description in the abelian case, is free of the embedding into $\mathcal{T}$.

EXAMPLE 1.10. Let $H : \mathcal{T} \to \mathcal{A}$ be a homological functor. We define $\mathcal{T}_{(H;[n,m])} \subset \mathcal{T}$ to be the full subcategory of $\mathcal{T}$ whose objects are

$$Ob(\mathcal{T}_{(H;[n,m])}) = \{X \in \mathcal{T} | H(\Sigma^q X) = 0 \text{ unless } n \leq q \leq m\}.$$

The long exact sequence for H immediately establishes that $\mathcal{T}_{(H;[n,m])}$ is exact, in the sense of Definition 1.8. If H is understood from the context, we will sometimes omit it. Thus $\mathcal{K}(\mathcal{A})_{[n,m]}$ will mean $\mathcal{K}(\mathcal{A})_{(H;[n,m])}$ where H is the cohomology functor of Example 1.7; similarly for $D(\mathcal{A})_{[n,m]}$. We will also use the notation

$$\begin{aligned} D^b(\mathcal{A}) &= \bigcup_{(n,m)\in\mathbb{Z}^2} D(\mathcal{A})_{[n,m]} \\ D^+(\mathcal{A}) &= \bigcup_{n\in\mathbb{Z}} D(\mathcal{A})_{[n,\infty)} \\ D^-(\mathcal{A}) &= \bigcup_{n\in\mathbb{Z}} D(\mathcal{A})_{(-\infty,n]} \end{aligned}$$

REMARK 1.11. Although in Example 1.10 we seemed to suppose that $\mathcal{A}$ be abelian, it suffices if $\mathcal{A}$ is exact. One can no longer define the cohomology functor directly, as a functor from $D(\mathcal{A}) \to \mathcal{A}$, but the problem is easily remedied.

Let us conclude by quoting one standard result which we shall heavily rely on:

LEMMA 1.12. *Let (X, Y, Z, u, v, w) and (X', Y', Z', u', v', w') be triangles. Let $(f; g; h)$ be a morphism of triangles. If f and g are isomorphisms, so is h. (Of course, h need not be unique).*

Proof. See [4], Proposition 1.1(c), p.23.

2. Mayer-Vietoris squares in triangulated categories. Let $\mathcal{T}$ be a triangulated category. A commutative square

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle g} \\ Y' & \xrightarrow{g'} & Z \end{array}$$

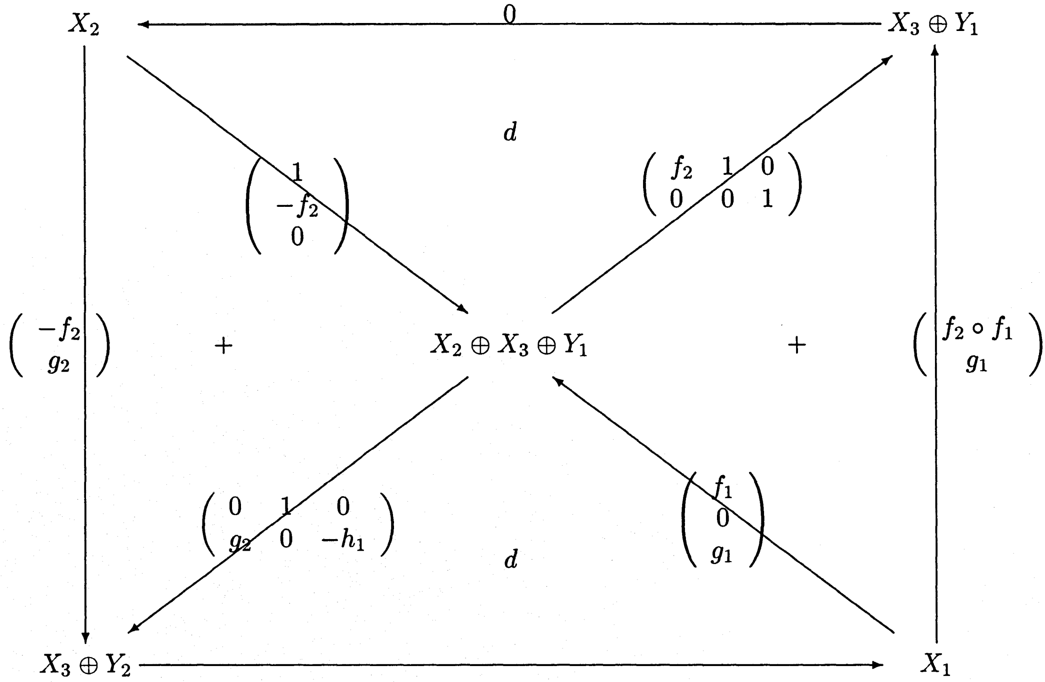
is called *Mayer-Vietoris*, or simply $M - V$, if $X \xrightarrow{\begin{pmatrix} f \\ f' \end{pmatrix}} Y \oplus Y' \xrightarrow{\begin{pmatrix} g & -g' \end{pmatrix}} Z$ is part of a triangle; i.e. if there exists $w : Z \to \Sigma X$ making the sextuple $\left(X, Y \oplus Y', Z, \begin{pmatrix} f \\ f' \end{pmatrix}, \begin{pmatrix} g & -g' \end{pmatrix}, w\right)$ a triangle.

LEMMA 2.1. *The composite of two $M - V$ squares is $M - V$.*

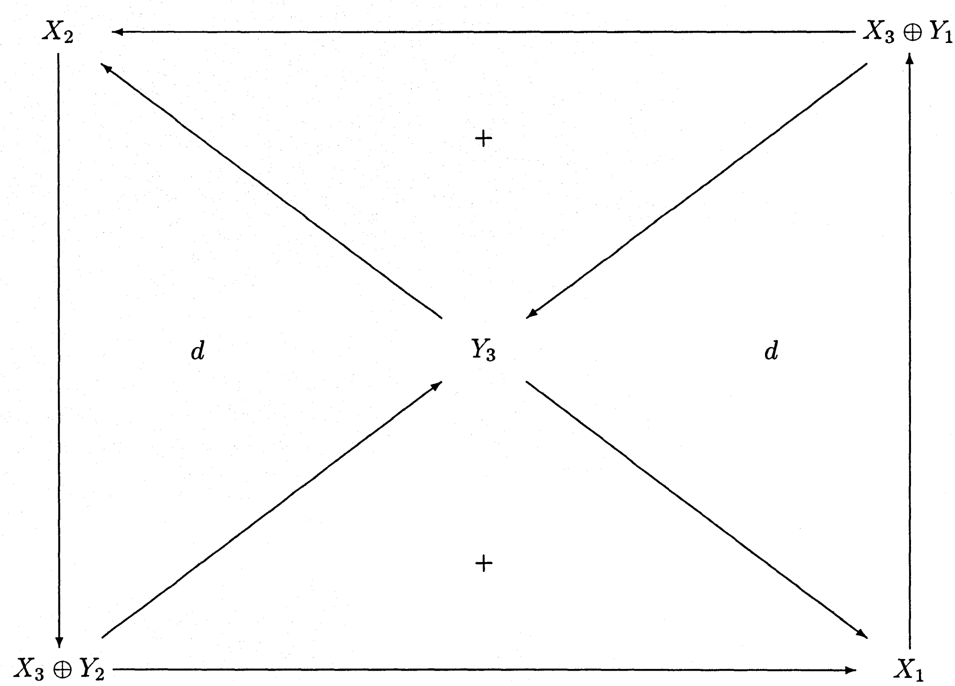
Proof. (The result seems well known, but for completeness we include the proof.) Let

$$\begin{array}{ccccc} X_1 & \xrightarrow{f_1} & X_2 & \xrightarrow{f_2} & X_3 \\ {\scriptstyle g_1}\downarrow & & & & \\ Y_1 & & & & \end{array}$$

be a diagram of morphisms. Consider the pentagon:



By [TR4] we may complete this diagram to an octahedron, whose back face is:



The study of this diagram easily yields

$$\begin{array}{ccccc} X_1 & \xrightarrow{f_1} & X_2 & \xrightarrow{f_2} & X_3 \\ {\scriptstyle g_1}\downarrow & & {\scriptstyle g_2}\downarrow & & {\scriptstyle g_3}\downarrow \\ Y_1 & \xrightarrow{h_1} & Y_2 & \xrightarrow{h_2} & Y_3 \end{array}$$

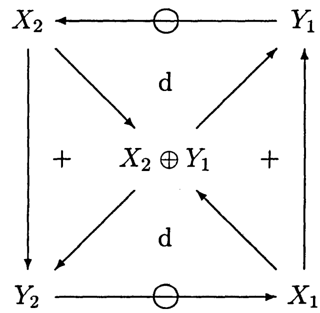
where all the squares are $M - V$. □

LEMMA 2.2. *Suppose we are given an $M - V$ square*

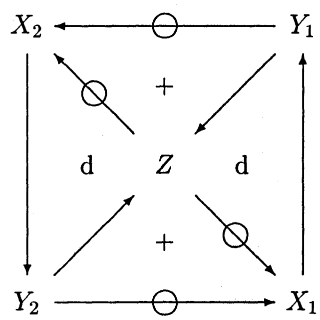
$$\begin{array}{ccc} X_1 & \longrightarrow & X_2 \\ {\scriptstyle f_1}\downarrow & & {\scriptstyle f_2}\downarrow \\ Y_1 & \longrightarrow & Y_2 \end{array}$$

Then the third edge of the triangle on f_1 is isomorphic to the third edge of the triangle on f_2.

Proof. (This result is also well known.) Consider the pentagon



It may be completed to an octahedron, with back face:



and this precisely establishes that Z is the common third vertex of f_1 and f_2. □

Let $\mathcal{T}$ be a triangulated category, and let $\mathcal{S} \subset \mathcal{T}$ be a exact subcategory. We define:

DEFINITION 2.3. *A morphism $f : X \to Y$ in $\mathcal{S}$ is* mono *if in the triangle $X \to Y \to Z \to \Sigma X$, $Z \in \mathcal{S}$.*

DEFINITION 2.4. *A morphism $f : X \to Y$ in $\mathcal{S}$ is* epi *if in the triangle $Z \to X \to Y \to \Sigma Z$, $Z \in \mathcal{S}$.*

Monos will be denoted by arrows $X \rightarrowtail Y$, epis by arrows $X \twoheadrightarrow Y$. Note that monos and epis are generally not monomorphisms and epimorphisms in the usual sense of category theory.

PROPOSITION 2.5. *Monos and epis are stable by (Mayer-Vietoris) pushouts and pullbacks; i.e. if*

$$\begin{array}{ccc} X & \longrightarrow & X' \\ {\scriptstyle f}\Big\downarrow & & \Big\downarrow{\scriptstyle f'} \\ Y & \longrightarrow & Y' \end{array}$$

is an $M - V$ square, then f' is mono (resp. epi) if and only if f is.

Proof. This is a corollary of Lemma 2.2. □

LEMMA 2.6. *Let $f : X_1 \to X_2$, $g : X_2 \to X_3$ be morphisms in a triangulated category $\mathcal{T}$. If Y_1, Y_2 and Y_3 are given by the distinguished triangles:*

$$X_1 \longrightarrow X_2 \longrightarrow Y_1 \longrightarrow \Sigma X_1$$

$$X_2 \longrightarrow X_3 \longrightarrow Y_2 \longrightarrow \Sigma X_2$$

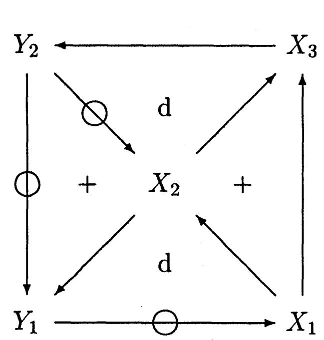
and

$$X_1 \longrightarrow X_3 \longrightarrow Y_3 \longrightarrow \Sigma X_1,$$

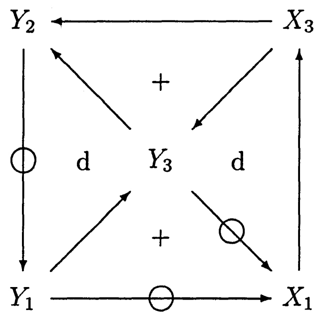
then there is a triangle

$$Y_1 \longrightarrow Y_3 \longrightarrow Y_2 \longrightarrow \Sigma Y_1.$$

Proof. Apply [TR4] to



getting an octahedron with missing vertex Y_3, as in

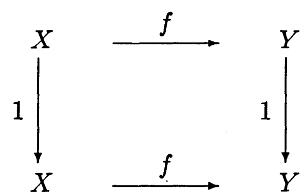


In particular, $Y_1 \to Y_3 \to Y_2 \to \Sigma Y_1$ is a triangle. □

COROLLARY 2.7. *$f : X_1 \to X_2$ and $g : X_2 \to X_3$ are monos (resp. epis) so is $g \circ f : X_1 \to X_3$.* □

Before proceeding, let us note:

TRIVIALITY 2.8. *If $f : X \to Y$ is any map, then the square*



is $M - V$. □

In the next Section we will define the Q-construction of a exact subcategory $\mathcal{S}$ of a triangulated category $\mathcal{T}$. The main example we should keep in mind is Example 1.10. Let us end this Section by explicitly working out what the definitions of monos and epis mean in this case.

If $H : \mathcal{T} \to \mathcal{A}$ is a cohomological functor, we defined $\mathcal{T}_{[n,m]}$ to be the exact subcategory of $X \in \mathcal{T}$ such that $H(\Sigma^q X) = 0$ unless $n \leq q \leq m$. Write $H^q(X)$ for $H(\Sigma^q X)$. Then what are the epis and what are the monos in $\mathcal{T}_{[n,m]}$?

The reader will easily verify that $f : X \to Y$ is epi if and only if $H^m(X) \to H^m(Y)$ is epi. Similarly, $f : X \to Y$ is mono if and only if $H^n(X) \to H^n(Y)$ is mono. If $\mathcal{T}$ is $D(\mathcal{A})$, the derived category of an abelian category $\mathcal{A}$, and H is the cohomology functor, then $D(\mathcal{A})_{[0,0]} \cong \mathcal{A}$. In this case, the equivalence of categories preserves epis and monos.

3. Two Q-constructions. Let $\mathcal{T}$ be a triangulated category, $\mathcal{S} \subset \mathcal{T}$ an exact subcategory. It is completely standard that one can define a Grothendieck group $K_0(\mathcal{S})$. We recall: let $F(\mathcal{S})$ be the free abelian group on the objects of $\mathcal{S}$. Put

$$K_0(\mathcal{S}) = F(\mathcal{S})/R(\mathcal{S})$$

where $R(\mathcal{S}) \subset F(\mathcal{S})$ is generated by the elements $X - Y + Z$, whenever X, Y, and Z are objects of $\mathcal{S}$, and there is a triangle in $\mathcal{T}$ $\quad X \to Y \to Z \to \Sigma X$.

It is also completely standard that if $\mathcal{S} = \mathcal{T} = D^b(\mathcal{E})$, where $\mathcal{E}$ is an exact category, then $K_0(D^b(\mathcal{E})) = K_0(\mathcal{E})$.

We wish to define $K_i(\mathcal{S})$ for $i \geq 0$. We will give two constructions. One of the constructions will be the one that the theorems will deal with; about the other, we know nearly nothing. The purpose of the second construction is to serve as a caution to the unwary.

CONSTRUCTION 3.1. **Simple-Minded Approach.**

Observe that $\mathcal{S}$ has the structure of a bicategory. Define $\mathcal{S}^{\uparrow}_{\longrightarrow}$ to be the bicategory which horizontally and vertically is simply $\mathcal{S}$, while the distinguished squares in $\mathcal{S}^{\uparrow}_{\longrightarrow}$ are the $M-V$ squares. The fact that $\mathcal{S}^{\uparrow}_{\longrightarrow}$ is a bicategory is Triviality 2.8 and Lemma 2.1. Let $\mathcal{N}..(\mathcal{S})$ be the bisimplicial nerve of the bicategory $\mathcal{S}^{\uparrow}_{\longrightarrow}$, and $\mathcal{B}..(\mathcal{S})$ its geometric realization. We define

$$K_i(\mathcal{S}) = \Pi_{i+1}\mathcal{B}..(\mathcal{S}).$$

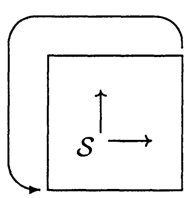
REMARK 3.2. For readers unfamiliar with nerves of bicategories, let us recall that $\mathcal{N}..(\mathcal{S})$ is a bisimplicial set whose (p,q)–simplices are diagrams of $M-V$ squares:

$$\begin{array}{ccccccc}
X_{p0} & \longrightarrow & X_{p1} & \longrightarrow & \cdots & \longrightarrow & X_{pq} \\
\uparrow & & \uparrow & & & & \uparrow \\
\vdots & & \vdots & & & & \vdots \\
\uparrow & & \uparrow & & & & \uparrow \\
X_{10} & \longrightarrow & X_{11} & \longrightarrow & \cdots & \longrightarrow & X_{1q} \\
\uparrow & & \uparrow & & & & \uparrow \\
X_{00} & \longrightarrow & X_{01} & \longrightarrow & \cdots & \longrightarrow & X_{0q}
\end{array}$$

There are horizontal and vertical face and degeneracy maps: the i^{th} horizontal (resp. vertical) face map is deleting the i^{th} column (resp. row) and composing the i^{th} and $(i+1)^{th}$ columns (resp. rows) of horizontal (resp. vertical) maps. The degeneracy maps are induced by inserting identities.

CONSTRUCTION 3.3. **The Approach That Almost Works.**

Once again we consider a bisimplicial set, which we will denote by



A (p,q)-simplex in here is a diagram

$$\begin{array}{ccccccc} X_{p0} & \longrightarrow & X_{p1} & \longrightarrow & \cdots & \longrightarrow & X_{pq} \\ \uparrow & & \uparrow & & & & \uparrow \\ \vdots & & \vdots & & & & \vdots \\ \uparrow & & \uparrow & & & & \uparrow \\ X_{10} & \longrightarrow & X_{11} & \longrightarrow & \cdots & \longrightarrow & X_{1q} \\ \uparrow & & \uparrow & & & & \uparrow \\ X_{00} & \longrightarrow & X_{01} & \longrightarrow & \cdots & \longrightarrow & X_{0q} \end{array}$$

as before, but this time it comes together with a compatible choice of differentials. By this we mean a morphism $\phi : X_{pq} \to \Sigma X_{00}$ in $\mathcal{T}$ such that for any $0 \leq i \leq i' \leq p$, $0 \leq j \leq j' \leq q$, the sequence

$$X_{ij} \to X_{i'j} \oplus X_{ij'} \to X_{i'j'} \xrightarrow{\rho} \Sigma X_{ij}$$

is a triangle in $\mathcal{T}$, where $\rho : X_{i'j'} \to \Sigma X_{ij}$ is the composite

$$X_{i'j'} \to X_{pq} \xrightarrow{\phi} \Sigma X_{00} \to \Sigma X_{ij}.$$

The face and degeneracy maps are as in Construction 3.1, but remembering the differential.

Once again, $\mathcal{B}..(\mathcal{S})$ is the geometric realization of this bisimplicial set, and

$$K_i(\mathcal{S}) \stackrel{def}{\equiv} \Pi_{i+1}\mathcal{B}..(\mathcal{S}).$$

This approach gives a K–theory, but it is not quite the one we will study. See Remark 3.5.

REMARK 3.4. **Symmetries of the above construction.** It seems right at this point to discuss the symmetries of the construction. There is a transposing map which takes an (p, q)–simplex to a (q, p)–simplex, simply by transposing the diagram. The very observant reader will notice that the sign of the differential must be switched. Given a (p, q) simplex, we have a diagram

$$\begin{array}{ccccccc} X_{p0} & \longrightarrow & X_{p1} & \longrightarrow & \cdots & \longrightarrow & X_{pq} \\ \uparrow & & \uparrow & & & & \uparrow \\ \vdots & & \vdots & & & & \vdots \\ \uparrow & & \uparrow & & & & \uparrow \\ X_{10} & \longrightarrow & X_{11} & \longrightarrow & \cdots & \longrightarrow & X_{1q} \\ \uparrow & & \uparrow & & & & \uparrow \\ X_{00} & \longrightarrow & X_{01} & \longrightarrow & \cdots & \longrightarrow & X_{0q} \end{array}$$

possibly together with a coherent differential $\phi : X_{pq} \to \Sigma X_{00}$. (The "possibly" is because we leave it open whether we are working with Construction 3.1 or Consruction 3.3.) Anyway, let $\partial : X_{i'j'} \to \Sigma X_{ij}$ be a differential. If it is given, fine; if not, choose one. That is, choose a map ∂ making the sequence

$$X_{ij} \to X_{i'j} \oplus X_{ij'} \to X_{i'j'} \xrightarrow{\partial} \Sigma X_{ij}$$

a triangle in $\mathcal{T}$. Recall that an $M - V$ square is a commutative square

$$\begin{array}{ccc} X_{ij} & \xrightarrow{\ f\ } & X_{ij'} \\ {\scriptstyle f'}\big\downarrow & & \big\downarrow{\scriptstyle g} \\ X_{i'j} & \xrightarrow{\ g'\ } & X_{i'j'} \end{array}$$

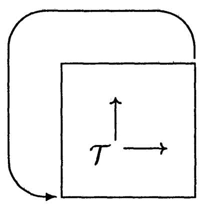
such that $X \overset{\begin{pmatrix} f \\ f' \end{pmatrix}}{\longrightarrow} X_{ij'} \oplus X_{i'j} \overset{\begin{pmatrix} g & -g' \end{pmatrix}}{\longrightarrow} X_{i'j'}$ is part of a triangle; i.e. if there exists $\partial : X_{i'j'} \to \Sigma X_{ij}$ making the sextuple $\left(X_{ij}, X_{ij'} \oplus X_{i'j}, X_{i'j'}, \begin{pmatrix} f \\ f' \end{pmatrix}, \begin{pmatrix} g & -g' \end{pmatrix},\right.$ $\left.\partial\right)$ a triangle. And if we are lucky, the choice for ∂ has already been made for us.

Be this as it may, transposition switches $X_{ij'}$ with $X_{i'j}$, so it switches the roles of g and g', and we find ourselves wondering whether

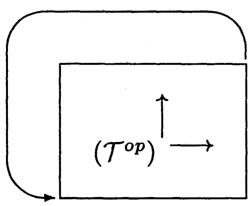
$$\left(X_{ij}, X_{ij'} \oplus X_{i'j}, X_{i'j'}, \begin{pmatrix} f \\ f' \end{pmatrix}, \begin{pmatrix} -g & g' \end{pmatrix}, w\right)$$

is a triangle for some choice of w, and if so, what on earth this w has to do with our original ∂. But all we have done is switched the sign of $\begin{pmatrix} g & -g' \end{pmatrix}$. The reader will easily show that by switching the sign of ∂ as well, we obtain a triangle. Replacing ∂ by $-\partial$ is a very coherent thing to do, so transposition does indeed define a simplicial map.

The other symmetry of the construction is the observation that a simplex in



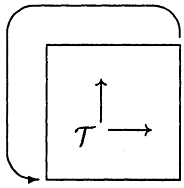
is nothing other than a simplex in



the point being that $\mathcal{T}^{op}$ is a triangulated category having the same set of triangles as $\mathcal{T}$. The dual of an exact subcategory of $\mathcal{T}$ is an exact subcategory of $\mathcal{T}^{op}$.

All the homotopies of the article come equipped with transposes and duals, and we use these without explicit mention.

REMARK 3.5. As we said in Construction 3.3, the simplicial set given there is not quite the one we will study. There is a difficulty here which I do not fully understand. When we construct homotopies in the simplicial set



we need to have a way of building new triangles out of old ones. The main technique I know is by the mapping cone construction. Given a map of triangles

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & \Sigma X \\ {\scriptstyle f}\downarrow & & {\scriptstyle g}\downarrow & & {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle \Sigma f} \\ X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z' & \xrightarrow{w'} & \Sigma X' \end{array}$$

one can build from it the complex

$$Y \oplus X' \xrightarrow{\begin{pmatrix} -v & 0 \\ g & u' \end{pmatrix}} Z \oplus Y' \xrightarrow{\begin{pmatrix} -w & 0 \\ h & v' \end{pmatrix}} \Sigma X \oplus Z' \xrightarrow{\begin{pmatrix} -\Sigma u & 0 \\ \Sigma f & w' \end{pmatrix}} \Sigma Y \oplus \Sigma X'.$$

The problem with this construction is that the mapping cone on a map of triangles is not itself necessarily a triangle. The only triangulated categories for which the mapping cone is automatically a triangle are the extremely dull ones, for instance the derived category of the category of vector spaces over a field.

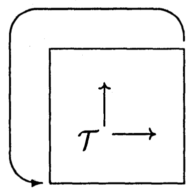
There are two ways to resolve this problem; one is to relax the hypothesis that $X_{ij} \to X_{i'j} \oplus X_{ij'} \to X_{i'j'} \to \Sigma X_{ij}$ in Construction 3.3 is a triangle. One introduces the category of semi–triangles which, roughly speaking, are all direct summands of successive mapping cones on triangles. This approach is elaborated in Appendix A. The homotopies in this article are then all well–defined. The subtlety comes from the fact that the simplicial sets are not quite what one expects. For some (sketchy) detail, see Appendix A.

The second approach is to restrict the set of simplices one allows, to ensure that all reasonable mapping cones are triangles. This is an approach which I understand only partly. The only clean way I see to describe the restrictions to be placed on the simplices is by changing the axiomatic formalism of triangulated categories. I once promised to write down such a formalism, but never did. The reader can find some of the ideas in [7] and [8].

Without getting drawn into axiomatic questions regarding the foundations of triangulated categories, one can restrict the allowable simplices by demanding that they lift to some model for $\mathcal{T}$. This is not pretty, but works. Here is my advice to the reader:

Group 1 Reader: Ignore this point. The only theorem you are advised to read is Theorem 4.8, which says that the K–theory of an abelian category $\mathcal{A}$ is a retract of the K–theory of $D^b(\mathcal{A})$. This is true without any modifications to the construction.

Group 2 Reader: The simplest modification of the simplicial set in Construction 3.3 that I know to work is the following. A simplex in



should be a diagram

$$\begin{array}{ccccc} X_{m0} & \longrightarrow & \cdots & \longrightarrow & X_{mn} \\ \uparrow & & & & \uparrow \\ \vdots & & & & \vdots \\ \uparrow & & & & \uparrow \\ X_{00} & \longrightarrow & \cdots & \longrightarrow & X_{0n} \end{array}$$

as before, together with a coherent differential. But we make a further assumption. The diagram should be a finite direct sum of diagrams of the same form, *each of which has a lifting to a model category for $\mathcal{T}$*. In fact, we will also assume that the model category is a biWaldhausen complicial category, as in [10]. This means concretely that it is a category of complexes over some abelian (or even exact) category, possibly with restrictions on the cohomology. To make my life slightly easier, I will assume that the only model categories we allow are the categories of bounded complexes over an abelian category.
Note that we allow the simplex to lift to *any* model for $\mathcal{T}$. The construction depends only on the triangulated category $\mathcal{T}$. However, it has problems. It is, for instance, not functorial in $\mathcal{T}$. It is possible to give better versions of this simplicial set, but not quite so easily.
Perhaps we should also say what we mean when we assert that a simplex has a lifting to some model category for $\mathcal{T}$. Suppose we are given a simplex

$$\begin{array}{ccccc} X_{m0} & \longrightarrow & \cdots & \longrightarrow & X_{mn} \\ \uparrow & & & & \uparrow \\ \vdots & & & & \vdots \\ \uparrow & & & & \uparrow \\ X_{00} & \longrightarrow & \cdots & \longrightarrow & X_{0n} \end{array}$$

in $\mathcal{T}$. A lifting to a model category consists of an embedding of categories $\mathcal{T} \subset D^b(\mathcal{Q})$ for some abelian category $\mathcal{Q}$, a diagram of bicartesian squares of chain complexes in $C(\mathcal{Q})$

$$\begin{array}{ccccc}
\tilde{X}_{m0} & \longrightarrow & \cdots & \longrightarrow & \tilde{X}_{mn} \\
\uparrow & & & & \uparrow \\
\vdots & & & & \vdots \\
\uparrow & & & & \uparrow \\
\tilde{X}_{00} & \longrightarrow & \cdots & \longrightarrow & \tilde{X}_{0n}
\end{array}$$

and for each i and j an isomorphism

$$\tilde{X}_{ij} \xrightarrow{\rho_{ij}} X_{ij}$$

in the category $\mathcal{T}$. We assume further that the isomorphisms ρ_{ij} define an isomorphism of the diagrams

$$\begin{array}{ccccc}
\tilde{X}_{m0} & \longrightarrow & \cdots & \longrightarrow & \tilde{X}_{mn} \\
\uparrow & & & & \uparrow \\
\vdots & & & & \vdots \\
\uparrow & & & & \uparrow \\
\tilde{X}_{00} & \longrightarrow & \cdots & \longrightarrow & \tilde{X}_{0n}
\end{array}$$

and

$$\begin{array}{ccccc}
X_{m0} & \longrightarrow & \cdots & \longrightarrow & X_{mn} \\
\uparrow & & & & \uparrow \\
\vdots & & & & \vdots \\
\uparrow & & & & \uparrow \\
X_{00} & \longrightarrow & \cdots & \longrightarrow & X_{0n}
\end{array}$$

In other words, the isomorphisms ρ_{ij} commute with the structure maps of the two diagrams, that means even the differential. Perhaps this needs a little explanation.

Recall that the diagram

$$\begin{array}{ccccc}
\tilde{X}_{m0} & \longrightarrow & \cdots & \longrightarrow & \tilde{X}_{mn} \\
\uparrow & & & & \uparrow \\
\vdots & & & & \vdots \\
\uparrow & & & & \uparrow \\
\tilde{X}_{00} & \longrightarrow & \cdots & \longrightarrow & \tilde{X}_{0n}
\end{array}$$

is a diagram of bicartesian squares in $C(\mathcal{Q})$. Thus all the squares must be genuine bicartesian squares. There is then a canonical choice for the coherent differentials. We have an exact sequence in $C^b(\mathcal{Q})$

$$0 \longrightarrow X_{00} \stackrel{f}{\longrightarrow} X_{0n} \oplus X_{m0} \longrightarrow X_{mn} \longrightarrow 0$$

which gives a quasi–isomorphism

$$C\left(X_{00} \stackrel{f}{\longrightarrow} X_{0n} \oplus X_{m0}\right) \stackrel{\beta}{\longrightarrow} X_{mn}$$

where $C\left(X_{00} \stackrel{f}{\longrightarrow} X_{0n} \oplus X_{m0}\right)$ is the mapping cone on the map f. There is also a natural map

$$C\left(X_{00} \stackrel{f}{\longrightarrow} X_{0n} \oplus X_{m0}\right) \stackrel{\alpha}{\longrightarrow} \Sigma X_{00}$$

and the composite $\alpha \circ \beta^{-1}$ is the coherent choice of differentials we consider natural. To be a lifting of

$$\begin{array}{ccccc}
X_{m0} & \longrightarrow & \cdots & \longrightarrow & X_{mn} \\
\uparrow & & & & \uparrow \\
\vdots & & & & \vdots \\
\uparrow & & & & \uparrow \\
X_{00} & \longrightarrow & \cdots & \longrightarrow & X_{0n}
\end{array}$$

to $C^b(\mathcal{Q})$, the given coherent differentials must agree, via the isomorphisms ρ_{ij}, with the natural ones in

$$\begin{array}{ccccc} \tilde{X}_{m0} & \longrightarrow & \cdots & \longrightarrow & \tilde{X}_{mn} \\ \uparrow & & & & \uparrow \\ \vdots & & & & \vdots \\ \uparrow & & & & \uparrow \\ \tilde{X}_{00} & \longrightarrow & \cdots & \longrightarrow & \tilde{X}_{0n} \end{array}$$

Group 3 Reader: This is a good problem to work on. It is at present an unsatisfactory aspect of the theory. There are a number of simplicial sets for which the proofs go through, and I do not understand well the relation among them.

LEMMA 3.6. *With either Construction 3.1 or Construction 3.3,*

$$K_0(\mathcal{S}) = \Pi_1\mathcal{B}..(\mathcal{S})$$

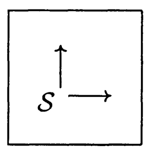
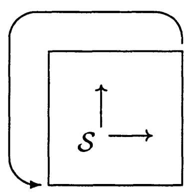
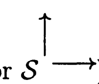
agrees with the usual Grothendieck group.

Proof. Consider the diagonal realization of the bisimplicial set. It has the homotopy type of $\mathcal{B}..(\mathcal{S})$. We need to compute its fundamental group. More explicitly, we will produce maps

$$\phi : \Pi_1\mathcal{B}..(\mathcal{S}) \to K_0(\mathcal{S})$$

$$\psi : K_0(\mathcal{S}) \to \Pi_1(\mathcal{B}..(\mathcal{S})),$$

and we will prove that $\psi \circ \phi$ and $\phi \circ \psi$ are the identity. (For the rest of the proof, $K_0(\mathcal{S})$ stands for the Grothendieck group of $\mathcal{S}$. Once the lemma has been proved, no confusion should arise.)

A 1-simplex in the diagonal realization of the bisimplicial set  (which here is allowed to stand for either  or ) is an $M - V$ square

$$\begin{array}{ccc} Y & \longrightarrow & Y' \\ \uparrow & & \uparrow \\ X & \longrightarrow & X' \end{array}$$

(possibly together with a differential $Y' \to \Sigma X$). We let its image in $K_0(\mathcal{S})$ be $Y - X = Y' - X'$ (recall that $X \to X' \oplus Y \to Y'$ is part of a triangle, hence $X - X' - Y + Y' = 0$ in $K_0(\mathcal{S})$). Let

$$\begin{array}{ccccc} Z & \longrightarrow & Z' & \longrightarrow & Z'' \\ \uparrow & & \uparrow & & \uparrow \\ Y & \longrightarrow & Y' & \longrightarrow & Y'' \\ \uparrow & & \uparrow & & \uparrow \\ X & \longrightarrow & X' & \longrightarrow & X'' \end{array}$$

be a 2-simplex. Its boundary is

$$\begin{array}{ccc} Y & \longrightarrow & Y' \\ \uparrow & & \uparrow \\ X & \longrightarrow & X' \end{array} \; - \; \begin{array}{ccc} Z & \longrightarrow & Z'' \\ \uparrow & & \uparrow \\ X & \longrightarrow & X'' \end{array} \; + \; \begin{array}{ccc} Z' & \longrightarrow & Z'' \\ \uparrow & & \uparrow \\ Y' & \longrightarrow & Y'' \end{array}$$

This makes it clear that the boundary of a 2-simplex maps to zero, via the map we have just defined. Thus we have given a map

$$\Pi_1(\mathcal{B}..(\mathcal{S})) \to H_1(\mathcal{B}..(\mathcal{S})) = \frac{1\text{–cycles}}{\text{boundaries}} \subset \frac{1\text{–chains}}{\text{boundaries}} \to K_0(\mathcal{S}).$$

This is the map ϕ. Now observe that $\mathcal{B}..(\mathcal{S})$ is an H–space, hence $\Pi_1\mathcal{B}..(\mathcal{S})$ is abelian. Therefore, the natural map $\Pi_1\mathcal{B}..(\mathcal{S}) \to H_1\mathcal{B}..(\mathcal{S})$ is an isomorphism. Thus, we need only define an inverse

$$K_0(\mathcal{S}) \to H_1\mathcal{B}..(\mathcal{S}) = \frac{1\text{–cycles}}{\text{boundaries}}.$$

We define ψ by

$$\psi(Y) \;=\; \begin{array}{ccc} Y & \longrightarrow & Y \\ \uparrow & & \uparrow \\ 0 & \longrightarrow & 0 \end{array} \; + \; \begin{array}{ccc} Y & \longrightarrow & 0 \\ \uparrow & & \uparrow \\ Y & \longrightarrow & 0 \end{array} .$$

The first difficulty is to establish that ψ is well defined; then we will prove that ϕ and ψ are inverse to each other.

3.6.1. *ψ is well defined.*

To prove this, we have to establish that if X, Y, and Z are objects of $\mathcal{S}$, and if $X \to Y \to Z \to \Sigma X$ is a triangle in $\mathcal{T}$, then $\psi(X - Y + Z) = 0$. To do this, compute the boundary of the 2–chain

$$\begin{array}{ccccc} Y & \longrightarrow & Y & \longrightarrow & Y \\ \uparrow & & \uparrow & & \uparrow \\ X & \longrightarrow & X & \longrightarrow & X \\ \uparrow & & \uparrow & & \uparrow \\ 0 & \longrightarrow & 0 & \longrightarrow & 0 \end{array} \; + \; \begin{array}{ccccc} Y & \longrightarrow & Z & \longrightarrow & 0 \\ \uparrow & & \uparrow & & \uparrow \\ Y & \longrightarrow & Z & \longrightarrow & 0 \\ \uparrow & & \uparrow & & \uparrow \\ Y & \longrightarrow & Z & \longrightarrow & 0 \end{array}$$

$$+ \; \begin{array}{ccccc} Y & \longrightarrow & Z & \longrightarrow & Z \\ \uparrow & & \uparrow & & \uparrow \\ X & \longrightarrow & 0 & \longrightarrow & 0 \\ \uparrow & & \uparrow & & \uparrow \\ X & \longrightarrow & 0 & \longrightarrow & 0 \end{array} \; - \; \begin{array}{ccccc} Y & \longrightarrow & Y & \longrightarrow & Z \\ \uparrow & & \uparrow & & \uparrow \\ Y & \longrightarrow & Y & \longrightarrow & Z \\ \uparrow & & \uparrow & & \uparrow \\ X & \longrightarrow & X & \longrightarrow & 0 \end{array} .$$

The reader will easily show that this boundary is nothing other than $\psi(X - Y + Z)$. This computation proves that $\psi(X - Y + Z)$ is a boundary, and hence vanishes in $\frac{\text{1–chains}}{\text{boundaries}}$. The fact that $\psi(Y)$ is a cycle for all Y is immediate, and is left to the reader. The argument works both in the simplicial set $\begin{array}{l}\uparrow\\ \mathcal{S}\longrightarrow\end{array}$, and in the simplicial set $\boxed{\begin{array}{l}\uparrow\\ \mathcal{S}\longrightarrow\end{array}}$ (with a looping arrow around the box). Although we have written out the simplices in the computation without indicating the differentials, the reader can easily provide those.

3.6.2. $\phi \circ \psi = 1$. This is the obvious identity, and we leave it to the reader.

3.6.3. $\psi \circ \phi = 1$. This is a little more delicate, so we will discuss it.

We need to show that the composite

$$H_1\mathcal{B}..(\mathcal{S}) = \frac{\text{1–cycles}}{\text{boundaries}} \xrightarrow{\phi} K_0(\mathcal{S}) \xrightarrow{\psi} \frac{\text{1–cycles}}{\text{boundaries}}$$

is the identity. It is easier to consider the composite

$$\frac{\text{1–cycles}}{\text{boundaries}} \to \frac{\text{1–chains}}{\text{boundaries}} \to K_0(\mathcal{S}) \to \frac{\text{1–cycles}}{\text{boundaries}} \subset \frac{\text{1–chains}}{\text{boundaries}} .$$

Let f be the composite

$$\frac{\text{1–chains}}{\text{boundaries}} \to K_0(\mathcal{S}) \to \frac{\text{1–chains}}{\text{boundaries}} .$$

We will prove that $1 - f$ vanishes on $\frac{\text{1–cycles}}{\text{boundaries}}$. This clearly suffices. Let

$$\begin{array}{ccc} Y & \longrightarrow & Y' \\ \uparrow & & \uparrow \\ X & \longrightarrow & X' \end{array}$$

be a 1-chain. Then

$$(1-f)\left(\begin{array}{ccc} Y & \longrightarrow & Y' \\ \uparrow & & \uparrow \\ X & \longrightarrow & X' \end{array}\right)$$

$$= \begin{array}{ccc} Y & \longrightarrow & Y' \\ \uparrow & & \uparrow \\ X & \longrightarrow & X' \end{array} - \begin{array}{ccc} Y' & \longrightarrow & Y' \\ \uparrow & & \uparrow \\ 0 & \longrightarrow & 0 \end{array} - \begin{array}{ccc} Y' & \longrightarrow & 0 \\ \uparrow & & \uparrow \\ Y' & \longrightarrow & 0 \end{array}$$

$$+ \begin{array}{ccc} X' & \longrightarrow & X' \\ \uparrow & & \uparrow \\ 0 & \longrightarrow & 0 \end{array} + \begin{array}{ccc} X' & \longrightarrow & 0 \\ \uparrow & & \uparrow \\ X' & \longrightarrow & 0 \end{array} .$$

Now, studying the boundary of the 2–chain

$$\begin{array}{ccccc} X & \longrightarrow & X' & \longrightarrow & 0 \\ \uparrow & & \uparrow & & \uparrow \\ X & \longrightarrow & X' & \longrightarrow & 0 \\ \uparrow & & \uparrow & & \uparrow \\ X & \longrightarrow & X' & \longrightarrow & 0 \end{array} + \begin{array}{ccccc} Y' & \longrightarrow & Y' & \longrightarrow & Y' \\ \uparrow & & \uparrow & & \uparrow \\ X' & \longrightarrow & X' & \longrightarrow & X' \\ \uparrow & & \uparrow & & \uparrow \\ 0 & \longrightarrow & 0 & \longrightarrow & 0 \end{array}$$

$$- \begin{array}{ccccc} Y & \longrightarrow & Y' & \longrightarrow & Y' \\ \uparrow & & \uparrow & & \uparrow \\ X & \longrightarrow & X' & \longrightarrow & X' \\ \uparrow & & \uparrow & & \uparrow \\ X & \longrightarrow & X' & \longrightarrow & X' \end{array} ,$$

the reader will prove the identity in $\frac{\text{1–chains}}{\text{boundaries}}$

$$\begin{array}{ccc} Y & \longrightarrow & Y' \\ \uparrow & & \uparrow \\ X & \longrightarrow & X' \end{array} = \begin{array}{ccc} Y' & \longrightarrow & Y' \\ \uparrow & & \uparrow \\ 0 & \longrightarrow & 0 \end{array} - \begin{array}{ccc} X' & \longrightarrow & X' \\ \uparrow & & \uparrow \\ 0 & \longrightarrow & 0 \end{array}$$

$$- \begin{array}{ccc} X' & \longrightarrow & 0 \\ \uparrow & & \uparrow \\ X' & \longrightarrow & 0 \end{array} + \begin{array}{ccc} X & \longrightarrow & 0 \\ \uparrow & & \uparrow \\ X & \longrightarrow & 0 \end{array} .$$

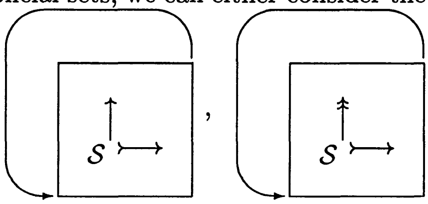
Combining these identities, we have proved

$$(1-f)\left(\begin{array}{ccc} Y & \longrightarrow & Y' \\ \uparrow & & \uparrow \\ X & \longrightarrow & X' \end{array}\right) = \begin{array}{ccc} X & \longrightarrow & 0 \\ \uparrow & & \uparrow \\ X & \longrightarrow & 0 \end{array} - \begin{array}{ccc} Y' & \longrightarrow & 0 \\ \uparrow & & \uparrow \\ Y' & \longrightarrow & 0 \end{array} .$$

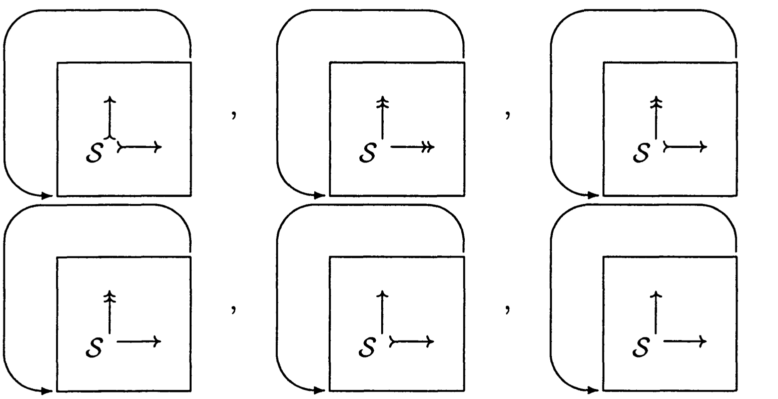
In particular, $(1-f)\left(\begin{array}{ccc} Y & \longrightarrow & Y' \\ \uparrow & & \uparrow \\ X & \longrightarrow & X' \end{array}\right)$ depends only on the boundary of the 1-cell $\begin{array}{ccc} Y & \longrightarrow & Y' \\ \uparrow & & \uparrow \\ X & \longrightarrow & X' \end{array}$. It is given by the formula

$$(1-f)(c) = \sum_{\substack{Z \in \partial c \\ \text{with multiplicity}}} \left(\begin{array}{ccc} Z & \longrightarrow & 0 \\ \uparrow & & \uparrow \\ Z & \longrightarrow & 0 \end{array}\right)$$

This formula initially holds for the generators $\begin{array}{ccc} Y & \longrightarrow & Y' \\ \uparrow & & \uparrow \\ X & \longrightarrow & X' \end{array}$ of $\frac{\text{1–chains}}{\text{boundaries}}$, but extends by linearity to the whole group. In particular, if $\partial c = 0$, then $(1-f)(c) = 0$. □

Now let us return to more general considerations. There are many ways to view $\mathcal{S}$ as a bicategory, and we want to consider several of these. The squares in $\mathcal{S}$ will always be the $M-V$ squares; but we want to permit ourselves some flexibility in the horizontal and vertical categories $h(\mathcal{S})$ and $v(\mathcal{S})$. For instance, we may insist that all the morphisms in $h(\mathcal{S})$ be mono. Our notation for the resulting bicategory is $\mathcal{S}^{\uparrow}_{\rightarrowtail}$. The arrows indicate that the horizontal morphisms are mono, while the vertical ones are free. In the same vein, we consider also bicategories $\mathcal{S}^{\twoheaduparrow}_{\longrightarrow}$, $\mathcal{S}^{\twoheaduparrow}_{\rightarrowtail}$, etc. Of course, there are associated simplicial sets; we can either consider the nerve of the bicategory or the bisimplicial sets  etc., where a simplex comes equipped with a choice of compatible differentials. We observe

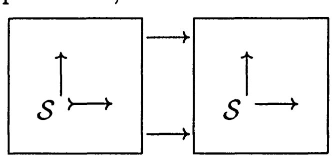
THEOREM 3.7. *The natural inclusions induce homotopy equivalences among the bicategories* $\mathcal{S}^{\rotatebox{90}{$\rightarrowtail$}}_{\rightarrowtail}$, $\mathcal{S}^{\twoheaduparrow}_{\twoheadrightarrow}$, $\mathcal{S}^{\twoheaduparrow}_{\rightarrowtail}$, $\mathcal{S}^{\twoheaduparrow}_{\longrightarrow}$, $\mathcal{S}^{\uparrow}_{\rightarrowtail}$ *and* $\mathcal{S}^{\uparrow}_{\longrightarrow}$. *(Respectively, the bisimplicial sets*



are also homotopy equivalent via the natural maps).

Proof. By way of illustration, we will prove that the inclusion $\mathcal{S}^{\uparrow}_{\rightarrowtail} \hookrightarrow \mathcal{S}^{\uparrow}_{\longrightarrow}$ induces a homotopy equivalence. The remaining statements have analogous proofs.

We introduce the trisimplicial set, which we denote

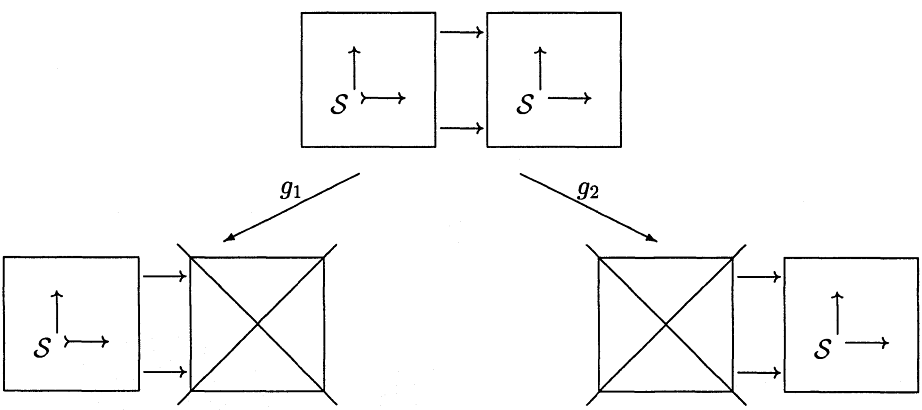


By definition, an (n, m, p)-simplex consists of a diagram of $M - V$ squares:

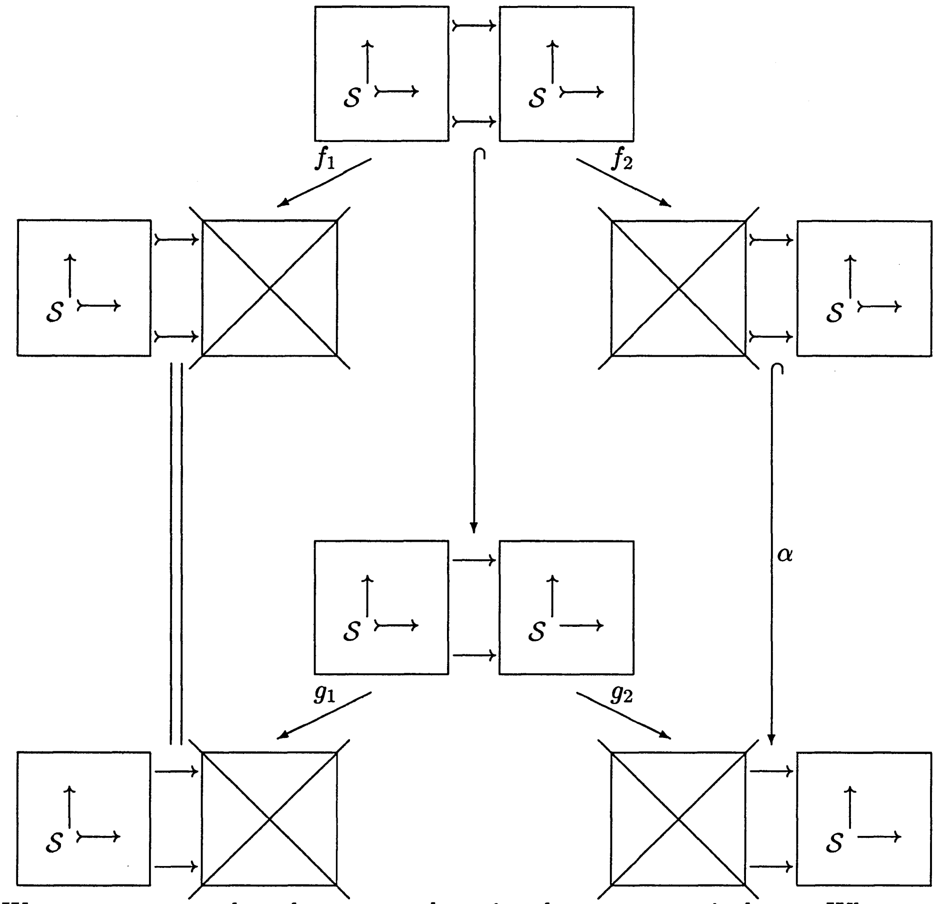
$$\begin{array}{ccccccccccccc}
X_{p0} & \rightarrowtail & \cdots & \rightarrowtail & X_{pn} & \longrightarrow & Y_{p0} & \longrightarrow & \cdots & \longrightarrow & Y_{pm} \\
\uparrow & & & & \uparrow & & \uparrow & & & & \uparrow \\
\vdots & & & & \vdots & & \vdots & & & & \vdots \\
\uparrow & & & & \uparrow & & \uparrow & & & & \uparrow \\
X_{00} & \rightarrowtail & \cdots & \rightarrowtail & X_{0n} & \longrightarrow & Y_{00} & \longrightarrow & \cdots & \longrightarrow & Y_{0m}
\end{array}$$

(If we want to consider the simplicial sets of Construction 3.3, then we also give a map $Y_{pm} \to \Sigma X_{00}$ as part of the structure.) The i^{th} face maps are deleting the i^{th} column in the X square, the i^{th} column in the Y square, or the i^{th} row. There are three different commuting simplicial structures, hence a trisimplicial set. The degeneracy maps are the corresponding insertions of identities.

There are two projections out of our trisimplicial set. The first "forgets" the $X's$, the second "forgets" the $Y's$. We will denote these maps

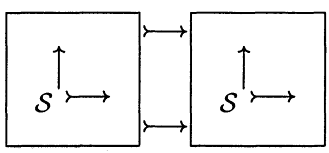


We will show that these maps are homotopy equivalences. This immediately establishes that there is some homotopy equivalence $\mathcal{S}\,{}^{\uparrow}\!\!\rightarrowtail \;\sim\; \mathcal{S}\,{}^{\uparrow}\!\!\longrightarrow$, but it is not completely clear that it is induced by the inclusion. The standard way to remedy this is to consider the commutative diagram



We want to prove that the map α above is a homotopy equivalence. What we will show is that each of the maps f_1, f_2, g_1 and g_2 is a homotopy equivalence. But then the commutativity forces α to also be.

3.7.1. **Notation**. I hope that the notation is self explanatory. For example, the symbol

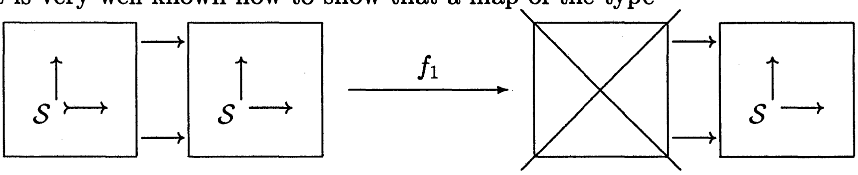


stands for the trisimplicial set which consists of diagrams

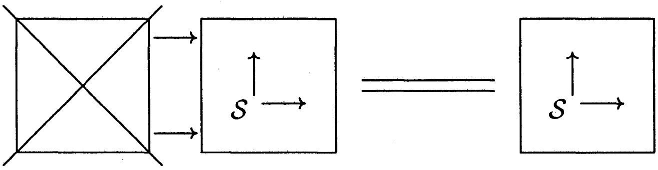
$$\begin{array}{ccccccccccc}
X_{p0} & \rightarrowtail & \cdots & \rightarrowtail & X_{pn} & \rightarrowtail & Y_{p0} & \rightarrowtail & \cdots & \rightarrowtail & Y_{pm} \\
\uparrow & & & & \uparrow & & \uparrow & & & & \uparrow \\
\vdots & & & & \vdots & & \vdots & & & & \vdots \\
\uparrow & & & & \uparrow & & \uparrow & & & & \uparrow \\
X_{00} & \rightarrowtail & \cdots & \rightarrowtail & X_{0n} & \rightarrowtail & Y_{00} & \rightarrowtail & \cdots & \rightarrowtail & Y_{0m}
\end{array}$$

Thus, the symbols inside the squares indicate the bicategory in which the corresponding simplex is formed, whereas the arrows connecting the squares describe the connecting morphisms $X \to Y$.

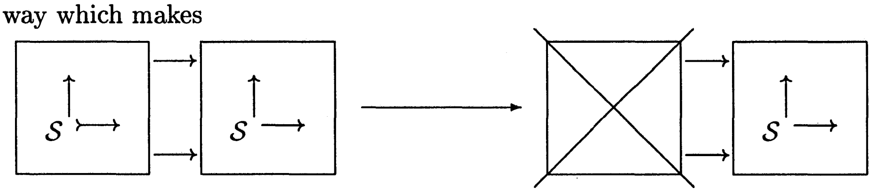
It is very well known how to show that a map of the type



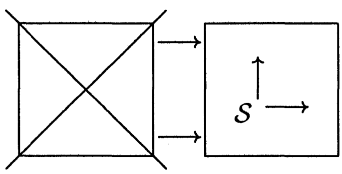
is a homotopy equivalence. The criterion is Quillen's Theorem A; it suffices to show that each fiber is contractible. But, because this section is written for the non-specialist, let us explain this carefully once. The term



is naturally a bisimplicial set. But out of perversity, we view it as a trisimplicial set, in a way which makes



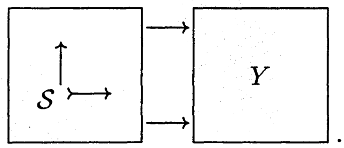
a map of trisimplicial sets. By Tornhave's theorem, we can geometrically realize the map in any order we wish. We wish to realize first the simplicial structure which is trivial on



i.e. where the face maps are deleting a column in the left-hand square. There is an induced map of bisimplicial spaces. By a theorem of Segal, it suffices to show that this map is a homotopy equivalence. But on the right we get a discrete space consisting of the squares

$$\begin{array}{ccccc} Y_{p0} & \longrightarrow & \cdots & \longrightarrow & Y_{pm} \\ \uparrow & & & & \uparrow \\ \vdots & & & & \vdots \\ \uparrow & & & & \uparrow \\ Y_{00} & \longrightarrow & \cdots & \longrightarrow & Y_{0m} \end{array}$$

and the fiber over this square is the geometric realization of a simplicial set, which we will denote

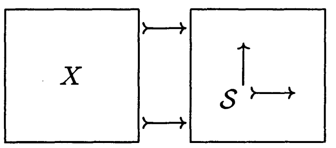


This is our notation for the simplicial set of diagrams

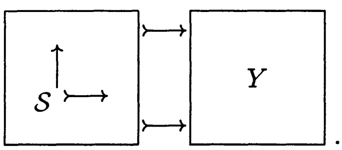
$$\begin{array}{ccccccccccccc}
X_{p0} & \rightarrowtail & \cdots & \rightarrowtail & X_{pn} & \longrightarrow & Y_{p0} & \longrightarrow & \cdots & \longrightarrow & Y_{pm} \\
\uparrow & & & & \uparrow & & \uparrow & & & & \uparrow \\
\vdots & & & & \vdots & & \vdots & & & & \vdots \\
\uparrow & & & & \uparrow & & \uparrow & & & & \uparrow \\
X_{00} & \rightarrowtail & \cdots & \rightarrowtail & X_{0n} & \longrightarrow & Y_{00} & \longrightarrow & \cdots & \longrightarrow & Y_{0m}
\end{array}$$

where the array of Y's is held fixed. Thus, if we argue similarly for the other maps in our diagram, we deduce that to prove our theorem it suffices to establish the contractibility of four simplicial sets.

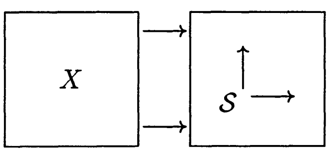
3.7.2. *To prove that f_1 is a homotopy equivalence, it suffices to establish the contractibility of*



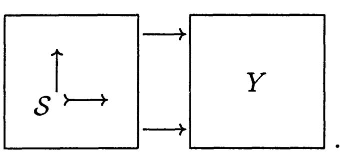
3.7.3. *To prove that f_2 is a homotopy equivalence, it suffices to establish the contractibility of*



3.7.4. *To prove that g_1 is a homotopy equivalence, it suffices to establish the contractibility of*

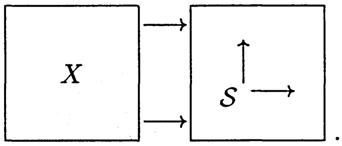


3.7.5. *To prove that g_2 is a homotopy equivalence, it suffices to establish the contractibility of*

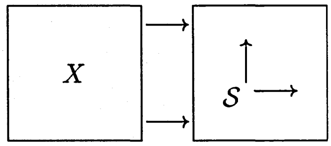


The reader will undoubtedly notice that the contractibility statements in 3.7.2, 3.7.3 and 3.7.4 are trivial. Nevertheless, the author wishes to be permitted to include a proof of the contractibility in 3.7.4. This is done mostly to introduce some handy notation and illustrate it in a particularly simple case.

3.7.6. Proof of the contractibility of



We remind the reader that a simplex in



is a diagram of $M - V$ squares

$$\begin{array}{ccccccccc}
X_{p0} & \rightarrowtail & \cdots & \rightarrowtail & X_{pn} & \longrightarrow & Y_{p0} & \longrightarrow & \cdots & \longrightarrow & Y_{pm} \\
\uparrow & & & & \uparrow & & \uparrow & & & & \uparrow \\
\vdots & & & & \vdots & & \vdots & & & & \vdots \\
\uparrow & & & & \uparrow & & \uparrow & & & & \uparrow \\
X_{00} & \rightarrowtail & \cdots & \rightarrowtail & X_{0n} & \longrightarrow & Y_{00} & \longrightarrow & \cdots & \longrightarrow & Y_{0m}
\end{array}$$

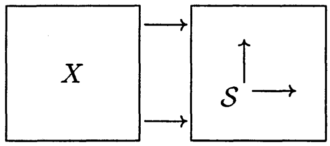
where the X's are held fixed. We will denote this by putting a frame around the X's, as shown below

$$\begin{array}{|ccccc|cccccc}
\hline
X_{p0} & \rightarrowtail & \cdots & \rightarrowtail & X_{pn} & \longrightarrow & Y_{p0} & \longrightarrow & \cdots & \longrightarrow & Y_{pm} \\
\uparrow & & & & \uparrow & & \uparrow & & & & \uparrow \\
\vdots & & & & \vdots & & \vdots & & & & \vdots \\
\uparrow & & & & \uparrow & & \uparrow & & & & \uparrow \\
X_{00} & \rightarrowtail & \cdots & \rightarrowtail & X_{0n} & \longrightarrow & Y_{00} & \longrightarrow & \cdots & \longrightarrow & Y_{0m} \\
\hline
\end{array}$$

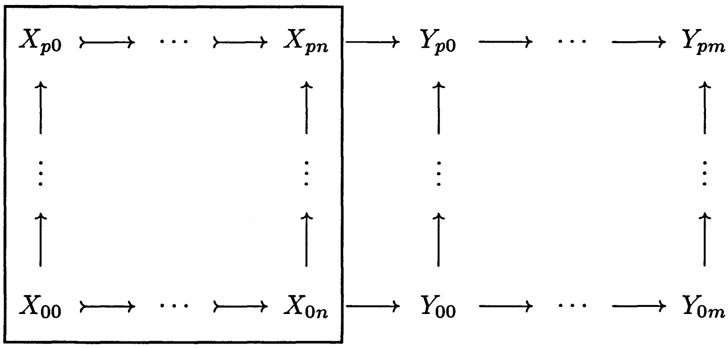
There is only one simplicial structure, obtained by varying the value of the integer m. A simplicial homotopy is a procedure for constructing $m+1$–simplices out of m–simplices. If we are given two topological spaces X and Y and two maps $f : X \to Y$, $g : X \to Y$, then a homotopy $H : f \Longrightarrow g$ is a map $H : X \times I \longrightarrow Y$. A simplicial version of this is as follows. A simplicial map from one simplicial set to another takes m–simplices to m–simplices. A simplicial homotopy should take an m–simplex s_m in

X to the image of $s_m \times I$. But there is a canonical decomposition of the prism $\Delta_m \times I$ as a union of $m+1$ simplices. Thus, a simplicial homotopy should take a simplex in X to an ordered set of $m+1$ simplices in Y. This assignment of $m+1$ simplices must, of course, be compatible with face maps, in a way that is excellently described elsewhere.

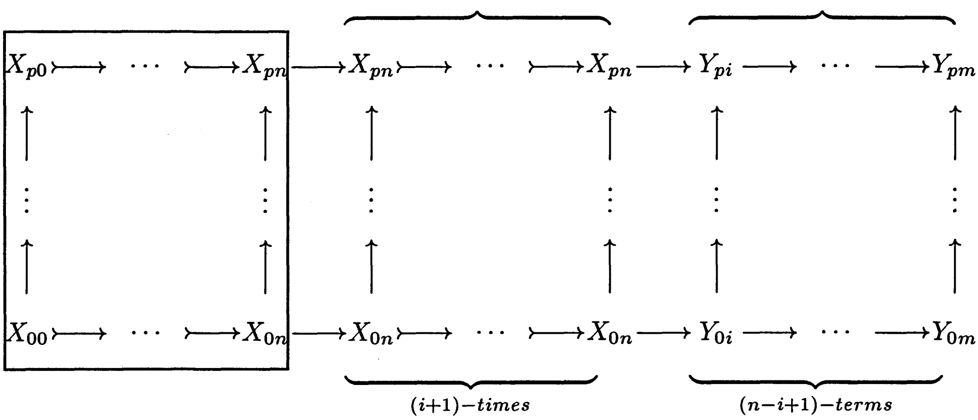
We want to prove the contractibility of



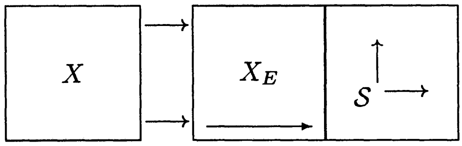
and we must therefore construct a homotopy of the identity map with a collapsing map. To every simplex



we must assign $m+1$ simplices, providing a contracting homotopy. The i^{th} simplex we construct in this ordered set of $m+1$ is

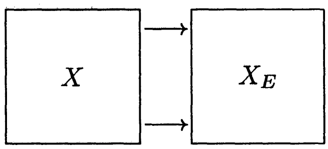


and as i increases, $0 \leq i \leq m$, this homotopy gradually replaces the Y's by X's, contracting the simplex. This is the "contraction to the initial object", which is so useful in the classical version of K-theory. The column of X's on the right hand side will be denoted X_E, where the E stands for East. This is the east face of the square of X's. Our shorthand for the homotopy is



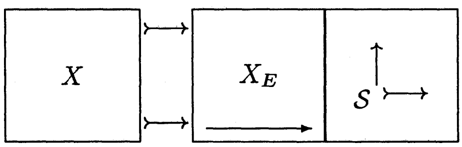
3.7.7. We will use the subscripts E, W, N and S to refer to the East, West, North and South faces of a (fixed) simplex. The arrow in the above square is to remind us that the homotopy is creeping in from the left, gradually replacing the God–fearing Y's with the godless X's.

The homotopy displayed above connects the identity with a map we will denote

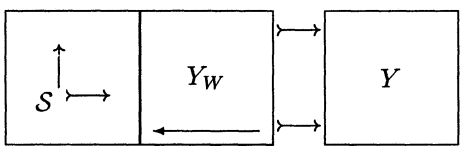


and this last map is clearly a contraction. □

The proofs of the contractibility in 3.7.2 and 3.7.3 are closely parallel. One uses the contracting homotopies



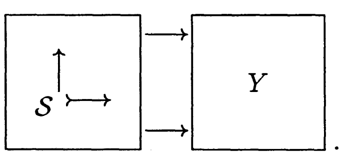
and



respectively to provide the contractions.The heart of the proof is therefore to establish the contractibility statement in 3.7.5.

3.7.8. *Proof of the contractibility in 3.7.5.*

We need to prove the contractibility of the simplicial set



As before, starting with the simplex

$$\begin{array}{ccc} \begin{array}{ccccc} X_{p0} & \rightarrowtail & \cdots & \rightarrowtail & X_{pn} \\ \uparrow & & & & \uparrow \\ \vdots & & & & \vdots \\ \uparrow & & & & \uparrow \\ X_{00} & \rightarrowtail & \cdots & \rightarrowtail & X_{0n} \end{array} & \begin{array}{c} \longrightarrow \\ \\ \\ \\ \longrightarrow \end{array} & \boxed{\begin{array}{ccccc} Y_{p0} & \longrightarrow & \cdots & \longrightarrow & Y_{pm} \\ \uparrow & & & & \uparrow \\ \vdots & & & & \vdots \\ \uparrow & & & & \uparrow \\ Y_{00} & \longrightarrow & \cdots & \longrightarrow & Y_{0m} \end{array}} \end{array}$$

we need to constuct homotopies. This time we will use a string of two simplicial homotopies, connecting the identity to a contracting map. In each case, we need to give the cells of the homotopy. Thus for the simplex

$$\begin{array}{ccc} \begin{array}{ccccc} X_{p0} & \rightarrowtail & \cdots & \rightarrowtail & X_{pn} \\ \uparrow & & & & \uparrow \\ \vdots & & & & \vdots \\ \uparrow & & & & \uparrow \\ X_{00} & \rightarrowtail & \cdots & \rightarrowtail & X_{0n} \end{array} & \begin{array}{c} \longrightarrow \\ \\ \\ \\ \longrightarrow \end{array} & \boxed{\begin{array}{ccccc} Y_{p0} & \longrightarrow & \cdots & \longrightarrow & Y_{pm} \\ \uparrow & & & & \uparrow \\ \vdots & & & & \vdots \\ \uparrow & & & & \uparrow \\ Y_{00} & \longrightarrow & \cdots & \longrightarrow & Y_{0m} \end{array}} \end{array}$$

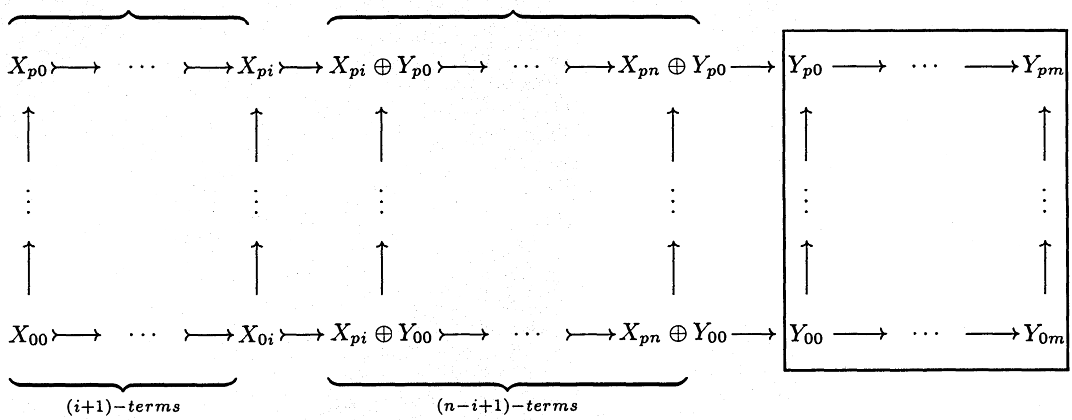
which is an n–simplex, we need to give an ordered set of $n+1$ $(n+1)$–simplices.

3.7.8.1. The first homotopy

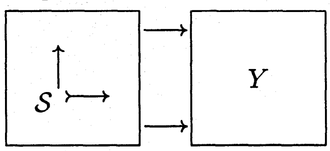
The first homotopy takes the simplex

$$\begin{array}{ccc} \begin{array}{ccccc} X_{p0} & \rightarrowtail & \cdots & \rightarrowtail & X_{pn} \\ \uparrow & & & & \uparrow \\ \vdots & & & & \vdots \\ \uparrow & & & & \uparrow \\ X_{00} & \rightarrowtail & \cdots & \rightarrowtail & X_{0n} \end{array} & \begin{array}{c} \longrightarrow \\ \\ \\ \\ \longrightarrow \end{array} & \boxed{\begin{array}{ccccc} Y_{p0} & \longrightarrow & \cdots & \longrightarrow & Y_{pm} \\ \uparrow & & & & \uparrow \\ \vdots & & & & \vdots \\ \uparrow & & & & \uparrow \\ Y_{00} & \longrightarrow & \cdots & \longrightarrow & Y_{0m} \end{array}} \end{array}$$

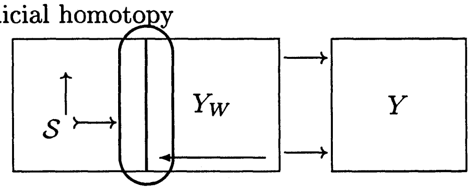
to a string of $n+1$ $(n+1)$–simplices, the $(n-i)^{th}$ of which is given by the diagram



The real point is that these simplices are well–defined; they lie in the simplicial set

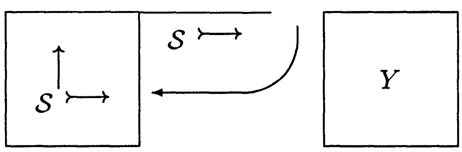


because the map $X_{ji} \longrightarrow X_{pi} \oplus Y_{j0}$ must be a mono, simply on the ground that the third edge of the triangle on this map is Y_{p0}, which unmistakeably is an object of $\mathcal{S}$. So although the simplicial homotopy

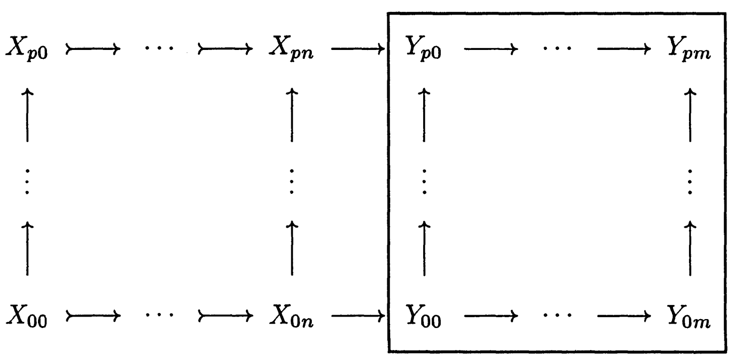


is not well–defined because the maps in the highlighted oval need not be mono, the new homotopy rectifies the problem by adding an X. This homotopy is so important in the remainder of the article, that we need a shorthand notation for it. In a sense that will be made precise in Sections II.1 and III.1, this is the only non–trivial homotopy in the article.

The shorthand we adopt for this homotopy is the rather curious–looking diagram

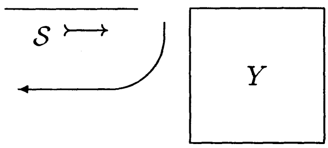


The idea of the notation is that cells of the homotopy are constucted out of

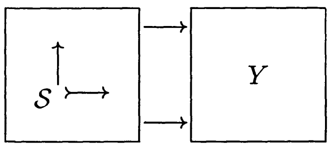


by using only the East face of Y and the North face of X.

Of course, the homotopy connects the identity with a map for which we also need a name. We will denote this map



and the notation is supposed to remind us that the map depends on the fixed Y and on the North face of X, and nothing else. Up to homotopy, we have factored the identity map on



through the simplicial set which I will denote

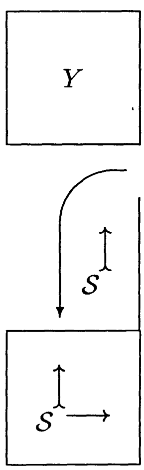
$$\overline{\mathcal{S}^{\rightarrowtail}}$$

My name for this simplicial set is one of my eccentricities. Most people would refer to it simply as the nerve of the category of monos in $\mathcal{S}$. An n–simplex is nothing other than a chain of composable morphisms

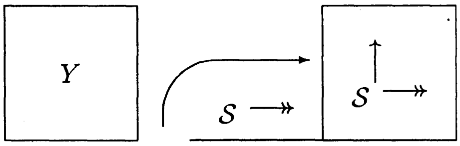
$$X_0 \rightarrowtail \cdots \rightarrowtail X_n$$

3.7.9. As I have said before, the homotopy of 3.7.8.1 is perhaps the only non-trivial homotopy in the article. Of course, the homotopy has a transpose, and both it and its transpose have duals. We will denote these:

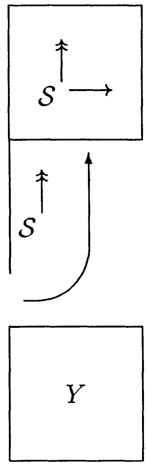
3.7.9.1. The transpose is denoted by



3.7.9.2. The dual and transpose of the dual are denoted

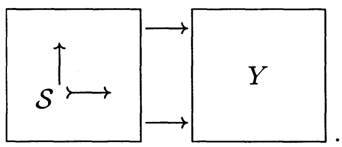


and

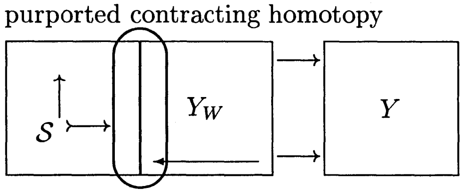


The reason I have not said which is the dual and which is the transpose of the dual is very simple; it is not clear to me whether there is a sensible way to distinguish between the two.

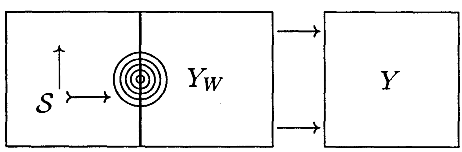
3.7.10. **A notation for warnings.** In the discussion of the first homotopy 3.7.8.1, I wanted to stress that the contraction to the terminal object is not a well–defined homotopy on



To do this, I drew the purported contracting homotopy

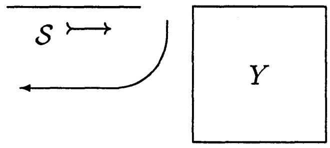


with an oval around the trouble spot. In the future, from time to time I will want to write down fake homotopies, to explain what is wrong with them and therefore perhaps clarify why our argument has to be so circuitous. The oval I drew above is rather ugly, so the notation we will adopt is that a trouble spot will be highlighted by five concentric circles. Thus the future notation for the fake homotopy above will be

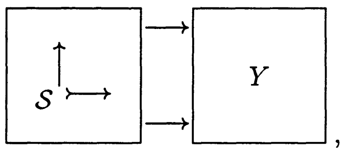


The motivation for the notation is that a well–defined homotopy should be thought of as being square, and curves are dangerous; especially an accumulation of five on top of one another.

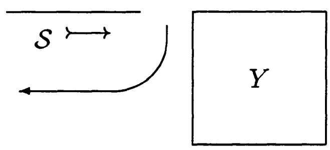
To complete the proof of 3.7.8 we need a second homotopy, showing us that the map



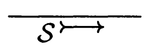
which we already know to be homotopic to the identity on



is in turn null homotopic. But by the discussion above, the map

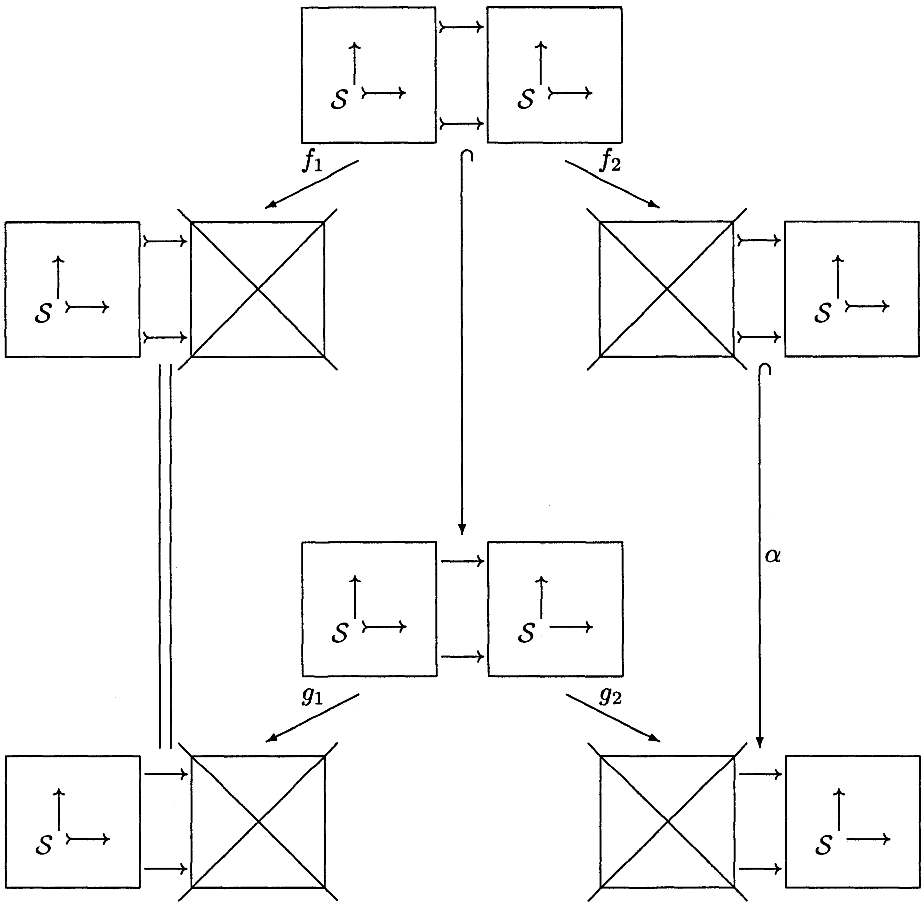


factors through the nerve of the subcategory of monos in $\mathcal{S}$, which in my somewhat idiosyncratic notation I write

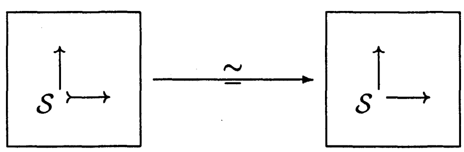


and this simplicial set is clearly contractible, by the contraction to the initial object. □

REMARK 3.8. In the future we will never give proofs as complete as the one you have just seen. Following the customs in the subject, we will leave some amount to the reader. In future, we will usually establish that in the analogues of the diagram



the maps g_1 and g_2 are homotopy equivalences. Thus we will honestly show that there is a homotopy equivalence

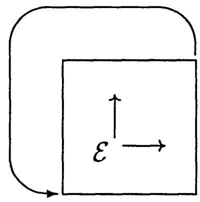


but we will leave entirely to the reader the proof that the homotopy equivalence is induced by some obvious map. Indeed, in general we will leave to the reader the construction of the analogues of the maps f_1, f_2 and α.

EXAMPLE 3.9. Suppose $\mathcal{T} = D^b(\mathcal{E})$, where $\mathcal{E}$ is an exact category, and $\mathcal{S} = D_{[0,0]}(\mathcal{E}) = \mathcal{E}$. Then Theorem 3.6 says, in particular, that $\mathcal{E}\overset{\uparrow}{\lfloor}\!\longrightarrow$ has the same homotopy type as $\mathcal{E}\overset{\twoheaduparrow}{\lfloor}\!\rightarrowtail$. But $\mathcal{E}\overset{\twoheaduparrow}{\lfloor}\!\rightarrowtail$ is well known to be just Quillen's Q-construction; the diagonal realization of $\mathcal{E}\overset{\twoheaduparrow}{\lfloor}\!\rightarrowtail$ is the simplicial set whose n-simplices are diagrams of cartesian squares

$$\begin{array}{ccccc}
X_{00} & \rightarrowtail & \cdots & \rightarrowtail & X_{0n} \\
\uparrow & & & & \uparrow \\
\vdots & & & & \vdots \\
\uparrow & & & & \uparrow \\
X_{n0} & \rightarrowtail & \cdots & \rightarrowtail & X_{nn}
\end{array}$$

and this diagram may be viewed as a sequence of maps in $Q(\mathcal{E})$, $X_{00} \bullet\!\!\to X_{11} \bullet\!\!\to \cdots \bullet\!\!\to X_{nn}$.

If we were dealing with  , that is still Quillen's K-theory. The point is that the differential $X_{pq} \to \Sigma X_{00}$ is unique. (See [1], Corollary 1.1.10.)

REMARK 3.10. The above statement about the uniqueness of differentials fails to hold if we replace triangles by semi–triangles. Several points in the argument become more delicate when dealing with semi–triangles. More will be said about this in Appendix A.

4. Two constructions for homological functors. Let $\mathcal{T}$ be a triangulated category, $\mathcal{A}$ an abelian category, and $H : \mathcal{T} \to \mathcal{A}$ a homological functor. Then it is well known that, provided H is bounded (i.e. for all objects X of $\mathcal{T}$, $H^n(X) = 0$ except for finitely many n), then H induces a "natural" map $K_0(H) : K_0(\mathcal{T}) \to K_0(\mathcal{A})$ on the Grothendieck groups. The map is given by the formula:

$$K_0(H)(X) = \sum_{n=-\infty}^{\infty} (-1)^n H^n(X).$$

We wish to show that such a map exists in higher K–theory. Precisely, we will prove:

THEOREM 4.1. *Let $H : \mathcal{T} \to \mathcal{A}$ be a homological functor. There exists a map $K_i(H) : K_i(\mathcal{T}) \to K_i(\mathcal{A})$ with the following naturality properties:*

4.1.1. *If $f : \mathcal{S} \to \mathcal{T}$ is a triangulated functor of triangulated categories, then $K_i(H \circ f) = K_i(H) \circ K_i(f)$.*

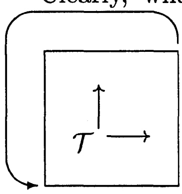
4.1.2. *If $g : \mathcal{A} \to \mathcal{B}$ is an exact functor of abelian categories, then $K_i(g \circ H) = K_i(g) \circ K_i(H)$.*

4.1.3. *If $h : \mathcal{B} \to \mathcal{T}$ is an exact functor from the abelian category $\mathcal{B}$ to the triangulated category $\mathcal{T}$, and if $H^n \circ h = 0$ except when $n = 0$, then $K_i(H^0 \circ h) = K_i(H) \circ K_i(h)$.*

REMARK 4.2. Part 3 of Theorem 4.1 requires elaboration. An exact functor $h : \mathcal{B} \to \mathcal{T}$ is taken to mean a functor such that if $A \rightarrowtail B \twoheadrightarrow C$ is an exact sequence in $\mathcal{B}$, then $h(A) \to h(B) \to h(C)$ is part of a triangle in $\mathcal{T}$: there exists a map $\phi : h(C) \to \Sigma h(A)$ which turns it into a triangle. If we take Construction 3.1 as the definition of the K–theory of $\mathcal{T}$, then such an h gives rise to a map of bicategories $\mathcal{E}^{\uparrow}_{\longrightarrow} \to \mathcal{T}^{\uparrow}_{\longrightarrow}$, hence to a map $K_i(h) : K_i(\mathcal{E}) \to K_i(\mathcal{T})$. However, if we use Construction 3.3, then to define $K_i(h)$ we need to assume for instance that $\phi : h(C) \to \Sigma h(A)$ is unique (see Example 3.9).

We only know how to prove Theorem 4.1 when the K–theory of $\mathcal{T}$ is defined via Construction 3.3. Nevertheless, we will proceed with two parallel arguments so that we can pinpoint where Construction 3.1 fails.

Clearly, what we need is some map out of the bisimplicial set $\mathcal{T}^{\uparrow}_{\longrightarrow}$ (resp.



). Let $H : \mathcal{T} \to \mathcal{A}$ be our homological functor. For each object $X \in \mathcal{T}$, we have a functor $H_X : \mathbb{Z} \to \mathcal{A}$ given by $H_X(n) = H^n(X)$. Thus, H induces a functor, which we also call H, $H : \mathcal{T} \to Hom(\mathbb{Z}, \mathcal{A})$. We will call the category $Hom(\mathbb{Z}, \mathcal{A})$ by the name $Gr(\mathcal{A})$. Thus we think of the functor $H_X : \mathbb{Z} \to \mathcal{A}$ as a graded object in $\mathcal{A}$.

The category $Gr(\mathcal{A})$ comes complete with a suspension functor $\Sigma : Gr(\mathcal{A}) \to Gr(\mathcal{A})$ (namely, the left shift). It also comes equipped with obvious subcategories $Gr_{[m,n]}(\mathcal{A})$ of complexes supported between degrees m and n, and

$$Gr^b(\mathcal{A}) = \bigcup_{m \le n} Gr_{[m,n]}(\mathcal{A}).$$

The fact that $H : \mathcal{T} \to \mathcal{A}$ is assumed bounded translates to mean that the induced functor $H : \mathcal{T} \to Gr(\mathcal{A})$ factors through $Gr^b(\mathcal{A}) \subset Gr(\mathcal{A})$.

DEFINITION 4.3. *A sequence in $Gr(\mathcal{A})$ of the form $X \xrightarrow{f} Y \xrightarrow{g} Z$ is called exact if*

4.3.1. *It is exact in the middle in every degree;*

4.3.2. *$Coker(g)$ and $\Sigma(Ker(f))$ agree up to filtrations. Write $H^n(X)$ for the degree n part of the graded object $X \in Gr(\mathcal{A})$. Then $H^n(Ker(f))$ is the kernel of $H^n(f) : H^n(X) \to H^n(Y)$. Similarly, $H^n(Coker(g))$ is the cokernel of $H^n(g) : H^n(Y) \to H^n(Z)$. For $X \xrightarrow{f} Y \xrightarrow{g} Z$ to be exact we assume, in addition to 4.3.1, that there is a finite filtration of $H^n(Coker(f))$ and a finite filtration of $H^{n+1}(Ker(g))$ such that the subquotients are isomorphic in pairs. In particular, after taking the graded module associated to suitable filtrations of $Coker(g)$ and $\Sigma(Ker(f))$, they must become isomorphic.*

DEFINITION 4.4. *A commutative square*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle g} \\ Y' & \xrightarrow{g'} & Z \end{array}$$

in $Gr(\mathcal{A})$ is called $M-V$ if

$$X \xrightarrow{\begin{pmatrix} f \\ f' \end{pmatrix}} Y \oplus Y' \xrightarrow{(\, g \ , \ -g' \,)} Z$$

is exact.

LEMMA 4.5. *Let*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow \\ X' & \xrightarrow{f'} & Y' \end{array} \quad \textit{and} \quad \begin{array}{ccc} Y & \xrightarrow{g} & Z \\ \downarrow & & \downarrow \\ Y' & \xrightarrow{g'} & Z' \end{array}$$

be $M-V$ squares in $Gr(\mathcal{A})$; then so is

$$\begin{array}{ccc} X & \xrightarrow{g \circ f} & Z \\ \downarrow & & \downarrow \\ X' & \xrightarrow{g' \circ f'} & Z' \end{array}$$

Proof. The lemma follows at once from a simple observation about the abelian category $\mathcal{A}$. Given two commutative squares of objects of $\mathcal{A}$

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ a\downarrow & & \downarrow b \\ X' & \xrightarrow{f'} & Y \end{array} \quad \text{and} \quad \begin{array}{ccc} Y & \xrightarrow{g} & Z \\ b\downarrow & & \downarrow c \\ Y' & \xrightarrow{g'} & Z' \end{array}$$

such that

$$X \xrightarrow{\begin{pmatrix} f \\ a \end{pmatrix}} Y \oplus X' \xrightarrow{(\, -b \ , \ f' \,)} Y'$$

and

$$Y \xrightarrow{\begin{pmatrix} g \\ b \end{pmatrix}} Z \oplus Y' \xrightarrow{(\, -c \ , \ g' \,)} Z'$$

are exact in the middle, then so is

$$X \xrightarrow{\begin{pmatrix} g \circ f \\ a \end{pmatrix}} Z \oplus X' \xrightarrow{(\, -c \ , \ g' \circ f' \,)} Z'$$

Furthermore, $Coker(-c \ , \ g' \circ f')$ is an extension of $Coker(-c \ , \ g')$ by $Coker(-b \ , \ f')$, while $Ker\begin{pmatrix} g \circ f \\ a \end{pmatrix}$ is an extension of $Ker\begin{pmatrix} g \\ b \end{pmatrix}$ by $Ker\begin{pmatrix} f \\ a \end{pmatrix}$. This statement about abelian categories is easy, but for the sake of completeness we include a proof.

Consider the three chain complexes

$$X \xrightarrow{\begin{pmatrix} f \\ a \end{pmatrix}} Y \oplus X' \xrightarrow{(\,-b\ ,\ f'\,)} Y'$$

$$Y \xrightarrow{\begin{pmatrix} g \\ b \end{pmatrix}} Z \oplus Y' \xrightarrow{(\,-c\ ,\ g'\,)} Z'$$

and

$$X \xrightarrow{\begin{pmatrix} g\circ f \\ a \end{pmatrix}} Z \oplus X' \xrightarrow{(\,-c\ ,\ g'\circ f'\,)} Z'.$$

If the objects X, X', Y, Y', Z, and Z' are all in $\mathcal{A}$, then these three complexes may be viewed as elements of $K(\mathcal{A})$, or even $D(\mathcal{A})$, simply by extending to infinite complexes all of whose other terms are zero. We have natural maps of complexes

$$\begin{array}{ccccccccc}
0 & \longrightarrow & X & \xrightarrow{\begin{pmatrix} f \\ a \end{pmatrix}} & Y \oplus X' & \xrightarrow{(\,-b\ ,\ f'\,)} & Y' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & X & \xrightarrow{\begin{pmatrix} g\circ f \\ a \end{pmatrix}} & Z \oplus X' & \xrightarrow{(\,-c\ ,\ g'\circ f'\,)} & Z' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Y & \xrightarrow{\begin{pmatrix} g \\ b \end{pmatrix}} & Z \oplus Y' & \xrightarrow{(\,-c\ ,\ g'\,)} & Z' & \longrightarrow & 0.
\end{array}$$

and it is trivial to check that this is part of a triangle; there is a map

$$\begin{array}{ccccccc}
0 & \longrightarrow & Y & \xrightarrow{\begin{pmatrix} g \\ b \end{pmatrix}} & Z \oplus Y' & \xrightarrow{(\,-c\ ,\ g'\,)} & Z' \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
X & \xrightarrow{\begin{pmatrix} f \\ a \end{pmatrix}} & Y \oplus X' & \xrightarrow{(\,-b\ ,\ f'\,)} & Y' & \longrightarrow & 0
\end{array}$$

which completes the above to a triangle in $K(\mathcal{A})$.

But then there is a long exact sequence in homology, namely

$$0 \longrightarrow Ker\begin{pmatrix} f \\ a \end{pmatrix} \longrightarrow Ker\begin{pmatrix} g\circ f \\ a \end{pmatrix} \longrightarrow Ker\begin{pmatrix} g \\ b \end{pmatrix} \longrightarrow$$

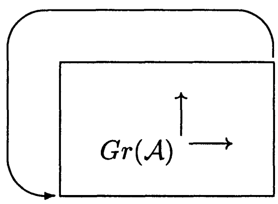
$$\longrightarrow \frac{Ker(-b\ ,\ f')}{Im\begin{pmatrix} f \\ a \end{pmatrix}} \longrightarrow \frac{Ker(-c\ ,\ g'\circ f')}{Im\begin{pmatrix} g\circ f \\ a \end{pmatrix}} \longrightarrow \frac{Ker(-c\ ,\ g')}{Im\begin{pmatrix} g \\ b \end{pmatrix}} \longrightarrow$$

$$\longrightarrow Coker(-b\ ,\ f') \longrightarrow Coker(-c\ ,\ g'\circ f') \longrightarrow Coker(-c\ ,\ g') \longrightarrow 0$$

and by hypothesis, the two outside terms in the middle row are zero. The assertion is now immediate. □

CONSTRUCTION 4.6. $Gr(\mathcal{A})$ is therefore a bicategory, with squares the $M-V$ squares. We can also define bicategories $Gr^b(\mathcal{A})$, $Gr_{[m,n]}(\mathcal{A})^{\uparrow}_{\rightarrowtail}$, $Gr_{[m,n]}(\mathcal{A})^{\twoheaduparrow}_{\rightarrow}$, $Gr_{[m,n]}(\mathcal{A})^{\twoheaduparrow}_{\rightarrowtail}$, etc. A mono in $Gr_{[m,n]}(\mathcal{A})$ is by definition a map which is mono in degree m; an epi is a map which is epi in degree n. Furthermore, the map $H : \mathcal{T} \to Gr^b(\mathcal{A})$ is a bifunctor of bicategories. If we define $K_i(Gr^b(\mathcal{A}))$ to be Π_{i+1} of the geometric realization of $Gr^b(\mathcal{A})$, there is an induced map $K_i(\mathcal{T}) \to K_i(Gr^b(\mathcal{A}))$.

CONSTRUCTION 4.7. We define a bisimplicial set



to consist of diagrams of $M-V$ squares with a compatible differential; i.e. a (p,q)-simplex is a diagram

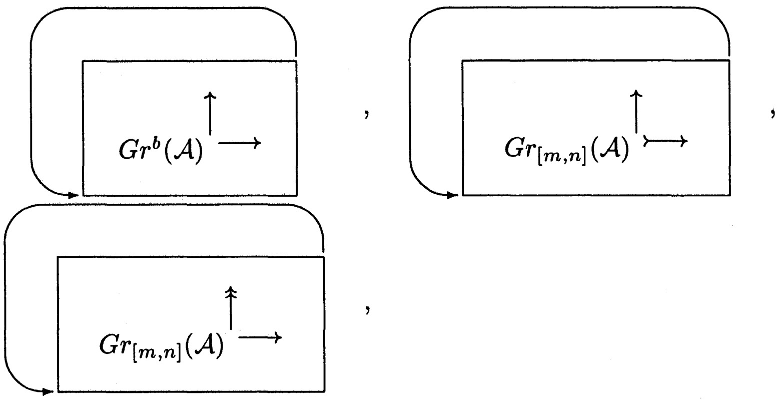
$$\begin{array}{ccccc} X_{p0} & \longrightarrow & \cdots & \longrightarrow & X_{pq} \\ \uparrow & & & & \uparrow \\ \vdots & & & & \vdots \\ \uparrow & & & & \uparrow \\ X_{00} & \longrightarrow & \cdots & \longrightarrow & X_{0q} \end{array}$$

together with a map $X_{pq} \to \Sigma X_{00}$, such that for every $0 \le i \le i' \le p$, $0 \le j \le j' \le q$, the sequence

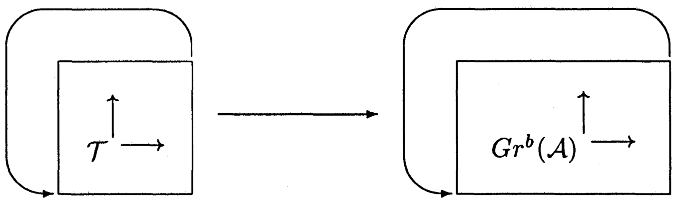
$$\Sigma^{-1}X_{i'j'} \to X_{ij} \to X_{i'j} \oplus X_{ij'} \to X_{i'j'} \to \Sigma X_{ij}$$

is exact degree by degree.

Once again, it is easy to define the constructions with restricted arrows, or where the objects are required to lie in some restricted subcategory. Predictably, we denote them

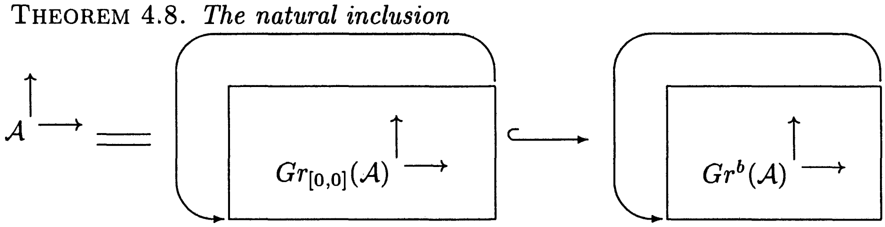


etc. It is also clear that the bounded homological functor $H : \mathcal{T} \to Gr^b(\mathcal{A})$ induces a map



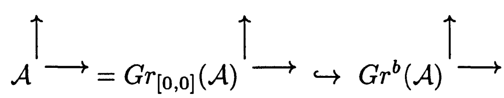
Theorem 4.1 will immediately follow once we establish

THEOREM 4.8. *The natural inclusion*



induces a homotopy equivalence.

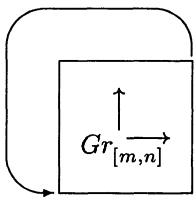
REMARK 4.9. The proof of Theorem 4.8 will not be given until later. For now, let us observe that we know relatively little about the map



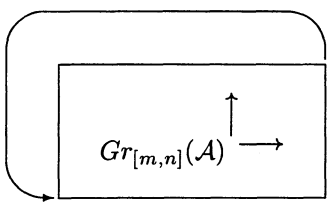
of Construction 4.6. This is remarkable because it should, by rights, be easier to construct homotopies in the nerve of the bicategory $Gr^b(\mathcal{A})\overset{\uparrow}{\longrightarrow}$. After all, a simplicial homotopy requires us to construct $(n+1)$–simplices out of n–simplices, and it is self–evident that keeping track of a coherent differential could only be a pain in the butt. Thus, it is hard to imagine a proof that works for the construction with differentials, but not for the one without. The argument will therefore have to be quite subtle. We will in fact try to highlight this point in the remainder of the article. And it is not so much that the author finds Construction 4.6 especially intelligent. Quite the contrary. But studying it helps one appreciate the subtlety of the proof.

In fact, one reason we will spend so much effort studying Construction 4.6 is as a caution to the reader. There are a number of reasons to suspect that Construction 4.6 yields a completely ridiculous K–theory. This makes it important to observe how close Construction 4.7 is to Construction 4.6, and just how much of the proof goes through for both simplicial sets.

REMARK 4.10. As long as no confusion can arise, that is as long as only one abelian category $\mathcal{A}$ is being considered, we will allow ourselves to omit explicit mention of $\mathcal{A}$ in the notation. We will write $Gr_{[m,n]}$ for $Gr_{[m,n]}(\mathcal{A})$, and

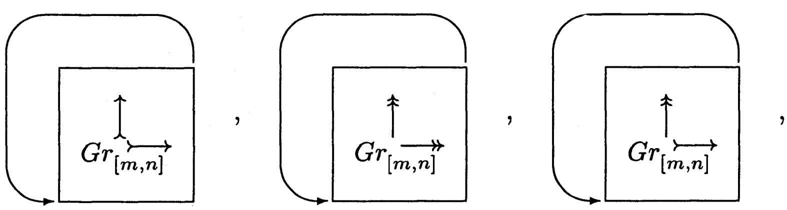


for the simplicial set



Before we end this section, let us observe that the proof of Theorem 3.7 works also in $Gr(\mathcal{A})$. Thus we have

THEOREM 4.11. *The bicategories* $Gr_{[m,n]}^{\rightarrowtail,\, \uparrowtail}$, $Gr_{[m,n]}^{\twoheadrightarrow,\, \twoheaduparrow}$, $Gr_{[m,n]}^{\rightarrowtail,\, \twoheaduparrow}$, *etc. are homotopy equivalent. Similarly, the bisimplicial sets*



etc. are also homotopy equivalent to each other. □

REMARK 4.12. It should be observed that the subtle difficulties mentioned in Remark 3.5 do not affect Construction 4.7. Precisely, given a map of "triangles" in $Gr(\mathcal{A})$

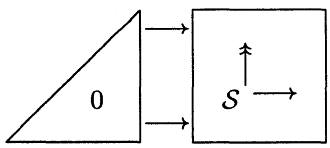
$$\begin{array}{ccccccc}
X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & \Sigma X \\
{\scriptstyle f}\downarrow & & {\scriptstyle g}\downarrow & & {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle \Sigma f} \\
X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z' & \xrightarrow{w'} & \Sigma X'
\end{array} .$$

i.e. a map of long exact sequences, then the mapping cone

$$Y \oplus X' \xrightarrow{\begin{pmatrix} -v & 0 \\ g & u' \end{pmatrix}} Z \oplus Y' \xrightarrow{\begin{pmatrix} -w & 0 \\ h & v' \end{pmatrix}} \Sigma X \oplus Z' \xrightarrow{\begin{pmatrix} -\Sigma u & 0 \\ \Sigma f & w' \end{pmatrix}} \Sigma Y \oplus \Sigma X'$$

is also a "triangle," i.e. a long exact sequence. This is because a "triangle" in $Gr(\mathcal{A})$ is a long exact sequence, i.e. the zero object in $D(\mathcal{A})$. The mapping cone on a map of "triangles" in $Gr(\mathcal{A})$ is the third vertex of a triangle in $D(\mathcal{A})$ where the other two vertices are zero; hence it is zero in $D(\mathcal{A})$, i.e. a long exact sequence.

5. Waldhausen-Style rigidifications. Following Waldhausen, it is standard to consider "rigidified" Q-constructions, where some data associated with a simplex is made to be part of the simplex. For instance, the bisimplicial set



has for its (p,q)-simplices diagrams

$$\begin{array}{ccccccccccc}
 & & & & 0 & \rightarrowtail & X_{p0} & \longrightarrow & \cdots & \longrightarrow & X_{pq} \\
 & & & . & \uparrow & & \uparrow & & & & \uparrow \\
 & & . & & \vdots & & \vdots & & & & \vdots \\
 & . & & & \uparrow & & \uparrow & & & & \uparrow \\
0 & \rightarrowtail & \cdots & \rightarrowtail & Y_{0p} & \rightarrowtail & X_{00} & \longrightarrow & \cdots & \longrightarrow & X_{0q}
\end{array}$$

This diagram is usually thought of as a (p,q)-simplex in $\overset{\uparrow}{\mathcal{S}}\longrightarrow$, but where $Y_{ij} = \ker(X_{ik} \to X_{jk})$ has been chosen. (For a triangulated category, Y_{ij} is the third edge of the triangle $Y_{ij} \to X_{ik} \to X_{jk}$). By Lemma 2.2, this is not unreasonable. We know that up to a (non-canonical) isomorphism, the object Y_{ij} is independent of k.

Because the map $X_{ik} \to X_{jk}$ are constrained to be epi, Y_{ij} must be in $\mathcal{S}$. Because $Y_{ij} \to X_{ik} \to X_{jk}$ is part of a triangle with $X_{jk} \in \mathcal{S}$, the map $Y_{ij} \to X_{ik}$ must be mono. In the notation, we feel free to suppress conditions on the objects and morphisms that are forced. Thus the diagrams

$$\begin{array}{ccccccccccc}
 & & & & 0 & \rightarrowtail & X_{p0} & \longrightarrow & \cdots & \longrightarrow & X_{pq} \\
 & & & & \twoheaduparrow & & \twoheaduparrow & & & & \twoheaduparrow \\
 & & & \cdot & \vdots & & \vdots & & & & \vdots \\
 & & \cdot & & \twoheaduparrow & & \twoheaduparrow & & & & \twoheaduparrow \\
0 & \rightarrowtail & \cdots & \rightarrowtail & Y_{0p} & \rightarrowtail & X_{00} & \longrightarrow & \cdots & \longrightarrow & X_{0q}
\end{array}$$

and

$$\begin{array}{ccccccccccc}
 & & & & 0 & \longrightarrow & X_{p0} & \longrightarrow & \cdots & \longrightarrow & X_{pq} \\
 & & & & \twoheaduparrow & & \twoheaduparrow & & & & \twoheaduparrow \\
 & & & \cdot & \vdots & & \vdots & & & & \vdots \\
 & & \cdot & & \twoheaduparrow & & \twoheaduparrow & & & & \twoheaduparrow \\
0 & \rightarrowtail & \cdots & \rightarrowtail & Y_{0p} & \longrightarrow & X_{00} & \longrightarrow & \cdots & \longrightarrow & X_{0q}
\end{array}$$

are really the same, since the apparently unrestricted arrows from the Y's to the X's are in fact forced to be restricted. Furthemore, both agree with

$$\begin{array}{ccccccccccc}
 & & & & 0 & \rightarrowtail & X_{p0} & \longrightarrow & \cdots & \longrightarrow & X_{pq} \\
 & & & & \uparrow & & \twoheaduparrow & & & & \twoheaduparrow \\
 & & & \cdot & \vdots & & \vdots & & & & \vdots \\
 & & \cdot & & \uparrow & & \twoheaduparrow & & & & \twoheaduparrow \\
0 & \rightarrowtail & \cdots & \rightarrowtail & Y_{0p} & \rightarrowtail & X_{00} & \longrightarrow & \cdots & \longrightarrow & X_{0q}
\end{array}$$

and

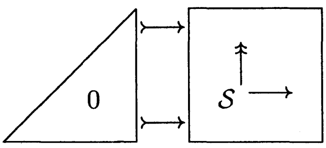
$$\begin{array}{ccccccccccc}
 & & & & 0 & \longrightarrow & X_{p0} & \longrightarrow & \cdots & \longrightarrow & X_{pq} \\
 & & & & \uparrow & & \twoheaduparrow & & & & \twoheaduparrow \\
 & & & \cdot & \vdots & & \vdots & & & & \vdots \\
 & & \cdot & & \uparrow & & \twoheaduparrow & & & & \twoheaduparrow \\
0 & \rightarrowtail & \cdots & \rightarrowtail & Y_{0p} & \longrightarrow & X_{00} & \longrightarrow & \cdots & \longrightarrow & X_{0q}
\end{array}$$

and

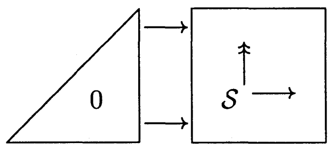
$$\begin{array}{ccccccccccc}
 & & & & 0 & \longrightarrow & X_{p0} & \longrightarrow & \cdots & \longrightarrow & X_{pq} \\
 & & & \cdot & \uparrow & & \twoheaduparrow & & & & \twoheaduparrow \\
 & & \cdot & & \vdots & & \vdots & & & & \vdots \\
 & \cdot & & & \uparrow & & \twoheaduparrow & & & & \twoheaduparrow \\
0 & \longrightarrow & \cdots & \longrightarrow & Y_{0p} & \longrightarrow & X_{00} & \longrightarrow & \cdots & \longrightarrow & X_{0q}
\end{array};$$

there is no need to explicitly insist that the vertical arrows in the Y's be epi. Of course, the careful reader will note there are still three more descriptions of the same simplex.

Therefore the simplicial sets

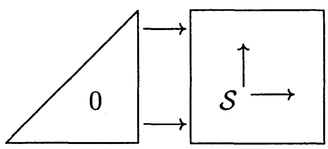


and



are really one and the same. Since the author has been too lazy to explicitly write out the contents of the triangle labeled 0, we need not in the notation concern ourselves about what arrows are permissible there. We permit all the arrows that could conceivably go there, keeping the diagram a diagram of $M - V$ squares.

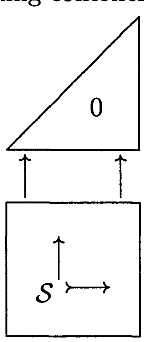
Until now, we have insisted that the morphisms $X_{ik} \to X_{jk}$ be constrained to be epi. This guaranteed that all the Y's are objects of $\mathcal{S}$. However, when convenient, we will do the unprejudiced thing. We will, when necessary, feel free to consider the bisimplicial set



,

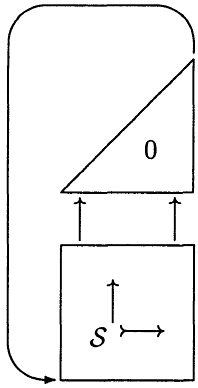
where for a general (p,q)-simplex, the objects in the left hand triangle need not be in $\mathcal{S}$.

Similarly, we can rigidify by adding cokernels of maps. Thus the bisimplicial set

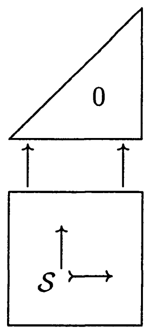


consist of simplices of $\mathcal{S}^{\uparrow}\!\rightarrowtail$, together with choices of the cokernels of horizontal maps.

There is, of course, a version of this construction which comes "with compatible differentials." Thus, a (p,q)-simplex in the bisimplicial set



is a simplex in



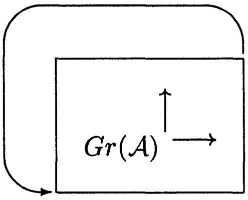
together with compatible choices of differentials. That is, for any two squares, one of which is embedded in the other

$$\begin{array}{ccccccc}
Z & \longrightarrow & Z' & \longrightarrow & Z'' & \longrightarrow & Z''' \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
Y & \longrightarrow & Y' & \longrightarrow & Y'' & \longrightarrow & Y''' \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
X & \longrightarrow & X' & \longrightarrow & X'' & \longrightarrow & X''' \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
W & \longrightarrow & W' & \longrightarrow & W'' & \longrightarrow & W'''
\end{array}$$

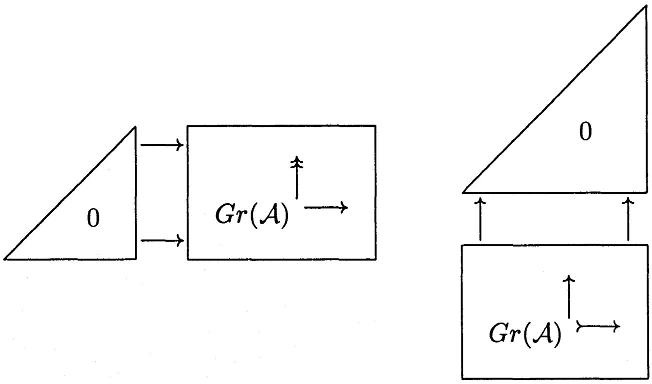
The differential $Z''' \to \Sigma W$ "induces" the differential $\phi : Y'' \to \Sigma X'$; precisely, the map ϕ is the composite

$$Y'' \to Z''' \to \Sigma W \to \Sigma X'.$$

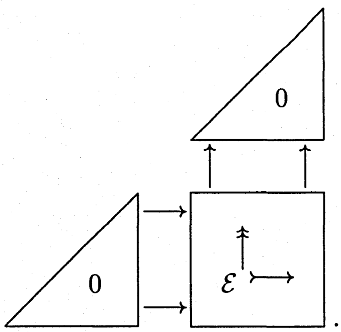
We have been considering how to "rigidify" the simplex by fixing kernels and cokernels, with or without compatible differentials. But until now we considered only the case of an exact subcategory $\mathcal{S}$ of a triangulated category $\mathcal{T}$. Of course, we could perform the same "rigidification" construction on the bisimplicial sets $Gr(\mathcal{A})^{\uparrow}_{\longrightarrow}$ and



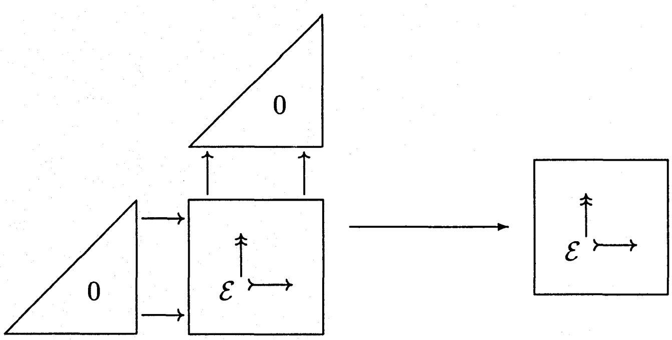
. We obtain, among others, the simplicial sets



Waldhausen's usual model for the K–theory of an exact category $\mathcal{E}$ is a simplicial set whose edgewise subdivision is the diagonal realization of the bisimplicial set

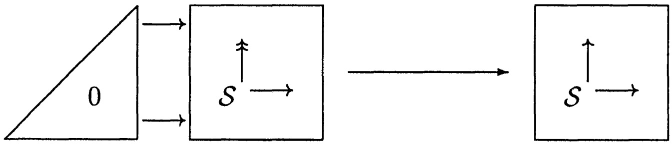


For an exact category, the choices of kernels and cokernels are rigid. Thus, it is easy to show that the "forgetful" map



induces a homotopy equivalence. The point of this section is that the same is true in a triangulated category; but the proof will clearly need to be more subtle.

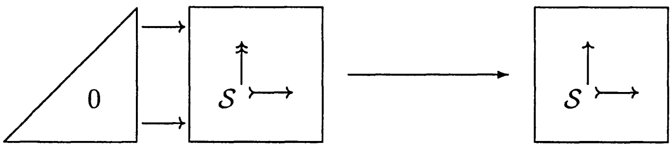
THEOREM 5.1. . *The natural map*

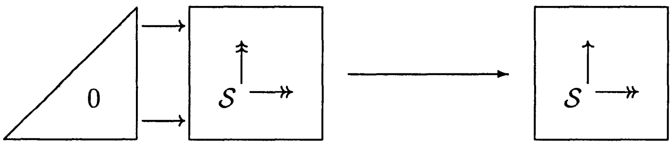


is a homotopy equivalence.

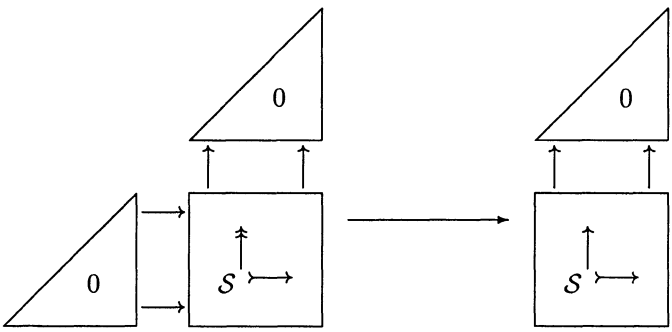
REMARK 5.2. . I do not want to write down all the variants of this theorem which follow by the same proof. The reader will observe some easy facts.

5.2.1. The nature of the horizontal map does not enter the proof: we will have proved that



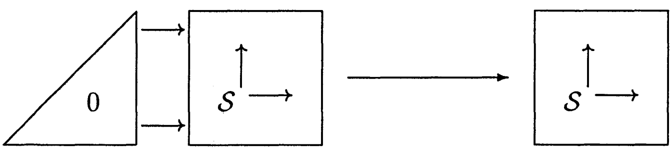


and

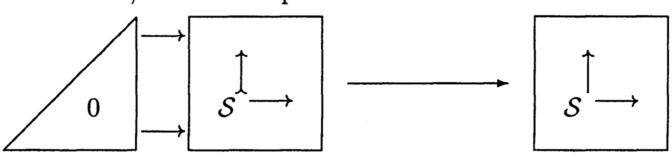


are all homotopy equivalences.

5.2.2. The same proof can be used to show that



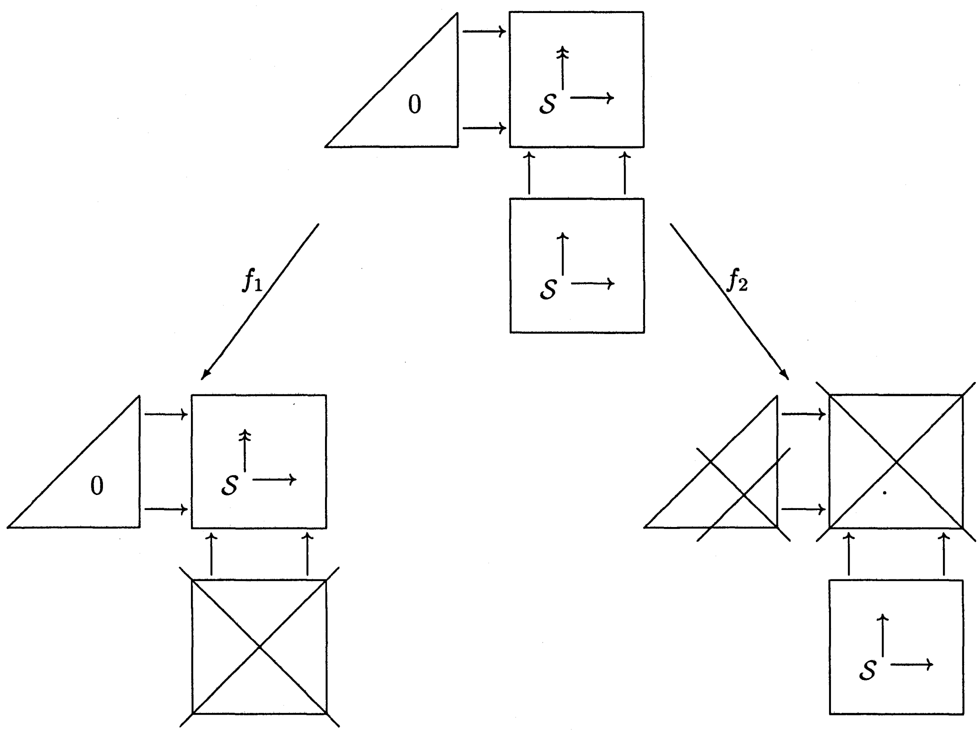
is a homotopy equivalence; in other words, it is inessential that the objects Y, the kernels for the vertical maps, lie in the category $\mathcal{S}$. But the proof does not allow the vertical map to be mono; we will not prove that



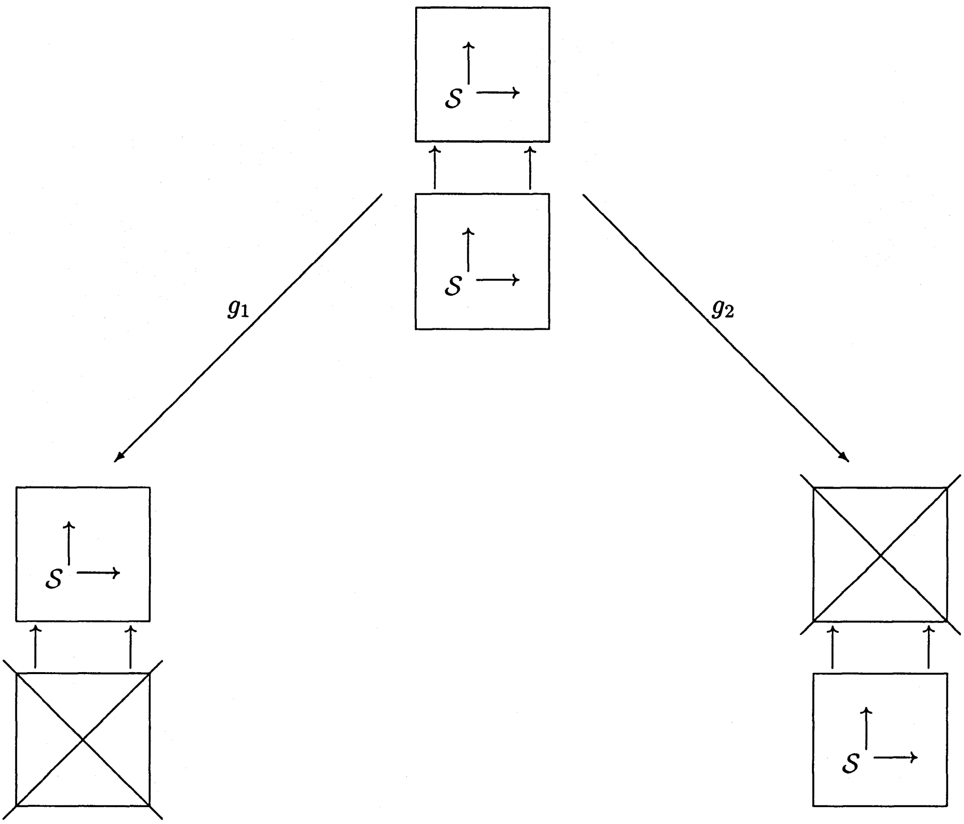
is a homotopy equivalence.

5.2.3. The proof is formal enough to work in all our constructions. It works for triangulated categories, for $Gr(\mathcal{A})$, and with or without compatible differentials.

Proof of Theorem 5.1. Consider the trisimplicial set and two projections

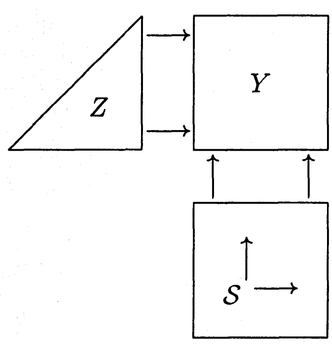


We will prove that f_1 and f_2 are homotopy equivalences. As we have discussed in Section 3, this suffices to prove that the simplicial sets are homotopy equivalent, but is not quite enough to establish that the homotopy equivalence is induced by the natural map. As in Section 3, the way to get around this is to consider the closely related diagram

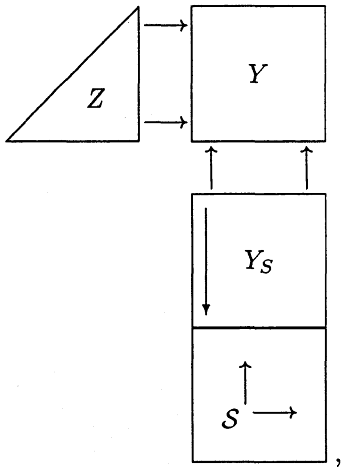


and the natural map between the diagrams. The reader should think over this point now; in the future we will assume without mention that the reader will complete our diagrams in the obvious way.

The serious part of the proof is therefore showing that f_1 and f_2 are homotopy equivalences. For f_1, this is clear; by Segal's Theorem it suffices to establish the contractibility of

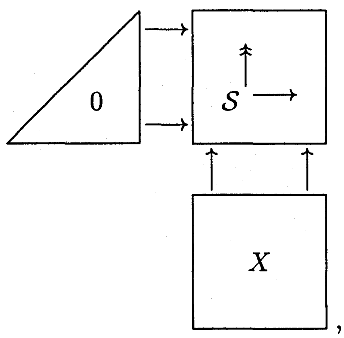


and this follows immediately from the contracting homotopy



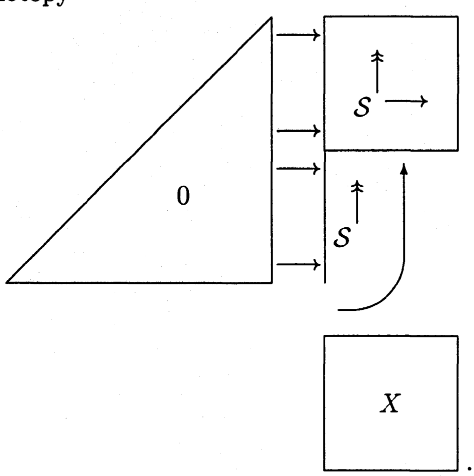
that is, by contracting to the "terminal object", or the south face of Y.

For f_2, the argument closely parallels that of Theorem 3.7. We must prove the contractibility of the simplicial set



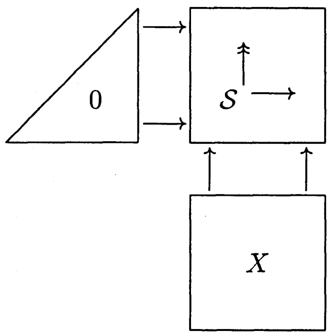
and the idea is to use essentially the same homotopy as in the proof of Theorem 3.7. It is, after all, our only non–trivial homotopy. It is about time we make it begin working for us.

We write the homotopy

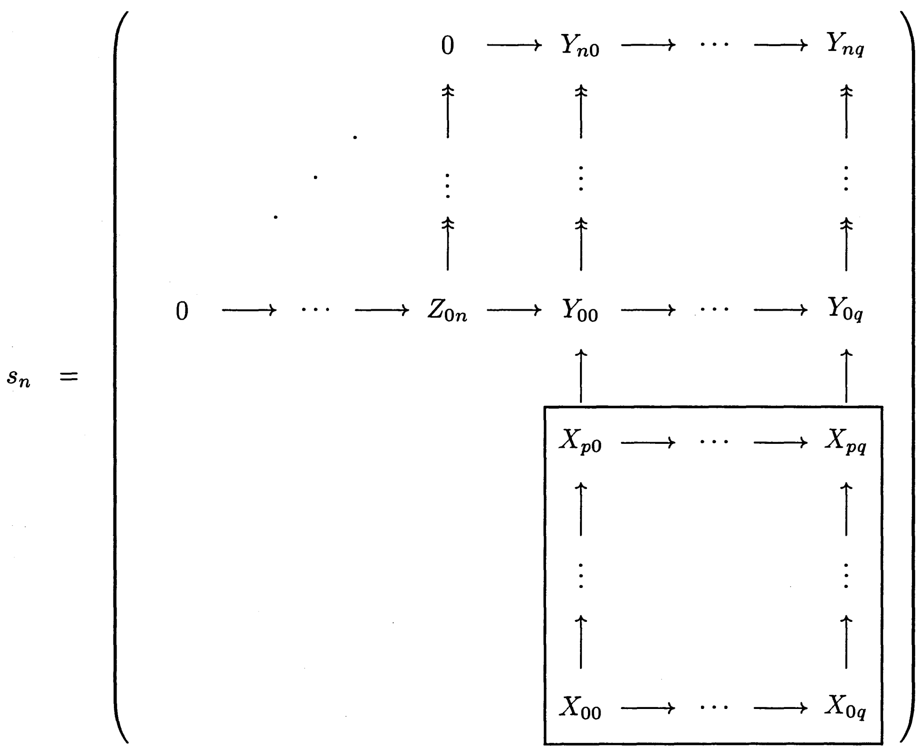


And now it would be only fair to say what the notation means.

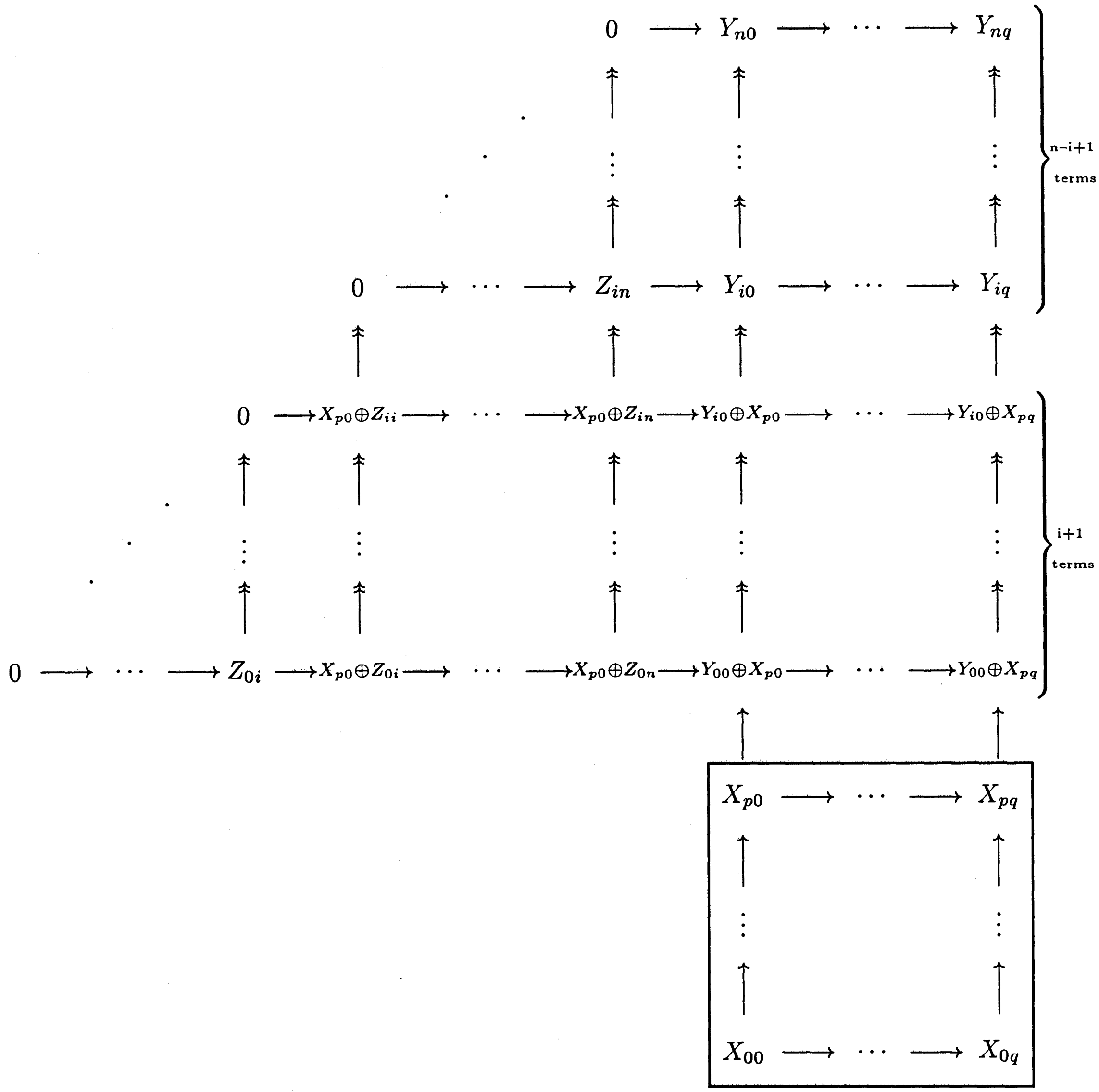
We begin by reminding the reader that the simplicial set



has only one simplicial structure. An n–simplex s_n is a diagram of $M - V$ squares

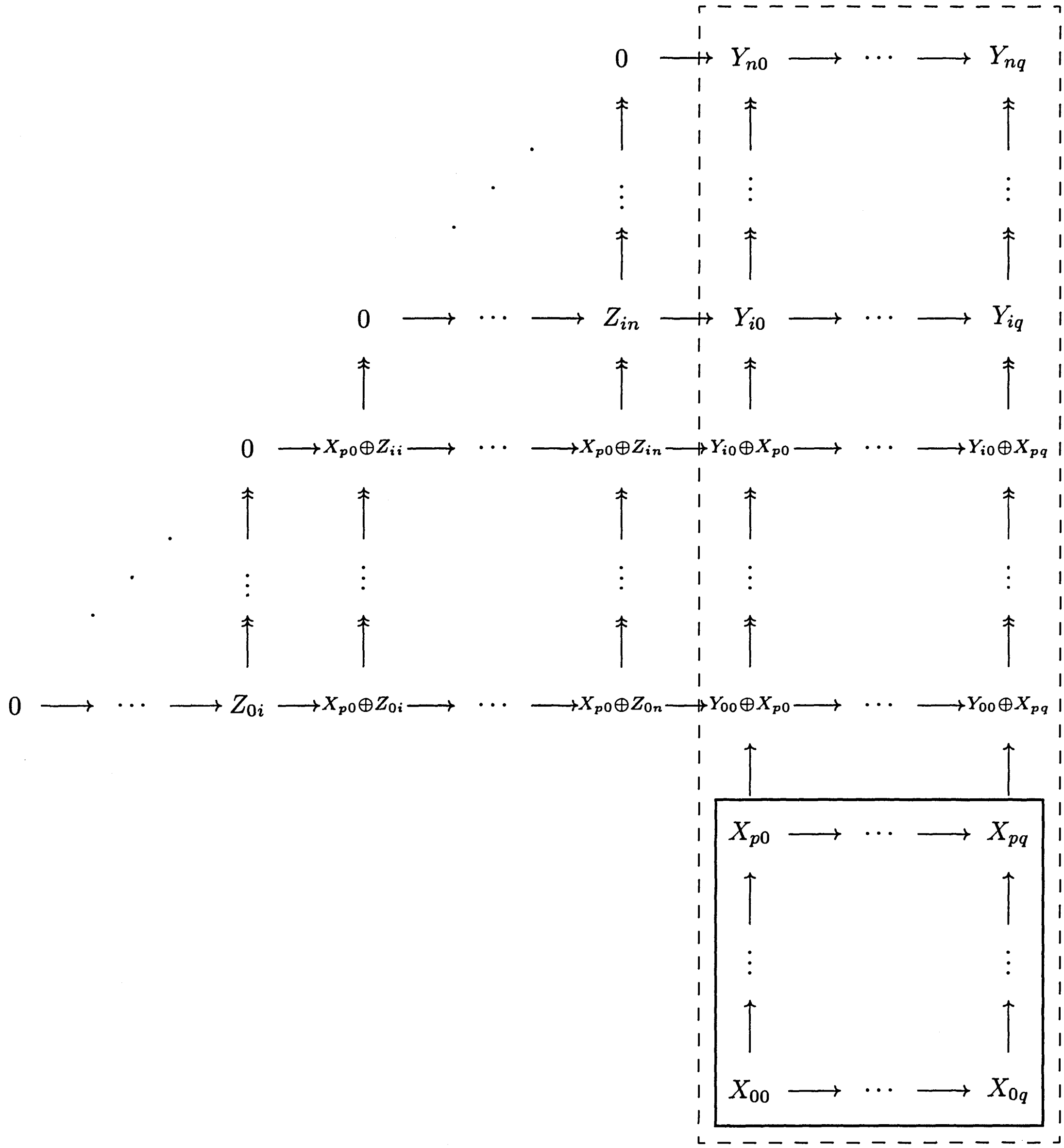


and as before, the framed rectangle is fixed. The unique simplicial structure comes from varying the integer n. To construct a homotopy we need to assign to every n–simplex an ordered set of $n+1$ different $(n+1)$–simplices. The homotopy analogous to the non–trivial one in Section 3 takes s_n to an ordered set of $n+1$ $(n+1)$–simplices, the i^{th} of which is

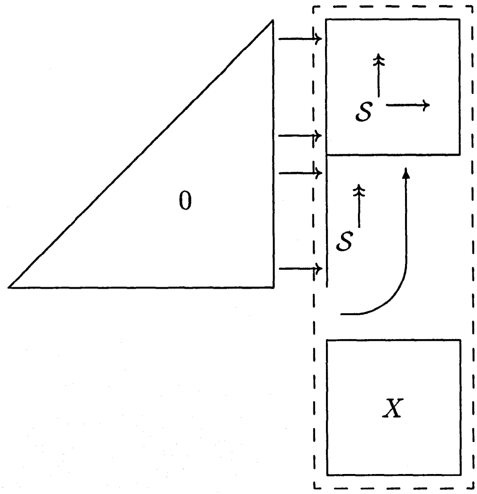


We will have occasion to refer back to this homotopy. When we do, we will refer to it as $**$.

The homotopy, and its relation with the homotopy we have seen in the proof of Theorem 3.7, are best described by noticing that in the following diagram, the part surrounded by the box of broken lines is nothing other than the dual of the homotopy of 3.7.8.1

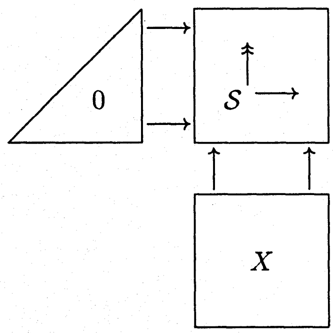


our shorthand for the homotopy is also intended to be suggestive; what is inside the box of broken lines below

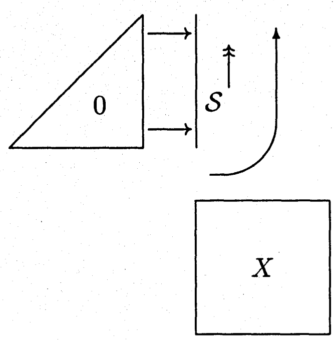


is also simply dual to 3.7.8.1. The only new point here is that in our favorite old homotopy it is relatively easy to keep track of the kernels.

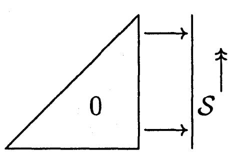
Of course, this means that the identity on



is homotopic to the simplicial map

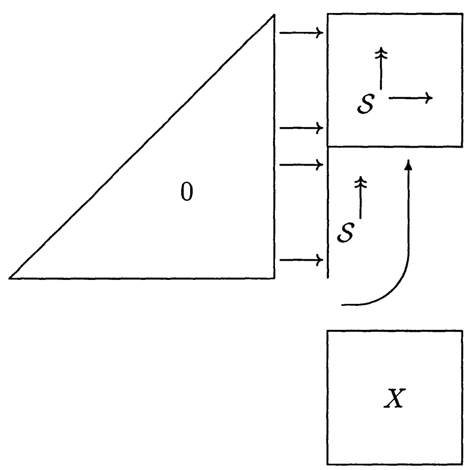


which, as the notation is meant to suggest, factors through

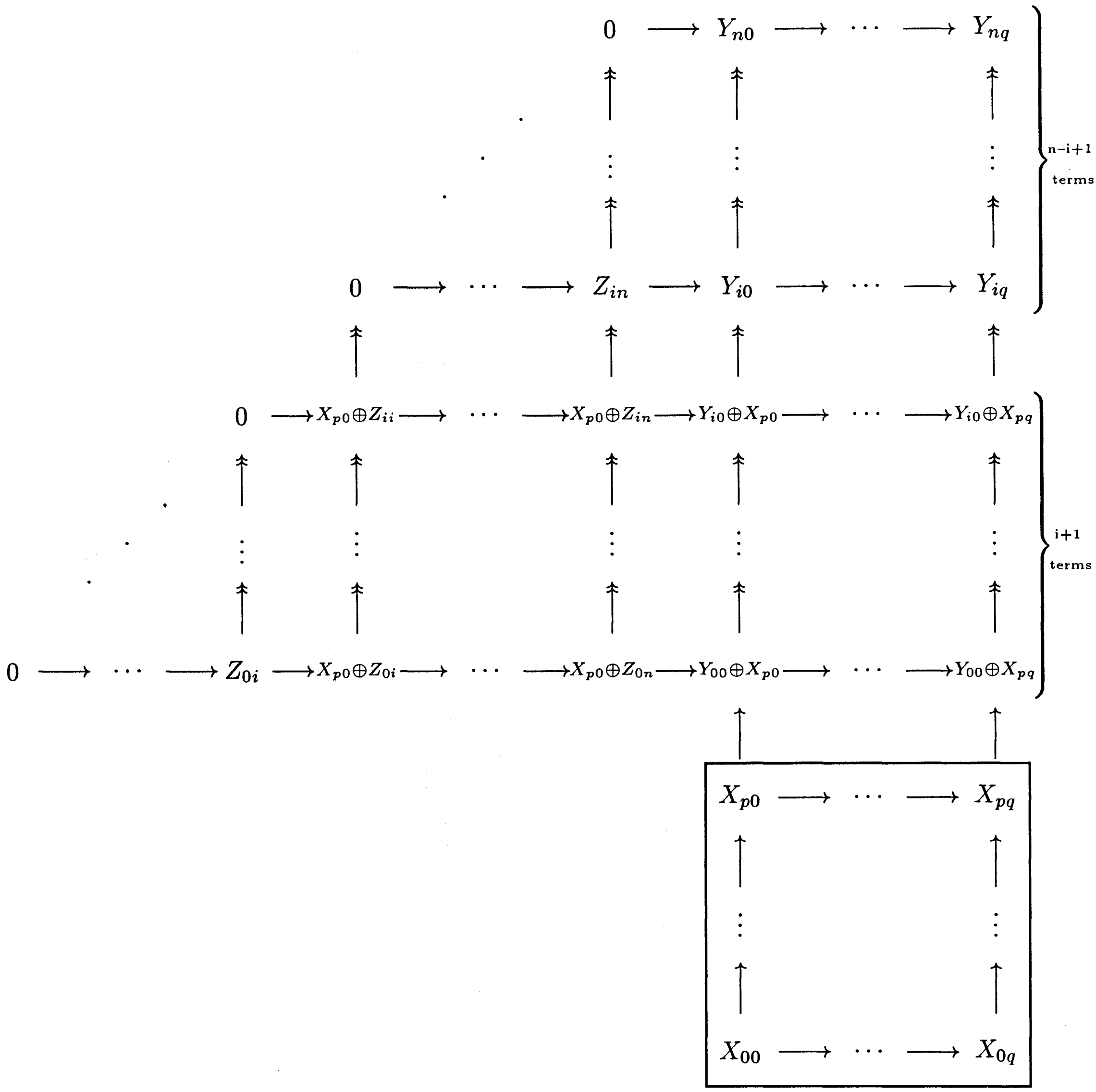


This last simplicial set is my notation for the slight modification of the nerve of the category of epis in $\mathcal{S}$, where one keeps track of the kernels. This category is clearly contractible, by the contraction to the terminal object. ◻

REMARK 5.3. Here is the first place where we need to use Remark 3.5. The problem is the following: the homotopy



is given by asigning to every n–simplex s_n a great many $(n+1)$–simplices, given by the diagrams



Unfortunately, without some extra hypotheses it is not true that every square need give rise to a triangle. In other words, taken at face value, what I have written above need not be a simplex. The reader will easily convince himself that the difficulty arises with squares of the form

$$\begin{array}{ccc} Z_{k'l} & \longrightarrow & Y_{k'l'} \\ \uparrow & & \uparrow \\ X_{p0} \oplus Z_{kl} & \longrightarrow & Y_{k0} \oplus X_{pl'} \end{array}$$

The other squares in the homotopy are very clearly $M - V$, being just direct sums of "triangles" (the reason for the inverted commas is that if we are working in the construction with no differentials, this means sequences which admit a differential completing them to triangles). The difficult square pictured above is difficult. Let us analyse it.

We start with a simplex

$$\begin{array}{ccccc} Z_{k'l} & \longrightarrow & Y_{k'0} & \longrightarrow & Y_{k'l'} \\ \uparrow & & \uparrow & & \uparrow \\ Z_{kl} & \longrightarrow & Y_{k0} & \longrightarrow & Y_{kl'} \\ & & \uparrow & & \uparrow \\ & & X_{p0} & \longrightarrow & X_{pl'} \end{array}$$

and out of it we somehow want to naturally construct a triangle

$$Z_{kl} \oplus X_{p0} \to Z_{k'l} \oplus Y_{k0} \oplus X_{pl'} \to Y_{k'l} \to \Sigma(Z_{kl} \oplus X_{p0}).$$

Let me explain why such a triangle should exist, leaving the details to the reader. We have three triangles

5.3.1. $Z_{kl} \to Z_{k'l} \oplus Y_{kl'} \to Y_{k'l'} \to \Sigma Z_{kl}$,

5.3.2. $X_{p0} \to Y_{k'0} \oplus X_{pl'} \to Y_{k'l'} \to \Sigma X_{p0}$,

and

5.3.3. $Y_{k0} \to Y_{k'0} \oplus Y_{kl'} \to Y_{k'l'} \to \Sigma Y_{k0}$.

Therefore, there is a map of triangles 5.3.1 $\oplus$ 5.3.2 $\to$ 5.3.3. If this map were good, its mapping cone would be a triangle

$$\begin{aligned} \to Z_{kl} \oplus X_{p0} \oplus \Sigma^{-1}Y_{k'l'} \to (Z_{k'l} \oplus Y_{kl'}) \oplus (Y_{k'0} \oplus X_{pl'}) \oplus Y_{k0} \\ \to Y_{k'l'} \oplus Y_{k'l'} \oplus (Y_{k'0} \oplus Y_{kl'}) \to \end{aligned}$$

There is a map of triangles where the top triangle is contractible

$$\begin{array}{ccccc} \Sigma^{-1}Y_{k'l'} & \to & Y_{k'l} \oplus Y_{k'0} & \to & (Y_{kl'} \oplus Y_{k'0}) \oplus Y_{k'l'} \\ \downarrow & & \downarrow & & \downarrow \\ Z_{kl} \oplus X_{p0} \oplus \Sigma^{-1}Y_{k'l'} & \to & (Z_{k'l} \oplus Y_{kl'}) \oplus (Y_{k'0} \oplus X_{pl'}) \oplus Y_{k0} & \to & Y_{k'l'} \oplus Y_{k'l'} \oplus (Y_{k'0} \oplus Y_{kl'}) \end{array}$$

This may be completed (uniquely) to a 3×3 square: the uniqueness is because each column is split. We deduce a triangle

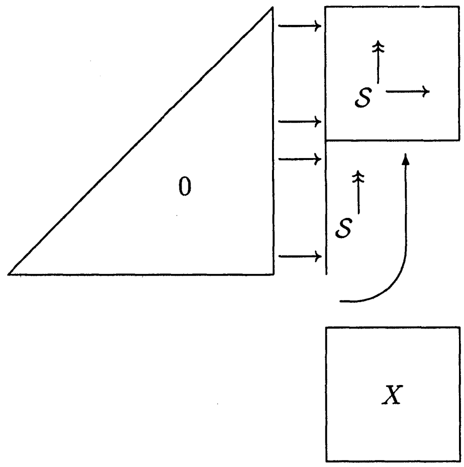
$$\to Z_{kl} \oplus X_{p0} \to Z_{k'l} \oplus X_{pl'} \oplus Y_{k0} \to Y_{k'l'} \to$$

which was what we wanted. The uniqueness allows us to compute every arrow of this triangle and prove it to be the obvious candidate.

When we work with the construction with no differentials, or alternatively with $Gr(\mathcal{A})$, the mapping cone on a map of "triangles" is always a triangle. For Construction 3.3 one needs to modify it somewhat to guarantee that the only permissible simplices behave well with respect to the iterated formation of mapping cones. This can be done in a number of ways. As in Remark 3.5, the way the reader should read this remark depends on which group he belongs to.

Group 1: The reader should ignore the point. For the Gr construction, mapping cones of exact sequences are exact.

Group 2: Every simplicial set we write down should be interpreted as consisting of diagrams, each of which splits in some way as a direct sum of other diagrams which lift to model categories. It therefore suffices to show that the homotopy



takes a simplex with a lifting to another simplex with a lifting. This is obvious.

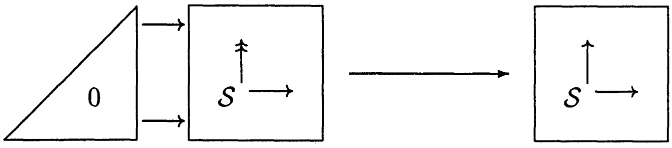
Group 3: Depending on your favorite choice of simplicial set representing the K–theory, you will need to provide your own argument at this point. I will keep this article fairly topological; foundational questions of homological algebra which affect the proof will be elaborated elsewhere.

As I have now repeated *ad nauseum,* the homotopy of Theorem 5.1 is the only homotopy I know in the triangulated setting. Thomason was very concerned about how one sees, in the many applications of this homotopy that follow, that all the choices of differentials can be made coherently. There is a meta-theorem which establishes that. The point is the following. If the $X's$, $Y's$, and $Z's$ were all inside $\mathcal{S} = D(\mathcal{A})_{[0,0]} = \mathcal{A} \subset D^b(\mathcal{A})$, then of course the homotopy of Lemma 5.1 is still well defined, having the cells described. But by [1] the differential is then unique. So the differentials one obtains from the mapping cone construction above must be the unique differentials making the various squares $M - V$. But then, of course, the differentials must be extremely compatible. This works initially only under the assumption that the $X's$, $Y's$, and $Z's$ are in $\mathcal{A} \subset D^b(\mathcal{A})$, but by the principle of the extension of algebraic identities, the differentials obtained from mapping cones must be highly compatible.

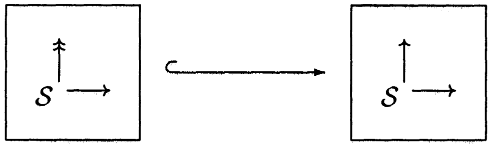
Caution 5.4. We noted in Remark 5.3 that the homotopy of Theorem 5.1 will never give us trouble, if by trouble we mean destroying triangles or giving incompatible differentials. But this might lull us into a false sense of security. The homotopy is very treacherous, and it is best to realize this.

What the homotopy rarely preserves are monos and epis. It is perhaps best to illustrate this with a cautionary example.

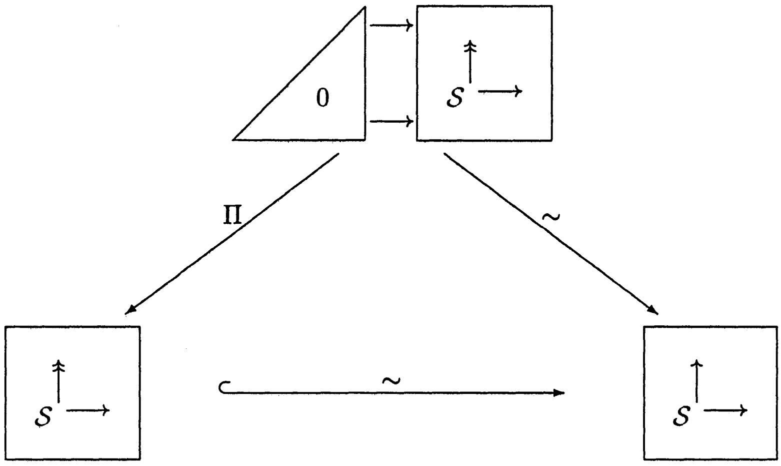
By Theorem 5.1 we know that the map



induces a homotopy equivalence. By Theorem 3.7 we know that the inclusion

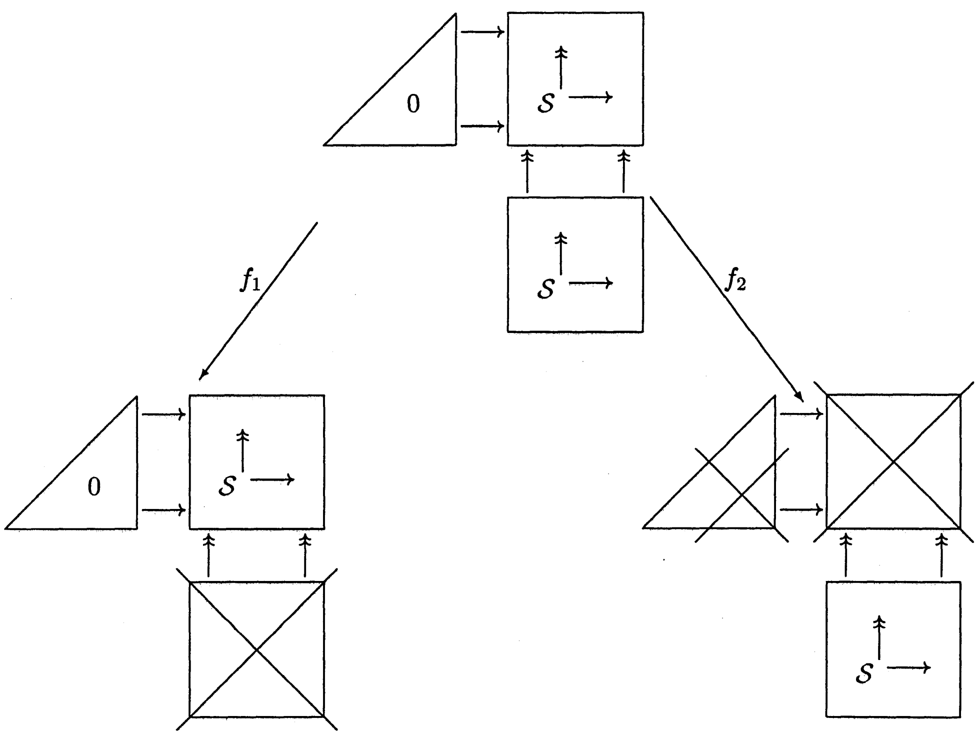


also induces a homotopy equivalence. From the commutative diagram

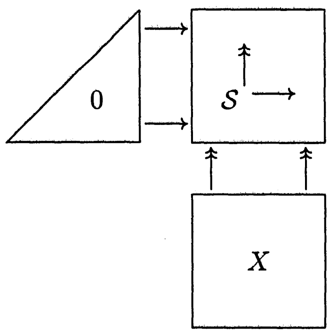


we easily deduce that the projection Π also induces a homotopy equivalence. Let us now see what goes wrong if we try to prove this directly, using the homotopy of Theorem 5.1.

Consider therefore the trisimplicial set and two projections

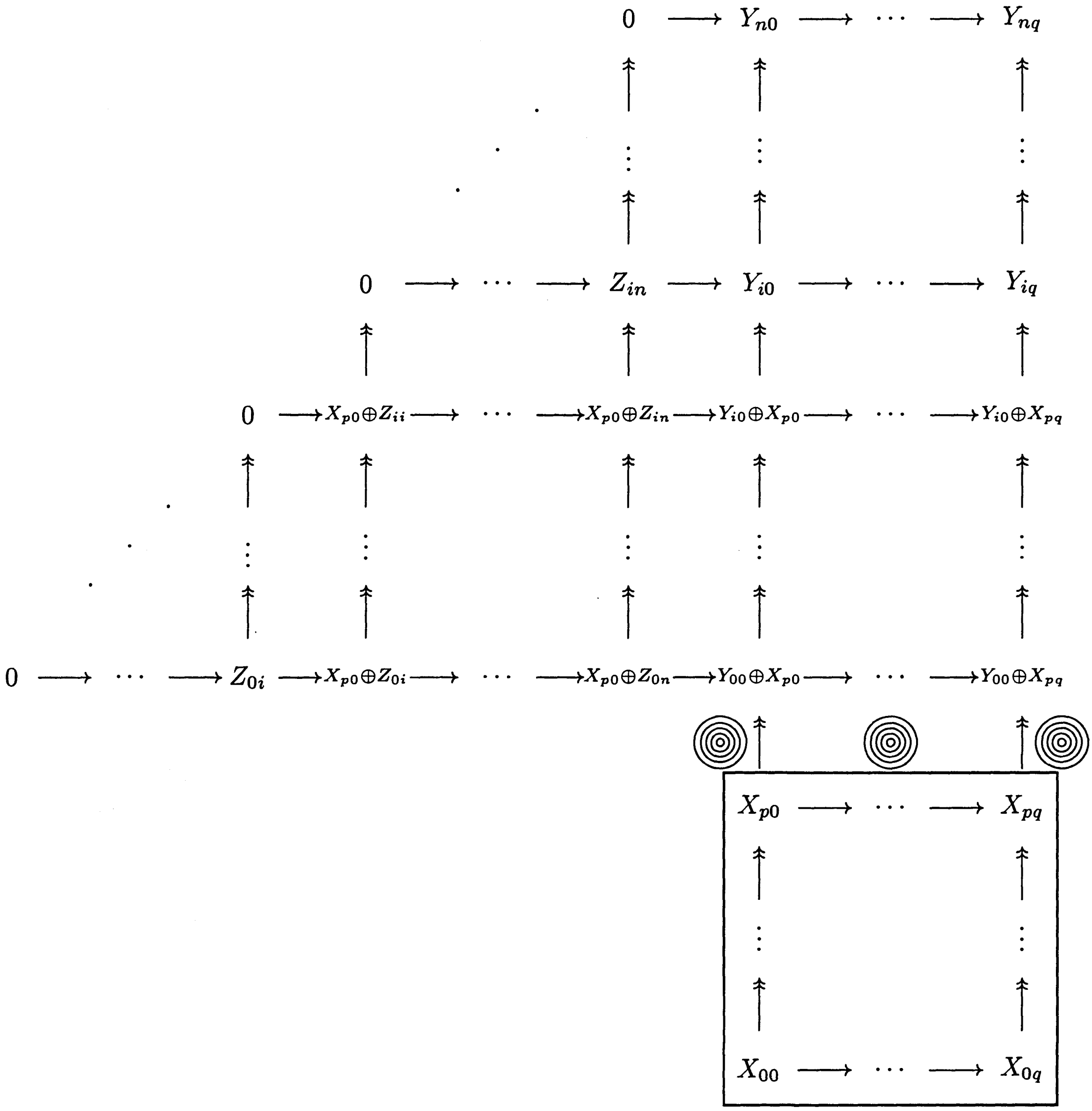


Clearly, f_1 is a homotopy equivalence. We know, by indirect arguments, that so is f_2. But we can also ask whether the argument of Theorem 5.1 works. Is it true that f_2 becomes a homotopy equivalence of bisimplicial spaces once we realize the simplicial structure degenerate on the target? More concisely, is the fiber:



necessarily contractible?

I do not know. The contracting homotopy of Theorem 5.1 certainly does not work. A typical cell in the homotopy would be a diagram

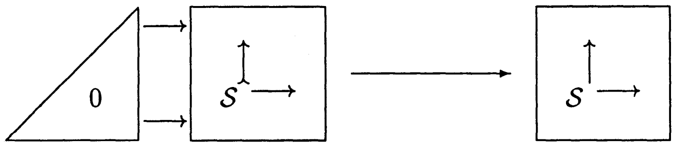


and the problem, as highlighted by the warning circles, is that the map $X_{pi} \to Y_{00} \oplus X_{pi}$ is patently only rarely epi.

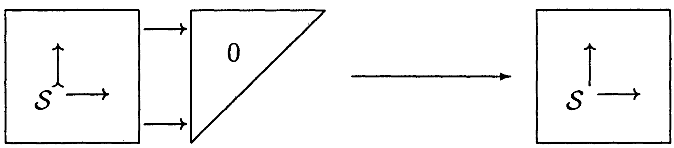
I cannot overemphasize the importance of care at this point. I no longer remember all the many times I "proved" a disconnected space contractible. The mistake invariably came down to some homotopy which wandered outside its simplicial set. This is, in fact, the main reason I have written the proof in such deail. Writing down the cells of each homotopy forced me to slow down and check that no homotopy goes astray, leaving its simplicial set. I know no general argument that substitutes for a careful check of each homotopy.

Originally, it seemed to me dangerous to even adopt a shorthand for the homotopy. A shorthand is an invitation for carelessness. A number of people, several referees included, complained that the original version was nearly unreadable; it was hard to tell which homotopy is new, and what part of the argument is repetitive. The new version of the article goes in for shorthand in a big way. This should make it clear that the homotopies are all the same.

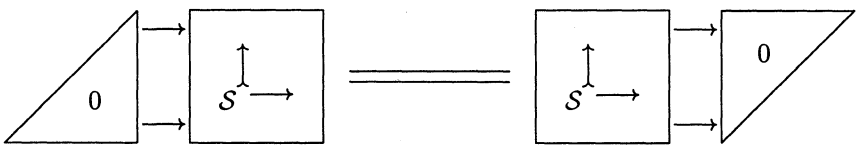
REMARK 5.5. Already in 5.2.2, it was observed that this proof does not establish the natural map



to be a homotopy equivalence. On the other hand, by the dual of Theorem 5.1, the projection

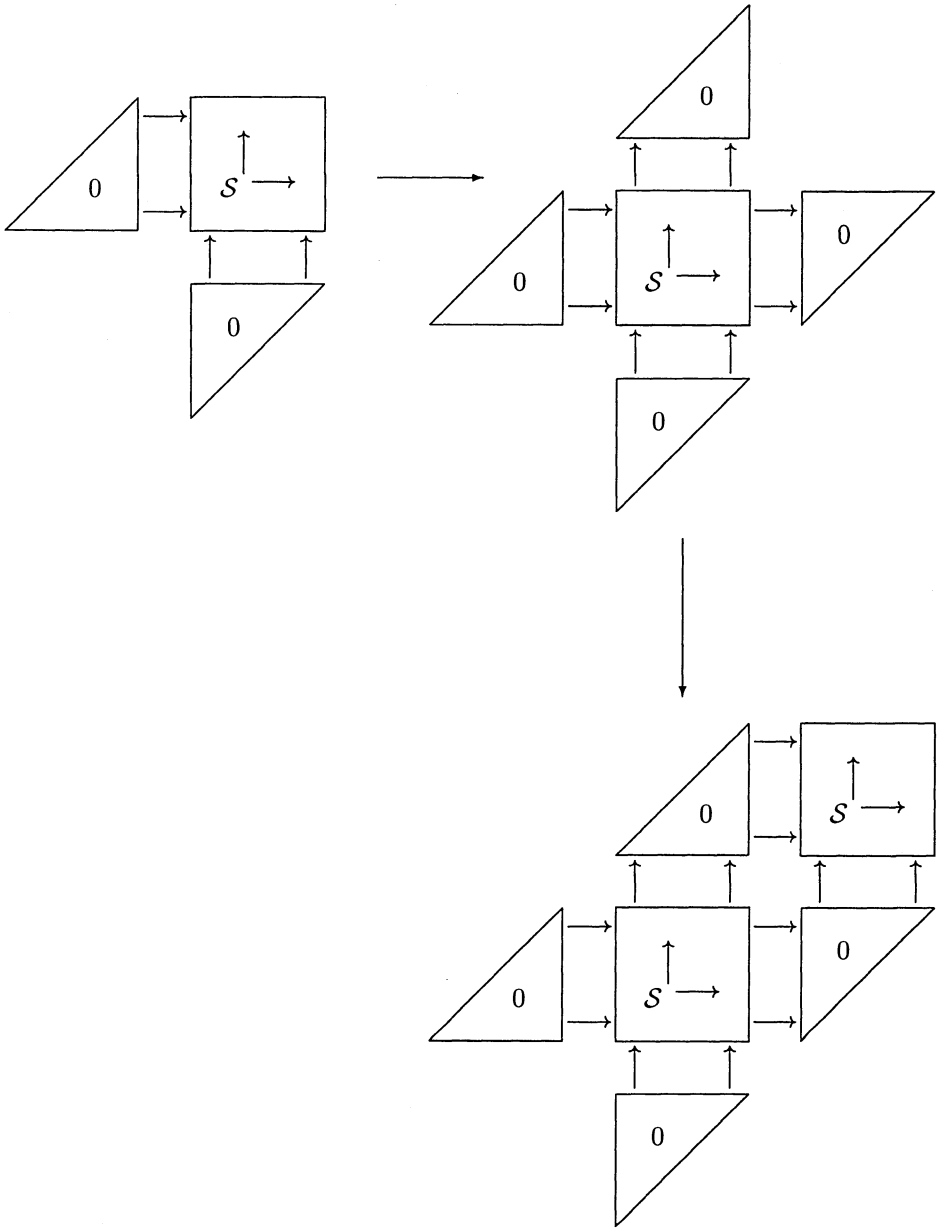


clearly is a homotopy equivalence. And here we come to the first key difference between the constructions with compatible differentials and the constructions without. With Construction 3.3 or 4.7, we have



and

The point is that in a triangulated category, the choice of a third edge Y_{ij} in the triangle $Y_{ij} \to X_{ik} \to X_{jk}$ is the same as a choice of a third edge in the triangle $X_{ik} \to X_{jk} \to \Sigma Y_{ij}$. This phenomenon is actually crucial. We deduce maps, for constructions with compatible differentials:



and so on. In other words, we can keep pushing out because the pushouts involve information which is already determined by the differentials. Similarly, we can keep pulling back.

6. Triangulated categories with a t–structure. A triangulated category is said to have a t-structure (sometimes one refers to such categories simply as t-categories) if they come equipped with two full subcategories, $\mathcal{T}^{\leq 0}$ and $\mathcal{T}^{\geq 0}$. We define $\mathcal{T}^{\leq n} = \Sigma^{-n}(\mathcal{T}^{\leq 0})$, $\mathcal{T}^{\geq n} = \Sigma^{-n}(\mathcal{T}^{\geq 0})$.

We suppose

6.0.1. $\mathcal{T}^{\leq 0} \subset \mathcal{T}^{\leq 1}$, $\mathcal{T}^{\geq 1} \subset \mathcal{T}^{\geq 0}$;

6.0.2. $Hom(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 1}) = 0$; *and*

6.0.3. *for all objects X of $\mathcal{T}$, there exist objects $X^{\leq 0} \in \mathcal{T}^{\leq 0}$, $X^{\geq 1} \in \mathcal{T}^{\geq 1}$, and a distinguished triangle $X^{\leq 0} \to X \to X^{\geq 1} \to \Sigma X^{\leq 0}$.*

It turns out that $X^{\leq 0}$, $X^{\geq 1}$, and the maps defining the triangle are all canonically unique. The assignments $X \longmapsto X^{\leq 0}$, $X \longmapsto X^{\geq 1}$ define functors.

EXAMPLE 6.1. Let $\mathcal{A} \subset \mathcal{B}$ be abelian categories, $\mathcal{A}$ a full subcategory of $\mathcal{B}$. Put $\mathcal{T} = D_{\mathcal{A}}(\mathcal{B})$, the derived category of complexes in $\mathcal{B}$ with $\mathcal{A}$-cohomology. Put $\mathcal{T}^{\leq 0} = D_{(-\infty,0]}(\mathcal{B}) \cap D_{\mathcal{A}}(\mathcal{B})$, $\mathcal{T}^{\geq 0} = D_{[0,\infty)}(\mathcal{B}) \cap D_{\mathcal{A}}(\mathcal{B})$. Conditions 6.0.1 and 6.0.2 are trivial; for 6.0.3 let X be an object of $\mathcal{T}$. Then X is a chain complex $\cdots \to X^i \to X^{i+1} \to \cdots$. Define $X^{\leq 0}$ to be the chain complex

$$\cdots \longrightarrow X^{-2} \longrightarrow X^{-1} \longrightarrow X^0 \longrightarrow Im(X^0 \to X^1) \longrightarrow 0 \longrightarrow \cdots,$$

while $X^{\geq 1}$ is the complex

$$\cdots \longrightarrow 0 \longrightarrow coker(X^0 \to X^1) \longrightarrow X^2 \longrightarrow X^3 \longrightarrow \cdots.$$

Clearly, this satisfies hypothesis 6.0.3.

Suppose $\mathcal{T}$ is a triangulated category with a t-structure. Define $\mathcal{C} = \mathcal{T}^{\leq 0} \cap \mathcal{T}^{\geq 0}$. Then $\mathcal{C}$ is a full subcategory of $\mathcal{T}$, and it may be proved that $\mathcal{C}$ is abelian. In Example 6.1, $\mathcal{C} = \mathcal{A}$ (See [1], Theorem 1.3.6).

It is also true that the functor $H : \mathcal{T} \to \mathcal{C}$ given by the formula $H(X) = (X^{\geq 0})^{\leq 0}$ is a homological functor. Suppose that H detects non–zero objects; that is, if $H^i(X) = 0$ for all i, then $X = 0$. It may be checked that $\mathcal{T}^{\leq n} = \mathcal{T}_{(H;(-\infty,n])}$, while $\mathcal{T}^{\geq n} = \mathcal{T}_{(H;[n,\infty))}$ in the notation of Example 1.10. Thus, an object X of $\mathcal{T}$ is in $\mathcal{T}^{\leq n}$ if and only if $H^i(X) = 0$ for $i > n$, and the dual statement is true of $\mathcal{T}^{\geq n}$. In particular, $\mathcal{T}^{\geq n}$, $\mathcal{T}^{\leq n}$ are exact subcategories of $\mathcal{T}$; so is $\mathcal{T}_{[m,n]} = \mathcal{T}^{\geq m} \cap \mathcal{T}^{\leq n}$. Details of the proof are in [1], Section 1.3.

We write $\mathcal{T}^b$ for $\bigcup_{m \leq n} \mathcal{T}_{[m,n]}$. If $\mathcal{T} = \mathcal{T}^b$, the t-structure is called nondegenerate.

REFERENCES

[1] A. A. BEILINSON, J. BERNSTEIN AND P. DELIGNE. *Analyse et topologie sur les éspaces singuliers.* Astérisque 100, Soc. Math. France (1982).

[2] A. BOREL AND J. P. SERRE. Le théorème de Riemann–Roch, *Bull. Soc. Math. France*, 86 (1958), 97-136.

[3] C. GIFFEN, AND A. NEEMAN. *K–theory for triangulated categories.* Preprint.

[4] R. HARTSHORNE. *Residues and duality*, SLN 20 (1966).

[5] V. A. HINICH AND V. V. SCHECHTMAN. Geometry of a category of complexes and algebraic K–theory. *Duke Math. J.*, 52 (1985), 399-430.

[6] J. F. JARDINE. The multiple Q–constructions. *Canad. J. Math.*, 39 (1987), 1174-1209.

[7] A. NEEMAN. Some new axioms for triangulated categories. *J. of Algebra*, 139 (1992) 221–255.

[8] A. NEEMAN. The Brown Representability Theorem and phantomless triangulated categories. *J. of Algebra*, 151 (1992) 118–155.

[9] D. QUILLEN. *Higher algebraic K–theory I*, SLN 341 (1973), 85-147.

[10] R. THOMASON AND T. TROBAUGH. *Higher algebraic K-theory of schemes and of derived categories.* In: The Grothendieck Festschrift (a collection of papers to honor Grothendieck's 60'th birthday) Volume 3 pp. 247–435, Birkhäuser 1990.

[11] J. L. VERDIER. *Catégories dérivées, état 0. SGA* $4\frac{1}{2}$, 262-308 (SLN 569, 1977).

[12] F. WALDHAUSEN. Algebraic K–theory of generalized free products I, II. *Ann. of Math.*, 108 (1978), 135-256.

[13] F. WALDHAUSEN. *Algebraic K-theory of spaces.* SLN 1126 (1985), 318-419.