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THE 14-DIMENSIONAL KERVAIRE INVARIANT AND THE SPORADIC GROUP M12 *

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There is a 14 dimensional aspect to the sporadic group M_{12}. For one thing

$$Syl_2(M_{12}) = Syl_2(G_2(q))$$

for $q \equiv 3, 5 \bmod (8)$ and the exceptional Lie group G_2 is 14 dimensional. Secondly, in [AMM1] we showed that the Poincaré series for $H^*(M_{12}; \mathbb{F}_2)$ is

$$\frac{1 + t^2 + 3t^3 + t^4 + 3t^5 + 4t^6 + 2t^7 + 4t^8 + 3t^9 + t^{10} + 3t^{11} + t^{12} + t^{14}}{(1 - t^4)(1 - t^6)(1 - t^7)}$$

which suggests, since $\frac{1}{(1-t^4)(1-t^6)(1-t^7)}$ is the Poincaré series for the classifying space B_{G_2}, that there should be a fibering of the form

$$F \longrightarrow B^+_{M_{12}} \longrightarrow B_{G_2}$$

with the fiber a 14 dimensional Poincaré duality space. Indeed, in [M], (see also [BW]), I constructed just such a fibration.

There is one more way in which one might expect the sporadic nature of M_{12} to manifest itself. According to [R], the natural homotopy class in

$$\pi^s_{14}(S^0) = 2^2 = \langle \bar{\kappa}, \sigma^2 \rangle$$

associated to the reframing of G_2 is $\bar{\kappa}$, so one would expect that there was some way in which a homotopy class in $\pi^s_{14}(S^0)$ could be associated to F, and this class should be σ^2 or $\sigma^2 + \bar{\kappa}$. In other words, this class should represent an element in $\pi^s_{14}(S^0)$ of Kervaire invariant one. Indeed, it is the object of this note to show just that.

One connection between homotopy theory and group theory lies in the well known equivalence $\mathbb{Z} \times B^+_{\mathcal{S}_\infty} \simeq Q(S^0) = \lim_{n \mapsto \infty}(\Omega^n \Sigma^n)$, and there is a natural embedding $h: M_{12} \subset \mathcal{S}_{12}$, embedding M_{12} as a maximal subgroup of $\mathcal{S}_{12}$, that gives rise to a composition map

$$B_h: B^+_{M_{12}} \longrightarrow B^+_{\mathcal{S}_{12}} \hookrightarrow B^+_{\mathcal{S}_\infty} \simeq Q(S^0)$$

which, at the homotopy level induces a map $B_{h,*}: \pi_*(B^+_{M_{12}}) \longrightarrow \pi^s_*(S^0)$. Then the main result of this note is

THEOREM. *There is an element $A \in \pi^s_{14}(B_{M_{12}})$ with Hurewicz image $[F]$, the orientation class of the fiber F above, and every such A has image $B_h(A) = \sigma_7^2$ or $\sigma_7^2 + \bar{\kappa}$ under the composition above.*

The way in which the class A is constructed may have independent interest. The construction is based on the use of the stable transfer,

$$tr: Q(B_{G_2(q)}) \longrightarrow Q(B_{Syl_2(M_{12})}),$$

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the existence of a fibration

$$G_2 \longrightarrow B^+_{G_2(q)} \longrightarrow B_{G_2},$$

and the fact that G_2 is parallelizable to construct the homotopy class A, while the inclusions $Syl_2(M_{12}) \hookrightarrow M_{12} \hookrightarrow \mathcal{S}_{12}$ give rise to an *exotic* way of mapping the homotopy class associated to G_2 into stable homotopy.

There is yet one more connection we should mention. In [MO] we construct stable splittings for $B_{M_{12}}$ and $B_{Syl_2(M_{12})}$. In particular, $B_{M_{12}}$ splits into irreducible summands as follows:

$$B_{M_{12}} \simeq_s B_{G_2(3)} \vee B_{L_3(2)} \vee X_{E,S_1}$$

where $E = Q_8 * Q_8 \subset Syl_2(M_{12})$ and X_{E,S_1} is one of the three dominant summands of this group. Then, by looking at the proof of our main result it becomes clear that the result would not have been possible without the summand $B_{L_3(2)}$ being part of the decomposition. In particular this tends to indicate that if there are any further direct constructions of Kervaire invariant one classes, the group G involved must, at the very least, have non-trivial Schur multiplier since $H_2(B_{L_3(2)}; \mathbb{F}_2) \neq 0$.

All homology and cohomology groups in the remainder of this note will have coefficients $\mathbb{F}_2$ and so the coefficients will not appear in the expressions $H^*(X)$ or $H_*(X)$.

In §1 we review the basic facts we need about M_{12}, $G_2(q)$, and $Syl_2(M_{12})$, as well as the cohomology of these groups. In particular, Theorem 1.8 gives the structure of $H^*(F)$ and proves the important relation $\langle \alpha^7, [F] \rangle = 1$ where α is the two dimensional generator in $H^*(M_{12})$.

THEOREM 1.8. *$H^*(F)$ has generators α, β, c, m, and n, subject to the following relations: $m^4 = n^2 = 0$, $\alpha^2\beta^2 = (\alpha^3 + \beta^2)\beta = 0$, $\alpha^5 = \alpha^2\beta c$, $\alpha^7 = m^3 n$. The Steenrod operations are given on these generators by $Sq^1(\alpha) = \beta$, $Sq^1(c) = \alpha^2$, $Sq^2(\beta) = \alpha\beta$, $Sq^2(c) = \alpha(\beta + c)$, $Sq^1(m) = 0$, $Sq^2(m) = n$, $Sq^4(n) = 0$. A table of the generators is given as follows*

Dim	2	3	4	5	6	7	8	9	10	11	12	13	14
	α	β	α^2	$\alpha\beta$	α^3		α^4						
		c		αc	βc	$\alpha^2 c$	$\alpha\beta c$	$\alpha^3 c$		$\alpha^4 c$			
					β^2	$\alpha^2\beta$	$\alpha\beta^2$	$\alpha^3\beta$	α^5	$\alpha^4\beta$	α^6		
		m		n	m^2		mn	m^3		$m^2 n$			$\alpha^7 = m^3 n$

and these generators also form a basis for $H^*(M_{12})$ as a module over the Dickson algebra $D_3 = \mathbb{F}_2[d_4, d_6, d_7]$.

It is exactly the fact that α^7 is dual to $[F]$ which twists the homotopy class A to a Kervaire invariant one class. The relation between G_2 and F reminds one of a degree one surgery problem, but I don't know if F can be replaced by a closed manifold.

In §2 and §3 we review the basic facts we need from iterated loop space theory. Then, in §4 we determine the cohomology map $H^*(S_{12}) \longrightarrow H^*(M_{12})$ induced from the inclusion above. This also may have independent interest, though the idea is very direct. Since both groups are detected by restriction to maximal elementaries it suffices to analyze the conjugacy classes of the images of the three maximal 2^3's in M_{12}. The result is given in table (4.5) and the explicit images of the generators are then given exactly in the remarks following the table.

Finally, in §5 we put all this together to prove our main result. Here, the key step is an explicit determination of a transfer map $tr^*(H^*(Q(B_{M_{12}})) \longrightarrow H^*(Q(B_{G_2(q)}))$ as far as we need it. Again this is achieved by restricting to 2-elementaries. The key result, Lemma 5.3, which gives an explicit method for determining such transfer-restriction may also have independent interest.

1. The groups M_{12} and $G_2(q)$ and their common 2-Sylow subgroup. The cohomology of M_{12} is discussed in [AM, pp. 261-263]. It is detected by restriction to the three 2^3's which represent the three conjugacy classes of maximal elementary 2-groups in M_{12}. The result is

THEOREM 1.1. *$H^*(M_{12};\mathbb{F}_2)$ has 8 generators, α_2, β_3, c_3, m_3, d_4, n_5, d_6 and d_7, (where the subscripts denote the dimension) with restriction images given as follows:*

(1.2)

gen.\group	V_1	V_3	V_5
α	0	h^2	d_2
β	0	0	d_3
c	0	h^3	$\tau^3+\tau^2\lambda+\lambda^3$
m	d_3	0	0
d_4	d_4	d_4	d_4
n	d_2d_3	0	0
d_6	d_6	d_6	d_6
d_7	d_7	d_7	d_7

where

$$
\begin{aligned}
H^*(2^3) &= \mathbb{F}_2[h,\tau,\lambda],\\
d_2 &= \tau^2+\tau\lambda+\lambda^2,\\
d_3 &= \tau^2\lambda+\lambda^2\tau,\\
d_4 &= h^4+h^2d_2+hd_3+d_2^2,\\
d_6 &= h^4d_2+h^2d_2^2+hd_2d_3+d_3^2,\\
d_7 &= h^4d_3+h^2d_2d_3+hd_3^2.
\end{aligned}
\tag{1.3}
$$

The structure of $H^*(B_{G_2(q)})$ with $q\equiv 3,5 \bmod (8)$ is described in [M] similarly. The groups share the same 2-Sylow subgroup but $G_2(q)$ has exactly two maximal 2-elementaries, the first represented by V_3 while V_1 fuses with V_5 to give the second. In particular, $H^*(G_2(q);\mathbb{F}_2)$ is also represented by restriction to V_1 and V_3, and we have the result of [M]

THEOREM 1.4. *$H^*(B_{G_2(q)})$ with $q\equiv 3,5 \bmod (8)$ has 5 generators $\bar{m}_3$, d_4, $\bar{n}_5$, d_6, d_7, with $\bar{m}_3$ restricting to d_3 in $H^*(V_1)$, $H^*(V_5)$, and 0 in $H^*(V_3)$, $\bar{n}=Sq^2(\bar{m})$, and the d_i as above.*

There is an identification of $\lim_{n\mapsto\infty} B^+_{G_2(q^n)}$ localized at 2 with B_{G_2} localized at 2 via etale theory and consequently a (2-local) fibration

$$G_2 \longrightarrow B^+_{G_2(q)} \longrightarrow B_{G_2}$$

using a theorem of J. Harper which identifies G_2 (2-locally) with any simply connected space which has the same $\mathbb{F}_2$-cohomology ring as G_2 as a module over the Steenrod algebra. Moreover we obtain the cohomology of the fiber of the map $B^+_{G_2(q)}\longrightarrow B_{G_2}$ by noting that since $H^*(B_{G_2(q)};\mathbb{F}_2)$ is Cohen-Macaulay over $H^*(B_{G_2};\mathbb{F}_2) = \mathbb{F}_2[d_4, d_6, d_7]$, it follows that

$$(1.5) \qquad H^*(Fiber) = H^*(B_{G_2(q)}) \otimes_{\mathbb{F}_2[d_4,d_6,d_7]} \mathbb{F}_2$$

and this is seen to give the desired isomorphism (see e.g. the discussion below, specifically the remarks preceeding 1.8).

Relating the two restriction images we see that $\bar{m} \mapsto m + \beta$ in $H^*(M_{12})$, $\bar{n} \mapsto n + \alpha\beta$ and $H^*(G_2(q)) \subset H^*(M_{12})$ though there is no homomorphism $M_{12} \to G_2(q)$ which will induce this cohomology map. However, in [M] a 2-local map of +-constructions

$$B^+_{M_{12}} \xrightarrow{f} B^+_{G_2(q)}$$

was constructed which does, in fact, realize the cohomology map above. In particular we get a diagram of fiberings

$$(1.6) \qquad \begin{array}{ccccc} F & \xrightarrow{j} & B^+_{M_{12}} & \xrightarrow{\pi} & B_{G_2} \\ \downarrow{\scriptstyle j_1} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle =} \\ G_2 & \xrightarrow{j} & B^+_{G_2(3)} & \xrightarrow{\pi} & B_{G_2} \end{array}$$

Again, in the upper fibration, since $H^*(B^+_{M_{12}};\mathbb{F}_2)$ is Cohen-Macaulay over $H^*(B_{G_2})$ from [AM], we have

$$(1.7) \qquad H^*(F) = H^*(B_{M_{12}}) \otimes_{\mathbb{F}_2[d_4,d_6,d_7]} \mathbb{F}_2$$

It is well known that $H^*(G_2) = \mathbb{F}_2[\bar{m}_3]/(\bar{m}^4) \otimes E(\bar{n})$ with $Sq^2(\bar{m}) = \bar{n}$, $Sq^1(\bar{n}) = \bar{m}^2$.

We also need the structure of $H^*(F)$ as a module over the Steenrod algebra. The result is

THEOREM 1.8. *$H^*(F)$ has generators α, β, c, m, and n, subject to the following relations: $m^4 = n^2 = 0$, $\alpha^2\beta^2 = (\alpha^3 + \beta^2)\beta = 0$, $\alpha^5 = \alpha^2\beta c$, $\alpha^7 = m^3 n$. The Steenrod operations are given on these generators by $Sq^1(\alpha) = \beta$, $Sq^1(c) = \alpha^2$, $Sq^2(\beta) = \alpha\beta$, $Sq^2(c) = \alpha(\beta + c)$, $Sq^1(m) = 0$, $Sq^2(m) = n$, $Sq^4(n) = 0$. A table of the generators is given as follows*

Dim	2	3	4	5	6	7	8	9	10	11	12	13	14
	α	β	α^2	$\alpha\beta$	α^3		α^4						
		c		αc	βc	$\alpha^2 c$	$\alpha\beta c$	$\alpha^3 c$		$\alpha^4 c$			
					β^2	$\alpha^2\beta$	$\alpha\beta^2$	$\alpha^3\beta$	α^5	$\alpha^4\beta$	α^6		
		m		n	m^2		mn	m^3		$m^2 n$			$\alpha^7 = m^3 n$

and these generators also form a basis for $H^*(M_{12})$ as a module over the Dickson algebra $D_3 = \mathbb{F}_2[d_4, d_6, d_7]$.

This result does not appear in the published literature, though it is given in [O]. We defer the proof to §6 since the details of the proof are not needed in proving our main result.

REMARK 1.9. The element in $H^*(G_2(q))$ which restricts to $(0, d_7^2, 0)$ is particularly important in what follows. We have the following decompositions in terms of the generators α, β, m, n, d_4, d_6 and d_7:

$$\begin{aligned}(d_7^2, 0, 0) &= m^3n + mnd_6 + md_4d_7 \\ (0, 0, d_7) &= \beta d_4 + \alpha^2\beta \\ (0, d_7^2, 0) &= d_7^2 + (m+\beta)d_4d_7 + \alpha^2\beta d_7 + mnd_6 + m^3n.\end{aligned} \tag{1.10}$$

We also have a second description of this element as

$$(0, d_7^2, 0) = \alpha(d_6 + \alpha d_4 + \alpha^3 + \beta^2)^2$$

where the first description gives the expression for this class in $H^*(F)$ as m^3n while the second expression gives it as $\alpha^7 + \alpha\beta^4$. But since, in $H^*(F)$ we have $\beta^3 = \alpha^3\beta$ it follows that $\beta^4 = \alpha(\alpha\beta)^2 = 0$ which establishes the cup product relation $\alpha^7 = m^3n$ in the theorem. Also, this shows that α^7 evaluates as 1 on the top dimensional class of the fiber F and hence also the fiber G_2.

For later use we also need to recall some of the detailed structure of $Syl_2(M_{12})$, which from here on we denote H.

$$H \cong 4^2{:}2^2 = \langle c, d\rangle{:}\langle s, k\rangle$$

where the action is given by $c^s = d$, $d^s = c$, $c^k = c^{-1}$, $d^k = d^{-1}$. Then H has five conjugacy classes of maximal 2-elementary subgroups

$$\begin{aligned}V_1 &= \langle c^2, d^2, k\rangle \\ V_2 &= \langle c^2, d^2, ck\rangle \\ V_3 &= \langle c^2, d^2, cdk\rangle \\ V_4 &= \langle (cd)^2, k, s\rangle \\ V_5 &= \langle (cd)^2, cdk, s\rangle\end{aligned} \tag{1.11}$$

with V_1 and V_3 normal in H. Moreover, as already noted, they fuse as follows: in M_{12}, $\{V_1, V_2\}$, $\{V_3, V_4\}$, $\{V_5\}$, and in $G_2(q)$, $\{V_1, V_2, V_5\}$, $\{V_2, V_3\}$.

2. Filtration arguments. We briefly review the homology structure of $Q(S^0)$ as far as we need it.

First, $\lim_{n\mapsto\infty} B_{\mathcal{S}_n} = B_{\mathcal{S}_\infty}$, and there is a map $e\colon B_{\mathcal{S}_\infty} \to Q(S^0)_0$, the 0-component of $Q(S^0)$, which is a homology equivalence. Moreover, the inclusion $B_{\mathcal{S}_n} \hookrightarrow B_{\mathcal{S}_\infty}$ induces an inclusion in homology.

The loop sum operation

$$*\colon Q(S^0) \times Q(S^0) \longrightarrow Q(S^0) \tag{2.1}$$

makes $H_*(Q(S^0))$ into a graded commutative ring, while the inclusions

$$+_{m,n}\colon \mathcal{S}_m \times \mathcal{S}_n \hookrightarrow \mathcal{S}_{n+m}$$

induce maps of classifying spaces

$$B_+\colon B_{\mathcal{S}_m} \times B_{\mathcal{S}_n} \hookrightarrow B_{\mathcal{S}_{n+m}} \tag{2.2}$$

which make $H_*(B_{\mathcal{S}_\infty})$ into a commutative ring. They fit together in a commutative diagram

$$\begin{array}{ccc} B_{\mathcal{S}_m} \times B_{\mathcal{S}_n} & \xrightarrow{e \times e} & Q(S^0) \times Q(S^0) \\ \Big\downarrow B_+ & & \Big\downarrow * \\ B_{\mathcal{S}_{n+m}} & \xrightarrow{e} & Q(S^0). \end{array} \tag{2.3}$$

We can put in more structure. There are inclusions $p\colon \mathcal{S}_n \wr 2 \longrightarrow \mathcal{S}_{2n}$ which, at the level of classifying spaces lead to maps

$$(B_{\mathcal{S}_n})^2 \times_{\mathbb{Z}/2} E_{\mathbb{Z}/2} \xrightarrow{B_p} B_{\mathcal{S}_{2n}} \tag{2.4}$$

and hence, for each class $\alpha \in H_j(\mathcal{S}_n)$, there is an associated class in $H_{2j+m}(B_{\mathcal{S}_n \wr 2})$ which, at the chain level is $\alpha \otimes \alpha \otimes e_m$. The image of this class in $H_{2j+m}(B_{\mathcal{S}_{2n}})$ is called $Q_m(\alpha)$ and we can iterate this operation obtaining $Q_{(i_1,\dots,i_r)}(\alpha)$ in dimension $2^r j + 2^{r-1} i_r + \cdots + i_1$.

REMARK. The construction above actually has an intrinsic description. Let $C_k(\mathbb{R}^n)$ be the configuration space of k-tuples of ordered, distinct points of $\mathbb{R}^n$. Let $f_1,\dots,f_k$ be based maps $S^n = \mathbb{R}^n \cup \infty \xrightarrow{f_i} X$, then define, for $(z, f_1, \dots, f_k) \in C_k(\mathbb{R}^n) \times (\Omega^n X)^k$, the map $g\colon \mathbb{R}^n \cup \infty \longrightarrow X$ as f_i (suitably rescaled) in a disk neighborhood of x_i, the i^{th} point in z and ∞ in the complement of these neighborhoods. Now pass to limits over n and note that the construction actually factors through $C_k(\mathbb{R}^k) \times_{\mathcal{S}_k} (\Omega^n X)^k$.

The operation $\alpha \mapsto Q_i(\alpha)$ is linear and there is a Cartan formula which connects it with the loop sum:

$$Q_i(\alpha * \beta) = \sum_{j=0}^{i} Q_j(\alpha) * Q_{i-j}(\beta).$$

Let $I = (i_1, \dots, i_r)$. We say that I is admissible if and only if $0 \le i_j \le i_{j+1}$ for each $j = 1, \dots, r-1$. Note that $Q_0(\alpha) = \alpha * \alpha$ the loop sum product. For this reason we suppress those I which contain 0's and say I is generating if it is admissible and $i_1 > 0$.

THEOREM 2.5. *$H_*(B_{\mathcal{S}_\infty}) = \mathbb{F}_2[1, Q_1(J)*2(-J), \dots, Q_I(J)*2^r(-J), \dots]$ where the ring structure is generated by loop sum, J represents the limit $J = \lim_{n\mapsto\infty}(J_n\colon S^n \longrightarrow S^n)$ with J_n the identity map, and $I = (i_1, \dots, i_r)$ runs over all generating sequences.*

A filtration degree in $H_*(B_{\mathcal{S}_\infty})$ is given by setting $deg(a * b) = deg(a) + deg(b)$, while $deg(Q_I(J)) = 2^r$ for $I = (j_1, \dots, j_r)$ and $deg(a + b) = max(deg(a), deg(b))$. Then $\mathcal{F}_n$ is the subgroup generated by all the elmements of degree $\le n$ in $H_*(B_{\mathcal{S}_\infty})$. It turns out that $H_*(B_{\mathcal{S}_n})$ is exactly the subgroup consisting of the elements above of filtration $\le n$.

Dually, let $\mathcal{F}^n$ be the kernel of the cohomology surjection $H^*(B_{\mathcal{S}_\infty}) \longrightarrow H^*(B_{\mathcal{S}_n})$. Then $\mathcal{F}^n$ and $\mathcal{F}_n$ are mutual annhilators, and $\mathcal{F}^n$ is closed under the action of the Steenrod algebra $\mathcal{A}(2)$.

Also, there is a natural compliment to $\mathcal{F}_n$, the ideal in $H_*(B_{\mathcal{S}_\infty})$ generated by the elements above with second degree $> n$. In view of the linearity of the Q_i-operations and the Cartan formula this ideal is actually intrinsic, and independent of the perhaps different choices for embedding $B_{\mathcal{S}_n}$ to construct $Q(S^0)$. We call this ideal $\mathcal{H}_n$, and this gives an associated graded algebra, $\coprod \mathcal{H}_n/\mathcal{H}_{n+1}$.

Additionally, the composite

$$\mathcal{F}_n \hookrightarrow H_*(B_{\mathcal{S}_\infty}) \xrightarrow{p} H_*(B_{\mathcal{S}_\infty})/\mathcal{H}_n$$

is an isomorphism.

Finally, note that $\mathcal{H}_n$ and $\mathcal{F}^n$ are dual to each other.

REMARK 2.6. The associated graded ring $\coprod_j \mathcal{H}_j/\mathcal{H}_{j+1}$ is naturally isomorphic to $H_*(B_{\mathcal{S}_\infty})$ as an algebra but *not as a coalgebra*. Indeed, as a coalgebra the associated graded ring is primitively generated.

3. The construction of $Q(X)$. We recall the well known May-Milgram construction of $Q(X) = \lim_{n\mapsto\infty} \Omega^n\Sigma^n X$ for X any connected CW-complex with base point $*$.

$$Q(X) \simeq \coprod_1^\infty E_{\mathcal{S}_n} \times_{\mathcal{S}_n} X^n/\sim \tag{3.1}$$

where $\sim$ is an equivalence relation that identifies points of the form $\{e, x_1, \ldots, x_n\}$ with $\{p_i(e), x_1, \ldots, \widehat{x_i}, \ldots, x_n\}$ if $x_i = *$, where $p_i\colon E_{\mathcal{S}_n} \to E_{\mathcal{S}_{n-1}}$ is a suitable map. A good model for $E_{\mathcal{S}_n}$ is the configuration space $C_n(\mathbb{R}^\infty)$ which consists of all ordered n-tuples $(x_1, \ldots, x_n) \in (\mathbb{R}^\infty)^n$ where the x_i are all distinct. Of course $\mathcal{S}_n$ acts by permuting coordinates. With this model p_i simply forgets x_i.

The natural map $Q(Q(X)) \to Q(X)$ given as $\Omega^\infty(eval)$ where

$$eval\colon\ \Sigma^n\Omega^n X \to X$$

is the usual evaluation map is realized in this model by the inclusion

$$E_{\mathcal{S}_n} \times_{\mathcal{S}_n} (E_{\mathcal{S}_m} \times_{\mathcal{S}_m} X^m)^n = E_{\mathcal{S}_n} \times (E_{\mathcal{S}_n})^n \times_{\mathcal{S}_m \wr \mathcal{S}_n} X^{nm} \hookrightarrow E_{\mathcal{S}_{nm}} \times_{\mathcal{S}_{nm}} X^{nm}$$

obtained by regarding the product of the E's as a free $\mathcal{S}_m \wr \mathcal{S}_n$ space and including the wreath product as as subgroup of $\mathcal{S}_{nm}$.

Consequently, given a subgroup $G \subset \mathcal{S}_n$ there is a map $B_G \overset{e}{\hookrightarrow} B_{\mathcal{S}_n} \hookrightarrow Q(S^0)$, which prolongs to a map $Q(e)\colon Q(B_G) \to Q(S^0)$.

The homology of $Q(X)$ is given as

$$\mathbb{F}_2[\alpha, \cdots Q_I(\alpha), \cdots] \tag{3.2}$$

as I runs over all generating sequences as above and the α run over a basis for $H_*(X; \mathbb{F}_2)$. Moreover the homology map $H_*(Q(B_G)) \to H_*(Q(S^0))$ is determined by observing that it is natural with respect to loop sums and $Q_I(\alpha) \mapsto Q_I(e(\alpha))$.

As a basic example, suppose that $H \subset G$ has index $k < \infty$. Then there is the Frobenius homomorphism

$$f\colon G \longrightarrow H \wr \mathcal{S}_k \tag{3.3}$$

defined for example in [AM, pp. 73-75] which gives rise to $B_f\colon B_G \to E_{\mathcal{S}_k} \times_{\mathcal{S}_k} (B_H)^k$, and by prolongation, a map

$$Q(B_G) \xrightarrow{Q(B_f)} Q(B_H)$$

which is one of the variants and generalizations of the well known cohomology transfer map.

In particular $H^*(H)$ embeds into $H^*(Q(B_H))$ via the liftings

$$\alpha \mapsto \alpha \otimes 1 \otimes \cdots \otimes 1 + 1 \otimes \alpha \otimes \cdots \otimes 1 + \cdots + 1 \otimes \cdots \otimes 1 \otimes \alpha$$

taking $H^*(H)$ to $H^*(E_{\mathcal{S}_k} \times_{\mathcal{S}_k} (B_H)^k)$, and the composite $B_G \hookrightarrow Q(B_G) \xrightarrow{tr} Q(B_H)$ gives rise to the ordinary cohomology transfer $H^*(H) \xrightarrow{tr} H^*(G)$.

4. The inclusion $M_{12} \hookrightarrow \mathcal{S}_{12}$. It is well known, see e.g. [AM, p. 185], that $H^*(\mathcal{S}_n, \mathbb{F}_p)$ is detected by restriction to conjugacy class representatives of the maximal elementary abelian p-groups, and since the same is true for $H^*(M_{12})$ it suffices to study the induced maps on elementary 2-groups in order to determine the cohomology map $H^*(\mathcal{S}_{12}; \mathbb{F}_2) \to H^*(M_{12}; \mathbb{F}_2)$.

The conjugacy classes of maximal elementary 2-groups in $\mathcal{S}_n$ are also completely determined in [AM, pp. 185-186]. For $\mathcal{S}_{12}$ they are $(W_1)^6$, $W_2 \times (W_1)^4$, $(W_2)^2 \times (W_1)^2$, $(W_3) \times (W_1)^2$, $(W_2)^3$ and $W_2 \times W_3$ where $W_i \cong 2^i \subset \mathcal{S}_{2^i}$ is the regular representation of 2^i.

The embedding of M_{12} in $\mathcal{S}_{12}$ is given explicitly in [AM, pp. 254-255]. There, if we set $c = db^{-1}$, we find that a copy of $Syl_2(M_{12})$ is given as $\langle c, d, k, s \rangle$ and the map into $\mathcal{S}_{12}$ restricts to this subgroup as

$$\begin{array}{rcl} c & \mapsto & (1,10,11,12)(2,9,3,5) \\ d & \mapsto & (1,10,11,12)(4,8,7,6) \\ k & \mapsto & (1,12)(5,9)(6,8)(10,11) \\ s & \mapsto & (1,11)(2,7)(3,4)(5,8)(6,9)(10,12). \end{array}$$

Thus the image of V_1 is given as

$$\begin{array}{r|c|c|c|c|c|} c^2 & (1,11)(10,12) & (2,3) & & (5,9) & \\ d^2 & (1,11)(10,12) & & (4,7) & & (6,8) \\ k & (1,12)(10,11) & & & (5,9) & (6,8) \end{array} \tag{4.1}$$

which shows that $V_1 \hookrightarrow W_2 \times (W_1)^4 \subset \mathcal{S}_{12}$. For the second, V_3, we have

$$\begin{array}{r|c|c|c|} c^2 & 1,11)(10,12) & (2,3)(5,9) & \\ d^2 & (1,11)(10,12) & & (4,7)(6,8) \\ cdk & (1,10)(11,12) & (2,5)(3,9) & (4,6)(7,8) \end{array} \tag{4.2}$$

which shows that $V_3 \hookrightarrow (W_2)^3 \subset \mathcal{S}_{12}$. Finally, the last group V_5 gives

$$\begin{array}{r|c|c|} c^2d^2 & & (2,3)(4,7)(5,9)(6,8) \\ c^{-1}dk & (1,12)(10,11) & (2,9)(3,5)(4,6)(7,8) \\ ks & (1,10)(11,12) & (2,7)(3,4)(5,6)(8,9) \end{array} \tag{4.3}$$

so $V_5 \hookrightarrow W_2 \times W_3 \subset \mathcal{S}_{12}$.

In [AMM2], see also [AM, pp. 213-214], the cohomology ring $H^*(\mathcal{S}_{12}; \mathbb{F}_2)$ was determined, and it was shown that over the Steenrod algebra $\mathcal{A}(2)$ it has five generators

σ_1, σ_2, σ_3, σ_4, and σ_6. Under restriction we get the following table

(4.4)

gen.\group	$W_2 \times (W_1)^4$	$(W_2)^3$	$W_2 \times W_3$
σ_1	$1 \otimes \sigma_1$	0	0
σ_2	$d_2 \otimes 1 + 1 \otimes \sigma_2$	$S(d_2 \otimes 1 \otimes 1)$	$d_2 \otimes 1$
σ_3	$d_2 \otimes \sigma_1 + 1 \otimes \sigma_3$	0	0
σ_4	$d_2 \otimes \sigma_2 + 1 \otimes \sigma_4$	$S(d_2 \otimes d_2 \otimes 1)$	$1 \otimes d_4$
σ_6	$d_2 \otimes \sigma_4$	$d_2 \otimes d_2 \otimes d_2$	$d_2 \otimes d_4$

where σ_i denotes the i^{th} symmetric monomial in the generators for the individual summands, while d_i denotes the appropriate Dickson element in the cohomology of W_i.

In terms of the filtration discussed in §3 σ_1^i is dual to the element $Q_i(J)*(-2)$, the unique element in dimension i and filtration 2 but not 4, while $\sigma_2^i\sigma_1^k$ lies in filtration 4 but not 6 and is dual to $e_i * e_{i+k}$. Also, $Sq^1(\sigma_2) + \sigma_2\sigma_1$ is dual to $Q_{(1,1,)}(J)$ and has filtration 4 but not 6 as do the elements $(\sigma_2)^i(Sq^1(\sigma^2)^j$ which corresponds to $Q_{j,i+j}(J)$. All the remaining elements of the image have filtration degree at least 6.

The data above together with these restrictions of the σ_i generators now determines the restriction image explicitly. As an example we do the cases of σ_3 and σ_6 restricting to $H^*(V_1)$. Let x, y generate $H^*(W_2)$, z_1, z_2, z_3, z_4 generate $H^*((W_1)^4)$, in the order above, then we see that

$$\begin{aligned} c^2 &\mapsto x^* z_1^* z_3^* \\ d^2 &\mapsto x^* z_2^* z_4^* \\ k &\mapsto y^* z_3^* z_4^* \end{aligned}$$

so, dually,

$$\begin{aligned} x &\mapsto \lambda + \tau \\ y &\mapsto h \\ z_1 &\mapsto \lambda \\ z_2 &\mapsto \tau \\ z_3 &\mapsto \lambda + h \\ z_4 &\mapsto \tau + h \end{aligned}$$

where h is dual to k, λ is dual to c^2, and τ is dual to d^2. It follows that we have

$$\begin{aligned} d_2 \otimes \sigma_1 + 1 \otimes \sigma_3 &\mapsto (\lambda\tau(\lambda + \tau) + (\tau + \lambda)(\lambda\tau + (\lambda + \tau)h + h^2) \\ &= (\lambda + \tau)^2 h + (\lambda + \tau)h^2 \\ &= d_3 \\ d_2 \otimes \sigma_4 &\mapsto ((\lambda + \tau)^2 + h(\lambda + \tau) + h^2)\lambda\tau(\lambda + h)(\tau + h) \\ &= \lambda\tau(h^4 + \lambda\tau h^2 + (\lambda + \tau)(\lambda^2 + \tau^2 + \lambda\tau)h + \lambda\tau(\lambda^2 + \tau^2)) \\ &= d_6 + d_3^2. \end{aligned}$$

The calculations in the remaining cases are similar. Specifically, σ_1 restricts to zero in all three, and for the rest we have

(4.5)

gen.\group	V_1	V_3	V_5
σ_1	0	0	0
σ_2	0	h^2	d_2
σ_3	d_3	0	0
σ_4	d_4	d_4	d_4
σ_6	$d_6 + d_3^2$	$h^6 + d_6$	$d_6 + d_2^3 + d_3^2$

Hence $\alpha = res^*(\sigma_2)$, $m = res^*(\sigma_3)$, $d_4 = res^*(\sigma_4)$, $res^*(\sigma_6) = \sigma^3 + \beta^2 + m^2 + d_6$. Moreover, since $\beta = Sq^1(\alpha)$, $n = Sq^2(m)$, $d_6 = Sq^2(d_4)$ and $d_7 = Sq^1(d_6)$ it follows that among the generators for $H^*(M_{12})$ only c is not in the image of restriction.

5. The connection of M_{12} with the Kervaire class in dimension 14. Recall that $H \subset M_{12}$ is the 2-Sylow subgroup of M_{12}. The sequence of inclusions

$$H \hookrightarrow M_{12} \xrightarrow{h} \mathcal{S}_{12}$$

gives rise to the sequence of inclusions of classifying spaces

$$B_H \hookrightarrow B_{M_{12}} \xrightarrow{B_h} B_{\mathcal{S}_{12}} \hookrightarrow Q(S^0)$$

which prolongs to

$$Q(B_H) \hookrightarrow Q(B_{M_{12}}) \xrightarrow{Q(B_h)} Q(S^0)$$

in the manner explained in §3. On the other hand, as we pointed out in §1, H is also the 2-Sylow subgroup of $G_2(q)$ for $q \equiv 3, 5 \bmod (8)$. Thus we have the transfer map $Q(B_{G_2(q)}) \to Q(B_H)$, and, since $Q(B_{G_2(q)}) = Q(B^+_{G_2(q)})$ because the Whitehead theorem already shows that $\Sigma B_G \simeq \Sigma B_G^+$, we obtain the following sequence of maps:

$$Q(G_2) \longrightarrow Q(B^+_{G_2(q)}) \xrightarrow{tr} Q(B_H) \longrightarrow Q(B_{M_{12}}) \longrightarrow Q(S^0). \tag{5.1}$$

But G_2, being a Lie group, is parallelizable, so the suspension $\Sigma^{14} G_2 \simeq S^{28} \vee \Sigma^{14} Y$, with Y the 11-skeleton of G_2, and there is a corresponding spherical class of the form $[G_2] + D \in H_{14}(Q(G_2); \mathbb{Z})$ where D is a sum of loop sum decomposable elements. Consequently we finally obtain the composition

$$S^{14} \longrightarrow Q(G_2) \longrightarrow Q(B^+_{G_2(q)}) \xrightarrow{tr} Q(B_H) \longrightarrow Q(B_{M_{12}}) \longrightarrow Q(S^0) \tag{5.2}$$

and we want to determine the image of this S^{14} in $\pi_{14}(Q(S^0)) = \pi_{14}^s(S^0)$.

What we need to understand to begin is the map $tr^*: H^*(H) \to H^*(G_2(q))$, at least when restricted to the image of $H^*(M_{12})$. In $G_2(q)$ we have pointed out that for the 5 conjugacy classes of 2^3's in H, V_1, V_2 and V_5 fuse to V_{II} with normalizer quotient $N_{G_2(3)}(V_{II})/(V_{II}) = \mathcal{S}_4$ (fixing a point) while V_3 and V_4 fuse to V_I with normalizer quotient $L_3(2)$. (In particular, the centralizers of V_{II} and V_I are just the V_j, $j = I, II$ themselves in $G_2(3)$.)

LEMMA 5.3. *The image of a general element of $H^*(H)$ under transfer to the cohomology ring $H^*(G_2(q))$ and then restriction to $H^*(V_{II})$ is the sum*

$$c^*_{g_1}\sum_{\tau_1}\tau_1^*(res_H^{V_1}(a)) + c^*_{g_2}\sum_{\tau_2}\tau_2^*(res_H^{V_2}(a)) + \sum_{\tau_5}\tau_5^*(res_H^{V_5}(a)) \tag{5.4}$$

where τ_i runs over coset representatives for $N_G(V_i)/N_H(V_i)$ while $g_1 \in G_2(3)$ satisfies $g_1^{-1}V_5g_1 = V_1$, and $g_2^{-1}V_5g_2 = V_2$. Similarly, for transfer through $G_2(a)$ followed by restriction to V_I we get the following sum

$$c_{g_3}\sum_{\tau_3}\tau_3^*(res_H^{V_3}(a)) + \sum_{\tau_4}\tau_4^*(res_H^{V_4}(a)) \tag{5.5}$$

with $g_3^{-1}V_4g_3 = V_3$, and τ_i as above.

Proof. We do the case of V_5, the case of V_4 being identical. Let $G_2(q) = \coprod V_5g_iH$ be the double coset decomposition. From the Mackey formula we have that the composition $res^* \cdot tr$ above is the sum over the g_i of the compositions

$$H^*(H) \xrightarrow{res} H^*(H \cap g_i^{-1}V_5g_i) \xrightarrow{c_g^*} H^*(g_i^{-1}Hg_i \cap V_5) \xrightarrow{tr} H^*(V_5)$$

but the last transfer is identically zero unless $g_i^{-1}V_5g_i \subset H$, that is to say, $g_i^{-1}V_5g_i$ is equal to one of V_5, V_1 or V_2 (this last since we can vary g_i by right multiplication by any element in H without changing the double coset).

On the other hand, if $g_i^{-1}V_5g_i \subset H$ then $V_5g_iH = g_i(g_i^{-1}V_5g_i)H = g_iH$ is just an ordinary coset of H, and conversely, if $V_5gH = g'H$ then $g^{-1}V_5g \subset H$ so the set of terms in the Mackey formula which are non-zero correspond exactly to the set of ordinary cosets of H in the double coset decompostion of $G_2(q)$. We now identify these double cosets more precisely.

Suppose that $g_i^{-1}V_5g_i = g_j^{-1}V_5g_j \subset H$. Then

$$g_j^{-1}g_i\colon g_i^{-1}V_5g_i \longrightarrow g_i^{-1}V_5g_i$$

is an automorphism of this subgroup of H. Varying g_i on the left by elements of V_5 has no effect on the automorphism, while varying g_i on the right by elements of $N_H(g_i^{-1}V_5g_i)$ varies the automorphism by an element in $N_G(g_i^{-1}V_5g_i)$ which lies in the same coset of H as $g_j^{-1}g_i$ in $G_2(q)$. Consequently, the set of cosets g_iH corresponding to V_i is in one to one correspondence with $N_G(V_i)/N_H(V_i)$, and the result follows. □

REMARK 5.6. Actually the lemma above is a special case of (the evident) general formula for the effect of the composition of transfer with restriction to an elementary abelian p-group.

COROLLARY 5.7. *The composite $H^*(H) \xrightarrow{tr^*} H^*(G_2(3)) \rightarrow H^*(V_{II})$ on an element a which restricts to zero in $H^*(V_1)$ and $H^*(V_2)$ is the same as the composition*

$$H^*(H) \xrightarrow{res^*} H^*(V_5) \xrightarrow{\Sigma} H^*(V_5)$$

where $\Sigma(res^*(a)) = \sum_{t\in\mathcal{S}_3} t^*(res^*(a))$ and $\mathcal{S}_3 = \mathcal{S}_4/N_H(V_5)$.

EXAMPLE 5.8. Note that $c \in H^*(M_{12})$ restricts to an element in $H^*(V_5)$ which is invariant under $\mathbb{Z}/3$ and restricts to 0 in V_1, V_2. Thus, $tr_H^{G_2(q)}(c) = (1+\tau^*)c$ where τ^* exchanges λ and τ. But this is just d_3. Hence, $tr^*(c) = m$.

EXAMPLE 5.9. A second special case of (5.4) is the transfer for $c\alpha^2$ in dimension 7. In V_{II} the restriction of $tr(c\alpha^2)$ is $d_2^2 d_3 = d_7 + d_3 d_4$ while in V_I the restriction is the sum over the cosets of $D_8 \subset L_3(2)$ of $\tau(h^7)$. This can be evaluated as follows: the cosets are the set of flags $W_1 \subset W_2 \subset V_I$ where W_i is a vector subspace of V_I having dimension i. In particular we can regard h as representing the non-zero vector in the dual V_I^* which annihilates W_2, so, since there are exactly three flags with the same W_2 we see that

$$\begin{aligned} res^{V_I}_{G_2(3)} tr^*(c\alpha^2) &= 3 \sum_{v \in V_I} v^7 \\ &= d_7 \end{aligned}$$

so it follows that $tr^*(c\alpha^2) = md_4 + d_7 \in H^*(G_2(3))$. Similarly, we have

$$tr^*(\alpha^7) = m^3 n \bmod (I(\mathbb{F}_2[d_4, d_6, d_7])), \tag{5.10}$$

where $I(\mathbb{F}_2[d_4, d_6, d_7])$ is the augmentation ideal in the Dickson algebra, since it restricts to 0 in $H^*(V_{II})$ and $d_7^2 \in H^*(V_I)$.

Finally, we can determine the image of transfer on each of the remaining generators of $H^*(M_{12}) \subset H$ in $H^*(G_2(3))$ as a module over the Dickson algebra. This is all we need to determine the cohomology map $tr \circ res^*\colon H^*(M_{12}) \to H^*(G_2(q))$ since

$$tr^G_H(a \cup res^H_G(b)) = tr^G_H(a) \cup b$$

for any $b \in H^*(G)$. Of course, this does not determine the cohomology map

$$H^*(Q(B_{G_2(q)})) \xrightarrow{tr^*} H^*(Q(B_H)),$$

but it does determine it up to terms involving elements dual to loop sums and terms of the form $Q_I(\beta)$.

Note that the image of $H_*(Q(B_{G_2(q)})$ in $\mathcal{H}_4/\mathcal{H}_6$ in $H_*(Q(S^0))$ is detected by the elements $\sigma_2^i Sq^1(\sigma_2)^j$. Thus, we need to study the composite map on the $\sigma_2^i(Sq^1(\sigma_2)^j$. The transfer does not preserve cup products but it is natural with respect to the action of the Steenrod algebra, $\mathcal{A}(2)$, and, in low dimensions, the module over $\mathcal{A}(2)$ spanned by these classes is generated by σ_2, σ_2^3, σ_2^7. For our purposes it suffices to determine the transfer just on the images of these elements in $H^*(M_{12})$. But $\sigma_2 \mapsto \alpha \in H^*(M_{12})$ as we have seen. We have

LEMMA 5.11. *$tr \circ res^*(\alpha) = tr \circ res^*(\alpha^3) = 0$ in $H^*(G_2(q))$, while $tr \circ res^*(\alpha^7) = m^3 n$.*

Proof. We have already verified the map on α^7. It remains to check the others. Since $H^2(G_2(q)) = 0$ the result follows for α. It remains to check α^3. Since $d_2^3 \in H^*(V_5)$ is invariant under $\mathcal{S}_3$ it follows that the restriction of $tr^*(\alpha^3)$ to $H^*(V_{II})$ is zero. Also, $\sum v \in H^1(2^3)v^3 = 0$ so the restriction to $H^*(V_I)$ is also zero and the result follows. □

COROLLARY 5.12: *The image of $[S^{14}]$ in diagram (d) represents either the class σ^2 or $\sigma^2 + \bar{\kappa}$ in $\pi_{14}^s(S^0)$.*

Proof. The projection of the image of $H_*(Q(B_{G_2(q)}))$ to $\mathcal{H}_4/\mathcal{H}_6$ via the composition in (5.1) or (5.2) is zero in dimensions < 14 abd 5.11 shows that it takes the class $[G_2]$ to $\{Q_7(J) * Q_7(J) * 4(-J)\}$.

As we have already seen, the Hurewicz image of the sphere in (5.2) has the form $[G_2] + D$ where D is a sum of loop sum decomposables. Moreover, by naturality with respect to loop sum, the image of D is contained in $\mathcal{H}_6$. Consequently, the spherical class in $\pi_{14}(Q(S^0)$ obtained by the composition (5.2) has non-trivial Hurewicz image in $H_{14}(Q(S^0))$ of the form $Q_7(J)*Q_7(J)*4(-J)$ summed with terms of higher filtration. Consequently, it can only be σ_7^2 or $\bar{\kappa}+\sigma_7^2$, since $\sigma_{14}^s(S^0) = 2^2$ and the Hurewicz image of $\bar{\kappa}$ is trivial in $\mathcal{H}_4/\mathcal{H}_6$. □

6. The proof of Theorem 1.8. We begin by determining the structures of the various images under projection

$$H^*(M_{12}) \xrightarrow{\pi_i^*} H^*(2^3)$$

as 2^3 runs over V_1, V_3, V_5 and $i = 1,3,5$ respectively.

LEMMA 6.1. *The image of π_5^* in $H^*(V_5)$ is*

$$\mathbb{F}_2[d_2, d_3, d_4] = \mathbb{F}_2[d_4, d_6, d_7](1, d_2, d_3, d_2^2, d_2d_3, d_3^2, d_2^4)(1, c)$$

with relations

$$\begin{aligned} c^2 &= d_3c + d_2^3 + d_3^2 \\ d_2^2d_3 &= d_7 + d_3d_4, \\ d_2^3 + d_3^2 &= d_2d_4 + d_6, \end{aligned}$$

In particular it is Cohen-Macaluay over $\mathbb{F}_2[d_4, d_6, d_7]$ on the stated generators.

Proof. First we note that $Sq^1(c) = \tau^4 + \tau^2\lambda^2 + \lambda^4 = d_2^2$, $Sq^2(c) = \tau^5 + \tau^4\lambda + \lambda^5$ which in turn is seen to be $d_2(c + d_3)$. Thus we have

$$c^2 = Sq^3(c) = Sq^1Sq^2(c) = Sq^1(d_2c + d_2d_3) = d_3c + d_2^3 + d_3^2$$

as asserted. This proves the first relation.

The expressions for the d_i $i = 4,6,7$, given in (1.3) give the relations $d_2d_4 + d_6 = d_2^3 + d_3^2$, $d_3d_4 + d_7 = d_2^2d_3$ which are sufficient to show that the algebra spanned by d_2, d_3, d_4, d_6, d_7 is just the polynomial algebra $\mathbb{F}_2[d_2, d_3, d_4]$. Consequently the Poincaré series of this algebra is

$$\begin{aligned} \frac{1}{(1-x^4)(1-x^3)(1-x^2)} &= \frac{(1-x+x^2)(1+x+x^2+x^3+x^4+x^5+x^6)}{(1-x^4)(1-x^6)(1-x^7)} \\ &= \frac{1+x^2+x^3+x^4+x^5+x^6+x^8}{(1-x^4)(1-x^6)(1-x^7)}. \end{aligned}$$

Next, if we factor out by the ideal (d_4, d_6, d_7) this polynomial ring becomes the algebra

$$\mathbb{F}_2[d_2, d_3]/(d_2^3 + d_3^2, d_2^2d_3)$$

which has generators (at most) $1, d_2, d_3, d_2^2, d_2d_3, d_3^2, d_2^4$. Hence, checking the Poincaré series the result follows. □

COROLLARY 6.2. *The image of π_1^* in $H^*(V_1)$ is generated by $d_4, d_6, d_7, d_3, d_2d_3$ and as a module over $\mathbb{F}_2[d_4, d_6, d_7]$ is free on the generators*

$$1, d_3, d_2d_3, d_3^2, d_2d_3^2, d_3^3, d_2d_3^3$$

Proof. We embed this algebra into the algebra above. Note that d_3, d_2d_3, d_3^2, are already generators and

$$d_2d_3^2 \equiv d_2^4 \bmod (d_4, d_6, d_7).$$

On the other hand we have $d_3^3 = d_3d_6 + d_2d_7$, $d_2d_3^3 = d_2d_3d_6 + d_2^2d_7$ which are independent of the previous generators over $\mathbb{F}_2[d_4, d_6, d_7]$. Finally,

$$\begin{aligned} d_2d_3^4 &= d_2d_3^2d_6 + (d_2^2d_3)d_7 \\ &= (d_2d_3)d_3d_6 + d_3d_4d_7 + d_7^2, \end{aligned}$$

which shows that this element is already present in the image, so the process stops and we are done. □

Finally, we look at the structure of the image of π_3^* in $H^*(V_3)$.

LEMMA 6.3. *The subalgebra of $H^*(V_3)$ generated by d_4, d_6, d_7, h has the form $\mathbb{F}_2[h, d_4, d_6]$ and is Cohen-Macaulay over $\mathbb{F}_2[d_4, d_6, d_7]$. It can be written*

$$\mathbb{F}_2[d_4, d_6, d_7](1, h, h^2, h^3, h^4, h^5, h^6)$$

with relation $d_7 = h^7 + h^3d_4 + hd_6$.

Proof. We first verify the relation:

$$\begin{aligned} h^3d_4 &= h^7 + h^5d_2 + h^4d_3 + h^3d_2^2 \\ hd_6 &= h^5d_2 + h^3d_2^2 + h^2d_2d_3 + hd_3^2 \end{aligned}$$

so summing with h^7 directly gives d_7. This relation shows that the algebra spanned by d_4, d_6, d_7, h is already spanned by d_4, d_6, and h. But it is easy to check that these three elements are transcendentally independent.

Hence, the algebra we are considering is, indeed, $\mathbb{F}_2[h, d_4, d_6]$ with Poincaré series

$$\begin{aligned} P &= \frac{1}{(1-x)(1-x^4)(1-x^6)} \\ &= \frac{1+x+x^2+x^3+x^4+x^5+x^6}{(1-x^4)(1-x^6)(1-x^7)} \end{aligned}$$

On the other hand, from our relation, the algebra is surely spanned as a module over $\mathbb{F}_2[d_4, d_6, d_7]$ by the elements 1, h, h^2, h^3, h^4, h^5 and h^6, so the Poincaré series above shows that it must be free on these generators and we are done. □

LEMMA 6.4. *Let P_i, $i = 1, 3, 5$ be the kernel of projection $H^*(M_{12}) \to H^*(V_i)$, then we have*

(1) A generating set for P_5 over $H^*(M_{12})$ is

$$\begin{array}{ccccc} (& d_6 & d_6 + h^2d_4 + h^6 & 0 &) \\ (& d_7 & d_7 & 0 &) \\ (& d_3 & 0 & 0 &) \\ (& d_2d_3 & 0 & 0 &) \end{array}$$

(2) The image of P_5 under $\pi_1 : H^*(M_{12}) \to H^*(V_1)$ is

$$(d_6, d_7) \oplus \mathbb{F}_2[d_4, d_6, d_7](d_3, d_2d_3, d_3^2, d_2d_3^2, d_3^3, d_2d_3^3)$$

where (d_6, d_7) is the ideal in $\mathbb{F}_2[d_4, d_6, d_7]$ generated by d_6 and d_7.

Proof. We check (1) first. The defining relations in the image of π_5 have been seen to be $d_7 + d_4d_3 + d_2^2d_3 = 0$, and $d_6 + d_4d_2 + d_3^2 + d_2^3 = 0$. Thus, the elements

$d_7+d_4d_3+d_2^2d_3$, $d_6+d_4d_2+d_3^2+d_2^3$ together with the remaining generators of $H^*(M_{12})$ which map to zero in $H^*(V_5)$ generate P_5. This gives the list in (1).

Next we check (2). But this is entirely direct from (1). □

Now we study P_5 in more detail. The generators given above are generators over $H^*(M_{12})$ so the generators as a module over $\mathbb{F}_2[d_4, d_6, d_7]$ are given by the elements

$$\begin{array}{cccc} (& d_6 & d_6+h^2d_4+h^6 & 0 \ ) \\ (& d_7 & d_7 & 0 \ ) \\ (& d_3 & 0 & 0 \ ) \\ (& d_2d_3 & 0 & 0 \ ) \\ (& d_3^2 & 0 & 0 \ ) \\ (& d_2d_3^2 & 0 & 0 \ ) \\ (& d_3^3 & 0 & 0 \ ) \\ (& d_2d_3^3 & 0 & 0 \ ) \end{array}$$

together with products of these elements with powers of the remaining generators,

$$(0, h^2, d_2), \qquad (0, 0, d_3) \qquad (0, h^3, c)$$

where we can clearly ignore the middle generator. From the relation in $\mathbb{F}_2[h, d_4, d_6]$ we see that

$$\begin{aligned} (0, h^2, d_2)(d_6, d_6+h^2d_4+h^6, 0) &= (0, h^8+h^2d_6+h^4d_4, 0) \\ &= (0, hd_7, 0) \end{aligned}$$

so the remaining generators (over $\mathbb{F}_2[d_4, d_6, d_7]$) for P_5 are contained in the following list

$$\begin{array}{ccccc} (& 0 & hd_7 & 0 &) \\ (& 0 & h^2d_7 & 0 &) \\ (& 0 & h^3d_7 & 0 &) \\ (& 0 & h^4d_7 & 0 &) \\ (& 0 & h^5d_7 & 0 &) \\ (& 0 & h^6d_7 & 0 &). \end{array}$$

But we easily see that the three generators $(d_6, h^6+h^2d_4+d_6, 0)$, $(d_7, d_7, 0)$ and

$$(0, h^6d_7+d_4h^2d_7, 0)$$

together give just two copies of $\mathbb{F}_2[d_4, d_6, d_7]$ since we have an exact sequence

(6.5) $$0 \longrightarrow \mathbb{F}_2[d_4, d_6, d_7](d_7A+d_6B) \hookrightarrow \mathbb{F}_2[d_4, d_6, d_7](A, B) \longrightarrow (d_6, d_7) \longrightarrow 0$$

where $A \mapsto d_6$ and $B \mapsto d_7$ in the ideal $(d_6, d_7) \subset \mathbb{F}_2[d_4, d_6, d_7]$. Thus the last generator in the list above may be suppressed Moreover, by the same imbedding technique as was used in the second lemma we see that the remaining needed generators $(0, h^id_7, 0)$, $1 \leq i \leq 6$ are independent and generate a free module over $\mathbb{F}_2[d_4, d_6, d_7]$. However, since the generators $(d_6, h^6+h^2d_4+d_6, 0)$ and $(d_7, d_7, 0)$ are not needed, we see that $(0, h^7d_7, 0) \sim (0, d_7^2, 0)$ is also a free generator, and adding this in gives a complete list of generators for $H^*(M_{12})$ over $\mathbb{F}_2[d_4, d_6, d_7]$. □

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