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A CHARACTERIZATION OF RATIONAL SINGULARITIES*

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1. Introduction. Recall that if $(R, \mathbf{m})$ is a local ring which is essentially of finite type over a field k of characteristic 0, then R is said to be of *F-injective type* (in the sense of Hochster-Roberts [HR]) if, loosely speaking, for 'almost all reductions of R modulo primes p', the Frobenius morphism acts injectively on the local cohomology modules of the reduction. If $(R, \mathbf{m})$ is normal and Cohen-Macaulay, and the top local cohomology of 'almost all' such reductions have no proper F-stable submodules, we say that R has *F-rational type* (in the sense of Huneke-Smith [HS] and Fedder-Watanabe [FW]). We will explain later in a more precise way as to what is meant by 'almost all reductions modulo primes p'; this discussion is however a little technical, and so we avoid it here.

In her thesis, Smith proved that if $(R, \mathbf{m})$ has F-rational type, then R has a *rational singularity.* Recall that this means that if $X \to \operatorname{Spec} R$ is a resolution of singularities, then $H^i(X, \mathcal{O}_X) = 0$ for $i > 0$, and $H^0(X, \mathcal{O}_X) = R$. She conjectured that the converse is true; this yields an intrinsic, algebraic *characterization* of rational singularities which *does not involve construction of a resolution of singularities.*

We present below a proof of the conjecture:

THEOREM 1.1. *Let $(R, \mathbf{m})$ be a local ring essentially of finite type over a field k of characteristic 0, such that R has a rational singularity. Then R has F-rational type.*

We should state right away that this result has already been obtained earlier by N. Hara [HaW], though our work was done independently, and we were not aware of Hara's work while working out our ideas.

In order to obtain the above result, we will consider below another pair of notions, related to the notions of F-injective type and F-rational type. Let R be a normal local ring, essentially of finite type over a field of characteristic 0. Let $X \to \operatorname{Spec} R$ be a resolution of singularities, such that the inverse image of the singular locus is a divisor with normal crossings. Let Z be the fibre of X over the maximal ideal of R. Since R is essentially of finite type over k, one can make sense of the following statement: for 'almost all reductions of X modulo a prime p', if $\overline{X}$ is such a reduction, and $\overline{Z} \subset \overline{X}$ is the corresponding reduction of Z, then the Frobenius morphism is injective on the local cohomology groups $H^i_{\overline{Z}}(\overline{X}, \mathcal{O}_{\overline{X}})$. If this is the case, we say that $(R, \mathbf{m})$ has *resolved F-injective type.* Similarly, if $H^i_{\overline{Z}}(\overline{X}, \mathcal{O}_{\overline{X}}) = 0$ for $i < d = \dim X$, and $H^d_{\overline{Z}}(\overline{X}, \mathcal{O}_{\overline{X}})$ has no proper $\overline{R}$-submodule stable under the Frobenius (where $\overline{R}$ is the corresponding 'reduction modulo p' of R), we say that $(R, \mathbf{m})$ has *resolved F-rational type* (see §2 for a more detailed discussion of this notion).

As will be seen below (lemma 7.1), for a rational singularity, it is easy to check that these notions agree with the notions of F-injective type and F-rational type, since $H^d_{\overline{Z}}(\overline{X}, \mathcal{O}_{\overline{X}})$ is then identified with the top local cohomology of the corresponding local ring.

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Hence, Theorem 1.1 is a special case of the following general result.

THEOREM 1.2. *Let $(R, \mathbf{m})$ be any normal local ring, essentially of finite type over a field of characteristic 0. Then R is of resolved F-rational type.*

In fact, the proof yields the statement that if $(R, \mathbf{m})$ is any reduced local ring of dimension d, essentially of finite type over a field of characteristic 0, then for 'almost all' reductions $\overline{R}$ as above, the submodule $0 \subset H^d_{\overline{Z}}(\overline{X}, \mathcal{O}_{\overline{X}})$ is *tightly closed*, in the sense of Hochster-Huneke. Note that, if R is not equidimensional, the lower resolved local cohomology modules $H^i_{\overline{Z}}(\overline{X}, \mathcal{O}_{\overline{X}})$ for $i < d$ need not all vanish. However, this will be true if R is equidimensional.

The proof given will also yield, as a by-product, another proof of the Grauert-Riemenschneider Vanishing Theorem on vanishing of higher direct images of the canonical bundle under a birational proper morphism from a smooth variety. An equivalent formulation is that on a smooth projective variety X in characteristic 0, $H^i(X, \mathcal{L}^{-N}) = 0$ for all $i < d = \dim X$, and all large N, if $\mathcal{L}$ is a semi-ample and big line bundle[1]. To our minds, this seems to suggest that the notion of resolved F-injective type is an analogue of this vanishing theorem, but for the d-th cohomology.

The technique of proofs is to make use of certain complexes of differentials with logarithmic poles along normal crossing divisors. On the one hand, this relates to topology over $\mathbb{C}$, and on the other hand, in characteristic p, can be analysed following Deligne and Illusie [DI]. Our treatment is influenced by the exposition of these ideas in the book [EV], but we do not need any results on branched coverings, and make no explicit mention of $\mathbb{Q}$-divisors, or residues of logarithmic connections, though they are present in spirit.

2. Reductions modulo almost all primes. Let $(R, \mathbf{m})$ be a local ring essentially of finite type over a field k of characteristic 0. We first make more precise the notion that some property is valid for 'reductions of R modulo almost all primes p'. To be specific, we first consider the property that R is of F-injective type (compare with the definition of *F-pure type* given in [S], which is in turn a convenient reformulation of the notion originally introduced by Hochster and Roberts [HR]).

Since R is essentially of finite type over k, we may (after enlarging k if necessary) assume that $R/\mathbf{m}$ is a finite algebraic extension of k. Then we can find a finitely generated $\mathbb{Z}$-subalgebra $A \subset k$, a finitely generated flat A-algebra R_A, together with a prime ideal $I \subset R_A$, with R_A/I finite and flat over A, such that (with respect to the natural inclusion of A into k) the localization $(R_A \otimes_A k)_{I R_A \otimes_A k}$ is just R. Then for any maximal ideal $\mathbf{n}$ of A, if $\overline{R} = R_A \otimes_A A/\mathbf{n}$, then $\overline{R}/I\overline{R}$ is semilocal; if $\overline{\mathbf{n}}$ is any prime of $\overline{R}$ lying over $I\overline{R}$, then we may consider the local ring $(\overline{R}_{\overline{\mathbf{n}}}, \overline{\mathbf{n}}\overline{R}_{\overline{\mathbf{n}}})$ as obtained by 'reduction modulo p' from R, if $A/\mathbf{n}$ has characteristic p. Now R is said to be of *F-injective type* if for all maximal ideals $\mathbf{n}$ in some non-empty Zariski open subset of $\operatorname{Spec} A$, the Frobenius morphism F on $\overline{R}$ (given by $a \mapsto a^p$) acts injectively on the local cohomology modules $H^i_{\overline{\mathbf{n}}}(\overline{R})$ for all i, for any $\overline{\mathbf{n}}$ as above.

In a similar way, if R is normal and Cohen-Macaulay of dimension d, we say that R is of *F-rational type* (see [HS], [FW]) if for all $\mathbf{n}$ in some non-empty Zariski open subset of $\operatorname{Spec} A$, the $\overline{R}_{\overline{\mathbf{n}}}$-module $H^d_{\overline{\mathbf{n}}}(\overline{R})$ has no proper $\overline{R}_{\overline{\mathbf{n}}}$-submodule stable under Frobenius, for any $\overline{\mathbf{n}}$ as above.

Using standard properties of local cohomology, it is easy to see that the notions of F-injective type and F-rational type are independent of the choice of the rings A,

[1] *i.e.*, some tensor power of $\mathcal{L}$ is generated by global sections, which birationally embed X in a projective space.

R_A and the ideal I.

Let $(R, \mathbf{m})$ be a *reduced* local ring essentially of finite type over a field k of characteristic 0. Then we can make sense of a resolution of singularities of $\operatorname{Spec} R$, *i.e.*, a proper morphism $\pi : X \to \operatorname{Spec} R$ which is an isomorphism over the regular locus of $\operatorname{Spec} R$, such that $X \to \operatorname{Spec} k$ is smooth. In general, such an X will be a disjoint union of smooth proper k-varieties, possibly of different dimensions. However, if R is normal, then X is irreducible. By results of Hironaka, we can in fact find a projective resolution of singularities $\pi : X \to \operatorname{Spec} R$ (*i.e.*, a resolution as above with π projective). *Henceforth we will always assume that R is reduced; thus such a resolution of singularities will exist.*

Then, given any resolution of singularities $\pi : X \to \operatorname{Spec} R$, we can find the $\mathbb{Z}$-algebra A, the flat A-algebra R_A, etc. such that in addition, there is a smooth A-scheme $X_A \to \operatorname{Spec} A$, and there is a proper morphism $\pi_A : X_A \to \operatorname{Spec} R_A$ of A-schemes, which induces $\pi : X \to \operatorname{Spec} R$ after the base change $\operatorname{Spec} R \to \operatorname{Spec} R_A$. All this may not be possible with the original rings A and R_A, but can be done after replacing them by suitable extensions.

Let $Z_A \subset X_A$ be the subscheme defined by the ideal sheaf generated by $I \subset A$. For any maximal ideal $\mathbf{n}$ of $\operatorname{Spec} A$, and any maximal ideal $\overline{\mathbf{n}}$ of $\overline{R}$ as before, let $\overline{\pi} : \overline{X} \to \operatorname{Spec} \overline{R}_{\overline{\mathbf{n}}}$ be the resulting morphism. Then $\overline{Z} = Z \times_{\operatorname{Spec} R_A} \operatorname{Spec} \overline{R}_{\overline{\mathbf{n}}}$ is the fibre of $\overline{\pi}$ over the closed point $\overline{\mathbf{n}}$.

We say that $(R, \mathbf{m})$ is of *resolved F-injective type* if R is normal, and for all maximal ideals $\mathbf{n}$ in a nonempty Zariski open set of $\operatorname{Spec} A$, and all $\overline{\mathbf{n}}$ as above, the Frobenius morphism on the scheme $\overline{X}$ is injective on the local cohomology groups $H^i_{\overline{Z}}(\overline{X}, \mathcal{O}_{\overline{X}})$. We say that $(R, \mathbf{m})$ is of *resolved F-rational type* if for a non-empty open set of such $\mathbf{n}$, and any corresponding $\overline{\mathbf{n}}$, we have that $H^i_{\overline{Z}}(\overline{X}, \mathcal{O}_{\overline{X}}) = 0$ for $i < d = \dim R$, and $H^d_{\overline{Z}}(\overline{X}, \mathcal{O}_{\overline{X}})$ has no proper $\overline{R}_{\overline{\mathbf{n}}}$-submodule stable under the Frobenius map. It is easy to see that if $H^d_{\overline{Z}}(\overline{X}, \mathcal{O}_{\overline{X}})$ has no proper $\overline{R}_{\overline{\mathbf{n}}}$-submodule stable under the Frobenius map, then in particular the Frobenius map is injective (else the submodule generated by the kernel of Forbenius would be such a proper submodule). So, in fact the notion of resolved F-injective type is in a sense redundant, for normal local rings, as a consequence of Theorem 1.2.

If $f : X' \to X$ is a birational projective (or even proper) morphism, where X' is also smooth, then one has that $f_*\mathcal{O}_{X'} = \mathcal{O}_X$, and $R^i f_* \mathcal{O}_{X'} = 0$ for $i > 0$ (this is 'well known', but we discuss below a proof of this from our perspective). If $Z' \subset X'$ is the inverse image of Z, then we may similarly form the schemes $\overline{X'}$, $\overline{Z'}$ over a Zariski open subset of $\operatorname{Spec} A$, such that we have a morphism $\overline{f} : \overline{X'} \to \overline{X}$, and $\overline{Z'}$ is the inverse image of $\overline{Z}$. Now for $\mathbf{n}$ in a perhaps smaller open subset of $\operatorname{Spec} A$, there are natural isomorphisms (compatible with the Frobenius endomorphism) $H^i_{\overline{Z}}(\overline{X}, \mathcal{O}_{\overline{X}}) \cong H^i_{\overline{Z'}}(\overline{X'}, \mathcal{O}_{\overline{X'}})$ for all i, since (by semicontinuity) $R^j \overline{f}_* \mathcal{O}_{\overline{X'}} = 0$ for all $j > 0$, and $\overline{f}_* \mathcal{O}_{\overline{X'}} = \mathcal{O}_{\overline{X}}$. Since by results of Hironaka, any two resolutions of singularities of $\operatorname{Spec} R$ are dominated by a third, we see that the notion of resolved F-injective type and resolved F-rational type do not depend on the particular resolution of singularities chosen *in characteristic 0.*

The local cohomology $H^i_{\overline{Z}}(\overline{X}, \mathcal{O}_{\overline{X}})$ will depend only on the local ring $\overline{R}_{\overline{\mathbf{n}}}$ in characteristic p, provided one has an analogue of Hironaka's results on resolution of singularities in characteristic p, in a sufficiently strong form. Thus, if we assume such results, the notion of *resolved local cohomology* for a local ring in characteristic p will be well-defined. However, this is at present only conjectural.

Therefore, in the sequel, when we refer to resolved local cohomology, we will mean the local cohomology of the 'reduction modulo p' of a specific resolution $X \to \operatorname{Spec} R$ of a local ring in characteristic 0, such that the inverse image of the singular locus is a divisor with normal crossings (*c.f.* the discussion after Theorem 1.1). However, we stress again that the notion of a normal local ring R of characteristic 0 being of resolved F-injective type or F-rational type is *independent* of the resolution of singularities $X \to \operatorname{Spec} R$ (see Corollary 6.6 and Proposition 7.3 below).

We note also that if $\overline{R}^{\wedge}$ denotes the completion of $\overline{R}$, and $\overline{X}^{\wedge} = \overline{X} \times_{\operatorname{Spec} \overline{R}} \overline{R}^{\wedge}$, $\overline{Z}^{\wedge} \cong Z$ the fibre of $\overline{X}^{\wedge} \to \operatorname{Spec} \overline{R}^{\wedge}$ the fibre over the maximal ideal, then we also have 'excision' isomorphisms $H^i_{\overline{Z}}(\overline{X}, \mathcal{O}_{\overline{X}}) \cong H^i_{\overline{Z}^{\wedge}}(\overline{X}^{\wedge}, \mathcal{O}_{\overline{X}^{\wedge}})$ for all i. These are also compatible with the Frobenius action.

Finally, we note that we may 'globalize' the situation above, as follows. There is a projective k-variety V and a closed point $x \in V$ such that $R = \mathcal{O}_{x,V}$. We may also resolve the singularities $\pi : W \to V$, and let $Z \subset W$ be the inverse image of x; we may choose such a resolution such that $E = \pi^{-1}(V_{sing})$ is a divisor in W with (global) normal crossings. Then $X = W \times_V \operatorname{Spec} \mathcal{O}_{x,V}$. If R is normal, we may (and will) further assume that V, W are irreducible, and V is normal.

Now we may similarly 'spread out' $\pi : W \to V$ to a projective scheme V_A over the ring A, and a projective morphism $\pi_A : W_A \to V_A$, with $W_A \to \operatorname{Spec} A$ a smooth projective morphism, such that there is a subscheme $T \subset V_A$, finite over $\operatorname{Spec} A$, and inverse image $Z_A \subset W_A$, such that $T \times_A k = \{x\}$. Now for any closed point $s \in \operatorname{Spec} A$, and any $t \in T$ lying over s, we have a diagram of $k(s)$-schemes $W_s \to V_s$, the corresponding local ring $\mathcal{O}_{t,V_s}$ in some characteristic $p > 0$, and fibre scheme Z_t over t. We also have $\overline{X} = W_s \times_{V_s} \operatorname{Spec} \mathcal{O}_{t,V_s}$, and $\overline{Z}$ is identified with Z_t. There are also natural 'excision' isomorphisms (compatible with Frobenius) $H^i_{Z_t}(W_s, \mathcal{O}_{W_s}) \cong H^i_{\overline{Z}}(\overline{X}, \mathcal{O}_{\overline{X}})$. Thus, the properties of a normal local ring $(R, \mathbf{m})$ being of resolved F-injective type, and resolved F-rational type, may be formulated instead using the 'global' schemes W_A, V_A and $T \subset V_A$ (which may be taken to be irreducible and normal, since R is normal).

Further, we may make a base change $T \to \operatorname{Spec} A$, replacing $W_A \to V_A$ by $W_T \to V_T$, so that $W_T \to V_T$ is still projective, and $W_T \to T$ smooth and projective. Now there is a natural section $\sigma : T \to V_T$ of $V_T \to T$, corresponding to the original inclusion $T \subset V_A$. We may assume, if necessary after localizing A to A_h for some h, that $T \to \operatorname{Spec} A$ is etale and faithfully flat. Then one sees that for any $t \in T$ with image $s \in \operatorname{Spec} A$, the morphism $W_t \to V_t$ is obtained from $W_s \to V_s$ by the base extension $\operatorname{Spec} k(t) \to \operatorname{Spec} k(s)$, and that $\mathcal{O}_{t,V_s} \to \mathcal{O}_{\sigma(t),V_t}$ is an etale extension, inducing an isomorphism on completions. This implies that the natural maps $H^i_{Z_t}(W_s, \mathcal{O}_{W_s}) \to H^i_{Z_{\sigma(t)}}(W_{\sigma(t)}, \mathcal{O}_{W_{\sigma(t)}})$ are isomorphisms, compatible with Frobenius. Hence we see that the properties of injectivity of Frobenius, and the non-existence of F-stable proper submodules, is equivalent for $W_s \to V_s$ and $W_{\sigma(t)} \to V_{\sigma(t)}$.

Thus, in the above context, we may also assume without loss of generality that $T \to \operatorname{Spec} A$ is an isomorphism.

3. A reformulation of resolved F-injectivity. Let $(R, \mathbf{m})$ be a reduced local ring essentially of finite type over a field k of characteristic 0, which is purely of dimension d. Let $W_A \to V_A$ be constructed as in the previous section, where now each fiber of $V_A \to T = \operatorname{Spec} A$ may be assumed to be reduced and purely of dimension d, and $W_A \to T$ is smooth of relative dimension d.

Let $s \in \operatorname{Spec} A = T$ be a closed point, $\pi_s : W_s \to V_s$ the corresponding morphism, and Z_s the fibre of π_s over s. In order to study the injectivity of the Frobenius morphism on $H^i_{Z_s}(W_s, \mathcal{O}_{W_s})$, we will make use of the following Duality Theorem, which we recall below (see [Ha1], III, Theorem 3.3 and [Ha2], III, Sect. 11; in fact this combines the Formal Duality Theorem as stated in [Ha1] with the formal function theorem of [Ha2]).

THEOREM 3.1. *Let Y be a non-singular projective variety (possibly not connected) of pure dimension d over a field k, $\pi : Y \to \overline{Y}$ a projective morphism of k-varieties, $y \in \overline{Y}$ a closed point, $Z = \pi^{-1}(y)$. Let $\widehat{\mathcal{O}}_{y,\overline{Y}}$ denote the complete local ring of y on $\overline{Y}$, and $Y^\wedge = Y \times_{\overline{Y}} \operatorname{Spec} \widehat{\mathcal{O}}_{y,\overline{Y}}$. Let $f : Y^\wedge \to Y$ be the projection.*

Then for any locally free sheaf $\mathcal{E}$ on Y, the dual k-vector space of $H^i_Z(Y, \mathcal{E})$ is naturally isomorphic to $H^{d-i}(Y^\wedge, f^*(\omega_Y \otimes_{\mathcal{O}_Y} \mathcal{E}^*))$, where $\mathcal{E}^* = \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{E}, \mathcal{O}_Y)$ is the dual locally free sheaf, and ω_Y is the dualizing sheaf (i.e., the sheaf of d-forms $\Omega^d_{Y/k}$).

In our context, taking $\mathcal{E} = \mathcal{O}_{W_s}$, we get that the $k(s)$-dual of $H^i_{Z_s}(W_s, \mathcal{O}_{W_s})$ is $H^{d-i}(W_s^\wedge, f_s^*\omega_{W_s})$, where $W_s^\wedge = W_s \times_{V_s} \operatorname{Spec} \widehat{\mathcal{O}}_{s,V_s}$, and $f_s : W_s^\wedge \to W_s$ is the natural map. Next, if $F : W_s \to W_s$ is the absolute Frobenius morphism, then taking $\mathcal{F} = F_*\mathcal{O}_{W_s}$, we get that the $k(s)$-linear dual of $H^i_{Z_s}(W_s, F_*\mathcal{O}_{W_s})$ is naturally identified with $H^{d-i}(W_s^\wedge, f^*(F_*\omega_{W_s}))$. The Frobenius map on $H^i_{Z_s}(W_s, \mathcal{O}_{W_s})$ may be viewed as the $k(s)$-linear map on local cohomology associated to the morphism of locally free $\mathcal{O}_{W_s}$ modules $\mathcal{O}_{W_s} \to F_*\mathcal{O}_{W_s}$. By the naturality of the duality isomorphisms above, the $k(s)$-linear dual to the Frobenius map $H^i_{Z_s}(W_s, \mathcal{O}_{W_s}) \to H^i_{Z_s}(W_s, F_*\mathcal{O}_{W_s})$ is a map

$$H^{d-i}(W_s^\wedge, f_s^*((F_*\mathcal{O}_{W_s})^* \otimes_{\mathcal{O}_{W_s}} \omega_{W_s})) \to H^{d-i}(W_s^\wedge, f_s^*\omega_{W_s}).$$

This map is induced by the map of locally free sheaves

$$\psi : \mathcal{H}om_{\mathcal{O}_{W_s}}(F_*\mathcal{O}_{W_s}, \omega_{W_s}) \to \mathcal{H}om_{\mathcal{O}_{W_s}}(\mathcal{O}_{W_s}, \omega_{W_s}) = \omega_{W_s}$$

dual to the Frobenius morphism $\mathcal{O}_{W_s} \to F_*\mathcal{O}_{W_s}$. But Grothendieck Duality for the Frobenius morphism (which is a finite flat morphism) identifies $\mathcal{H}om_{\mathcal{O}_{W_s}}(F_*\mathcal{O}_{W_s}, \omega_{W_s})$ with $F_*\omega_{W_s}$, such that the above sheaf map ψ is identified with the *Cartier operator* on the sheaf of d-forms on W_s (see [Ha2], III, Ex. 6.10, and also [MS]).

The duality pairing between the Frobenius and Cartier operators is expressed by the formula

$$(\alpha, F(\beta)) = (C(\alpha), \beta)^p,$$

for all $\alpha \in H^{d-i}(W_s^\wedge, f_s^*((F_*\mathcal{O}_{W_s})^* \otimes_{\mathcal{O}_{W_s}} \omega_{W_s}))$ and $\beta \in H^i_{Z_s}(W_s, \mathcal{O}_{W_s})$. We recall also that the Frobenius mapping is p-linear, while the Cartier operator C is p^{-1}-linear. Thus $F(f) = f^p$ for any rational function f, while $C(f^p\omega) = fC(\omega)$ for any rational function f and rational differential d-form ω.

Hence, the desired injectivity of the Frobenius map on the local cohomology $k(s)$-vector spaces $H^i_{Z_s}(W_s, \mathcal{O}_{W_s})$ is equivalent to the *surjectivity* of the 'completed' Cartier operator

$$f_s^*(C) : H^{d-i}(W_s^\wedge, f_s^*(F_*\omega_{W_s})) \to H^{d-i}(W_s^\wedge, f_s^*\omega_{W_s}).$$

But

$$H^j(W_s^\wedge, f_s^*\omega_{W_s}) \cong (R^j(\pi_s)_*\omega_{W_s})_s \otimes_{\mathcal{O}_{s,V_s}} \widehat{\mathcal{O}}_{s,V_s},$$

$$H^j(W_s^\wedge, f_s^* F_* \omega_{W_s}) \cong (R^j(\pi_s)_* F_* \omega_{W_s})_s \otimes_{\mathcal{O}_{s,V_s}} \widehat{\mathcal{O}}_{s,V_s},$$

by the Formal Function Theorem ([Ha2], III, Sect. 11). Since $R^j(\pi_s)_*\omega_{W_s}$ and $R^j(\pi_s)_*F_*\omega_{W_s} \cong F_*R^j(\pi_s)_*\omega_{W_s}$ are coherent $\mathcal{O}_{V_s}$-modules, the stalks of these direct image sheaves at $s \in V_s$ are finite $\mathcal{O}_{s,V_s}$ modules, and the Cartier operator C induces an $\mathcal{O}_{s,V_s}$-linear map between them. By Nakayama's lemma, the induced map $f_s^*(C)$ on completions is surjective if and only if the original map on stalks is surjective.

To simplify notation, we will not distinguish below between the Cartier operator C and its completion $f_s^*(C)$.

Since $s \in V_s$ was essentially an arbitrary point, we may reformulate our problem in the following terms: to show that for a Zariski open subset of closed points $s \in T = \operatorname{Spec} A$, the induced maps of sheaves

$$C : R^j(\pi_s)_*F_*\omega_{W_s} \to R^j(\pi_s)_*\omega_{W_s}$$

are surjective, for all $j \geq 0$.

We note that by semicontinuity and the Grauert-Riemenschneider Vanishing Theorem, in fact $R^j(\pi_s)_*\omega_{W_s} = 0$ (and hence also $R^j(\pi_s)_*F_*\omega_{W_s} = F_*R^j(\pi_s)_*\omega_{W_s} = 0$), for all $j > 0$ for all s in a non-empty Zariski open subset of T. So the new result is the surjectivity of the Cartier operator on the (0-th) direct image sheaves. However, we will give a 'parallel' proof for the surjectivity and vanishing below; as stated in the introduction, this yields another proof of the Grauert-Riemenschneider Vanishing Theorem.

4. Logarithmic de Rham complexes. Our next step is to prove certain results on logarithmic de Rham complexes, which we will apply to the above situation. We first recall some facts and notation regarding *differentials with logarithmic poles.* If Y is a non-singular k-variety, where k is a field, $A \subset Y$ a reduced divisor with normal crossings, then $\Omega^a_{Y/k}(\log A)$ will denote the sheaf of Kähler a-forms with logarithmic poles along A. If $x \in Y$ is a closed point, then there exists a regular system of parameters $x_1, \ldots, x_n$ in $\mathcal{O}_{x,Y}$ (with $n = \dim Y$) such that the ideal of A in $\mathcal{O}_{x,Y}$ is generated by $\prod_{i=1}^r x_i$; then the stalk at x of $\Omega^1_{Y/k}(\log A)$ is the free $\mathcal{O}_{x,Y}$-module with basis $\frac{dx_1}{x_1}, \ldots, \frac{dx_r}{x_r}, dx_{r+1}, \ldots, dx_n$ (regarded as a submodule of the module of meromorphic differentials at x). The sheaf $\Omega^a_{Y/k}(\log A)$ is the a-th exterior power (over $\mathcal{O}_Y$) of $\Omega^1_{Y/k}(\log A)$; it is locally free of rank $\binom{n}{a}$. Finally, if B is any divisor on Y, then $\Omega^a_{Y/k}(\log A)(B)$ denotes $\Omega^a_{Y/k}(\log A) \otimes_{\mathcal{O}_Y} \mathcal{O}_Y(B)$. For any divisor B which is a linear combination of components of A, a local calculation shows that the exterior derivative gives a well-defined complex of sheaves

$$0 \to \mathcal{O}_Y(B) \to \Omega^1_{Y/k}(\log A)(B) \to \cdots \to \Omega^n_{Y/k}(\log A)(B) \to 0.$$

We write $\Omega^\bullet_{Y/k}(\log A)(B)$ for this complex; we refer to any complex of this type as a *logarithmic de Rham complex.*

Similarly, one can define a complex of coherent analytic sheaves

$$\Omega^\bullet_{Y_{an}}(\log A_{an})(B_{an}),$$

if $k = \mathbb{C}$, and Y_{an}, A_{an}, B_{an} are the analytic objects associated to Y, A, B respectively. We refer to these too as logarithmic de Rham complexes. However, to simplify notation, we will usually omit the subscript an, since it will be clear from the context when one is working in the analytic topology.

LEMMA 4.1. *Let Y be a non-singular variety over a field k, and let D_1, D_2 be effective Cartier divisors on Y such that D_1 is non-singular, and $D_1 + D_2$ is reduced with global normal crossings. Let D be a (not necessarily effective) divisor on Y such that for some integer m, $D - mD_1$ is supported on D_2. Then the natural inclusion map of complexes of sheaves on Y*

$$\Omega^{\bullet}_{Y/k}(\log(D_1 + D_2))(D - D_1) \to \Omega^{\bullet}_{Y/k}(\log(D_1 + D_2))(D)$$

is a quasi-isomorphism, if either (i) k has characteristic 0, and $m \neq 0$, or (ii) k has characteristic $p > 0$, and m is not a multiple of p. If $k = \mathbb{C}$, a similar result (to (i)) holds for the corresponding map of complexes of analytic sheaves.

Proof. Let $\mathcal{F}^{\bullet}$ denote the cokernel complex of the inclusion

$$\Omega^{\bullet}_{Y/k}(\log(D_1 + D_2))(D - D_1) \to \Omega^{\bullet}_{Y/k}(\log(D_1 + D_2))(D).$$

The terms $\mathcal{F}^a$ are coherent $\mathcal{O}_Y$-modules supported on the non-singular divisor $D_1 \subset Y$. The residue map along D_1 yields an exact sequence for each $a > 0$

$$0 \to \Omega^a_{Y/k}(\log D_2)(D) \to \Omega^a_{Y/k}(\log(D_1 + D_2))(D) \to \Omega^{a-1}_{D_1/k}(\log D_2 \mid_{D_1}) \otimes \mathcal{O}_Y(D) \to 0.$$

Further, there is an exact sequence for each $a \geq 0$

$$0 \to \Omega^a_{Y/k}(\log(D_1 + D_2))(D - D_1) \to \Omega^a_{Y/k}(\log D_2)(D) \to \Omega^a_{D_1/k}(\log(D_2 \mid_{D_1}) \otimes \mathcal{O}_Y(D) \to 0.$$

In other words, if we let

$$\mathcal{B}^a = \Omega^a_{D_1/k}(\log D_2 \mid_{D_1}) \otimes \mathcal{O}_Y(D),$$

then we have $\mathcal{B}^0 = \mathcal{F}^0$, and we have exact sequences for $a > 0$

$$0 \to \mathcal{B}^a \to \mathcal{F}^a \to \mathcal{B}^{a-1} \to 0.$$

Now locally on Y, we can choose coordinates $x_1, \ldots, x_d$ ($d = \dim Y$) such that D_1 is defined by $x_1 = 0$, and D_2 by $x_2 \cdots x_r = 0$, say. Then a local section of $\mathcal{F}^a$ lifts to a section of $\Omega^a_{Y/k}(\log(D_1 + D_2))(D)$ which has the form

$$\alpha = x_1^{-m} \frac{dx_1}{x_1} \wedge \omega + x_1^{-m} \eta,$$

where ω, η are local regular forms (of degrees $a-1$ and a respectively) which are local sections of $\Omega^{\bullet}_{Y/k}(\log D_2)(D - mD_1)$. Then the image of α in $\mathcal{B}^{a-1}$ is just $\omega \mid_{D_1} \otimes x_1^{-m}$; if $\omega = 0$, then α maps to the section $\eta \mid_{D_1} \otimes x_1^{-m}$ of $\mathcal{B}^a \subset \mathcal{F}^a$.

In particular, suppose $\alpha = x_1^{-m} \eta$ is a lift of a local section of $\mathcal{B}^a$ (where η is a regular function, in case $a = 0$). Then

$$d\alpha = (-m) x_1^{-m} \frac{dx_1}{x_1} \wedge \eta + x_1^{-m} d\eta.$$

Hence the residue along D_1 of $d\alpha$, *i.e.*, the image under $\mathcal{F}^{a+1} \to \mathcal{B}^a$, is just

$$(-m)\eta \mid_{D_1} \otimes x_1^{-m}.$$

In other words, the composite

$$\mathcal{B}^a \hookrightarrow \mathcal{F}^a \xrightarrow{d} \mathcal{F}^{a+1} \twoheadrightarrow \mathcal{B}^a$$

is multiplication by $(-m)$. This is an *isomorphism*, if m is not a multiple of the characteristic of k. This easily implies that $\mathcal{F}^\bullet$ is *acylic*: $\ker d : \mathcal{F}^a \to \mathcal{F}^{a+1}$ has trivial intersection with $\mathcal{B}^a \subset \mathcal{F}^a$, so that it maps injectively under $\mathcal{F}^a \to \mathcal{B}^{a-1}$. This implies the kernel vanishes, if $a = 0$. For $a > 0$, note that image $d : \mathcal{F}^{a-1} \to \mathcal{F}^a$ maps *onto* $\mathcal{B}^a$, since in fact $d(\mathcal{B}^{a-1})$ maps isomorphically onto $\mathcal{B}^{a-1}$. Since $\mathcal{F}^\bullet$ is a complex, we see that in fact it is exact at $\mathcal{F}^a$, for each $a > 0$.

It is clear that essentially the same proof goes through in the analytic case as well. □

COROLLARY 4.2. *For any $n > 0$, the natural inclusion map of complexes of sheaves on Y*

$$\Omega^\bullet_{Y/k}(\log(D_1 + D_2))(-D_2) \to \Omega^\bullet_{Y/k}(\log(D_1 + D_2))(nD_1 - D_2)$$

is a quasi-isomorphism, if either (i) k has characteristic 0, or (ii) k has characteristic $p > 0$, and $n < p$. If $k = \mathbb{C}$, a similar result holds for the corresponding map of complexes of analytic sheaves.

LEMMA 4.3. *In the preceeding lemma, let $k = \mathbb{C}$. Assume the divisor $D = mD_1 - D'$, where each component of D_2 occurs with a strictly positive coefficient in D'. Then on Y_{an}, the underlying complex manifold, the cohomology sheaves of*

$$\Omega^\bullet_{Y_{an}}(\log(D_1 + D_2))(D)$$

vanish when restricted to D_2.

Proof. By the preceeding lemma, we see that the inclusion

$$\Omega^\bullet_{Y_{an}}(\log(D_1 + D_2))(-D') \hookrightarrow \Omega^\bullet_{Y_{an}}(\log(D_1 + D_2))(D)$$

is a quasi-isomorphism, by induction on m. Similarly, $D' = D_2 + D''$ where D'' is effective and supported within D_2 (some, or even all, the components of D_2 may occur with 0 coefficient in D''). Then again from the previous lemma, the inclusion

$$\Omega^\bullet_{Y_{an}}(\log(D_1 + D_2))(-D') \hookrightarrow \Omega^\bullet_{Y_{an}}(\log(D_1 + D_2))(-D_2)$$

is a quasi-isomorphism.

So we are reduced to proving the lemma when $D = -D_2$. In this case, we have a short exact sequence of complexes

$$0 \to \Omega^\bullet_{Y_{an}}(\log D_2)(-D_2) \to \Omega^\bullet_{Y_{an}}(\log(D_1 + D_2))(-D_2) \\ \to \Omega^\bullet_{(D_1)_{an}}(\log D_2 |_{D_1})(-D_2 |_{D_1})[-1] \to 0.$$

However, it is well known (and a variant of the holomorphic Poincaré lemma, easily proved by local calculations) that the first term is a resolution of $j_!\mathbb{C}_{Y-D_2}$, where $j : Y_{an} - D_2 \hookrightarrow Y_{an}$, while the third complex is quasi-isomorphic to $j_!\mathbb{C}_{D_1-(D_2\cap D_1)}[-1]$. In both cases, the only non-zero cohomology sheaf of the complex vanishes when restricted to D_2. Hence the same is true for cohomology sheaves of the middle complex. □

COROLLARY 4.4. *Let $\pi : Y \to \overline{Y}$ be a birational morphism between complete varieties over $\mathbb{C}$, with Y non-singular, and let $S \subset \overline{Y}$ be a subvariety such that if*

$D_2 = \pi^{-1}(S)$ (with its reduced structure), then $Y - D_2 \to \overline{Y} - S$ is an ismomorphism. Let D_1 be a non-singular divisor on Y such that $D_1 + D_2$ is a reduced divisor on Y with normal crossings (then D_1, D_2 have no common component).

Then for any divisor $D = mD_1 - D'$ as above, the natural maps on hypercohomology

$$\pi^* : \mathbb{H}^i\left(\overline{Y}, \pi_*\left\{\Omega^\bullet_{Y/\mathbb{C}}(\log(D_1 + D_2))(D)\right\}\right) \to \mathbb{H}^i\left(Y, \Omega^\bullet_{Y/\mathbb{C}}(\log(D_1 + D_2))(D)\right)$$

are surjective, for all $i \geq 0$.

Proof. By Serre's GAGA, it suffices to prove a similar assertion for the corresponding analytic hypercohomology groups, since these are naturally isomorphic to the algebraic ones. We let

$$\mathcal{C}^\bullet = \Omega^\bullet_{Y_{an}}(\log(D_1 + D_2))(D).$$

Let $j : Y_{an} - D_2 \hookrightarrow Y_{an}$. From lemma 4.3, the map

$$j_! j^*(\mathcal{C}^\bullet) \to \mathcal{C}^\bullet$$

is a quasi-isomorphism. On the other hand, if $\overline{j} : \overline{Y}_{an} - S \hookrightarrow \overline{Y}_{an}$, then the canonical map

$$\mathbb{H}^i(\overline{Y}_{an}, \overline{j}_! \overline{j}^* \pi_*(\mathcal{C}^\bullet)) \to \mathbb{H}^i(Y_{an}, j_! j^*(\mathcal{C}^\bullet))$$

is an isomorphism, since $\overline{Y} - S \cong Y - D_2$.

Now consider the commutative diagram

$$\begin{array}{ccc}
\mathbb{H}^i(Y_{an}, j_! j^*(\mathcal{C}^\bullet)) & \xrightarrow{\cong} & \mathbb{H}^i(Y_{an}, \mathcal{C}^\bullet) \\
\cong\uparrow & & \uparrow \pi^* \\
\mathbb{H}^i(\overline{Y}_{an}, \overline{j}_! \overline{j}^* \pi_*(\mathcal{C}^\bullet)) & \to & \mathbb{H}^i(\overline{Y}_{an}, \pi_* \mathcal{C}^\bullet))
\end{array}$$

Since the left vertical arrow and the top horizontal arrows are isomorphisms, the right vertical arrow π^* is split, and hence surjective, as claimed. □

REMARK. By our earlier analysis of the cohomology sheaves of the complexes involved, the isomorphism

$$\mathbb{H}^i(\overline{Y}, \overline{j}_! \overline{j}^* \pi_*(\mathcal{C}^\bullet)) \to \mathbb{H}^i(Y, j_! j^*(\mathcal{C}^\bullet))$$

is in fact just an expression in terms of de Rham cohomology of the isomorphism $H^i(\overline{Y} - \pi(D_1), S - \pi(D_1); \mathbb{Z}) \cong H^i(Y - D_1, D_2 - D_1; \mathbb{Z})$ on singular cohomology, which is valid because $(Y - D_1, D_2 - D_1) \to (\overline{Y} - \pi(D_1), S - \pi(D_1))$ is a relative homeomorphism.

5. Surjectivity of the Cartier operator. We now return again to our situation (arising from the reduced, purely d-dimensional local ring) as in §3.

We may reformulate our desired vanishing/surjectivity statement again, as follows. Let D_A be an effective divisor which is the pull-back to W_A of an ample Cartier divisor on V_A, such that if $\mathcal{L}_A$ is the corresponding invertible sheaf on V_A, with restriction $\mathcal{L}_s$ to V_s, then $R^j\pi_*\omega_{W_s} \otimes \mathcal{L}_s^{\otimes n}$ is generated by global sections and has vanishing cohomology in positive degrees, for all $j \geq 0$, for all $s \in T$, for all $n \geq 1$. This implies that $H^j(W_s, \omega_{W_s}(D_s)) \cong H^0(V_s, (R^j\pi_*\omega_{W_s}) \otimes \mathcal{L}_s)$ for all $j \geq 0$. It thus suffices to prove that for a non-empty Zariski open set of $s \in T$, we have

(i) $H^j(W_s, \omega_{W_s}(D_s)) = 0$ for $j > 0$, and
(ii) the natural map of locally free sheaves $C \otimes 1 : (F_*\omega_{W_s}) \otimes \mathcal{O}_{W_s}(D_s) \to \omega_{W_s} \otimes \mathcal{O}_{W_s}(D_s)$ induces a surjection on global sections.

We may assume further (after localizing A) that (i) each irreducible component of D_A is smooth and projective over $T = \operatorname{Spec} A$, with connected fibres (ii) if E_A denotes the exceptional set of $\pi_A : W_A \to V_A$ (*i.e.*, the inverse image in W_A of the singular locus of $V_A \to T$), then each irreducible component of E_A is smooth and projective over T with connected fibres (iii) for each $s \in T$, the divisor $E_s + D_s$ in W_s has simple normal crossings. This follows from the corresponding assertions for the generic fiber W, which can be deduced from the Bertini theorem for base-point free linear systems ([Ha2], III, Cor. 10.9) applied to the pull-back to W of a very ample linear system on V, and to the restriction of this linear system to intersections of components of E.

In the following theorem, we will relax slightly the condition that $\pi : W \to V$ is a resolution of singularities. We assume instead that
(i) W is non-singular, and $\pi : W \to V$ is birational and proper
(ii) there is a subvariety $S \subset V$ such that $E = \pi^{-1}(S)_{red} \subset W$ is a divisor wth normal crossings, and $\pi^{-1}(V - S) \to V - S$ is an isomorphism
(iii) there is an effective Cartier divisor D on W, pulled back from V, such that $D+E$ is a reduced divisor with normal crossings, and $R^j\pi_*\omega_W(nD)$ is generated by global sections and has vanishing higher cohomology, for each $n \geq 1$, $j \geq 0$.

Then as before we may 'spread out' all of the above data over $T = \operatorname{Spec} A$.

We prove below the Grauert-Riemenschneider vanishing theorem, that $R^i\pi_*\omega_W = 0$ for $i > 0$, for morphisms π as above. As will be shown later, this easily implies the general case *i.e.*, for an arbitrary birational proper morphism from a smooth variety (see Theorem 7.4).

THEOREM 5.1. *With the above hypotheses (i)-(iii), there is a dense Zariski open subset of closed points $s \in T$ such that*
(i) (Grauert-Riemenschneider) $R^i\pi_*\omega_{W_s} = 0$ for all $i > 0$, and
(ii) the Cartier operator

$$C : F_*\pi_*\omega_{W_s} \to \pi_*\omega_{W_s}$$

is surjective.

Proof. In this proof, let k denote the quotient field of A. First note that we can find $n > 0$ such that if W, V, D, E are the generic fibres of $W_A \to T$, $V_A \to T$, etc., and $\pi : W \to V$ is the induced map, then

$$H^i(V, \pi_*\Omega^a_{W/k}(\log(D+E))(nD-E)) = 0 \ \forall\, i > 0,\ \forall a \geq 0.$$

Indeed,

$$\pi_*\Omega^a_{W/k}(\log(D+E))(nD-E) \cong \left(\pi_*\Omega^a_{Y/k}(\log(D+E))(-E)\right) \otimes_{\mathcal{O}_V} \mathcal{L}^{\otimes n},$$

where $\mathcal{L}$ is the invertible sheaf on V such that $\pi^*\mathcal{L} = \mathcal{O}_W(D)$. Since $\mathcal{L}$ is ample on V, the desired vanishing result is a consequence of Serre Vanishing.

This implies (from the first spectral sequence for hypercohomology) that

$$\mathbb{H}^i(V, \pi_*\Omega^\bullet_{W/k}(\log(D+E))(nD-E)) = 0 \,\forall\, i > d,$$

and the natural map

$$H^0(V, \pi_*\Omega^d_{W/k}(\log(D+E))(nD-E)) \to \mathbb{H}^d(V, \pi_*\Omega^\bullet_{W/k}(\log(D+E))(nD-E))$$

is surjective.

Now by embedding k into $\mathbb{C}$ and making the faithfully flat base change to $\mathbb{C}$, corollary 4.4 implies that

(i) $\mathbb{H}^i(W, \Omega^\bullet_{W/k}(\log(D+E))(nD-E)) = 0$ for $i > d$

(ii) $H^0(W, \Omega^d_{W/k}(\log(D+E))(nD-E)) \to \mathbb{H}^d(W, \Omega^\bullet_{W/k}(\log(D+E))(nD-E))$ is surjective.

By semicontinuity, it follows that both points are valid for W_s and

$$\Omega^\bullet_{W_s/k(s)}(\log(D_s+E_s))(nD_s-E_s)$$

for all s in some non-empty Zariski open subset of T. From corollary 4.2, it follows that similar claims hold for W_s and

$$\Omega^\bullet_{W_s/k(s)}(\log(D_s+E_s)((p-1)D_s-E_s),$$

if in addition $k(s)$ has characteristic $p > n$. From results of Deligne-Illusie [DI], combined with lemma 4.2, the natural maps induced by the Cartier operator on d-forms

$$\mathbb{H}^i\left(W_s, F_*\left\{\Omega^\bullet_{W_s/k(s)}(\log(D_s+E_s))((p-1)D_s-E_s)\right\}\right) \\ \to H^{i-d}(W_s, \Omega^d_{W_s/k(s)}(\log(D_s+E_s))(-E_s))$$

are (naturally split) *surjections*, since in fact the complex

$$F_*\left\{\Omega^\bullet_{W_s/k(s)}(\log(D_s+E_s))((p-1)D_s-E_s)\right\}$$

decomposes into the direct sum of its cohomology objects (shifted), in the derived category.

But $\Omega^d_{W_s/k(s)}(\log(D_s+E_s))(-E_s)$ is just $\omega_{W_s}(D_s)$. We conclude that

(i) $H^{i-d}(W_s, \omega_{W_s}(D_s)) = 0$ for $i > d$, and

(ii) the composite

$$H^0(W_s, (F_*\omega_{W_s})(D_s)) \to \\ \mathbb{H}^d\left(W_s, F_*\left\{\Omega^\bullet_{W_s/k(s)}(\log(D_s+E_s))((p-1)D_s-E_s)\right\}\right) \to H^0(W_s, \omega_{W_s}(D_s))$$

is surjective.

The first point is just the Grauert-Riemenschneider Vanishing theorem (in the special case), while the second point is the desired surjectivity of the Cartier operator. □

6. Resolved F-rational type. We continue with the study of the morphism $W_A \to V_A$ of $T = \operatorname{Spec} A$-schemes, arising from a reduced purely d-dimensional local ring, as in §3.

We now further study the action of the Cartier operator on $\Omega^d_{W_s/k(s)}$ and its twists by suitable divisors; this leads to a proof that normal singularities in characteristic 0 have resolved F-rational type. The main point of the argument is a slight modification of that of the previous section, where we now make a different choice of a divisor supported on $D+E$.

Let $\mathcal{I}$ be the ideal sheaf of the singular locus of V_A, and let $\mathcal{J} = \pi^{-1}\mathcal{I}\cdot\mathcal{O}_{W_A}$ be the ideal sheaf on W_A defined by $\mathcal{I}$.

LEMMA 6.1.

(i) *Let $f : Y \to X$ be a proper morphism between Noetherian schemes, $\mathcal{F}$ a coherent sheaf on Y, $\mathcal{I} \subset \mathcal{O}_X$ a coherent ideal sheaf on X, and $\mathcal{J} = f^{-1}\mathcal{I}\cdot\mathcal{O}_Y$ the inverse image ideal sheaf on Y. Then there exists $n_0 \geq 0$ such that for all $n \geq n_0$, we have an 'Artin-Rees formula'*

$$f_*(\mathcal{J}^n\mathcal{F}) = \mathcal{I}^{n-n_0} f_*(\mathcal{J}^{n_0}\mathcal{F}).$$

(ii) *In our situation above, there exists $n > 0$ such that*

$$\pi_*(\mathcal{J}^n\Omega^d_{W_A/A}) \subset \mathcal{I}(\pi_*\Omega^d_{W_A/A}).$$

Proof. Of course (ii) follows immediately from (i), which in turn is (presumably) 'well-known to experts'; we give the short proof below.

Since X is Noetherian, it is quasi-compact, and so (i) is local on the base (take the 'global' n_0 to be the maximum of a finite number of 'local' ones). So we may assume without loss of generality that X is affine. Now we have the following situation: $Y \to \operatorname{Spec} A$ is proper, $\mathcal{F}$ is coherent on Y, $I \subset A$ is an ideal, and we want to prove an Artin-Rees formula

$$H^0(Y, I^n\mathcal{F}) = I^{n-n_0}H^0(Y, I^{n_0}\mathcal{F}).$$

Let $Z = \operatorname{Proj}(\oplus_{n\geq 0}I^n)$ be the blow up of I, $g : Z \to X = \operatorname{Spec} A$ the structure map. Let $W = \mathbf{Proj}\oplus_{n\geq 0} I^n\mathcal{O}_Y$ be the blow up of the inverse image of I on Y, and $h : W \to Y$ the structure map (see [Ha2], II, §7). Then there is an obvious morphism $k : W \to Z$ giving a commutative diagram

$$\begin{array}{ccc} W & \xrightarrow{k} & Z \\ h\downarrow & & \downarrow g \\ Y & \xrightarrow{f} & X = \operatorname{Spec} A \end{array}$$

Further, if $\mathcal{O}_Z(1)$ and $\mathcal{O}_W(1)$ are the tautological ample invertible sheaves given by $\mathcal{O}_Z(1) = I\mathcal{O}_Z$, and $\mathcal{O}_W(1) = h^{-1}(I\mathcal{O}_Y)\mathcal{O}_W = I\mathcal{O}_W$, then evidently $k^*\mathcal{O}_Z(1) = \mathcal{O}_W(1)$.

There is a correspondence between coherent $\mathcal{O}_W$-modules and certain quasi-coherent graded sheaves of modules over $\oplus_{n\geq 0}I^n\mathcal{O}_Y$. Let $\widetilde{\mathcal{F}}$ be the $\mathcal{O}_W$-module corresponding to the 'Rees sheaf' $\oplus_{n\geq 0}I^n\mathcal{F}$. Then there are natural maps $I^n\mathcal{F} \to h_*\left(\widetilde{\mathcal{F}}(n)\right)$, which are isomorphisms for all large n.

From the diagram, it is immediate that since f and h are proper, the composite $W \to X$ is proper, and so k is proper (see [Ha2], II, Cor. 4.8 (e), for example). Then $k_*\widetilde{\mathcal{F}}$ is a coherent sheaf on Z, by Grothendieck's theorem on coherence of direct images (if f is projective, then so is k, and then this is [Ha2], III, Theorem 5.2). Hence the graded module $\oplus_{n\geq 0}\Gamma\left(Z, \left(k_*\widetilde{\mathcal{F}}\right)(n)\right)$ is finitely generated over the graded (Rees) ring $\oplus_{n\geq 0}I^n$ (see for example [Ha2], II, Ex. 5.9). But

$$\Gamma\left(Z, \left(k_*\widetilde{\mathcal{F}}\right)(n)\right) = \Gamma(W, \widetilde{\mathcal{F}}(n)) = \Gamma(Y, h_*\left(\widetilde{\mathcal{F}}(n)\right)),$$

since $k^*\mathcal{O}_Z(1) = \mathcal{O}_W(1)$. Hence

$$\oplus_{n\geq 0}\Gamma(Y, h_*\left(\widetilde{\mathcal{F}}(n)\right))$$

is a finitely generated graded $\oplus_{n\geq 0}I^n$ module. Since the map of graded $\oplus_{n\geq 0}I^n$-modules

$$\oplus_{n\geq 0}\Gamma(Y, I^n\mathcal{F}) \to \oplus_{n\geq 0}\Gamma\left(Y, h_*\left(\widetilde{\mathcal{F}}(n)\right)\right)$$

is an isomorphism in all large degrees, we deduce that

$$\oplus_{n\geq 0}\Gamma(Y, I^n\mathcal{F})$$

is a finitely generated graded $\oplus_{n\geq 0}I^n$-module. If this graded module is generated by elements of degree $\leq n_0$, then it is evident that for $n \geq n_0$, we have

$$\Gamma(Y, I^n\mathcal{F}) = I^{n-n_0}\Gamma(Y, I^{n_0}\mathcal{F}).$$

$\square$

COROLLARY 6.2. *In the above situation, for all $s \in \operatorname{Spec} A$ in a dense Zariski open subset,*

$$\pi_*(\mathcal{J}_s^n\Omega^d_{W_s/k(s)}) \subset \mathcal{I}_s(\pi_*\Omega^d_{W_s/k(s)}),$$

where $\mathcal{I}_s \subset \mathcal{O}_{V_s}$, $\mathcal{J}_s \subset \mathcal{O}_{W_s}$ are the ideal sheaves determined by $\mathcal{I}$ and $\mathcal{J}$ respectively.

PROPOSITION 6.3. *There is a dense Zariski open set of closed points $s \in \operatorname{Spec} A$ such that for each $N \geq 0$, the iterated Cartier operator map*

$$C^{N+1} : F_*^{N+1}\left(\mathcal{I}^{[p^N]}\pi_*\omega_{W_s}\right) \to \pi_*\omega_{W_s}$$

is surjective. (Here F^{N+1} is the $N+1$-fold iterate of the Frobenius mapping. The symbol F_*^{N+1} is put in to make the sheaf mapping $\mathcal{O}_{V_s}$-linear; recall that this is governed by the identity $C(f^p\omega) = fC(\omega)$. Finally, $\mathcal{I}^{[p^N]}$ is the ideal sheaf locally generated by p^N-th powers of local sections of $\mathcal{I}$; clearly $\mathcal{I}^{[p^N]} \subset \mathcal{I}^{p^N}$.)

Proof of the proposition. We will show the existence of a dense open set of $s \in T$ such that

$$C : F_*\left(\mathcal{I}\pi_*\omega_{W_s}\right) \to \pi_*\omega_{W_s}$$

is surjective. For such an s, of course

$$C : F_*\pi_*\omega_{W_s} \to \pi_*\omega_{W_s}$$

is surjective as well, since $\mathcal{I}\pi_*\omega_{W_s}$ is a subsheaf of $\pi_*\omega_{W_s}$. We then claim that the iterated Cartier operator

$$C^{N+1} : F_*^{N+1}\left(\mathcal{I}^{p^N}\pi_*\omega_{W_s}\right) \to \pi_*\omega_{W_s}$$

is automatically surjective. Indeed, since $C(f^p\omega) = fC(\omega)$, it follows that

$$C^N(F_*^{N+1}(\mathcal{I}^{[p^N]}\pi_*\omega_{W_s})) = F_*\left(\mathcal{I}C^N(F_*^N\pi_*\omega_{W_s})\right) = F_*(\mathcal{I}\pi_*\omega_{W_s}).$$

Applying C once more, we obtain the claimed surjectivity.

Hence, we are reduced to proving that

$$C : F_* \left(\mathcal{I}\pi_*\omega_{W_s}\right) \to \pi_*\omega_{W_s}$$

is surjective, for all closed points s in a dense open subset of T. This is a stronger version of our earlier surjectivity, with an additional factor on the left of the ideal sheaf $\mathcal{I}$.

We first choose an effective divisor D'_A supported on E_A such that

$$\mathcal{O}_{W_A}(-D'_A) \subset \mathcal{J}.$$

This can be done since the subscheme determined by $\mathcal{J}$ has E_A as its reduced subscheme (by definition, E_A is the reduced inverse image of the singular locus of V_A). Next, choose the effective divisor D_A on W_A (as before) to be the pull-back of an ample Cartier divisor on V_A, such that if $\mathcal{L}_A$ is the corresponding ample invertible sheaf on V_A, with restriction $\mathcal{L}_s$ to V_s, then $(R^j\pi_*\omega_{W_s})\otimes\mathcal{L}_s$ as well as $(R^j\pi_*\omega_{W_s}(-D'_s))\otimes\mathcal{L}_s$ are generated by global sections and have vanishing cohomology in positive degrees, for all $j \geq 0$, for all $s \in T$.

Now choose $n > 0$ such that if D' is the divisor determined by D'_A on the generic fibre W, then

$$H^i(V, R^j\pi_*\Omega^a_{W/k}(\log(D+E))(nD - D')) = 0 \;\; \forall\, i > 0,\ \forall a \geq 0.$$

As in the earlier situation, this can be done because of Serre Vanishing: indeed,

$$R^j\pi_*\left\{\Omega^a_{W/k}(\log(D+E))(nD - D')\right\} \cong \left(R^j\pi_*\Omega^a_{Y/k}(\log(D+E))(-D')\right) \otimes_{\mathcal{O}_V} \mathcal{L}^{\otimes n},$$

where $\mathcal{L}$ is the ample invertible sheaf on $\mathcal{V}$ determined by $\mathcal{L}_A$.

This implies, as before, that

$$\mathbb{H}^i(V, \pi_*\Omega^\bullet_{W/k}(\log(D+E))(nD - D')) = 0 \;\forall\, i > d,$$

and the natural map

$$H^0(V, \pi_*\Omega^d_{W/k}(\log(D+E))(nD - D')) \to \mathbb{H}^d(V, \pi_*\Omega^a_{W/k}(\log(D+E))(nD - D'))$$

is surjective. Making the base change to $\mathbb{C}$, corollary 4.4 implies that

(i) $\mathbb{H}^i(W, \Omega^\bullet_{W/k}(\log(D+E))(nD - D')) = 0$ for $i > d$

(ii) $H^0(W, \Omega^d_{W/k}(\log(D+E)(nD - D')) \to \mathbb{H}^d(W, \Omega^\bullet_{W/k}(\log(D+E))(nD - D'))$ is surjective.

By semicontinuity, it again follows that both points are valid for W_s and

$$\Omega^\bullet_{W_s/k(s)}(\log(D_s + E_s))(nD_s - D'_s)$$

for all s in some non-empty Zariski open subset of T. Now lemma 4.1 implies that similar claims hold for W_s and

$$\Omega^\bullet_{W_s/k(s)}(\log(D_s + E_s)((p-1)D_s - D'_s),$$

if $k(s)$ has characteristic $p > n$.

From lemma 4.1, we see further that if each component in D'_s occurs with a coefficient $\leq p$, then the d-th cohomology sheaf of

$$\Omega^\bullet_{W_s/k(s)}(\log(D_s + E_s))((p-1)D_s - D'_s)$$

is $\Omega^d_{W_s/k(s)}(D_s)$. Hence, using the results of Deligne and Illusie, we get that for s in a Zariski dense open subset of T,

(i) $H^{i-d}(W_s, \omega_{W_s}(D_s)) = 0$ for $i > d$, and
(ii) the composite

$$H^0(W_s, (F_*\omega_{W_s}(-D'_s))\,(D_s)) \to \\ \mathbb{H}^d(W_s, F_*\{\Omega^{\bullet}_{W_s/k(s)}(\log(D_s + E_s))((p-1)D_s - D'_s)\}) \to H^0(W_s, \omega_{W_s}(D_s))$$

is surjective.

The second point implies that the sheaf map, determined by the Cartier operator,

$$C : F_*\left(\pi_*\omega_{W_s}(-D'_s)\right) \to \pi_*\omega_{W_s}$$

is surjective. Corollary 6.2, together with our choice of D', implies that

$$C : F_*\left(\mathcal{I} \cdot \pi_*\omega_{W_s}\right) \to \pi_*\omega_{W_s}$$

is surjective, as desired. □

We now deduce some consequences. We first need a simple lemma. We use the following (more or less standard) terminology — if Y is a purely d-dimensional variety over a perfect field k, then a point $x \in Y$ (not necessarily closed) will be called *non-singular* if $\Omega^1_{\mathcal{O}_{x,Y}/k}$ is free of rank d.

LEMMA 6.4.

(i) *Let $U = \operatorname{Spec} B$ be a non-singular (irreducible) affine variety over a perfect field k of characteristic p. Then $\Gamma(U, \omega_U)$ has no proper B-submodule stable under the Cartier operator C.*

(ii) *Let Y be a variety of pure dimension d over a perfect field k of characteristic p, and $x \in Y$ a non-singular point (not necessarily closed). Then the dualizing module $\omega_{\mathcal{O}_{x,Y}}$ has no proper $\mathcal{O}_{x,Y}$-submodules stable under C.*

(iii) *In (ii), suppose $x \in Y$ is a point (possibly singular), $f \in \mathcal{O}_{x,Y}$ such that $\mathcal{O}_{x,Y}[1/f]$ is regular. Let $S = \widehat{\mathcal{O}}_{x,Y}[1/f]$. Then S is regular, and the dualizing module $\omega_S = \omega_{\mathcal{O}_{x,Y}} \otimes S$ has no proper S-submodules stable under the completed Cartier operator (also denoted C).*

Proof. We first consider (i), *i.e.*, the case of the non-singular variety $U = \operatorname{Spec} B$. A B-submodule of $\Gamma(U, \omega_U)$ corresponds to a coherent subsheaf of ω_U. So it suffices to prove that ω_U has no coherent subsheaf stable under C. By considering stalks, we are reduced to the assertion in (ii). Taking completions, (ii) follows from (iii) with $f = 1$.

In (iii), it is clear that S is regular, since $\mathcal{O}_{x,Y}[1/f]$ is regular, and $\mathcal{O}_{x,Y}$ has regular formal fibres (it is essentially of finite type over a field, hence excellent). Thus $\omega_S = \omega_{\mathcal{O}_{x,Y}} \otimes S$ is a projective S-module of rank 1. To show that ω_S has no proper S-submodule stable under C, it suffices to do it for all localizations of S at prime ideals (for example, because such a proper submodule would correspond to a coherent subsheaf of $\widetilde{\omega_S}$ on $\operatorname{Spec} S$, which means, by considering stalks, that there is a proper submodule after localization — or else, instead of using sheaf language, one can work with associated primes of the quotient module, to get the same conclusion).

Let $\mathbf{p} \in \operatorname{Spec} S \subset \operatorname{Spec} \widehat{\mathcal{O}}_{x,Y}$, and $\mathbf{q} \in \operatorname{Spec} \mathcal{O}_{x,Y}$ its image. Then

$$S_{\mathbf{p}} = (\widehat{\mathcal{O}}_{x,Y})_{\mathbf{p}}$$

is a localisation of

$$(\mathcal{O}_{x,Y})_{\mathbf{q}} \otimes \widehat{\mathcal{O}}_{x,Y},$$

and

$$\omega_{S_{\mathbf{p}}} = (\omega_{\widehat{\mathcal{O}}_{x,Y}})_{\mathbf{p}}$$

is a localization of

$$(\omega_{\mathcal{O}_{x,Y}})_{\mathbf{q}} \otimes \widehat{\mathcal{O}}_{x,Y}.$$

To simplify notation, let $\mathcal{O}$ denote $\mathcal{O}_{x,Y}$, and $\widehat{\mathcal{O}}$ its completion. By our choice of S, the rings $\widehat{\mathcal{O}}_{\mathbf{p}}$ and $\mathcal{O}_{\mathbf{q}}$ are regular. Now $\omega_{\mathcal{O}_{\mathbf{q}}} = \Omega^d_{\mathcal{O}_{\mathbf{q}}/k}$, where $\Omega^1_{\mathcal{O}_{\mathbf{q}}/k}$ denotes the module of Kahler differentials, which is a free module of rank d, and $\Omega^d_{\mathcal{O}_{\mathbf{q}}/k}$ is its d-th exterior power; here $d = \dim Y$.

Since $\mathcal{O}_{\mathbf{q}}$ is the local ring of a smooth (possibly non-closed) point of Y, $\mathcal{O}_{\mathbf{q}}$ is a free $\mathcal{O}^p_{\mathbf{q}}$-module of rank p^d. Further, one can find a p-basis $x_1, \ldots, x_d \in \mathcal{O}_{\mathbf{q}}$, so that the monomials $x_1^{i_1} \cdots x_d^{i_d}$, with $0 \le i_j \le p-1$, form a basis for $\mathcal{O}_{\mathbf{q}}$ over $\mathcal{O}^p_{\mathbf{q}}$. Then $\Omega^d_{\mathcal{O}_{\mathbf{q}}}$ is free of rank 1 with a basis element $\omega = dx_1 \wedge \cdots \wedge dx_d$. The Cartier operator is then given by the formula

$$C\left(\left\{\sum_{0 \le i_1, \ldots, i_d \le p-1} a^p_{i_1,\ldots,i_d} x_1^{i_1} \cdots x_d^{i_d}\right\}\omega\right) = a_{p-1,\ldots,p-1}\omega.$$

Since $\widehat{\mathcal{O}}_{\mathbf{p}}$ is obtained from $\mathcal{O}_{\mathbf{q}}$ by tensoring with $\widehat{\mathcal{O}}$ and localizing, we see that $\widehat{\mathcal{O}}_{\mathbf{p}}$ is étale over $\mathcal{O}_{\mathbf{q}}$, and $x_1, \ldots, x_d$ is also a p-basis for $\widehat{\mathcal{O}}_{\mathbf{p}}$; further, $\omega_{\widehat{\mathcal{O}}_{\mathbf{p}}}$ is also a free module of rank 1 with basis ω, and the completed Cartier operator is given by the same formula as above, where now the $a_{i_1,\ldots,i_d}$ are elements of $\widehat{\mathcal{O}}_{\mathbf{p}}$.

It is now easy to show that $\omega_{\widehat{\mathcal{O}}_{\mathbf{p}}}$ has no proper $\widehat{\mathcal{O}}_{\mathbf{p}}$-submodules stable under C — the computation is very similar to the case of a power series ring over a perfect field. By induction, we see that the monomials $x_1^{i_1} \cdots x_d^{i_d}$ with $0 \le i_1, \ldots, i_d \le p^n - 1$ form a basis for $\widehat{\mathcal{O}}_{\mathbf{p}}$ over its subring of p^n-th powers $\widehat{\mathcal{O}}^{p^n}_{\mathbf{p}}$. Given a non-zero element

$$\eta \in \omega_{\widehat{\mathcal{O}}_{\mathbf{p}}},$$

we can then uniquely write

$$\eta = \left(\sum_{0 \le i_1, \ldots, i_d \le p^n - 1} a^{p^n}_{i_1,\ldots,i_d} x_1^{i_1} \cdots x_d^{i_d}\right)\omega,$$

with $a_{i_1,\ldots,i_d} \in \widehat{\mathcal{O}}_{\mathbf{p}}$. If each of the $a_{i_1,\ldots,i_d}$ is a non-unit, then $\eta = a \cdot \omega$ with a lying in the p^n-th power of the maximal ideal $\mathbf{p}\widehat{\mathcal{O}}_{\mathbf{p}}$. Hence, by the Krull intersection theorem, we see that for large enough n, the above expression for η has a unit coefficient $a_{i_1,\ldots,i_d}$ for some $(i_1, \ldots, i_d)$. Multiplying η by $x_1^{p^n-1-i_1} \cdots x_d^{p^n-1-i_d}$, we obtain a similar element η' such that the coefficient of $(x_1 \cdots x_d)^{p^n-1}$ is the p^n-th power of a unit u. Then clearly $C^n(\eta') = u\,\omega$, which generates $\omega_{\widehat{\mathcal{O}}_{\mathbf{p}}}$ as an $\widehat{\mathcal{O}}_{\mathbf{p}}$-module. Thus, the smallest $\widehat{\mathcal{O}}_{\mathbf{p}}$-submodule of $\omega_{\widehat{\mathcal{O}}_{\mathbf{p}}}$ containing η, and stable under C, is $\omega_{\widehat{\mathcal{O}}_{\mathbf{p}}}$ itself. □

THEOREM 6.5. *Let $W_A \to V_A$ be a morphism of $T = \operatorname{Spec} A$-schemes, arising as above from a normal d-dimensional local ring $(R, \mathbf{m})$ over a field k of characteristic 0 (thus V_s is normal for all $s \in T$). Let $s \in T$ be a closed point so that the conclusion of Proposition 6.3 holds for s.*

(a) *Let $U \subset V_s$ be an affine open subset, with coordinate ring $\mathcal{O}(U)$. Then $\Gamma(\pi^{-1}(U), \omega_{W_s})$ has no proper $\mathcal{O}(U)$-submodules which are stable under the Cartier operator C.*

(b) *Let $x \in V_s$ be any point. Then the $\mathcal{O}_{x,V_s}$-module $(\pi_*\omega_{W_s})_x$ has no proper $\mathcal{O}_{x,V_s}$-submodules stable under C.*

(c) *In the situation of (b), the $\widehat{\mathcal{O}}_{x,V_s}$-module*

$$(\pi_*\omega_{W_s})_x^\wedge = (\pi_*\omega_{W_s})_x \otimes \widehat{\mathcal{O}}_{x,V_s}$$

has no proper $\widehat{\mathcal{O}}_{x,V_s}$-submodule stable under C.

Proof. If $U \subset V_s - (V_s)_{sing}$ is contained in the non-singular locus of V_s, then (a) reduces to (i) of the preceeding lemma. More generally, if $I \subset \mathcal{O}(U)$ is the ideal of the singular locus of $V_s \cap U$, then by (i) of the preceeding lemma, we have that for any $f \in I$,

$$\Gamma(\pi^{-1}(U), \omega_{W_s}) \otimes \mathcal{O}(U)[1/f] = \omega_{\mathcal{O}(U)[1/f]}$$

has no proper $\mathcal{O}(U)[1/f]$-submodule stable under C. Since V_s is normal, $\mathcal{O}(U)$ is an integral domain, and $\omega_{\mathcal{O}(U)}$ is a torsion-free $\mathcal{O}(U)$-module, and for any non-zero $f \in \mathcal{O}(U)$, the map $\omega_{\mathcal{O}(U)} \to \omega_{\mathcal{O}(U)}[1/f]$ is injective . Hence if

$$M \subset \Gamma(\pi^{-1}(U), \omega_{W_s})$$

is a non-zero $\mathcal{O}(U)$-submodule stable under C, then for every non-zero $f \in I$, we have $M[1/f] \neq 0$, and so must have

$$M[1/f] = \Gamma(\pi^{-1}(U), \omega_{W_s})[1/f] = \omega_{\mathcal{O}(U)[1/f]}.$$

Hence for some $r > 0$, we have that

$$I^r \cdot \Gamma(\pi^{-1}(U), \omega_{W_s}) \subset M.$$

Increasing r if necessary, we may assume $r = p^N$. Now applying C^{N+1}, we get that

$$\Gamma(\pi^{-1}(U), \omega_{W_s}) \subset M,$$

by Proposition 6.3, and using that U is affine (to get an assertion about modules and an ideal, instead of sheaves). This means that M is not a proper submodule.

The proofs of (b) and (c) are very similar, using (ii) and (iii) of the preceeding lemma, and the surjectivity of the iterated Cartier operator

$$C^{N+1} : F_*^{N+1}\left(\mathcal{I}^{[p^N]}\pi_*\omega_{W_s}\right) \to \pi_*\omega_{W_s}$$

on stalks at x, and on the completions of the stalks at x. The normality of V_s implies that $\widehat{\mathcal{O}}_{x,V_s}$ is a domain, whose dualizing module is torsion-free. □

REMARK. In fact, instead of normality, one can make do with the weaker hypothesis of analytic irreducibility in the above result.

We now prove Theorem 1.2 in the following form.

COROLLARY 6.6. *For s as given by Proposition 6.3, the resolved local cohomology $H^d_{Z_s}(W_s, \mathcal{O}_{W_s})$ has no proper $\mathcal{O}_{s,V_s}$-submodules stable under the Frobenius mapping.*

Proof. Under the pairing of Formal Duality Theorem 3.1, the orthogonal of a proper F-stable $\mathcal{O}_{s,V_s}$-submodule of the resolved local cohomology $H^d_{Z_s}(W_s, \mathcal{O}_{W_s})$ is a C-stable proper $\widehat{\mathcal{O}}_{s,V_s}$-submodule of $(\pi_*\omega_{W_s})^\wedge$. But we have seen in theorem 6.5(c) that there are no such proper submodules. □

7. Some further remarks. In this section, we make some further remarks on the results obtained. First, we prove Theorem 1.1, characterizing rational singularities, and also consider the graded case. Next, we discuss some characterizations of the tight closure of 0 in $H^d_{\overline{\mathbf{m}}}(\overline{R})$, where $(\overline{R}, \overline{\mathbf{m}})$ is obtained by "reduction modulo $p >> 0$" from a reduced local ring $(R, \mathbf{m})$ which is essentially of finite type over a field of characteristic 0. Finally, we indicate how the results obtained in this paper have an application to invariant theory in positive characteristic; this was partly the motivation of the first author in considering these questions.

We first prove a lemma, which allows us to deduce Theorem 1.1 from our earlier results, and also has an application to graded rings.

Let $(R, \mathbf{m})$ be a local ring essentially of finite type over a field k of characteristic 0, and $\pi : X \to \operatorname{Spec} R$ a proper morphism, such that (i) for a dense open subset V of $\operatorname{Spec} R$, $\pi^{-1}(V) \to V$ is an isomorphism, and (ii) X has rational singularities. For example, if R has a rational singularity, we may take π to be the identity map. Let $Z = \pi^{-1}(\mathbf{m})$ be the fibre over the closed point of $\operatorname{Spec} R$. Let $\overline{X}$, $\overline{Z}$ be obtained from X, Z by 'reductions modulo a prime' $\overline{\mathbf{n}} \in \operatorname{Spec} A$ (as explained in §2).

LEMMA 7.1. *Under the above conditions, for a dense set of* $\overline{\mathbf{n}} \in \operatorname{Spec} A$, *the local cohomology modules* $H^i_{\overline{Z}}(\overline{X}, \mathcal{O}_{\overline{X}})$ *coincide with the resolved local cohomology modules. In particular, if R has a rational singularity, then the resolved local cohomology coincides with the local cohomology for 'almost all reductions modulo p' of R.*

Proof. By results of Hironaka, we can find a resolution of singularities $\psi : Y \to \operatorname{Spec} R$ which dominates X, by taking $f : Y \to X$ to be a resolution of singularities, and $\psi = \pi \circ f$. Let $T = f^{-1}(Z) = \psi^{-1}(\mathbf{m})$. Since X has rational singularities, $f_*\mathcal{O}_Y = \mathcal{O}_X$, and $R^i f_* \mathcal{O}_Y = 0$ for $i > 0$.

After 'spreading out' over $\operatorname{Spec} A$ as in Section 2, one may then form $\overline{f} : \overline{Y} \to \overline{X}$ and $\overline{T}$ over $\overline{\mathbf{n}}$ as well. By semicontinuity, over a dense open subset of $\operatorname{Spec} A$, we will have $\overline{f}_*\mathcal{O}_{\overline{Y}} = \mathcal{O}_{\overline{X}}$, and $R^i\overline{f}_*\mathcal{O}_{\overline{Y}} = 0$ for $i > 0$.

There is a Leray spectral sequence with supports (a particular case of Grothendieck's spectral sequence for the derived functors of a composition)

$$E_2^{p,q} = H^p_{\overline{Z}}(\overline{X}, R^q f_* \mathcal{O}_{\overline{Y}}) \Rightarrow H^{p+q}_{\overline{T}}(\overline{Y}, \mathcal{O}_{\overline{Y}}).$$

This degenerates at E_2, since $E_2^{p,q} = 0$ for $q > 0$. We deduce that the natural maps

$$H^i_{\overline{Z}}(\overline{X}, \mathcal{O}_{\overline{X}}) \to H^i_{\overline{T}}(\overline{Y}, \mathcal{O}_{\overline{Y}})$$

are isomorphisms. □

Proof of Theorem 1.1. We note that if $(R, \mathbf{m})$ has a rational singularity, then Theorem 1.1 is a consequence of lemma 7.1 combined with Theorem 5.1 and Corollary 6.6. □

PROPOSITION 7.2. *Let* $R = \oplus_{n \geq 0} R_n$ *be a (Noetherian) normal graded ring of dimension d, with* $R_0 = k$, *a field of characteristic 0, such that if* R_+ *is the 'irrelevant' maximal ideal, then* $\operatorname{Spec} R - \{R_+\}$ *has rational singularities. Then for 'almost all reductions modulo primes p', the i-th resolved local cohomology of* $\overline{R}$ *(with respect to the reduction* $\overline{R}_+$ *of* R_+*) is naturally isomorphic to*

$$\frac{H^i_{\overline{R}_+}(\overline{R})}{\textit{graded submodule generated by elements of degrees} \geq 0}.$$

Proof. This goes back to the paper [W] of Watanabe, where a construction of Grothendieck is studied, which is the analogue for a general graded ring R of the blow-up of the vertex of the affine cone over a projective variety. Let $Y = \operatorname{Proj} R$. Let $U = \operatorname{Spec} R - \{R_+\}$, where R_+ is the maximal ideal of positively graded elements.

Consider the graded ring

$$R^\natural = \oplus_{n\geq 0} R_n^\natural,$$

$$R_n^\natural = \oplus_{m\geq n} R_m,$$

where the multiplication is induced by that in R. Let $X = \operatorname{Proj} R^\natural$. There is a natural inclusion $R \to R^\natural$ as the subring of elements of degree 0, inducing a morphism $\pi : X \to \operatorname{Spec} R$.

As shown by Watanabe in [W], under the above hypotheses, $\pi : X \to \operatorname{Spec} R$ is projective and birational, π is an isomorphism over U, and X has only rational singularities. Hence from lemma 7.1, we may use 'reductions modulo primes p' of X to compute the resolved local cohomology of reductions of R.

There is also a graded homomorphism $R \to R^\natural$ induced by the natural inclusions $R_n \hookrightarrow R_n^\natural$, inducing a morphism $f : X \to \operatorname{Proj} R = Y$. Further, there is a graded ring homomorphism $R^\natural \to R$ given by the obvious projections $R_n^\natural \to R_n$; thus the morphism f has a section $s : Y \to X$, and we let $Z = s(Y)$. If $U^\natural = X - Z$, then π induces an isomorphism $U^\natural \to U$, and $\pi(Z) = \{R_+\}$ consists of the closed point corresponding to the 'irrelevant' maximal ideal R_+.

Since R is normal, its local cohomology vanishes in degrees < 2. For $i \geq 2$, the local cohomology $H^i_{R_+}(R)$ is identified with $H^{i-1}(U, \mathcal{O}_U)$, *i.e.*, with $H^{i-1}(U^\natural, \mathcal{O}_{U^\natural})$. There is a long exact sequence

$$\cdots \to H^{i-1}(X, \mathcal{O}_X) \to H^{i-1}(U^\natural, \mathcal{O}_{U^\natural}) \to H^i_Z(X, \mathcal{O}_X) \\ \to H^i(X, \mathcal{O}_X) \to H^i(U^\natural, \mathcal{O}_{U^\natural}) \to \cdots$$

Now f is affine, as is $f\,|_{U^\natural}$. Further,

$$f_*\mathcal{O}_X = \oplus_{n\geq 0}\mathcal{O}_Y(n), \quad (f\,|_{U^\natural})_*\mathcal{O}_{U^\natural} = \oplus_{n\in\mathbb{Z}}\mathcal{O}_Y(n),$$

where $\mathcal{O}_Y(n)$ is the sheaf on $Y = \operatorname{Proj} R$ associated to the graded R-module $R(n)$ (obtained by shifting the grading; note that in general, $\mathcal{O}_Y(n)$ need not be invertible, and we need not have $\mathcal{O}_Y(n) \otimes \mathcal{O}_Y(m) = \mathcal{O}_Y(m+n)$.) Hence for each $j \geq 0$,

$$H^j(X, \mathcal{O}_X) \cong \oplus_{n\geq 0} H^j(Y, \mathcal{O}_Y(n)),$$

$$H^j(U^\natural, \mathcal{O}_{U^\natural}) \cong \oplus_{n\in\mathbb{Z}} H^j(Y, \mathcal{O}_Y(n)),$$

and the natural map

$$H^j(X, \mathcal{O}_X) \to H^j(U^\natural, \mathcal{O}_{U^\natural})$$

is the obvious graded inclusion.

This implies that

$$H^i_{R_+}(R) \cong \oplus_{n\in\mathbb{Z}} H^{i-1}(Y, \mathcal{O}_Y(n)),$$

and in fact this is a graded isomorphism. From the above long exact sequence, we then identify

$$H^i_Z(X, \mathcal{O}_X) \cong \oplus_{n<0} H^{i-1}(Y, \mathcal{O}_Y(n))$$

$$\cong \frac{H^i_{R_+}(R)}{\text{graded submodule generated by elements of degrees} \geq 0}.$$

Finally, we remark that if we 'spread out' R to a flat graded A-algebra R_A, then we may similarly 'spread out' $R^\natural$ to a flat graded A-algebra $R^\natural_A$, construct $X_A = \operatorname{Proj} R^\natural_A$, etc. A calculation very similar to the above one then yields the desired expression for resolved local cohomology of the reduction modulo a maximal ideal $\mathbf{n} \in \operatorname{Spec} A$. □

Next, we discuss two characterizations of the tight closure of the submodule (0) of the d-th local cohomology, for 'almost all reductions modulo p' of a reduced local ring in characteristic 0. Let $(R, \mathbf{m})$ be a reduced local ring of dimension d which is essentially of finite type over a field k of characteristic 0. Let R_A be obtained by 'spreading out' R over a finitely generated $\mathbb{Z}$-subalgebra $A \subset k$, as before, and for maximal ideals $\mathbf{n} \in \operatorname{Spec} A$, let $(\overline{R}, \overline{\mathbf{n}})$ be one of the finitely many corresponding local rings over $A/\mathbf{n}$. Assume further that there is a resolution of singularities $X \to \operatorname{Spec} R$, with fibre Z over $\mathbf{m}$, which has also been 'spread out' to $X_A \to \operatorname{Spec} R_A$, $Z_A \to \operatorname{Spec} A$, etc. with corresponding resolutions $\overline{\pi} : \overline{X} \to \operatorname{Spec} \overline{R}$, and $\overline{Z} = \pi^{-1}(\overline{\mathbf{n}})$.

Let $\omega_{\overline{R}}$ be the dualizing module, *i.e.*, the $\overline{R}$-module, unique up to isomorphism, such that $H^d_{\overline{\mathbf{n}}}(M)$ is Matlis dual to $\operatorname{Hom}_{\overline{R}}(M, \omega_{\overline{R}})$ for any finite $\overline{R}$-module M. We may equivalently define $\omega_{\overline{R}}$ to be the 0-th cohomology of any dualizing complex of $\overline{R}$. We have then a well-defined submodule $H^0(\overline{X}, \omega_{\overline{X}}) \subset \omega_{\overline{R}}$. One sees easily that this submodule is independent of the resolution of singularities chosen (this *does not* need a version of Hironaka's results in characteristic p; one can instead reason as in [Ha2], II, Theorem 8.19).

PROPOSITION 7.3. *There is a dense Zariski open subset of maximal ideals* $\mathbf{n} \in \operatorname{Spec} A$ *such that for any* $(\overline{R}, \overline{\mathbf{n}})$ *as above, we have*

tight closure of (0) *in* $H^d_{\overline{\mathbf{n}}}(\overline{R}) =$

$$\ker \left(H^d_{\overline{\mathbf{n}}}(\overline{R}) \to H^d_{\overline{Z}}(\overline{X}, \mathcal{O}_{\overline{X}}) \right)$$

$=$ *perpendicular of* $H^0(\overline{X}, \omega_{\overline{X}})$ *with respect to the Matlis duality pairing*

between ω_R *and* $H^d_{\overline{\mathbf{n}}}(\overline{R})$.

Proof. From the thesis of K. Smith, one knows that the tight closure of (0) in $H^d_{\overline{\mathbf{n}}}(\overline{R})$ is the largest $\overline{R}$-submodule which is stable under the Frobenius mapping, and which maps to a proper submodule of the local cohomology modulo any d-dimensional minimal prime of $\overline{R}$ (Smith's thesis considers the case when $\overline{R}$ is a normal domain, but simple modifications of her reasoning yield this extension).

Since the natural map $H^d_{\overline{\mathbf{n}}}(\overline{R}) \to H^d_{\overline{Z}}(\overline{X}, \mathcal{O}_{\overline{X}})$ is a surjection of $\overline{R}$-modules compatible with the action of Frobenius, the kernel of this mapping is clearly contained in the tight closure of (0) (the condition on the minimal primes is obviously satisfied, since the d-th resolved local cohomology of any d-dimensional quotient of $\overline{R}$ is always non-zero, for example by Formal Duality). So the first equality in the proposition is equivalent to the claim that (0) *is tightly closed in the resolved local cohomology* $H^d_{\overline{Z}}(\overline{X}, \mathcal{O}_{\overline{X}})$.

First consider the case when $\overline{R}$ is normal. The tight closure of (0) is a strictly smaller submodule of $H^d_{\overline{Z}}(\overline{X}, \mathcal{O}_{\overline{X}})$. Then corollary 6.6 implies that it must in fact be

(0). In general, one first reduces to the case when R is purely of dimension d; then one applies (6.4)(iii), as in the proof of (6.5(c). We leave the details to the interested reader.

The equivalence of the two descriptions given of the tight closure follows using formal duality 3.1, and its compatibilty with Matlis duality (the intrepid reader can deduce this compatibility from [Ha3], VII, Prop. 3.5, pg. 386). □

We now discuss the proof of the Grauert-Riemenschneider theorem in the general case. We have already seen that if $\pi : W \to V$ is a birational proper morphism in characteristic 0, then $R^i\pi_*\omega_W = 0$ for $i > 0$, provided (i) V is projective, and (ii) there exists $S \subset V$ with $\pi^{-1}(S)$ a divisor with normal crossings, and $W - \pi^{-1}(S) \to V - S$ is an isomorphism. The general result is an easy deduction from this, given the results of Hironaka.

THEOREM 7.4. *Let $f : X \to Y$ be a birational proper morphism of varieties over a field k of characteristic 0, such that X is non-singular. Then $R^i f_*\omega_X = 0$ for $i > 0$.*

Proof. The result is local on Y. Hence we may assume without loss of generality that Y is projective (first replace the given Y by an affine neighbourhood U of a chosen point, then choose a projective compactification of the neighbourhood, then replace X by a resolution of singularities of some compactification of $f^{-1}(U)$). Next, let $S \subset Y$ be the locus outside which f is an isomorphism. Let $T = f^{-1}(S)$. By Hironaka's results, we can find an embedded resolution of singularities of T, *i.e.*, a composition of blow-ups at smooth centres $h : Z \to X$ such that $h^{-1}(T)$ is a divisor with normal crossings in Z.

Now by the case of the Grauert-Riemenschneider already proved, we know that $R^i(f \circ h)_*\omega_Z = 0$ for $i > 0$. For any invertible sheaf $\mathcal{L}$ on Y, we have a Leray spectral sequence

$$E_2^{p,q} = H^p(Y, (R^q(f \circ h)_*\omega_Z) \otimes_{\mathcal{O}_Y} \mathcal{L}^N) \Rightarrow H^{p+q}(Z, \omega_Z \otimes_{\mathcal{O}_Z} (f \circ h)^*\mathcal{L}^N).$$

By the Serre vanishing theorem on Y, this implies that for any ample invertible sheaf $\mathcal{L}$ on Y, we have $H^i(Z, \omega_Z \otimes \mathcal{L}^N) = 0$ for all $i > 0$, for all large N. By Serre duality on Z, this gives $H^i(Z, (f \circ h)^*\mathcal{L}^{-N}) = 0$ for all $i < \dim Z$, for all large N. Since h is a composition of blow-ups at smooth centres, $R^ih_*\mathcal{O}_Z = 0$ for $i > 0$, and $h_*\mathcal{O}_Z = \mathcal{O}_X$. Hence we deduce (from the Leray spectral sequence for h) that $H^i(X, f^*\mathcal{L}^{-N}) \cong H^i(Z, h^*f^*\mathcal{L}^{-N}) = 0$ for all $i < \dim X = \dim Z$, for all large N. By Serre duality on X, this gives $H^i(X, \omega_X \otimes f^*\mathcal{L}^N) = 0$ for $i > 0$ for all large N, and hence (by the Leray spectral sequence for f) the vanishing of $R^if_*\omega_X$ for $i > 0$. □

REMARK. We also see easily that for any proper birational morphism $f : X \to Y$ of non-singular varieties, we have $R^if_*\mathcal{O}_X = 0$ for $i > 0$. This is again local on Y, so we may assume Y is projective. If $\mathcal{L}$ is ample on Y, then it suffices to prove $H^i(X, f^*\mathcal{L}^N) = 0$ for all $i > 0$ and all large N. By Serre duality on X and Grauert-Riemenschneider vanishing for f, we need to equivalently prove that $H^i(Y, (f_*\omega_X) \otimes \mathcal{L}^{-N}) = 0$ for all $i < \dim Y = \dim X$, and all large N. But $f_*\omega_X = \omega_Y$ as X, Y are non-singular (there are maps $\omega_Y \to f_*\omega_X$ and $f_*\omega_X \to \omega_Y$, given by pulling back forms, and trace, which are inverse isomorphisms). Hence by Serre duality on Y, we reduce to showing that $H^i(Y, \mathcal{L}^N) = 0$ for $i > 0$ and large N, which is true since $\mathcal{L}$ is ample.

We end this section with the application to invariant theory mentioned earlier. Let G be a semi-simple almost simple simply connected algebraic group of one of the types A, B, C, D and let $G_{\mathbb{Z}} \to \operatorname{Spec}\mathbb{Z}$ be the Chevalley group scheme over $\mathbb{Z}$ of the same type. Let $\mathfrak{g}$ be the Lie algebra of G, and $\mathfrak{g}_{\mathbb{Z}}$ the corresponding Lie algebra scheme over $\operatorname{Spec}\mathbb{Z}$. We are interested in the $G_{\mathbb{Z}}$-representation

$$V_{\mathbb{Z}} := \mathfrak{g}_{\mathbb{Z}}^{\oplus n}, \ \ n > 0.$$

Denote the ring of polynomial functions on $V_{\mathbb{Z}}$ by $R_{\mathbb{Z}}$, so that

$$R_{\mathbb{Z}} = \operatorname{Sym}_{\mathbb{Z}}(V_{\mathbb{Z}}^{\vee})$$

is the symmetric algebra on the dual module (here $A^{\vee}$ denotes $\operatorname{Hom}_{\mathbb{Z}}(A, \mathbb{Z})$).We will denote by a subscript $\mathbb{Q}$ or p the result of tensoring by $\mathbb{Q}$ or $\mathbb{Z}/p\mathbb{Z}$. We are interested in comparing the rings of invariants

$$R_{\mathbb{Z}}^{G_{\mathbb{Z}}} \otimes \mathbb{Z}/p\mathbb{Z}, \quad R_p^{G_p}, \quad R_{\mathbb{Q}}^{G_{\mathbb{Q}}}.$$

We quote a result implicit in [Zu].

LEMMA 7.5. *For $p > 2$, the natural inclusion*

$$R_{\mathbb{Z}}^{G_{\mathbb{Z}}} \otimes \mathbb{Z}/p\mathbb{Z} \to R_p^{G_p}$$

is an isomorphism.

Proof. This is an immediate consequence of [Zu], where it is proved that $R_{\mathbb{Q}}^{G_{\mathbb{Q}}}$ and $R_p^{G_p}$ have the same Poincaré series, if $p > 2$. □

As $R_{\mathbb{Q}}^{G_{\mathbb{Q}}}$ has rational singularities, $R_p^{G_p}$ also has rational singularities for large enough p. Further, $R_p^{G_p}$ is F-injective and F-unstable for all large p, by Theorem 1.1, *i.e.*, the Frobenius acts injectively, and without non-zero fixed points, on the local cohomology modules of $R_p^{G_p}$ at any point. Hence $R_p^{G_p}$ is Gorenstein, F-injective and F-unstable, for large p. It follows easily now that $R_p^{G_p}$ is F-regular, in the sense of Hochster and Huneke, for all large p. A partial result was first proved for $G = \mathrm{SL}\,(3)$ and all primes $p > 3$ in [MRa], using the methods of geometric invariant theory and Frobenius splitting.

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