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THE DIFFERENTIAL EQUATION $\triangle u = 8\pi - 8\pi h e^u$ ON A COMPACT RIEMANN SURFACE*

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Abstract. Let M be a compact Riemann surface, $h(x)$ a positive smooth function on M. In this paper, we consider the functional

$$J(u) = \frac{1}{2}\int_M |\bigtriangledown u|^2 + 8\pi \int_M u - 8\pi \log \int_M h e^u.$$

We give a sufficient condition under which J achieves its minimum.

1. Introduction and main result. Let (M, ds^2) be a compact Riemann surface, $h(x)$ a smooth function on M. For simplicity, we assume in this paper that the volume of M equals 1. Twenty years ago, Kazdan and Warner ([KW]) asked, under what kind of conditions on h, the equation

$$\triangle u = 8\pi - 8\pi h e^u \tag{1.1}$$

has a solution. An obvious necessary condition is that $\max h > 0$.

If M is the standard sphere, the problem is called "Nirenberg problem". The geometric significance of this problem is that if g denotes a metric of constant curvature 4π on S^2, then the metric $e^u g$ has curvature equal to h. This problem has been studied by Moser ([M1], [M2]), Kazdan-Warner ([KW]), Hong ([H]), Chen-Ding ([CD1], [CD2]), Chang-Yang ([CY1], [CY2]), Chang-Liu ([CL]), and others.

For a compact Riemann surface other than S^2 or $\mathbb{RP}^2$, the preceding interpretation is no longer possible as such a surface does not carry a background metric of constant positive curvature. However, the differential equation (1.1) also arises in the so-called Chern-Simons Higgs theory. This is a classical field theory that is defined on (2+1) dimensional Minkowski space and believed to be relevant in high temperature superconductivity and in other areas of theoretical physics. Hong-Kim-Pac [HKP] and Jackiw-Weinberger [JW] observed that for a special choice of the Higgs potential, a sixth order polynomial, stationary vortex solutions satisfy certain first order selfduality equations. On a compact torus, these equations have been studied by Caffarelli-Yang [CaY] and Tarantello [T]. In particular, Tarantello showed that one may find a certain type of solution that corresponds to a symmetric vacuum. In the case of only one vortex p of multiplicity 1 she found that asymptotically, as the coupling parameter in the theory tends to zero, one obtains a solution of

$$\triangle u(x) = 4\pi - 4\pi \frac{e^{-G(x,p)+u(x)}}{\int_M e^{-G(y,p)+u(y)}dy}, \qquad \int_M u = 0, \qquad u \in L_1^2(M),$$

where $G(x,p)$ is the Green function defined below in equation (1.2). This result was shown with the help of the Moser-Trudinger inequality. For N vortices (counted with

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multiplicity), 4π in the preceding equation has to be replaced by $4\pi N$, and already for $N = 2$, the situation becomes more difficult, as the factor 8π represents the limiting case in the Moser-Trudinger inequality. We shall discuss this in detail in [DJLW].

Thus, the Kazdan-Warner problem becomes relevant for an area quite different from problems of prescribed Gauss curvature. Therefore, we shall address the problem of finding solutions (1.1) here on a general compact Riemann surface. We shall pursue a variational approach. Namely, we shall try to minimize the functional

$$J(u) = \frac{1}{2}\int_M |\nabla u|^2 + 8\pi \int_M u - 8\pi \log \int_M he^u.$$

We shall first show that the functional has a lower bound. This generalizes the Moser-Trudinger ([M1]) inequality to the case where M is an arbitrary compact Riemann surface.

THEOREM 1.1. *Let (M, ds^2) be a compact Riemann surface. For any $u \in L_1^2(M)$ with $\int_M u = 0$ one has*

$$\int_M e^u \leq C_M e^{\frac{1}{16\pi}\|\nabla u\|_2^2},$$

where C_M is a positive constant depending only on (M, ds^2).

To show that J is bounded from below, we consider

$$J_\epsilon(u) = \frac{1}{2}\int_M |\nabla u|^2 + (8\pi - \epsilon)\int_M u - (8\pi - \epsilon)\log \int_M he^u,$$

where $\epsilon > 0$. It is not hard to verify that J_ϵ achieves its minimum at some u_ϵ.

There are two possibilities: If a subsequence of the sequence of minimizers u_ϵ converges to some u_0 for $\epsilon \to 0$, then u_0 minimizes J. In order to show this convergence, it suffices to establish estimates for the u_ϵ in the Sobolev space $L_1^2(M)$ that do not depend on ϵ. If such estimates do not hold, then the sequence u_ϵ blows up, and after subtracting mean values, u_ϵ converges to some Green function $G(x, p)$ satisfying

$$\begin{cases} \triangle G = 8\pi - 8\pi\delta_p, \\ \int_M G = 0. \end{cases} \tag{1.2}$$

In a normal coordinate system around p we assume that

$$\begin{aligned} G(x,p) = &-4\log r + A(p) + b_1 x_1 + b_2 x_2 \\ &+ c_1 x_1^2 + 2c_2 x_1 x_2 + c_3 x_2^2 + O(r^3), \end{aligned} \tag{1.3}$$

where $r(x) = dist(x, p)$.

One should note that (1.2) is not conformally invariant, but depends on the metric ds^2 on M. Therefore, also the constants in the expansion (1.3) will depend on that metric. If the metric is homogeneous as on the standard sphere or on a flat torus, $b_1 = b_2 = 0$. For a more detailed discussion of the leading term $A(p)$ - which does not depend on p in the homogeneous case - on flat tori see section 4.

More precisely, in this step we show that, if the minimizing sequence u_ϵ of J_ϵ blows up,

$$\inf_{u \in L_1^2(M)} J(u) \geq -8\pi - 8\pi\log\pi - 4\pi(\max_{p\in M}(A(p) + 2\log h(p))). \tag{1.4}$$

In other words, if (1.4) does not hold, then no blow-up is possible, and we get convergence of the u_ϵ to a minimizer u_0 of J.

Inequality (1.4) and the results that have been obtained for the Nirenberg problem ([CD1], [CY1], [CY2], [CL]) indicate that it will depend on the asymptotic expansion of h near a potential blow-up point whether a blow-up is possible. In this sense, we shall obtain the following result.

THEOREM 1.2. *Let (M, ds^2) be a compact Riemann surface, let $K(x)$ be its Gauss curvature. Let $h(x)$ be a positive smooth function on M. Suppose that $A(p)+2\log h(p)$ achieves its maximum at p_0. Let $b_1(p_0)$ and $b_2(p_0)$ be the constants in the expression (1.3), and write $\nabla h(p_0) = (k_1(p_0), k_2(p_0))$ in the normal coordinate system. If*

$$\begin{aligned}&\Delta h(p_0) + 2(b_1(p_0)k_1(p_0) + b_2(p_0)k_2(p_0))\\ &> -(8\pi + (b_1^2(p_0) + b_2^2(p_0)) - 2K(p_0))h(p_0)\end{aligned}$$

the minimum of the functional J can be obtained, and consequently the equation (1.1) has a smooth solution.

REMARK 1.1. *The inequality in Theorem 1.2 is implied by the following one*

$$\Delta \log h(p_0) = \frac{\Delta h(p_0)}{h(p_0)} - \frac{|\nabla h(p_0)|^2}{h^2(p_0)} > -(8\pi - 2K(p_0)).$$

In the second step, we shall construct a blowing up sequence ϕ_ϵ with the property that

$$J(\phi_\epsilon) < -8\pi - 8\pi\log\pi - 4\pi(\max_{p\in M}(A(p) + 2\log h(p)))$$

for sufficiently small $\epsilon > 0$, assuming that h satisfies the hypotheses in Theorem 1.2. This contradicts (1.4), and Theorem 1.2 will follow.

Our methods are closely related to those used by Schoen ([Sc]) in his solution of the Yamabe problem and by Escobar-Schoen ([E-S]) for finding conformal metrics with prescribed curvatures in higher dimensions. However, our analysis is more delicate. In their work, they need only to compare the minimum of the corresponding functional to the minimum on the standard sphere. That is because their problems are conformally invariant. In our case, we have to compute the limit functional value of a blowing up minimizing sequence very carefully, and it turns out that the limit is not unique, it depends on the geometry of the surface (Theorem 1.2). On the other hand, while in their work to establish the existence result they need only the constant term in the expansion of the Green function of the conformal Laplacian to be positive (the positive mass theorem), in our case we need to consider a higher order term in the expansion of the usual Green function.

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2. The lower bound. In this section, we shall show that the functional $J(u)$ is bounded from below, and consequently, we shall prove the Moser-Trudinger inequality.

We shall consider the minimum of the functional J in the space $H_1 = \{ u \in L_1^2(M) \mid \int_M he^u = 1 \}$.

PROPOSITION 2.1. *Let M be a compact Riemann surface. Let $h(x)$ be a positive smooth function on M. Then there exists a positive constant C depending only on M and h such that*

$$\inf_{u \in H_1} J(u) \geq -C.$$

The following lemma will yield our proposition.

LEMMA 2.2. *There exists a positive constant C depending only on M and h, but not on ϵ, such that*

$$\inf_{u \in H_1} J_\epsilon(u) \geq -C.$$

Set

$$\Lambda_\epsilon = \inf_{u \in H_1} J_\epsilon(u).$$

Using Aubin's inequality (see [A]) one obtains $u_\epsilon \in H_1$ satisfying

$$J_\epsilon(u_\epsilon) = \Lambda_\epsilon$$

and

$$\Delta u_\epsilon = (8\pi - \epsilon) - (8\pi - \epsilon) h e^{u_\epsilon}. \tag{2.1}$$

If $u_\epsilon \to u_0$ in $L_1^2(M)$ as $\epsilon \to 0$, the lemma follows. And Theorem 1.2 also follows. Therefore we shall assume in the sequel that u_ϵ does not converge in $L_1^2(M)$. However we have

LEMMA 2.3. *For any $1 < q < 2$, $\| \nabla u_\epsilon \|_q \leq C_q$.*

Proof. Let $q' = \frac{q}{q-1} > 2$. Then

$$\| \nabla u_\epsilon \|_q \leq \sup\{ \, | \int_M \nabla u_\epsilon \cdot \nabla \varphi | \mid \varphi \in L_1^{q'}(M), \int_M \varphi = 0, \|\varphi\|_{L_1^{q'}(M)} = 1 \, \}.$$

By the Sobolev embedding theorem we have

$$\|\varphi\|_{L^\infty(M)} \leq C.$$

It is clear that

$$| \int_M \nabla u_\epsilon \cdot \nabla \varphi | = | \int_M \Delta u_\epsilon \varphi | \leq C.$$

This proves the lemma. □

Let $\overline{u}_\epsilon = \int_M u_\epsilon$. We set $\lambda_\epsilon = \max_{x \in M} u_\epsilon(x)$, assume that $u_\epsilon(x_\epsilon) = \lambda_\epsilon$ and that $x_\epsilon \to p$. We shall show

LEMMA 2.4. $\lambda_\epsilon \to \infty$ *as* $\epsilon \to 0$.

Proof. If λ_ϵ did not tend to ∞, e^{u_ϵ} would be bounded above (At least there would exist a subsequence u_{ϵ_k} such that $e^{u_{\epsilon_k}}$ is bounded. For simplicity, in this paper we do not distinguish this point.). We set $v_\epsilon = u_\epsilon - \overline{u}_\epsilon$. By Lemma 2.3 we have $\|v_\epsilon\|_p \leq C_p$ for any $p > 1$. Since $| \Delta v_\epsilon | \leq C$, by the elliptic estimates we can see that v_ϵ is bounded in $C^k(M)$. So, if $\overline{u}_\epsilon$ is bounded, then $\|u_\epsilon\|_{L^\infty(M)} \leq C$, which contradicts the assumption that u_ϵ blows up. But if $\overline{u}_\epsilon \to -\infty$, then v_ϵ converges to a smooth solution of the equation $\Delta v = 8\pi$, which is impossible. This proves the lemma. □

We choose a local normal coordinate system around p. Let $(\lambda_\epsilon^*)^2 = e^{\lambda_\epsilon}$, and

$$\varphi_\epsilon(x) = u_\epsilon(x_\epsilon + \frac{x}{\lambda_\epsilon^*}) - \lambda_\epsilon.$$

We shall show

LEMMA 2.5. *(i) For any $\Omega \subset\subset M \setminus \{p\}$, we have $\int_\Omega he^{u_\epsilon} \to 0$ as $\epsilon \to 0$. And $\overline{u}_\epsilon \to -\infty$ as $\epsilon \to 0$. (ii) For any $\Omega \subset\subset R^2$, we have $\varphi_\epsilon(x) \to \varphi_0(x)$ in $C^\infty(\Omega)$ as $\epsilon \to 0$, where $\varphi_0(x) = -2\log(1 + \pi h(p)|x|^2)$.*

Proof. For any $R > 0$, we have

$$\begin{aligned}\Delta\varphi_\epsilon &= \frac{1}{(\lambda_\epsilon^*)^2}(8\pi - \epsilon) - (8\pi - \epsilon)\frac{h(x_\epsilon + \frac{x}{\lambda_\epsilon^*})}{(\lambda_\epsilon^*)^2}e^{\varphi_\epsilon(x)+\lambda_\epsilon} \\ &= \frac{1}{(\lambda_\epsilon^*)^2}(8\pi - \epsilon) - (8\pi - \epsilon)h(x_\epsilon + \frac{x}{\lambda_\epsilon^*})e^{\varphi_\epsilon(x)}\end{aligned}$$

in $B_R(0) \subset R^2$ for $\epsilon > 0$ sufficiently small.

We consider the equation

$$\begin{cases} \Delta\varphi_\epsilon^1 = \frac{1}{(\lambda_\epsilon^*)^2}(8\pi - \epsilon) - (8\pi - \epsilon)h(x_\epsilon + \frac{x}{\lambda_\epsilon^*})e^{\varphi_\epsilon(x)} & x \in B_R(0), \\ \varphi_\epsilon^1|_{\partial B_R(0)} = 0 \end{cases}$$

Let $\varphi_\epsilon^2 = \varphi_\epsilon - \varphi_\epsilon^1$. Then $\Delta\varphi_\epsilon^2 = 0$ in $B_R(0)$. The elliptic estimates together with $L_2^p(B_R(0)) \subset C(\overline{B_R(0)})$ give $\sup_{B_R(0)} |\varphi_\epsilon^1| \leq C$. So $\sup_{B_R(0)} \varphi_\epsilon^2 \leq C$. The Harnack inequality yields that $\sup_{B_{\frac{R}{2}}(0)} |\varphi_\epsilon^2| \leq C$, because $\varphi_\epsilon^2(0)$ is bounded. Therefore $\sup_{B_{\frac{R}{2}}(0)} |\varphi_\epsilon| \leq C$.

By the elliptic estimates, we can show that $\varphi_\epsilon(x) \to \varphi_0(x)$ in $C^\infty(B_{\frac{R}{4}}(0))$. As $h(x_\epsilon + \frac{x}{\lambda_\epsilon^*}) \to h(p)$ in $C(B_R(0))$ we can see that φ_0 satisfies

$$\Delta_0\varphi_0(x) = -8\pi h(p)e^{\varphi_0},$$

$$\varphi_0(0) = 0,$$

and

$$\int_{R^2} h(p)e^{\varphi_0} \leq 1,$$

where Δ_0 is the Laplace operator on R^2.

However Ding's lemma ([D], c.f. [CL2] Lemma 1.1) yields that

$$\int_{R^2} h(p)e^{\varphi_0} = 1.$$

Since $\int_M he^{u_\epsilon} = 1$ we can see that, for any $\Omega \subset\subset M \setminus \{p\}$, we have $\int_\Omega he^{u_\epsilon} \to 0$ as $\epsilon \to 0$. By Jensen's inequality we have $\overline{u}_\epsilon \to -\infty$ as $\epsilon \to 0$. The uniqueness theorem in [CL2] implies that

$$\varphi_0(x) = -2\log(1 + \pi h(p)|x|^2).$$

This proves the lemma. □

We also need the following lemma.

LEMMA 2.6. *We have* $u_\epsilon - \overline{u}_\epsilon \to G(x,p)$ *weakly in* $L_1^q(M)$ $(1 < q < 2)$ *as* $\epsilon \to 0$, *where* G *is the Green function satisfying* (1.2), $p \in M$. *Furthermore* $u_\epsilon - \overline{u}_\epsilon \to G(x,p)$ *in* $C^\infty(\Omega)$ *for any* $\Omega \subset\subset M \setminus \{p\}$.

In order to show the lemma, we shall use a theorem proved by Brezis and Merle ([BM], Theorem 1), formulated as Lemma 2.7 below.

Let $\Omega \subset M$ be a domain. Suppose that u is a solution of the equation

$$\begin{cases} \Delta u = f(x) \\ u|_{\partial\Omega} = 0 \end{cases}$$

in Ω with $||f||_{L^1(\Omega)} < \infty$.

LEMMA 2.7. *For any* $0 < \delta < 4\pi$, *we have*

$$\int_\Omega \exp\{\frac{(4\pi - \delta)|u(x)|}{||f||_{L^1(\Omega)}}\} \leq C_\delta,$$

where $C_\delta > 0$ *is independent of* $||f||_{L^1(\Omega)}$.

Using Lemma 2.7 we can show

LEMMA 2.8. *Suppose that* $\Omega \subset M$ *is a domain. If*

$$\int_\Omega h e^{u_\epsilon} \leq (\frac{1}{2} - \delta)$$

for some $0 < \delta < \frac{1}{2}$, *then*

$$||u_\epsilon - \overline{u}_\epsilon||_{L^\infty(\Omega_0)} \leq C(\Omega_0, \Omega)$$

for any $\Omega_0 \subset\subset \Omega$.

Proof. Assume that u_ϵ^1 is a solution of the equation

$$\begin{cases} \Delta u_\epsilon^1 = -(8\pi - \epsilon) h e^{u_\epsilon}, \\ u_\epsilon^1|_{\partial\Omega} = 0. \end{cases}$$

Set $u_\epsilon^2 = u_\epsilon - u_\epsilon^1 - \overline{u}_\epsilon$, then $\Delta u_\epsilon^2 = (8\pi - \epsilon)$ in Ω. Harnack's inequality yields that

$$\begin{aligned} ||u_\epsilon^2||_{L^\infty(\Omega_1)} &\leq C(||u_\epsilon^2||_{L^1(\Omega)}) \\ &\leq C(||u_\epsilon - \overline{u}_\epsilon||_{L^1(\Omega)} + ||u_\epsilon^1||_{L^1(\Omega)}) \\ &\leq C(||\nabla u_\epsilon||_{L^q(M)} + ||u_\epsilon^1||_{L^1(\Omega)}) \end{aligned}$$

whenever $\Omega_0 \subset\subset \Omega_1 \subset\subset \Omega$.

By Lemma 2.7, one can see that $e^{|u_\epsilon^1|}$ is bounded in $L^p(\Omega)$ for some $p > 1$, which yields that

$$||u_\epsilon^1||_{L^1(\Omega)} \leq C.$$

We therefore have

$$||u_\epsilon^2||_{L^\infty(\Omega_1)} \leq C.$$

Note that

$$\begin{aligned}\int_{\Omega_1} e^{pu_\epsilon} &= \int_{\Omega_1} e^{p\overline{u}_\epsilon} e^{pu_\epsilon^2} e^{pu_\epsilon^1} \\ &\le C\int_{\Omega_1} e^{p|u_\epsilon^1|} \\ &\le C.\end{aligned}$$

By the standard elliptic estimates, we can obtain

$$\|u_\epsilon^1\|_{L^\infty(\Omega_0)} \le C.$$

This proves the lemma. □

Now we turn to the proof of Lemma 2.6.

Proof of Lemma 2.6. By Lemma 2.5 we can see that $(8\pi - \epsilon)he^{u_\epsilon}$ converges to $8\pi\delta_p$ in the sense of measures as $\epsilon \to 0$.

Therefore $u_\epsilon - \overline{u}_\epsilon \to G(x,p)$ weakly in $L_1^q(M)$ for any $1 < q < 2$, where G is the Green function satisfying (1.2), because G is the only solution of (1.2) in $L_1^q(M)$.

Lemma 2.5 and Lemma 2.8 yield that for any $\Omega \subset\subset M \setminus \{p\}$,

$$\|u_\epsilon - \overline{u}_\epsilon\|_{L^\infty(\Omega)} \le C. \tag{2.2}$$

The inequality (2.2) and the standard elliptic estimates yield that $u_\epsilon - \overline{u}_\epsilon \to G(x,p)$ in $C^\infty(\Omega)$ for any $\Omega \subset\subset M \setminus \{p\}$. This completes the proof of Lemma 2.6. □

For any $R > 0$, we set $r_\epsilon = \frac{R}{\lambda_\epsilon^*}$.

LEMMA 2.9. *In $M \setminus B_{r_\epsilon}(0)$, we have*

$$u_\epsilon \ge G - \lambda_\epsilon - 2\log(\frac{1+\pi h(p)R^2}{R^2}) - A(p) + o_\epsilon(1)$$

where $o_\epsilon(1) \to 0$ as $\epsilon \to 0$.

Proof. It is clear that we have $\triangle(u_\epsilon - G - C_\epsilon) \le 0$ for any constant C_ϵ. We choose C_ϵ such that

$$(G + C_\epsilon)|_{\partial B_{r_\epsilon}} \le u_\epsilon|_{\partial B_{r_\epsilon}}.$$

By Lemma 2.5 and (1.3) we get

$$C_\epsilon = -\lambda_\epsilon - 2\log(\frac{1+\pi h(p)R^2}{R^2}) - A(p) + o_\epsilon(1).$$

Then the lemma follows from the maximum principle. □

Now we are ready to finish the proof of Lemma 2.2.

Proof of Lemma 2.2. We let $\delta > 0$ small enough so that (1.3) holds in $B_\delta(p)$. We denote by $o_\epsilon(1)$ (resp. $o_R(1)$; $o_\delta(1)$) the terms which tend to 0 as $\epsilon \to 0$ (resp. $R \to \infty$; $\delta \to 0$).

We recall that $r_\epsilon = \frac{R}{\lambda_\epsilon^*}$ $(R > 0)$. We assume that ϵ is so small that $\delta > r_\epsilon$. We have

$$\int_M |\nabla u_\epsilon|^2 = \int_{M\setminus B_\delta(p)} |\nabla u_\epsilon|^2 + \int_{B_\delta(p)\setminus B_{r_\epsilon}(p)} |\nabla u_\epsilon|^2 + \int_{B_{r_\epsilon}(p)} |\nabla u_\epsilon|^2.$$

It is clear that

$$
\begin{aligned}
\int_{M\setminus B_\delta(p)} |\nabla u_\epsilon|^2 &= \int_{M\setminus B_\delta(p)} |\nabla (u_\epsilon - \overline{u}_\epsilon)|^2 \\
&= \int_{M\setminus B_\delta(p)} |\nabla G|^2 + o_\epsilon(1) \\
&= -\int_{\partial B_\delta} G \cdot \frac{\partial G}{\partial n} + o_\epsilon(1) + o_\delta(1)
\end{aligned}
\tag{2.3}
$$

and

$$
\begin{aligned}
\int_{B_{r_\epsilon}(p)} |\nabla u_\epsilon|^2 &= \int_{B_R(0)} |\nabla \varphi_0|^2 + o_\epsilon(1) \\
&= 16\pi \log(1 + \pi h(p) R^2) - 16\pi + o_\epsilon(1) + o_R(1),
\end{aligned}
\tag{2.4}
$$

by Lemma 2.6 and Lemma 2.5.

It remains to estimate $\int_{B_\delta(p)\setminus B_{r_\epsilon}(p)} |\nabla u_\epsilon|^2$.

Since u_ϵ satisfies (2.1), we have

$$
\begin{aligned}
\int_{B_\delta(p)\setminus B_{r_\epsilon}(p)} |\nabla u_\epsilon|^2 = &-(8\pi - \epsilon) \int_{B_\delta(p)\setminus B_{r_\epsilon}(p)} u_\epsilon \\
&+(8\pi - \epsilon) \int_{B_\delta(p)\setminus B_{r_\epsilon}(p)} h e^{u_\epsilon} u_\epsilon \\
&+ \int_{\partial B_\delta(p)} u_\epsilon \frac{\partial u_\epsilon}{\partial n} - \int_{\partial B_{r_\epsilon}(p)} u_\epsilon \frac{\partial u_\epsilon}{\partial n}.
\end{aligned}
$$

Using Lemma 2.9, we have

$$
\begin{aligned}
\int_{B_\delta(p)\setminus B_{r_\epsilon}(p)} h e^{u_\epsilon} u_\epsilon \geq &-\lambda_\epsilon \int_{B_\delta(p)\setminus B_{r_\epsilon}(p)} h e^{u_\epsilon} \\
&+ \int_{B_\delta(p)\setminus B_{r_\epsilon}(p)} h e^{u_\epsilon} G + o_\epsilon(1) + o_\delta(1).
\end{aligned}
$$

Using the equation (2.1) and the Green formula, one gets that

$$
\begin{aligned}
(8\pi - \epsilon) \int_{B_\delta(p)\setminus B_{r_\epsilon}(p)} h e^{u_\epsilon} G = &-8\pi \int_{B_\delta(p)\setminus B_{r_\epsilon}(p)} u_\epsilon - \int_{\partial B_\delta(p)} \frac{\partial u_\epsilon}{\partial n} G \\
&+ \int_{\partial B_\delta(p)} u_\epsilon \frac{\partial G}{\partial n} + \int_{\partial B_{r_\epsilon}(p)} \frac{\partial u_\epsilon}{\partial n} G \\
&- \int_{\partial B_{r_\epsilon}(p)} u_\epsilon \frac{\partial G}{\partial n} + (8\pi - \epsilon) \int_{B_\delta(p)\setminus B_{r_\epsilon}(p)} G.
\end{aligned}
$$

By Lemma 2.6 we have

$$
\begin{aligned}
(8\pi - \epsilon) \int_{B_\delta(p)\setminus B_{r_\epsilon}(p)} h e^{u_\epsilon} G = &-8\pi \int_{B_\delta(p)\setminus B_{r_\epsilon}(p)} u_\epsilon + \int_{\partial B_\delta(p)} \frac{\partial G}{\partial n} \overline{u}_\epsilon \\
&+ \int_{\partial B_{r_\epsilon}(p)} \frac{\partial u_\epsilon}{\partial n} G - \int_{\partial B_{r_\epsilon}(p)} u_\epsilon \frac{\partial G}{\partial n} \\
&+ o_\epsilon(1) + o_\delta(1).
\end{aligned}
$$

Using the equation (2.1) we also have

$$-(8\pi-\epsilon)\lambda_\epsilon\int_{B_\delta(p)\setminus B_{r_\epsilon}(p)} he^{u_\epsilon} = -(8\pi-\epsilon)(Vol(B_\delta(p)) - Vol(B_{r_\epsilon}(p)))\lambda_\epsilon - \lambda_\epsilon\int_{\partial B_{r_\epsilon}(p)}\frac{\partial u_\epsilon}{\partial n} + \lambda_\epsilon\int_{\partial B_\delta(p)}\frac{\partial u_\epsilon}{\partial n}.$$

We conclude that

$$\begin{aligned}\int_{B_\delta(p)\setminus B_{r_\epsilon}(p)} |\nabla u_\epsilon|^2 \geq & -(16\pi-\epsilon)\int_{B_\delta(p)\setminus B_{r_\epsilon}(p)} u_\epsilon + \int_{\partial B_\delta(p)} u_\epsilon\frac{\partial u_\epsilon}{\partial n}\\ & -\int_{\partial B_{r_\epsilon}(p)} u_\epsilon\frac{\partial u_\epsilon}{\partial n} + \int_{\partial B_\delta(p)}\frac{\partial G}{\partial n}\overline{u}_\epsilon\\ & +\int_{\partial B_{r_\epsilon}(p)}\frac{\partial u_\epsilon}{\partial n}G - \int_{\partial B_{r_\epsilon}(p)} u_\epsilon\frac{\partial G}{\partial n}\\ & -\lambda_\epsilon\int_{\partial B_{r_\epsilon}(p)}\frac{\partial u_\epsilon}{\partial n} + \lambda_\epsilon\int_{\partial B_\delta(p)}\frac{\partial u_\epsilon}{\partial n}\\ & -(8\pi-\epsilon)(Vol(B_\delta(p)) - Vol(B_{r_\epsilon}(p)))\lambda_\epsilon\\ & +o_\epsilon(1) + o_\delta(1).\end{aligned}$$

Applying Lemma 2.5 and Lemma 2.9 one has

$$\begin{aligned}& -\int_{\partial B_{r_\epsilon}(p)}\frac{\partial u_\epsilon}{\partial n}(u_\epsilon - (G-\lambda_\epsilon))\\ & \geq \frac{8\pi^2 h(p)R^2}{(1+\pi h(p)R^2)}(-A(p) - 2\log(\frac{1+\pi h(p)R^2}{R^2}))\\ & +o_\epsilon(1) + o_R(1).\end{aligned}$$

Using Lemma 2.5 we have

$$\begin{aligned}-\int_{\partial B_{r_\epsilon}(p)} u_\epsilon\frac{\partial G}{\partial n} &= -\lambda_\epsilon\int_{\partial B_{r_\epsilon}(p)}\frac{\partial G}{\partial n} - 16\pi\log(1+\pi h(p)R^2)\\ &\quad +o_\epsilon(1) + o_R(1)\\ &= 8\pi(1 - Vol(B_{r_\epsilon}(p)))\lambda_\epsilon - 16\pi\log(1+\pi h(p)R^2)\\ &\quad +o_\epsilon(1) + o_R(1).\end{aligned}$$

By the equation (2.1) one gets

$$\begin{aligned}\lambda_\epsilon\int_{\partial B_\delta(p)}\frac{\partial u_\epsilon}{\partial n} &= -(8\pi-\epsilon)(1 - Vol(B_\delta(p)))\lambda_\epsilon\\ &\quad +(8\pi-\epsilon)\lambda_\epsilon\int_{M\setminus B_\delta(p)} he^{u_\epsilon}\\ &\geq -(8\pi-\epsilon)(1 - Vol(B_\delta(p)))\lambda_\epsilon.\end{aligned}$$

Similarly

$$\begin{aligned}\overline{u}_\epsilon\int_{\partial B_\delta(p)}\frac{\partial u_\epsilon}{\partial n} &= -(8\pi-\epsilon)(1 - Vol(B_\delta(p)))\overline{u}_\epsilon\\ &\quad +(8\pi-\epsilon)\overline{u}_\epsilon e^{\overline{u}_\epsilon}\int_{M\setminus B_\delta(p)} he^{u_\epsilon - \overline{u}_\epsilon}\\ &= -(8\pi-\epsilon)(1 - Vol(B_\delta(p)))\overline{u}_\epsilon + o_\epsilon(1)\end{aligned}$$

and

$$\int_{\partial B_\delta(p)} \frac{\partial G}{\partial n}\overline{u}_\epsilon = -8\pi(1 - Vol(B_\delta(p)))\overline{u}_\epsilon.$$

We have

$$\begin{aligned}\int_{B_\delta(p)\setminus B_{r_\epsilon}(p)} u_\epsilon = &\int_{B_\delta(p)\setminus B_{r_\epsilon}(p)} (u_\epsilon - \overline{u}_\epsilon) \\ &+(Vol(B_\delta(p)) - Vol(B_{r_\epsilon}(p)))\overline{u}_\epsilon.\end{aligned}$$

By Lemma 2.3 we have

$$\int_{B_\delta(p)\setminus B_{r_\epsilon}(p)} u_\epsilon = (Vol(B_\delta(p)) - Vol(B_{r_\epsilon}(p)))\overline{u}_\epsilon + o_\delta(1).$$

It is clear that

$$\lambda_\epsilon Vol(B_{r_\epsilon}(p)) = o_\epsilon(1).$$

By Lemma 2.9 we also have

$$-\overline{u}_\epsilon Vol(B_{r_\epsilon}(p)) = o_\epsilon(1).$$

We therefore have

$$\begin{aligned}\int_{B_\delta(p)\setminus B_{r_\epsilon}(p)} |\nabla u_\epsilon|^2 \geq\ &\epsilon\lambda_\epsilon - (16\pi - \epsilon)\overline{u}_\epsilon - 16\pi\log(1 + \pi h(p)R^2) \\ &+ \int_{\partial B_\delta} G\cdot\frac{\partial G}{\partial n} - 8\pi A(p) - 16\pi\log\pi - 16\pi\log h(p) \\ &+o_\epsilon(1) + o_R(1) + o_\delta(1).\end{aligned} \tag{2.5}$$

It follows from (2.3), (2.4) and (2.5) that

$$\begin{aligned}\int_M |\nabla u_\epsilon|^2 \geq\ &\epsilon\lambda_\epsilon - (16\pi - \epsilon)\overline{u}_\epsilon \\ &-8\pi A(p) - 16\pi\log\pi - 16\pi - 16\pi\log h(p) \\ &+o_\epsilon(1) + o_R(1) + o_\delta(1).\end{aligned}$$

So,

$$\begin{aligned}J_\epsilon(u_\epsilon) \geq\ &\frac{\epsilon}{2}\lambda_\epsilon - \frac{\epsilon}{2}\overline{u}_\epsilon - 4\pi A(p) - 8\pi\log\pi \\ &-8\pi - 8\pi\log h(p) + o_\epsilon(1) + o_R(1) + o_\delta(1).\end{aligned}$$

Thus, we have

$$J_\epsilon(u_\epsilon) \geq -4\pi A(p) - 8\pi\log\pi - 8\pi - 8\pi\log h(p) + o_\epsilon(1) + o_R(1) + o_\delta(1).$$

Hence

$$\inf_{\epsilon>0} \Lambda_\epsilon \geq -8\pi - 8\pi\log\pi - 4\pi(\max_{p\in M}(A(p) + 2\log h(p))). \tag{2.6}$$

The lemma follows. □

Consequently, we have the following lemma which will be used in the proof of Theorem 1.2.

LEMMA 2.10. *Assume that the minimizing sequence u_ϵ of J_ϵ does not converge in $L_1^2(M)$. Then*

$$\inf_{H_1} J(u) \geq -8\pi - 8\pi\log\pi - 4\pi(\max_{p\in M}(A(p) + 2\log h(p))).$$

Proof. Otherwise, there would exist $u \in H_1$ and $\gamma > 0$ such that

$$J(u) < -8\pi - 8\pi\log\pi - 4\pi(\max_{p\in M}(A(p) + 2\log h(p))) - 2\gamma.$$

So,

$$J_\epsilon(u) < -8\pi - 8\pi\log\pi - 4\pi(\max_{p\in M}(A(p) + 2\log h(p))) - \gamma,$$

when ϵ is sufficiently small, which contradicts (2.6). □

3. Existence theorems. One can directly prove the following theorem using Lemma 2.10, because $J(0) = -8\pi\log\int_M h$.

THEOREM 3.1. *Let M be a compact Riemann surface. Let $h(x)$ be a positive smooth function on M. Suppose that*

$$\log\int_M h > (1 + \log\pi) + \frac{1}{2}\max_{p\in M}(A(p) + 2\log h(p)).$$

Then the equation (1.1) has a smooth solution.

REMARK 3.1. *If h is a positive constant, then the condition of Theorem 3.1 is satisfied precisely if*

$$\max A(p) < -2 - 2\log\pi.$$

If M is the standard sphere with volume 1, the constant A in the local expression of G (see (1.3)) is $-2 - 2\log\pi$, and so the preceding inequality does not hold. We shall see in Section 4, that it holds for some, but not for all flat tori with volume 1.

In the sequel, we shall use

PROPOSITION 3.2. *Let M be a compact Riemann surface. Let $K(p)$ be the Gauss curvature of M at p. Let $G(x,p)$ be the Green function on M satisfying* (1.2). *Let G be locally expressed by* (1.3). *Then*

$$c_1 + c_3 + \frac{2}{3}K(p) = 4\pi.$$

Proof. We denote by (r,θ) the chosen normal coordinate system around p. We write $ds^2 = dr^2 + g^2(r,\theta)d\theta^2$. It is well-known that

$$g(r,\theta) = r - \frac{K(p)}{6}r^3 + O(r^4).$$

By the divergence theorem, we have

$$\int_{\partial B_r} \frac{\partial G}{\partial n} = -8\pi(1 - Vol(B_r)).$$

So,

$$\int_0^{2\pi} (-\frac{4}{r} + 2c_1 r\cos^2\theta + 2c_3 r\sin^2\theta)(r - \frac{1}{6}K(p)r^3 + O(r^4))d\theta$$

$$= -8\pi(1 - \pi r^2 + O(r^3)).$$

Comparing the coefficients of r^2, we get

$$c_1 + c_3 + \frac{2}{3}K(p) = 4\pi.$$

This proves the proposition. □

We now turn to finish the proof of our main theorem, Theorem 1.2.

Proof of Theorem 1.2. We shall construct a blow up sequence ϕ_ϵ with

$$J(\phi_\epsilon) < -8\pi - 8\pi\log\pi - 4\pi(\max_{p\in M}(A(p) + 2\log h(p)))$$

for ϵ sufficiently small. Note that $J(u) = J(u + C)$ for any constant C. Combining the above fact and Lemma 2.10 one gets Theorem 1.2. Therefore, it only remains to construct the blow up sequence.

Suppose that $A(p) + 2\log h(p) = \max_{x\in M}(A(x) + 2\log h(x))$. Let $r = dist(x,p)$. We set

$$\omega_\epsilon = -2\log(r^2 + \epsilon),$$

$$G = -4\log r + A(p) + b_1 r\cos\theta + b_2 r\sin\theta + \beta(r,\theta),$$

where b_1 and b_2 are constants in (1.3).

$$\phi_\epsilon = \begin{cases} \omega_\epsilon + b_1 r\cos\theta + b_2 r\sin\theta + \log\epsilon, & r \le \alpha\sqrt{\epsilon}, \\ (G - \eta\beta(r,\theta)) + C_\epsilon + \log\epsilon, & \alpha\sqrt{\epsilon} \le r \le 2\alpha\sqrt{\epsilon}, \\ G + C_\epsilon + \log\epsilon, & r \ge 2\alpha\sqrt{\epsilon}. \end{cases}$$

Here $\eta \in C_0^\infty(B_{2\alpha\sqrt{\epsilon}}(p))$ is a cutoff function, $\eta = 1$ in $B_{\alpha\sqrt{\epsilon}}(p)$, $|\nabla\eta| \le \frac{C}{\alpha\sqrt{\epsilon}}$,

$$C_\epsilon = -2\log(\frac{\alpha^2+1}{\alpha^2}) - A(p)$$

and $\alpha = \alpha(\epsilon)$ will be fixed later on satisfying $\alpha \to \infty$ and $\alpha^2\epsilon \to 0$ as $\epsilon \to 0$.

By a simple calculation one has

$$\begin{aligned}\int_{B_{\alpha\sqrt{\epsilon}}} |\nabla\phi_\epsilon|^2 = {} & 16\pi\log(\alpha^2+1) - 16\pi\frac{\alpha^2}{\alpha^2+1} \\ & -\frac{16\pi}{6}K(p)\alpha^2\epsilon + \frac{32\pi}{6}K(p)\epsilon\log(\alpha^2+1) \\ & +\pi(b_1^2 + b_2^2)\alpha^2\epsilon + O(\epsilon) + O(\alpha^4\epsilon^2).\end{aligned}$$

It is clear that

$$\begin{aligned}\int_{M\setminus B_{\alpha\sqrt{\epsilon}}} |\nabla \phi_\epsilon|^2 &= \int_{M\setminus B_{\alpha\sqrt{\epsilon}}} |\nabla G|^2 + \int_{B_{2\alpha\sqrt{\epsilon}}\setminus B_{\alpha\sqrt{\epsilon}}} |\nabla (\eta\beta(r,\theta))|^2 \\ &\quad -2\int_{B_{2\alpha\sqrt{\epsilon}}\setminus B_{\alpha\sqrt{\epsilon}}} \nabla G\cdot\nabla(\eta\beta(r,\theta)) \\ &= -\int_{\partial B_{\alpha\sqrt{\epsilon}}} G\cdot\frac{\partial G}{\partial n} - 8\pi\int_{M\setminus B_{\alpha\sqrt{\epsilon}}} G \\ &\quad +2\int_{\partial B_{\alpha\sqrt{\epsilon}}} \frac{\partial G}{\partial n}(\eta\beta(r,\theta)) \\ &\quad +O(\alpha^4\epsilon^2).\end{aligned}$$

Using (1.3) one has locally

$$\begin{aligned}G(r,\theta) &= -4\log r + A(p) + b_1 r\cos\theta + b_2 r\sin\theta \\ &\quad +c_1 r^2\cos^2\theta + 2c_2 r^2\cos\theta\sin\theta + c_3 r^2\sin^2\theta + O(r^3)\end{aligned}$$

and

$$\begin{aligned}\frac{\partial G}{\partial r} &= \frac{-4}{r} + b_1\cos\theta + b_2\sin\theta \\ &\quad +2c_1 r\cos^2\theta + 4c_2 r\cos\theta\sin\theta + 2c_3 r\sin^2\theta + O(r^2).\end{aligned}$$

So,

$$\begin{aligned}-\int_{\partial B_{\alpha\sqrt{\epsilon}}} G\cdot\frac{\partial G}{\partial n} &= -16\pi\log(\alpha^2\epsilon) + 8\pi A(p) + 4\pi(c_1+c_3)\alpha^2\epsilon \\ &\quad +4\pi(c_1+c_3)\alpha^2\epsilon\log(\alpha^2\epsilon) - 2\pi A(p)(c_1+c_3)\alpha^2\epsilon \\ &\quad +\frac{16\pi}{6}K(p)\alpha^2\epsilon\log(\alpha^2\epsilon) - \frac{8\pi}{6}K(p)A(p)\alpha^2\epsilon \\ &\quad -\pi(b_1^2+b_2^2)\alpha^2\epsilon + O(\alpha^4\epsilon^2\log(\alpha^2\epsilon)).\end{aligned}$$

Similarly,

$$2\int_{\partial B_{\alpha\sqrt{\epsilon}}} \eta\beta(r,\theta)\cdot\frac{\partial G}{\partial n} = -8\pi(c_1+c_3)\alpha^2\epsilon + O(\alpha^4\epsilon^2).$$

Hence,

$$\begin{aligned}&-\int_{\partial B_{\alpha\sqrt{\epsilon}}} G\cdot\frac{\partial G}{\partial n} + 2\int_{\partial B_{\alpha\sqrt{\epsilon}}} \eta\beta(r,\theta)\cdot\frac{\partial G}{\partial n} \\ &= -16\pi\log(\alpha^2\epsilon) + 8\pi A(p) + 4\pi(c_1+c_3)\alpha^2\epsilon \\ &\quad +4\pi(c_1+c_3)\alpha^2\epsilon\log(\alpha^2\epsilon) - 2\pi A(p)(c_1+c_3)\alpha^2\epsilon \\ &\quad +\frac{16\pi}{6}K(p)\alpha^2\epsilon\log(\alpha^2\epsilon) - \frac{8\pi}{6}K(p)A(p)\alpha^2\epsilon \\ &\quad -8\pi(c_1+c_3)\alpha^2\epsilon - \pi(b_1^2+b_2^2)\alpha^2\epsilon + O(\alpha^4\epsilon^2\log(\alpha^2\epsilon)).\end{aligned}$$

Since $\int_M G = 0$, we have

$$\begin{aligned}-8\pi\int_{M\setminus B_{\alpha\sqrt{\epsilon}}} G &= 8\pi\int_{B_{\alpha\sqrt{\epsilon}}} G \\ &= -16\pi^2\alpha^2\epsilon\log(\alpha^2\epsilon) + 16\pi^2\alpha^2\epsilon \\ &\quad +8\pi A(p)\alpha^2\epsilon + O(\alpha^4\epsilon^2\log(\alpha^2\epsilon)).\end{aligned}$$

So,

$$\begin{aligned}\int_M |\nabla \phi_\epsilon|^2 = & 16\pi \log(\frac{\alpha^2+1}{\alpha^2}) - 16\pi \log \epsilon \\ & -16\pi + \frac{16\pi}{\alpha^2+1} + 8\pi A(p) \\ & +4\pi((c_1+c_3) + \frac{2}{3}K(p) - 4\pi)\alpha^2 \epsilon \log(\alpha^2 \epsilon) \\ & +4\pi(4\pi - (c_1+c_3) - \frac{2}{3}K(p))\alpha^2\epsilon \\ & +2\pi A(p)(4\pi - (c_1+c_3) - \frac{2}{3}K(p))\alpha^2\epsilon \\ & +\frac{32\pi}{6}K(p)\epsilon\log(\alpha^2+1) + O(\alpha^4\epsilon^2\log(\alpha^2\epsilon)).\end{aligned}$$

Applying Proposition 3.3, one has

$$\begin{aligned}\int_M |\nabla \phi_\epsilon|^2 = & 16\pi \log(\frac{\alpha^2+1}{\alpha^2}) - 16\pi \log \epsilon \\ & -16\pi + \frac{16\pi}{\alpha^2+1} + 8\pi A(p) \\ & +\frac{32\pi}{6}K(p)\epsilon\log(\alpha^2+1) + O(\alpha^4\epsilon^2\log(\alpha^2\epsilon)).\end{aligned}$$

Calculating directly, one has

$$\begin{aligned}\int_{B_{\alpha\sqrt{\epsilon}}} \omega_\epsilon = & -2\pi\alpha^2\epsilon\log(\alpha^2+1)\epsilon - 2\pi\epsilon\log(\alpha^2+1) \\ & +2\pi\alpha^2\epsilon + O(\alpha^4\epsilon^2\log(\alpha^2\epsilon)).\end{aligned}$$

It is also obvious that

$$\begin{aligned}\int_{M\setminus B_{\alpha\sqrt{\epsilon}}} \phi_\epsilon = & (1 - Vol(B_{\alpha\sqrt{\epsilon}}))\log\epsilon - \int_{B_{\alpha\sqrt{\epsilon}}} G \\ & +C_\epsilon(1 - Vol(B_{\alpha\sqrt{\epsilon}})) - \int_{B_{2\alpha\sqrt{\epsilon}}\setminus B_{\alpha\sqrt{\epsilon}}} \eta\beta(r,\theta) \\ = & 2\pi\alpha^2\epsilon\log(\alpha^2\epsilon) - 2\pi\alpha^2\epsilon - A(p)\pi\alpha^2\epsilon \\ & +(1 - Vol(B_{\alpha\sqrt{\epsilon}}))\log\epsilon + C_\epsilon(1 - Vol(B_{\alpha\sqrt{\epsilon}})) + O(\alpha^4\epsilon^2).\end{aligned}$$

Thus,

$$\begin{aligned}\int_M \phi_\epsilon = & \log\epsilon - 2\pi\alpha^2\epsilon\log(\frac{\alpha^2+1}{\alpha^2}) - 2\pi\epsilon\log(\alpha^2+1) \\ & -A(p) - 2\log(\frac{\alpha^2+1}{\alpha^2})(1 - Vol(B_{\alpha\sqrt{\epsilon}})) \\ & +O(\alpha^4\epsilon^2\log(\alpha^2\epsilon)).\end{aligned}$$

We have

$$\int_{B_{\alpha\sqrt{\epsilon}}} e^{\phi_\epsilon} = \epsilon \int_0^{\alpha\sqrt{\epsilon}} \frac{2\pi}{(r^2+\epsilon)^2}(r - \frac{1}{6}K(p)r^3)dr$$

$$
\begin{aligned}
&+\frac{\epsilon}{2}\int_0^{\alpha\sqrt{\epsilon}}\int_0^{2\pi}\frac{b_1^2r^2\cos^2\theta+b_2^2r^2\sin^2\theta}{(r^2+\epsilon)^2}d\theta r dr+O(\epsilon)\\
&=\pi(\frac{\alpha^2}{\alpha^2+1})-\frac{1}{6}K(p)\pi\epsilon\log(\alpha^2+1)\\
&+\frac{\pi}{4}(b_1^2+b_2^2)\epsilon\log(\alpha^2+1)+O(\epsilon).
\end{aligned}
$$

We choose $\delta>0$ sufficiently small so that G has the expression (1.3) in $B_\delta(p)$, then we have

$$
\begin{aligned}
\int_{M\backslash B_{\alpha\sqrt{\epsilon}}} e^{\phi_\epsilon} &= \epsilon\int_{M\backslash B_\delta} e^{G+C_\epsilon}+\epsilon\int_{B_\delta\backslash B_{\alpha\sqrt{\epsilon}}} e^{-4\log r+A(p)+C_\epsilon}\\
&+\epsilon\int_{B_{2\alpha\sqrt{\epsilon}}\backslash B_{\alpha\sqrt{\epsilon}}} e^{G+C_\epsilon}(e^{-\eta\beta(r,\theta)}-1)\\
&+\epsilon\int_{B_\delta\backslash B_{\alpha\sqrt{\epsilon}}} e^{-4\log r+A(p)+C_\epsilon}(e^{b_1r\cos\theta+b_2r\sin\theta+\beta(r,\theta)}-1).
\end{aligned}
$$

Calculating directly one gets

$$
\begin{aligned}
\epsilon\int_{B_\delta\backslash B_{\alpha\sqrt{\epsilon}}} e^{-4\log r+A(p)+C_\epsilon} &= \pi\frac{\alpha^2}{(\alpha^2+1)^2}\\
&+2\pi\frac{\alpha^4}{(\alpha^2+1)^2}\frac{1}{6}K(p)\epsilon\log(\alpha\sqrt{\epsilon})+O(\epsilon)
\end{aligned}
$$

and

$$
\begin{aligned}
&\epsilon\int_{B_\delta\backslash B_{\alpha\sqrt{\epsilon}}} e^{-4\log r+A(p)+C_\epsilon}(e^{b_1r\cos\theta+b_2r\sin\theta+\beta(r,\theta)}-1)\\
&=-\pi\frac{\alpha^4}{(\alpha^2+1)^2}(c_1+c_3)\epsilon\log(\alpha\sqrt{\epsilon})\\
&-\frac{\pi}{2}\frac{\alpha^4}{(\alpha^2+1)^2}(b_1^2+b_2^2)\epsilon\log(\alpha\sqrt{\epsilon})+O(\epsilon).
\end{aligned}
$$

We therefore have

$$
\begin{aligned}
\int_{M\backslash B_{\alpha\sqrt{\epsilon}}} e^{\phi_\epsilon} &= \pi\frac{\alpha^2}{(\alpha^2+1)^2}+2\pi\frac{\alpha^4}{(\alpha^2+1)^2}\frac{1}{6}K(p)\epsilon\log(\alpha\sqrt{\epsilon})\\
&-\pi\frac{\alpha^4}{(\alpha^2+1)^2}(c_1+c_3)\epsilon\log(\alpha\sqrt{\epsilon})\\
&-\frac{\pi}{2}\frac{\alpha^4}{(\alpha^2+1)^2}(b_1^2+b_2^2)\epsilon\log(\alpha\sqrt{\epsilon})+O(\epsilon).
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\int_M e^{\phi_\epsilon} &= \pi\frac{\alpha^2}{\alpha^2+1}(1+\frac{1}{\alpha^2+1}-\frac{\alpha^2+1}{\alpha^2}\frac{1}{6}K(p)\epsilon\log(\alpha^2+1)\\
&-\frac{\alpha^2}{\alpha^2+1}(c_1+c_3-\frac{1}{3}K(p))\epsilon\log(\alpha\sqrt{\epsilon})
\end{aligned}
$$

$$+\frac{1}{4}\frac{\alpha^2+1}{\alpha^2}(b_1^2+b_2^2)\epsilon\log(\alpha^2+1)$$
$$-\frac{1}{2}\frac{\alpha^2}{\alpha^2+1}(b_1^2+b_2^2)\epsilon\log(\alpha\sqrt{\epsilon}))+O(\epsilon).$$

It is clear that

$$\int_M he^{\phi_\epsilon} = h(p)\int_M e^{\phi_\epsilon} + \int_M (h-h(p))e^{\phi_\epsilon}.$$

Suppose that

$$h(x)-h(p) = k_1 r\cos\theta + k_2 r\sin\theta$$
$$+ k_3 r^2\cos^2\theta + 2k_4 r^2\cos\theta\sin\theta + k_5 r^2\sin^2\theta + O(r^3)$$

in $B_\delta(p)$.

By a simple computation, we obtain

$$\int_{B_{\alpha\sqrt{\epsilon}}}(h-h(p))e^{\phi_\epsilon} = \frac{\pi}{2}(k_3+k_5)\epsilon\log(\alpha^2+1)$$
$$+\frac{\pi}{2}(k_1b_1+k_2b_2)\epsilon\log(\alpha^2+1)+O(\epsilon)$$

and

$$\int_{M\backslash B_{\alpha\sqrt{\epsilon}}}(h-h(p))e^{\phi_\epsilon} = \int_{B_\delta\backslash B_{\alpha\sqrt{\epsilon}}}(h-h(p))e^{\phi_\epsilon} + \int_{M\backslash B_\delta}(h-h(p))e^{\phi_\epsilon}$$
$$= -\frac{\pi}{2}(k_3+k_5)(\frac{\alpha^2}{\alpha^2+1})^2\epsilon\log(\alpha^2\epsilon)$$
$$-\frac{\pi}{2}(k_1b_1+k_2b_2)(\frac{\alpha^2}{\alpha^2+1})^2\epsilon\log(\alpha^2\epsilon)+O(\epsilon).$$

So,

$$\int_M (h-h(p))e^{\phi_\epsilon} = \frac{\pi}{4}(\triangle h(p))\epsilon\log(\alpha^2+1)$$
$$+\frac{\pi}{2}(k_1b_1+k_2b_2)\epsilon\log(\alpha^2+1)$$
$$-\frac{\pi}{4}(\triangle h(p))(\frac{\alpha^2}{\alpha^2+1})^2\epsilon\log(\alpha^2\epsilon)$$
$$-\frac{\pi}{2}(k_1b_1+k_2b_2)(\frac{\alpha^2}{\alpha^2+1})^2\epsilon\log(\alpha^2\epsilon)+O(\epsilon).$$

Therefore,

$$\int_M he^{\phi_\epsilon} = h(p)\pi\frac{\alpha^2}{\alpha^2+1}(1+\frac{1}{\alpha^2+1}-\frac{\alpha^2+1}{\alpha^2}\frac{1}{6}K(p)\epsilon\log(\alpha^2+1)$$
$$+\frac{1}{4}\frac{\alpha^2+1}{\alpha^2}(b_1^2+b_2^2)\epsilon\log(\alpha^2+1)-\frac{1}{2}\frac{\alpha^2}{\alpha^2+1}(b_1^2+b_2^2)\epsilon\log(\alpha\sqrt{\epsilon})$$
$$-\frac{\alpha^2}{\alpha^2+1}(c_1+c_3-\frac{1}{3}K(p))\epsilon\log(\alpha\sqrt{\epsilon}))+\frac{\pi}{4}(\triangle h(p))\epsilon\log(\alpha^2+1)$$
$$-\frac{\pi}{4}(\triangle h(p))(\frac{\alpha^2}{\alpha^2+1})^2\epsilon\log(\alpha^2\epsilon)+\frac{\pi}{2}(k_1b_1+k_2b_2)\epsilon\log(\alpha^2+1)$$
$$-\frac{\pi}{2}(k_1b_1+k_2b_2)(\frac{\alpha^2}{\alpha^2+1})^2\epsilon\log(\alpha^2\epsilon)+O(\alpha^4\epsilon^2)+O(\epsilon).$$

Adding the terms in the functional, we get

$$\begin{aligned}
J(\phi_\epsilon)= & -8\pi - 8\pi\log\pi - 4\pi A(p) - 8\pi\log h(p) \\
& -16\pi^2(1-\frac{1}{4\pi}K(p))\epsilon\log(\alpha^2+1) + 4\pi(c_1+c_3-\frac{1}{3}K(p))\epsilon\log(\alpha^2\epsilon) \\
& -2\pi(b_1^2+b_2^2)\epsilon\log(\alpha^2+1) + 2\pi(b_1^2+b_2^2)\epsilon\log(\alpha^2\epsilon) \\
& -4\pi\frac{(k_1b_1+k_2b_2)}{h(p)}\epsilon\log(\alpha^2+1) + 4\pi\frac{(k_1b_1+k_2b_2)}{h(p)}\epsilon\log(\alpha^2\epsilon) \\
& -2\pi\frac{\Delta h(p)}{h(p)}\epsilon\log(\alpha^2+1) + 2\pi\frac{\Delta h(p)}{h(p)}\epsilon\log(\alpha^2\epsilon) \\
& +O(\frac{\epsilon\log(\alpha^2+1)}{\alpha^2}) + O(\frac{-\epsilon\log(\alpha^2\epsilon)}{\alpha^2}) \\
& +O((\epsilon\log(\alpha^2+1))^2) + O((-\epsilon\log(\alpha^2\epsilon))^2) \\
& +O(\frac{1}{\alpha^4}) + O(\alpha^4\epsilon^2\log(\alpha^2\epsilon)) + O(\epsilon).
\end{aligned}$$

Choosing α so that $\alpha^4\epsilon = \frac{1}{\log(-\log\epsilon)}$ and applying Proposition 3.3, we get

$$\begin{aligned}
J(\phi_\epsilon)= & -8\pi - 8\pi\log\pi - 4\pi A(p) - 8\pi\log h(p) \\
& -16\pi^2(1-\frac{1}{4\pi}K(p)+\frac{b_1^2+b_2^2}{8\pi}+\frac{\Delta h(p)}{8\pi h(p)}+\frac{(k_1b_1+k_2b_2)}{4\pi h(p)})\epsilon(-\log\epsilon) \\
& +o(\epsilon(-\log\epsilon)).
\end{aligned}$$

This proves theorem 1.2. □

4. The Green function on a flat torus. For details on the Green function, we refer to [L].

Let $z = x+iy$ be a variable in C (the complex plane) and let $\tau = u+iv$, $v>0$. Here for simplicity, we assume $u=0$. Let $q=e^{-2\pi v}$ and $q_z = e^{2\pi iz}$. Let $\Sigma_q = \Sigma_v = C^*/(q^Z)$, where Z is the set of integers, $C^* = C-\{0\}$ and q acts on C^* by the usual multiplication. In other words, Σ_q is the torus generated by the lattice $[1,\tau]$. Define a metric on Σ_q by

$$ds^2 = \frac{1}{v}dx\wedge dy.$$

The area of Σ_q with respect to ds^2 is 1. The corresponding Green function is

$$G(z,0) = -4\log|q^{B_2(y/v)/2}(1-q_z)\prod_{n=1}^{\infty}(1-q^nq_z)(1-q^{-n}q_z)|,$$

where $B_2(y) = y^2-y+\frac{1}{6}$ is the second Bernouli polynomial. Recall the definition of the Green function in the introduction.

Now the asymptotic expansion of the above Green function at the origin is

$$\begin{aligned}
& -4\log|z| - 4\log 2\pi + \frac{2v\pi}{3} - 8\log(\prod_{n=1}^{\infty}(1-e^{-2\pi nv})) + O(|z|^2) \\
& = -4\log(v^{1/2}|z|) + 2\log v - 4\log 2\pi + \frac{2v\pi}{3}
\end{aligned}$$

$$-8\log(\prod_{n=1}^{\infty}(1-e^{-2\pi nv}))+O(|z|^2)$$
$$= -4\log r + 2\log v - 4\log 2\pi + \frac{2v\pi}{3}$$
$$-8\log(\prod_{n=1}^{\infty}(1-e^{-2\pi nv}))+O(|r|^2)$$

where $r = v^{1/2}|z|$. The latter expansion is in normal coordinates. Therefore,

$$A_v = -2\log v - 4\log 2\pi + 2\frac{v\pi}{3} - 8\log(\prod_{n=1}^{\infty}(1-e^{-2\pi nv}))$$

Clearly, the asymptotic expansion of the Green function on Σ_v is independent of the base point 0. A_v is increasing between $[1,+\infty)$. Furthermore $A_1 < -2-2\log\pi = A_0$ and $\lim_{v\to+\infty} A_v = +\infty$. Hence there exists $v^* \in (1,+\infty)$ such that

(i) $A_v < A_0$, if $1 \le v < v^*$,

(ii) $A_v > A_0$, if $v^* < v < \infty$.

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