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Correction to "On the geometry of metric measure spaces. I"

by

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This is a correction to [11] (*Acta Math.*), as well as to the follow-up publications [3] and [5] (both in *J. Funct. Anal.*).

Theorem. *In the formulations of the* tensorization property *for*

(i) *the* $\mathsf{CD}(K,\infty)$ *condition* [11, Proposition 4.16],

(ii) *the* $\mathsf{CD}^*(K,N)$ *condition* [3, Theorem 4.1], *and*

(iii) *the* $\mathsf{CD}(K,N)$ *condition* [5, Theorem 1.1],

the assumption non-branching *has to be replaced by the stronger assumption that all the metric measure spaces* $(\mathsf{M}_i,\mathsf{d}_i,\mathsf{m}_i)$ *for* $i=1,...,\ell$ *are*

(a) infinitesimally Hilbertian spaces (*in the sense of* [1] *and* [7]), *or*

(b) smooth Finslerian spaces (*in the sense of* [9] *and* [10]).

Recall that a metric measure space $(\mathsf{M},\mathsf{d},\mathsf{m})$ is called *infinitesimally Hilbertian* if the Cheeger energy $\mathcal{E}$ on M is a quadratic functional on $L^2(\mathsf{M},\mathsf{m})$ or, in other words, if the associated heat flow is linear — and note that *Finslerian spaces* constitute the main class of metric measure spaces that are not infinitesimally Hilbertian.

The previous erroneous proofs relied on an argument which can not be made rigorous [11, p. 115, first two lines]: "According to Lemma 2.11 (iii), since M is non-branching and since the $\nu_{0,j}$ for $j=1,...,n$ are mutually singular, also the η_j for $j=1,...,n$ must be mutually singular".

As thankfully pointed out to the author first by G. Savaré and then also by N. Gigli, the composition of the partial optimal transports is not necessarily an optimal transport of the composed marginals, and thus mutual singularity of the partial marginals (and non-branching property) does not imply mutual singularity of the partial midpoints.

The corrected statements are derived as follows.

(a) Infinitesimally Hilbertian spaces

• If all the $(\mathsf{M}_i, \mathsf{d}_i, \mathsf{m}_i)$ are infinitesimally Hilbertian spaces and satisfy $\mathsf{CD}(K,\infty)$, then so does their tensor product $(\mathsf{M}, \mathsf{d}, \mathsf{m})$; [2, Theorems 5.2 and 4.17]. This is a consequence of the tensorization property for the Bakry–Émery condition $\mathsf{BE}(K,\infty)$ and the equivalence between Eulerian and Lagrangian curvature-dimension conditions $\mathsf{BE}(K,\infty)$ and $\mathsf{CD}(K,\infty)$.

(Note that the proof of the tensorization property in [1, Theorem 6.13] is based on the incorrect [11, Proposition 4.16]. Also notice that in [2], at some stage, a tensorization argument from [1] is used, but, as noted there, the lemma used was not involving curvature properties.)

• If all the $(\mathsf{M}_i, \mathsf{d}_i, \mathsf{m}_i)$ are infinitesimally Hilbertian spaces and satisfy $\mathsf{CD}^*(K,N)$, then so does their tensor product $(\mathsf{M}, \mathsf{d}, \mathsf{m})$; [6, Theorem 3.23].

• If all the $(\mathsf{M}_i, \mathsf{d}_i, \mathsf{m}_i)$ are infinitesimally Hilbertian spaces and satisfy $\mathsf{CD}(K,N)$, then, in particular, they all satisfy $\mathsf{CD}^*(K,N)$. Thus, by the previous assertion, their tensor product $(\mathsf{M}, \mathsf{d}, \mathsf{m})$, is infinitesimally Hilbertian and satisfies $\mathsf{CD}^*(K,N)$. According to the globalization theorem in [4], the latter implies that $(\mathsf{M}, \mathsf{d}, \mathsf{m})$ also satisfies the $\mathsf{CD}(K,N)$ condition. (Indeed, the original argument in [4] only applied to normalized mm-spaces but it is extended to σ-finite mm-spaces in [8].)

(b) Smooth Finslerian spaces

If all the $(\mathsf{M}_i, \mathsf{d}_i, \mathsf{m}_i)$ for $i=1,\dots,\ell$ are smooth Finslerian spaces (in the sense of [9] and [10]), then so is their tensor product

$$(\mathsf{M}, \mathsf{d}, \mathsf{m}) = \bigotimes_{i=1}^{\ell} (\mathsf{M}_i, \mathsf{d}_i, \mathsf{m}_i).$$

For smooth Finslerian spaces, the $\mathsf{CD}(K,\infty)$ condition is equivalent to the weighted flag Ricci curvature being bounded from below by K, [9, Theorem 1.2]. By construction, the weighted flag Ricci curvature bound has the tensorization property.

Moreover, the $\mathsf{CD}^*(K,N)$ condition, as well as the $\mathsf{CD}(K,N)$ condition, are both equivalent to the weighted flag N-Ricci curvature being bounded from below by K, [9, Theorem 1.2]. The tensorization property of the weighted flag N-Ricci curvature bound follows as in the Riemannian case: if each of the spaces M_i with dimension n_i and weight V_i for $i=1,...,\ell$ satisfies

$$\mathrm{Ric}_{N_i}(\xi_i) := \mathrm{Ric}(\xi_i) + \mathrm{Hess}V_i(\xi,\xi) - \frac{1}{N_i-n_i}\langle \nabla V_i, \xi_i\rangle^2 \geqslant K|\xi_i|^2$$

for all $\xi_i \in T\mathsf{M}_i$, then the space

$$\mathsf{M} = \bigotimes_i \mathsf{M}_i,$$

with dimension $n=\sum_i n_i$ and weight $V=\bigoplus_i V_i$, satisfies

$$\begin{aligned}\mathrm{Ric}(\xi) - \mathrm{Hess}V(\xi,\xi) - K|\xi|^2 &= \sum_i \mathrm{Ric}(\xi_i) - \sum_i \mathrm{Hess}V(\xi_i,\xi_i) - \sum_i K|\xi_i|^2 \\ &\geqslant \sum_i \frac{1}{N_i-n_i}\langle \nabla V_i, \xi_i\rangle^2 \\ &\geqslant \frac{1}{N-n}\langle \nabla V, \xi\rangle^2\end{aligned}$$

for all $\xi=(\xi_1,...,\xi_\ell)\in T\mathsf{M}$, with $N=\sum_i N_i$.

Indeed, in all these cases, the tensorization property in the Lagrangian picture is known only through its proof within the Eulerian picture.

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