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Every complete Pick space satisfies the column-row property

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1. Introduction

1.1. Background and main result

Complete Pick spaces are reproducing kernel Hilbert spaces that satisfy a version of the classical Nevanlinna–Pick interpolation theorem. The prototypical example of a complete Pick space is the Hardy space H^2 on the unit disc, but there are many others, such as the classical Dirichlet space on the unit disc, the Sobolev space W_1^2 on the unit interval, and the Drury–Arveson space H_d^2 on the unit ball $\mathbb{B}_d$ of $\mathbb{C}^d$ or of ℓ^2, also known as symmetric Fock space. This last space is a universal complete Pick space [1]; furthermore, it plays a key role in multivariable operator theory [9]. In addition to classical tools coming from complex and harmonic analysis, complete Pick spaces are amenable to operator theoretic and operator algebraic methods. For instance, the realization that the classical Dirichlet space is a complete Pick space has provided a new handle on this object, which has led to the solution of several open problems [3], [22].

In recent years, a new hypothesis, called the column-row property, has emerged in the theory of complete Pick spaces. It plays a role in the context of corona theorems [24], [34], weak products [7], [11], [22], interpolating sequences [5] and de Branges–Rovnyak spaces on the ball [17], [20], [21].

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To define the column-row property, recall that, for any n-tuple $T=(T_1,...,T_n)$ of bounded linear operators on a Hilbert space $\mathcal{H}$, one can consider the column operator

$$C=\begin{bmatrix} T_1 \\ \vdots \\ T_n \end{bmatrix} : \mathcal{H} \longrightarrow \mathcal{H}^n,$$

$$x \longmapsto \begin{bmatrix} T_1 x \\ \vdots \\ T_n x \end{bmatrix},$$

as well as the row operator

$$R=[\,T_1 \ \ldots \ T_n\,] : \mathcal{H}^n \longrightarrow \mathcal{H},$$

$$\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \longmapsto \sum_{j=1}^n T_j x_j.$$

Similar constructions are possible for infinite sequences, but then boundedness of the column or the row operator is no longer automatic.

Given a reproducing kernel Hilbert space $\mathcal{H}$, we will write $\operatorname{Mult}(\mathcal{H})$ for its multiplier algebra. Thus, $\varphi \in \operatorname{Mult}(\mathcal{H})$ if and only if $\varphi \cdot f \in \mathcal{H}$ whenever $f \in \mathcal{H}$. If $\varphi \in \operatorname{Mult}(\mathcal{H})$, we denote the associated multiplication operator on $\mathcal{H}$ by M_φ. The closed graph theorem implies that M_φ is automatically bounded.

Definition 1.1. A reproducing kernel Hilbert space $\mathcal{H}$ is said to satisfy the *column-row property* with constant $c \geqslant 1$ if whenever $(\varphi_n)_{n=1}^\infty$ is a sequence in $\operatorname{Mult}(\mathcal{H})$ with

$$\left\| \begin{bmatrix} M_{\varphi_1} \\ M_{\varphi_2} \\ \vdots \end{bmatrix} \right\| \leqslant 1,$$

then

$$\|[\, M_{\varphi_1} \ M_{\varphi_2} \ \ldots \,]\| \leqslant c.$$

In this definition, one can restrict to sequences of finite length, the key point being that the constant c should be independent of the length of the sequence. It is worth remarking that, for general n-tuples of Hilbert space operators, not much can be said about the relationship between the operator norm of the column operator and that of the row operator. The C^*-identity shows that, for R and C as above,

$$\|R\| = \|RR^*\|^{1/2} = \left\| \sum_{j=1}^n T_j T_j^* \right\|^{1/2} \leqslant \sqrt{n} \max_{1 \leqslant j \leqslant n} \|T_j\| \leqslant \sqrt{n} \|C\|,$$

and similarly $\|C\|\leqslant\sqrt{n}\|R\|$. Easy examples using matrices with only one non-zero entry demonstrate that the factor $\sqrt{n}$ is best possible in general.

Nonetheless, a number of complete Pick spaces were shown to satisfy the column-row property with some constant $c\geqslant 1$. The norm in the Hardy space H^2 can be expressed as an L^2-norm, from which it easily follows that, for any sequence of multipliers (φ_n) on H^2,

$$\|[\,M_{\varphi_1}\ \ M_{\varphi_2}\ \ \dots\,]\| = \sup_{z\in\mathbb{D}} \|(\varphi_n(z))\|_{\ell^2} = \left\| \begin{bmatrix} M_{\varphi_1} \\ M_{\varphi_2} \\ \vdots \end{bmatrix} \right\|.$$

In particular, H^2 satisfies the column-row property with constant 1. But the behavior of H^2 is not typical, as the multiplier norm is typically not even comparable to a supremum norm.

The first non-trivial example of a complete Pick space with the column-property is due to Trent [34], who showed that the Dirichlet space satisfies the column-row property with constant $\sqrt{18}$. Moreover, he showed that there are sequences of multipliers of the Dirichlet space that give bounded row multiplication operators, but unbounded column multiplication operators. Thus, the column-row property really is asymmetrical. Kidane and Trent [24] later showed that standard weighted Dirichlet spaces $\mathcal{D}_\alpha$ satisfy the column-row property with constant $\sqrt{10}$. In [7], Aleman, McCarthy, Richter and the author showed that standard weighted Besov spaces on the unit ball $\mathbb{B}_d$ in $\mathbb{C}^d$ satisfy the column-row property with some finite constant, possibly depending on d and on the Hilbert function space. In particular, this implies that, for each $d\in\mathbb{N}$, the Drury–Arveson space H^2_d satisfies the column-row property with some finite constant c_d, but the case $d=\infty$ remained open. In fact, the upper bound for c_d obtained in [7] grows exponentially in d. Also in the case of H^2_d for $d\geqslant 2$, there are sequences of multipliers that give unbounded column, but bounded row operators; see [7, §4.2]. In fact, the basic $\sqrt{n}$-bound between column norm and row norm is sharp in this case; see [11, Lemma 4.8]. The Drury–Arveson space, which is a universal complete Pick space, gives some idea of why the column-row property is asymmetrical. Indeed, it is a fundamental feature of the Drury–Arveson space that the coordinate functions form a row multiplication operator of norm 1, but the norm of the column multiplication operator is $\sqrt{d}$. The column-row property with some finite constant was extended to radially weighted Besov spaces on the ball in [6]. We also mention the recent paper of Pascoe [28], where it is shown that certain spaces satisfy the column-row property on average. Very recently, Augat, Jury and Pascoe [10] showed that the column-row property fails for the full Fock space, which can be regarded as the natural non-commutative analogue of the Drury–Arveson space.

The main result of this article shows that the column-row property is automatic for complete Pick spaces, with the optimal constant.

THEOREM 1.2. *Every normalized complete Pick space satisfies the column-row property with constant* 1.

The precise definition of a normalized complete Pick space is recalled in §2.2. The normalization hypothesis is not crucial and merely assumed for convenience. As a special case of Theorem 1.2, we also see that the constants mentioned above for the Dirichlet space, the standard weighted Dirichlet spaces $\mathcal{D}_\alpha$ and for the Drury–Arveson H^2_d can in fact be replaced by 1, which is important for some applications; see, for instance, Theorem 1.6 below.

The proof of Theorem 1.2 occupies §3. A sketch of the overall structure of the argument will be provided in §3.1. In fact, the proof yields the more general "column-matrix property"; see Corollary 3.16 for the precise statement. In Theorem 3.18, we will deduce a version of the column-row property for spaces with a complete Pick factor.

1.2. Applications

As alluded to above, the column-row property has appeared in a number of places in recent years. We review some of the applications that can be obtained by combining Theorem 1.2 with known results in the literature.

First, we consider weak product spaces, which play the role of the classical Hardy space H^1 in the theory of complete Pick spaces. These spaces go back to work of Coifman, Rochberg and Weiss [13]. If $\mathcal{H}$ is a reproducing kernel Hilbert space, the weak product space is

$$\mathcal{H}\odot\mathcal{H}=\left\{h=\sum_{n=1}^{\infty} f_n g_n : \sum_{n=1}^{\infty} \|f_n\|\,\|g_n\|<\infty\right\}.$$

This space is a Banach function space when equipped with the norm

$$\|h\|_{\mathcal{H}\odot\mathcal{H}}=\inf\left\{\sum_{n=1}^{\infty}\|f_n\|\,\|g_n\| : h=\sum_{n=1}^{\infty} f_n g_n\right\}.$$

(We adopt the convention that norms without subscripts denote norms in the Hilbert space $\mathcal{H}$.) If $\mathcal{H}=H^2$, then $\mathcal{H}\odot\mathcal{H}=H^1$, with equality of norms. In fact, in this case, each function $h\in H^1$ can be factored as $h=fg$, with $f,g\in H^2$ and

$$\|h\|_{\mathcal{H}\odot\mathcal{H}}=\|f\|\,\|g\|.$$

Jury and Martin [22, Theorem 1.3] showed that, if $\mathcal{H}$ is a normalized complete Pick space that satisfies the column-row property with constant c, then every $h \in \mathcal{H} \odot \mathcal{H}$ factors as $h = fg$, with

$$\|f\| \, \|g\| \leqslant c \|h\|_{\mathcal{H} \odot \mathcal{H}}.$$

Combining their result with Theorem 1.2, we therefore obtain the following description of $\mathcal{H} \odot \mathcal{H}$.

Theorem 1.3. *Let $\mathcal{H}$ be a normalized complete Pick space and let $h \in \mathcal{H} \odot \mathcal{H}$. Then, there exist $f, g \in \mathcal{H}$ with $h = fg$ and*

$$\|f\| \, \|g\| = \|h\|_{\mathcal{H} \odot \mathcal{H}}.$$

The multiplier algebra $\operatorname{Mult}(\mathcal{H} \odot \mathcal{H})$ of the weak product space is the algebra of all functions that pointwise multiply $\mathcal{H} \odot \mathcal{H}$ into itself, equipped with the norm of the multiplication operator. In the case of H^2, it is easy to see that

$$\operatorname{Mult}(H^2) = H^\infty = \operatorname{Mult}(H^1),$$

with equality of norms. Richter and Wick showed that, if $\mathcal{H}$ is a first order Besov space on $\mathbb{B}_d$, then

$$\operatorname{Mult}(\mathcal{H} \odot \mathcal{H}) = \operatorname{Mult}(\mathcal{H}),$$

with equivalence of norms [31]. For normalized complete Pick spaces $\mathcal{H}$, a description of $\operatorname{Mult}(\mathcal{H} \odot \mathcal{H})$ in terms of column multiplication operators of $\mathcal{H}$ was obtained by Clouâtre and the author in [11]. As observed there, in the presence of the column-row property, this leads to the equality

$$\operatorname{Mult}(\mathcal{H} \odot \mathcal{H}) = \operatorname{Mult}(\mathcal{H}).$$

Thus, combining [11, Corollary 1.3] with Theorem 1.2, we obtain the following result.

Theorem 1.4. *Let $\mathcal{H}$ be a normalized complete Pick space. Then,*

$$\operatorname{Mult}(\mathcal{H} \odot \mathcal{H}) = \operatorname{Mult}(\mathcal{H})$$

and

$$\|\varphi\|_{\operatorname{Mult}(\mathcal{H} \odot \mathcal{H})} = \|\varphi\|_{\operatorname{Mult}(\mathcal{H})}$$

for all $\varphi \in \operatorname{Mult}(\mathcal{H})$.

A subspace $\mathcal{M}$ of a Banach function space $\mathcal{B}$ is said to be multiplier invariant if $\varphi \cdot f \in \mathcal{M}$ whenever $f \in \mathcal{M}$ and φ is a function that multiplies $\mathcal{B}$ into itself. Multiplier invariant subspaces of H^2 and of H^1 are described by Beurling's theorem; in particular, they are in one-to-one correspondence. This one-to-one correspondence was extended to complete Pick spaces satisfying the column-row property in [7]. Now, combining [7, Theorem 3.7] with Theorem 1.2, we obtain the following result.

Theorem 1.5. *Let $\mathcal{H}$ be a normalized complete Pick space. Then, the mappings*

$$\mathcal{M} \longmapsto \overline{\mathcal{M}}^{\mathcal{H}\odot\mathcal{H}},$$
$$\mathcal{N}\cap\mathcal{H} \longleftarrow\!\shortmid \mathcal{N}$$

establish a bijection between closed multiplier invariant subspaces $\mathcal{M}$ of $\mathcal{H}$ and closed multiplier invariant subspaces $\mathcal{N}$ of $\mathcal{H}\odot\mathcal{H}$.

In §4, we will collect a few more applications to weak product spaces.

Let $\mathcal{H}$ be normalized complete Pick space of functions on X. A sequence (z_n) in X is said to be an *interpolating sequence* for $\mathrm{Mult}(\mathcal{H})$ if the evaluation map

$$\begin{aligned}\mathrm{Mult}(\mathcal{H}) &\longrightarrow \ell^\infty,\\ \varphi &\longmapsto (\varphi(z_n)),\end{aligned}$$

is surjective. Interpolating sequences for H^∞ were characterized by Carleson in terms of what are now known as Carleson measure and separation conditions. In [5], Carleson's theorem was extended by Aleman, McCarthy, Richter and the author to normalized complete Pick spaces, but the proof required the solution of the Kadison–Singer problem due to Marcus, Spielman and Srivastava [26]. It was also observed in [5] that in the presence of the column-row property, a simpler proof that is independent of the theorem of Marcus, Spielman and Srivastava can be given. For the convenience of the reader, we provide the essential part of the argument in which the column-row property enters in §4.

Our final application of Theorem 1.2 concerns de Branges–Rovnyak spaces on the ball. Classical de Branges–Rovnyak spaces on the unit disc play an important role in function theory and operator theory; see the book [33] for background. Much of the classical theory was extended to the multivariable setting by Jury [17] and by Jury and Martin [20], [21]. If b is an element of the closed unit ball of $\mathrm{Mult}(H^2_d)$, the space $\mathcal{H}(b)$ is the reproducing kernel Hilbert space on $\mathbb{B}_d$ with reproducing kernel

$$\frac{1-b(z)\overline{b(w)}}{1-\langle z,w\rangle}.$$

If $d=1$, then $\mathrm{Mult}(H^2_d)=H^\infty$, and we recover the de Branges–Rovnyak spaces on the unit disc. A key feature of the classical theory is a qualitative difference in the behavior of $\mathcal{H}(b)$, depending on whether or not b is an extreme point of the unit ball of H^∞. The results of [17], [20], [21] show a similar dichotomy in the multivariable setting, depending on whether or not b is column extreme. Here, a contractive multiplier b of a reproducing kernel Hilbert space $\mathcal{H}$ is said to be column extreme if there does not exist $a \in \mathrm{Mult}(\mathcal{H}) \setminus \{0\}$ such that

$$\left\| \begin{bmatrix} M_b \\ M_a \end{bmatrix} \right\| \leqslant 1.$$

In [21], Jury and Martin showed that every extreme point of the closed unit ball of $\mathrm{Mult}(H_d^2)$ is column extreme, and they asked if the converse holds. Combining Theorem 1.2 with a result of Jury and Martin, we obtain a positive answer.

THEOREM 1.6. *Let $\mathcal{H}$ be a normalized complete Pick space, and let b belong to the closed unit ball of* $\mathrm{Mult}(\mathcal{H})$. *Then, b is an extreme point of the closed unit ball of* $\mathrm{Mult}(\mathcal{H})$ *if and only if b is column extreme.*

This result will also be proved in §4. We remark that the proof crucially uses the fact that the column-row property holds with constant 1, as opposed to some larger constant.

In the final section, we exhibit counterexamples to some potential strengthenings of Theorem 1.2. In particular, we put the main results in the context of operator spaces and record the simple observation that the complete version of Theorem 1.2 fails, even though some matrix versions of Theorem 1.2 hold. Moreover, we show that Theorem 1.2 may fail in the absence of the complete Pick property. In fact, we construct a reproducing kernel Hilbert space of holomorphic functions on the unit disc that does not satisfy the column-row property with any constant (and is not a complete Pick space). Finally, we mention some open questions.

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2. Preliminaries

2.1. Kernels and multipliers

We briefly recall some preliminaries from the theory of reproducing kernel Hilbert spaces. For more background, the reader is referred to the books [2], [29]. A *reproducing kernel Hilbert space* is a Hilbert space $\mathcal{H}$ of complex-valued functions on a set X such that, for each $w \in X$, the evaluation functional

$$\begin{aligned} \mathcal{H} &\longrightarrow \mathbb{C}, \\ f &\longmapsto f(w), \end{aligned}$$

is bounded. The *reproducing kernel* of $\mathcal{H}$ is the unique function $K: X \times X \to \mathbb{C}$ satisfying

$$\langle f, K(\cdot, w) \rangle_{\mathcal{H}} = f(w)$$

for all $w \in X$ and $f \in \mathcal{H}$. We will assume for simplicity that all reproducing kernel Hilbert spaces are separable.

A function $\varphi: X \to \mathbb{C}$ is said to be a *multiplier* of $\mathcal{H}$ if $\varphi \cdot f \in \mathcal{H}$ whenever $f \in \mathcal{H}$. We write $\mathrm{Mult}(\mathcal{H})$ for the algebra of all multipliers of $\mathcal{H}$. More generally, if $\mathcal{E}$ is an auxiliary Hilbert space, we can think of elements of $\mathcal{H} \otimes \mathcal{E}$ as $\mathcal{E}$-valued functions on X. If $\mathcal{K}$ is another reproducing kernel Hilbert space on X and if $\mathcal{F}$ is another auxiliary Hilbert space, we define $\mathrm{Mult}(\mathcal{H} \otimes \mathcal{E}, \mathcal{K} \otimes \mathcal{F})$ to be the space of all $B(\mathcal{E}, \mathcal{F})$-valued functions on X that pointwise multiply $\mathcal{H} \otimes \mathcal{E}$ into $\mathcal{K} \otimes \mathcal{F}$. If $\mathcal{E}$ and $\mathcal{F}$ are understood from context, we simply call elements of $\mathrm{Mult}(\mathcal{H} \otimes \mathcal{E}, \mathcal{H} \otimes \mathcal{F})$ multipliers of $\mathcal{H}$. The closed graph theorem easily implies that every element of $\mathrm{Mult}(\mathcal{H} \otimes \mathcal{E}, \mathcal{K} \otimes \mathcal{F})$ induces a bounded multiplication operator $M_\Phi: \mathcal{H} \otimes \mathcal{E} \to \mathcal{K} \otimes \mathcal{F}$. The *multiplier norm* of Φ is the operator norm of M_Φ. In particular, we say that Φ is a *contractive multiplier* if M_Φ is a contraction, i.e. if $\|M_\Phi\| \leqslant 1$. We write $\mathrm{Mult}_1(\mathcal{H} \otimes \mathcal{E}, \mathcal{K} \otimes \mathcal{F})$ for the closed unit ball of $\mathrm{Mult}(\mathcal{H} \otimes \mathcal{E}, \mathcal{K} \otimes \mathcal{F})$, i.e. for the set of all contractive multipliers from $\mathcal{H} \otimes \mathcal{E}$ into $\mathcal{K} \otimes \mathcal{F}$.

An element $\Phi \in \mathrm{Mult}(\mathcal{H} \otimes \mathbb{C}^M, \mathcal{H} \otimes \mathbb{C}^N)$ can be identified with an $N \times M$ matrix of elements $\varphi_{ij} \in \mathrm{Mult}(\mathcal{H})$, and

$$\|\Phi\|_{\mathrm{Mult}(\mathcal{H} \otimes \mathbb{C}^M, \mathcal{H} \otimes \mathbb{C}^N)} = \left\| \begin{bmatrix} M_{\varphi_{11}} & \dots & M_{\varphi_{1M}} \\ \vdots & & \vdots \\ M_{\varphi_{N1}} & \dots & M_{\varphi_{NM}} \end{bmatrix} \right\|.$$

In particular, columns of N multiplication operators are identified with elements of $\mathrm{Mult}(\mathcal{H}, \mathcal{H} \otimes \mathbb{C}^N)$, and rows of M multiplication operators are identified with elements of $\mathrm{Mult}(\mathcal{H} \otimes \mathbb{C}^M, \mathcal{H})$.

A function $L: X \times X \to B(\mathcal{E})$ is said to be *positive* if, for all $n \in \mathbb{N}$ and all $z_1, \dots, z_n \in X$, the operator matrix

$$[L(z_i, z_j)]_{i,j=1}^n \tag{2.1}$$

defines a positive operator on $\mathcal{E}^n$. In this case, we write $L \geqslant 0$. It is easy to see that in this definition, it suffices to consider distinct points z_i. In particular, if X itself consists of n points, positivity of L can be checked by considering the single operator matrix (2.1).

We will heavily use the following well-known characterization of multipliers, which is essentially contained in [29, Theorem 6.28].

LEMMA 2.1. *Let $\mathcal{H}$ and $\mathcal{K}$ be reproducing kernel Hilbert spaces of functions on X with reproducing kernels K and L, respectively. Let $\mathcal{E}$ and $\mathcal{F}$ be auxiliary Hilbert spaces. Then, the following assertions are equivalent for a function $\Phi: X \to B(\mathcal{E}, \mathcal{F})$:*

(i) $\Phi \in \mathrm{Mult}_1(\mathcal{H} \otimes \mathcal{E}, \mathcal{K} \otimes \mathcal{F})$;

(ii) *the function*

$$\begin{aligned} X \times X &\longrightarrow B(\mathcal{E}), \\ (z, w) &\longmapsto L(z, w) I_{\mathcal{E}} - K(z, w) \Phi(z) \Phi(w)^* \end{aligned}$$

is positive.

Moreover, we will frequently use the following basic fact, which follows directly from the definition of positivity. For easier reference, we state it as a lemma.

LEMMA 2.2. *Let $\mathcal{E}$ and $\mathcal{F}$ be Hilbert spaces, let $L: X \times X \to B(\mathcal{E})$ be positive and let $f: X \to B(\mathcal{F}, \mathcal{E})$ be a function. Then,*

$$\begin{aligned} X \times X &\longrightarrow B(\mathcal{F}), \\ (z, w) &\longmapsto f(z) L(z, w) f(w)^*, \end{aligned}$$

is positive as well.

2.2. Complete Pick spaces and normalization

Complete Pick spaces are defined in terms of an interpolation property for multipliers that recovers the classical Pick interpolation theorem in the case of the Hardy space H^2. If $\mathcal{H}$ is a reproducing kernel Hilbert space of functions on X with kernel K and $Y \subset X$, then $\mathcal{H}|_Y$ denotes the reproducing kernel Hilbert space on Y with kernel $K|_{Y \times Y}$. The restriction map $\mathcal{H} \to \mathcal{H}|_Y$ is a co-isometry (see [29, Corollary 5.8]). On the level of multipliers, the restriction map $\mathrm{Mult}(\mathcal{H}) \to \mathrm{Mult}(\mathcal{H}|_Y)$ is a complete contraction ("complete contraction" means that all induced maps on matrices of multipliers are also contractive.) The space $\mathcal{H}$ is said to be a *complete Pick space* if restriction $\mathrm{Mult}(\mathcal{H}) \to \mathrm{Mult}(\mathcal{H}|_Y)$ is a complete exact quotient map for all finite sets $Y \subset X$ ("exact quotient map" means that the closed unit ball is mapped onto the closed unit ball, and "complete exact quotient map" means that the same is true for all induced maps on matrices of multipliers). A weak-$*$ compactness argument shows that the condition "complete exact quotient map" could be weakened to "complete quotient map". The multiplier characterization of Lemma 2.1 recovers the familiar formulation involving Pick matrices. Explicitly, $\mathcal{H}$ is a complete Pick space if whenever $N\in\mathbb{N}$ and $z_1, ..., z_n\in X$ and $\Lambda_1, ..., \Lambda_n\in M_N(\mathbb{C})$, positivity of the matrix

$$[K(z_i, z_j)(I-\Lambda_i\Lambda_j^*)]_{i,j=1}^n$$

is sufficient for the existence of a multiplier $\Phi\in\mathrm{Mult}_1(\mathcal{H}\otimes\mathbb{C}^N, \mathcal{H}\otimes\mathbb{C}^N)$ satisfying

$$\Phi(z_i)=\Lambda_i \quad \text{for } 1\leqslant i\leqslant n.$$

Theorems of McCullough [27], Quiggin [30] and Agler and McCarthy [1] give an equivalent characterization of the complete Pick property in terms of the reproducing kernel, which we might also take as the definition here. The reproducing kernel Hilbert space $\mathcal{H}$ (or its kernel K) is said to be *irreducible* if the underlying set X cannot be partitioned into two non-empty disjoint sets X_1 and X_2 such that

$$K(x_1, x_2)=0 \quad \text{for all } x_1\in X_1 \text{ and } x_2\in X_2.$$

The kernel K of an irreducible complete Pick space satisfies (see [1, Lemma 1.1])

$$K(z, w)\neq 0 \quad \text{for all } z, w\in X.$$

By [1, Theorem 3.1], the space $\mathcal{H}$ is an irreducible complete Pick space if and only if there exist a function $\delta: X\to\mathbb{C}\setminus\{0\}$, a number $d\in\mathbb{N}\cup\{\infty\}$ and a function $b: X\to\mathbb{B}_d$, where $\mathbb{B}_d$ denotes the open unit ball of $\mathbb{C}^d$ if $d<\infty$ and that of ℓ^2 if $d=\infty$, such that

$$K(z, w)=\frac{\delta(z)\overline{\delta(w)}}{1-\langle b(z), b(w)\rangle}, \quad z, w\in X. \tag{2.2}$$

We also refer the reader to [25] for a simple and elegant proof of necessity.

A kernel K is *normalized* at $z_0\in X$ if

$$K(z, z_0)=1 \quad \text{for all } z\in X.$$

Clearly, every normalized kernel is irreducible. By a *normalized complete Pick space*, we mean an irreducible complete Pick space whose kernel is normalized at a point. If K is the reproducing kernel of a complete Pick space that is normalized at $z_0\in X$, then one can achieve that in (2.2), the function δ is the constant function 1, and hence $b(z_0)=0$; see the discussion following [1, Theorem 3.1].

Given an irreducible complete Pick space $\mathcal{H}$ on X with kernel K and $z_0\in X$, one can consider the rescaled kernel

$$L(z, w)=\frac{K(z, w)K(z_0, z_0)}{K(z, z_0)K(z_0, w)},$$

which is normalized at z_0. The corresponding reproducing kernel Hilbert space $\mathcal{K}$ is a normalized complete Pick space whose multipliers agree with those of $\mathcal{H}$. More precisely,

$$\mathrm{Mult}(\mathcal{H}\otimes\mathcal{E},\mathcal{H}\otimes\mathcal{F})=\mathrm{Mult}(\mathcal{K}\otimes\mathcal{E},\mathcal{K}\otimes\mathcal{F})$$

for all Hilbert spaces $\mathcal{E}$ and $\mathcal{F}$, with equality of norms. This follows, for instance, from Lemmas 2.1 and 2.2. In particular, Theorem 1.2 for normalized complete Pick spaces implies the same result for irreducible complete Pick spaces. The main result was formulated for normalized spaces as this has become the standard setting, but, in the proof, it is convenient to work in the slightly more flexible class of irreducible complete Pick spaces. More background on rescaling kernels can be found in [2, §2.6].

Remark 2.3. Each general complete Pick space can be decomposed as an orthogonal direct sum of irreducible complete Pick spaces; see [1, Lemma 1.1]. Moreover, the direct summands are reducing for all multiplication operators, so if each summand satisfies the column-row property, then the entire space does. We omit the details as the theory of complete Pick spaces is usually only developed in the irreducible setting. This is somewhat analogous to studying holomorphic functions on connected open sets, rather than more general open sets.

We remark that, sometimes, irreducibility is assumed to include the condition that $K(\cdot,w_1)$ and $K(\cdot,w_2)$ are linearly independent if $w_1\neq w_2$; see [2, Definition 7.1]. However, we will not make that assumption here.

3. Proof of main result

3.1. Brief outline

We briefly discuss the main ideas that go into the proof of Theorem 1.2 (the column-row property of complete Pick spaces). First, we use the universality of the Drury–Arveson space and a straightforward approximation argument to reduce Theorem 1.2 to the case where the complete Pick space $\mathcal{H}$ is the restriction of the Drury–Arveson space H^2_d to a finite subset of $\mathbb{B}_d$, $d<\infty$, and the sequence of multipliers is finite.

To treat the case when $\mathcal{H}=H^2_d|_F$ for a finite set $F\subset\mathbb{B}_d$, we use a variant of the Schur algorithm. Classically, the Schur algorithm can be used to solve Pick interpolation problems on the disc; see [2, p. 8] for a sketch. It consists of two steps. In the first step, one applies conformal automorphisms of the disc to the interpolation nodes and to the targets to move one node and the corresponding target to the origin. In the second step, one factors the desired solution $f\in H^\infty$ as $f=zg$ for another function $g\in H^\infty$. This reduces an interpolation problem with n points for f to an interpolation problem with $n-1$ points for g, and hence makes an inductive approach possible.

Our approach to proving the column-row property of $H^2_d|_F$ follows a similar outline, proceeding by induction on $|F|$. Given a contractive column multiplier Φ of $H^2_d|_F$ with N components, we apply a conformal automorphism of $\mathbb{B}_d$ to the domain to arrange that $0\in F$, and a conformal automorphism of $\mathbb{B}_N$ to the range to achieve that $\Phi(0)=0$. In the factorization step, we use Leech's theorem to factor

$$\Phi = ([z_1 \;\; \dots \;\; z_d] \otimes I_N)\Psi,$$

where Ψ is a contractive column multiplier with dN components. Taking the transpose of Φ corresponds to rearranging the column Ψ into a $d\times N$ matrix. To deal with this process, we show, using a result of Jury and Martin, that the column-row property implies the "column-matrix property". Roughly speaking, this allows us to reduce the problem for Φ on the finite set F to a problem for Ψ on the set $F\setminus\{0\}$ with one fewer point, which makes it possible to apply induction.

Remark 3.1. A version of the Schur algorithm for the Drury–Arveson space appears in [8], but the results do not seem to be directly applicable to our problem.

3.2. Reductions

We carry out a few straightforward reductions for the proof of Theorem 1.2. Firstly, it is obvious that it suffices to consider finite sequences of multipliers in the definition of the column-row property (since the constant $c=1$ is independent of the length of the sequence).

Secondly, recall that the Drury–Arveson space H^2_d is the reproducing kernel Hilbert space on the open unit ball $\mathbb{B}_d$ of a d-dimensional Hilbert space, where $d\in\mathbb{N}\cup\{\infty\}$, with reproducing kernel

$$\frac{1}{1-\langle z, w\rangle}.$$

We will use the universality of the Drury–Arveson space H^2_d, which is essentially the representation (2.2), to replace a general irreducible complete Pick space with H^2_d. Moreover, an approximation argument can be used to reduce to $d<\infty$ (once again, this works if the column-row constant of H^2_d is independent of d). In fact, it is sufficient to consider restrictions of H^2_d to finite subsets of the ball, which yields the reduction to finite d at the same time.

If $F\subset\mathbb{B}_d$, we let $H^2_d|_F$ be the reproducing kernel Hilbert space on F whose reproducing kernel is the restriction of the kernel of H^2_d to $F\times F$.

LEMMA 3.2. *In order to prove Theorem 1.2, it suffices to show that, for each* $d\in\mathbb{N}$ *and all finite sets* $F\subset\mathbb{B}_d$, *the space* $H^2_d|_F$ *satisfies the column-row property with constant* 1.

Proof. Suppose that $H^2_d|_F$ satisfies the column-row property with constant 1 for all $d\in\mathbb{N}$ and all finite sets $F\subset\mathbb{B}_d$. Let $\mathcal{H}$ be a normalized complete Pick space on X. Thus, the reproducing kernel K of $\mathcal{H}$ is of the form

$$K(z,w)=\frac{1}{1-\langle b(z),b(w)\rangle}$$

for a function $b: X\to\mathbb{B}_e$ and a number $e\in\mathbb{N}\cup\{\infty\}$.

Let $N\in\mathbb{N}$ and let $\Phi\in\mathrm{Mult}_1(\mathcal{H},\mathcal{H}\otimes\mathbb{C}^N)$ be a contractive column multiplier of $\mathcal{H}$. Our goal is to show that the transposed function Φ^T is a contractive row multiplier of $\mathcal{H}$, i.e. that $\Phi^T\in\mathrm{Mult}_1(\mathcal{H}\otimes\mathbb{C}^N,\mathcal{H})$. By Lemma 2.1, this is equivalent to showing that, for any finite collection of points $x_1,...,x_n\in X$, the matrix

$$[K(x_i,x_j)(1-\Phi^T(x_i)(\Phi^T(x_j))^*)]_{i,j=1}^n \tag{3.1}$$

is positive.

Since $\Phi\in\mathrm{Mult}_1(\mathcal{H},\mathcal{H}\otimes\mathbb{C}^N)$, we know from Lemma 2.1 that the matrix

$$[K(x_i,x_j)(I_N-\Phi(x_i)\Phi(x_j)^*)]_{i,j=1}^n \tag{3.2}$$

is positive. Consider the points $b(x_1),...,b(x_n)\in\mathbb{B}_e$, which are contained in a subspace of dimension $d\leqslant n$. Thus, there exist points $\lambda_1,...,\lambda_n\in\mathbb{B}_d$ such that $\langle\lambda_i,\lambda_j\rangle=\langle b(x_i),b(x_j)\rangle$ for $1\leqslant i,j\leqslant n$ and such that $\lambda_i=\lambda_j$ if and only if $b(x_i)=b(x_j)$. Hence,

$$K(x_i,x_j)=\frac{1}{1-\langle\lambda_i,\lambda_j\rangle}$$

for $1\leqslant i,j\leqslant n$. Let $F=\{\lambda_1,...,\lambda_n\}\subset\mathbb{B}_d$. We wish to define

$$\Psi: F\longrightarrow B(\mathbb{C},\mathbb{C}^N)$$

by

$$\Psi(\lambda_j)=\Phi(x_j)\quad\text{for } 1\leqslant j\leqslant n.$$

To see that Ψ is well defined, we have to check that $\Phi(x_j)=\Phi(x_i)$ if $b(x_i)=b(x_j)$. But if $b(x_i)=b(x_j)$, then $K(\cdot,x_i)=K(\cdot,x_j)$, hence $f(x_i)=f(x_j)$ for all $f\in\mathcal{H}\otimes\mathbb{C}^N$, and therefore $\Phi(x_i)=\Phi(x_j)$, as $1\in\mathcal{H}$ by the normalization assumption. Thus, Ψ is well defined.

Having defined $\lambda_1, ..., \lambda_n$ and Ψ, we can now rewrite (3.2) as

$$\left[\frac{1}{1-\langle \lambda_i, \lambda_j\rangle}(I_N - \Psi(\lambda_i)\Psi(\lambda_j)^*)\right]_{i,j=1}^n \geqslant 0.$$

By Lemma 2.1, this means that $\Psi \in \mathrm{Mult}_1(H_d^2|_F, H_d^2|_F \otimes \mathbb{C}^N)$. Since $H_d^2|_F$ satisfies the column-row property with constant 1 by assumption, it follows that

$$\Psi^T \in \mathrm{Mult}_1(H_d^2|_F \otimes \mathbb{C}^N, H_d^2|_F),$$

so that, by the same lemma,

$$\left[\frac{1}{1-\langle \lambda_i, \lambda_j\rangle}(1 - \Psi^T(\lambda_i)(\Psi^T(\lambda_j))^*)\right]_{i,j=1}^n \geqslant 0.$$

Recalling the choice of λ_j and Ψ, we see that the matrix in (3.1) is positive, which finishes the proof. □

The proof above shows that we could assume that $|F| = d$, but it is convenient to keep the size of F and the dimension d independent.

3.3. Conformal automorphisms on the domain

For the remainder of this section, we assume that $d \in \mathbb{N}$. Our next task is to understand the action of biholomorphic automorphisms of $\mathbb{B}_d$ on the domain of multipliers, which is needed in the first step of the Schur algorithm. Let $\mathrm{Aut}(\mathbb{B}_d)$ denote the group of biholomorphic automorphisms of $\mathbb{B}_d$; see [32, §2.2] for more background. It is well known that $\mathrm{Aut}(\mathbb{B}_d)$ acts transitively on $\mathbb{B}_d$; see [32, Theorem 2.2.3]. We require the following basic facts about $\mathrm{Aut}(\mathbb{B}_d)$, which can be found in [32, Theorem 2.2.5].

LEMMA 3.3. *Let $\theta \in \mathrm{Aut}(\mathbb{B}_d)$ and let $a = \theta^{-1}(0)$.*

(a) *θ extends to a homeomorphism of $\overline{\mathbb{B}}_d$.*

(b) *The identity*

$$1 - \langle \theta(z), \theta(w)\rangle = \frac{(1-\langle a, a\rangle)(1-\langle z, w\rangle)}{(1-\langle z, a\rangle)(1-\langle a, w\rangle)}$$

holds for all $z, w \in \overline{\mathbb{B}}_d$.

It is well known that every conformal automorphism of $\mathbb{B}_d$ induces a completely isometric composition operator on $\mathrm{Mult}(H_d^2)$. This can be deduced from part (b) of Lemma 3.3. We require the following variant of this fact, which also easily follows from the same identity.

LEMMA 3.4. *Let $F\subset\mathbb{B}_d$, let $\mathcal{E}$ and $\mathcal{F}$ be Hilbert spaces and let $\Phi: F\to B(\mathcal{E},\mathcal{F})$ be a function. Let $\theta\in\operatorname{Aut}(\mathbb{B}_d)$. Then, Φ is a contractive multiplier of $H^2_d|_F$ if and only if $\Phi\circ\theta$ is a contractive multiplier of $H^2_d|_{\theta^{-1}(F)}$.*

Proof. Let

$$K(z,w)=\frac{1}{1-\langle z,w\rangle}$$

be the reproducing kernel of H^2_d. Suppose that Φ is a contractive multiplier of $H^2_d|_F$. Then, Lemma 2.1 shows that

$$(z,w)\longmapsto K(z,w)(I_{\mathcal{F}}-\Phi(z)\Phi(w)^*)\geqslant 0$$

on $F\times F$, and hence

$$(z,w)\longmapsto K(\theta(z),\theta(w))(I_{\mathcal{F}}-\Phi(\theta(z))\Phi(\theta(w))^*)\geqslant 0 \tag{3.3}$$

on $\theta^{-1}(F)\times\theta^{-1}(F)$. Let $a=\theta^{-1}(0)$. Part (b) of Lemma 3.3 implies that

$$K(\theta(z),\theta(w))=\frac{K(z,w)K(a,a)}{K(z,a)K(a,w)}$$

for all $z,w\in\mathbb{B}_d$, and hence

$$\begin{aligned}&K(z,w)(I_{\mathcal{F}}-\Phi(\theta(z))\Phi(\theta(w))^*)\\&\quad=\frac{1}{K(a,a)}K(z,a)K(\theta(z),\theta(w))(I_{\mathcal{F}}-\Phi(\theta(z))\Phi(\theta(w))^*)K(a,w),\end{aligned}$$

which is positive as a function of (z,w) on $\theta^{-1}(F)\times\theta^{-1}(F)$, by Lemma 2.2 and (3.3). This means that $\Phi\circ\theta$ is a contractive multiplier of $H^2_d|_{\theta^{-1}(F)}$, by Lemma 2.1. The converse follows by consideration of θ^{-1}. □

3.4. Conformal automorphisms on the range

Next, we study the action of conformal automorphisms on the range of multipliers. Observe that conformal automorphisms of the unit ball act on both row vectors and column vectors. More precisely, for $N\in\mathbb{N}$, the group $\operatorname{Aut}(\mathbb{B}_N)$ acts on the closed unit ball of $M_{1,N}(\mathbb{C})$ and on the closed unit ball of $M_{N,1}(\mathbb{C})$. The next lemma shows that this gives an action on contractive row and column multipliers.

LEMMA 3.5. *Let $\mathcal{H}$ be a reproducing kernel Hilbert space of functions on a set F and let $\theta\in\operatorname{Aut}(\mathbb{B}_N)$.*

(a) *Let $\Phi: F\to M_{1,N}$ be a function with $\|\Phi(z)\|\leqslant 1$ for all $z\in F$. Then, Φ is a contractive multiplier of $\mathcal{H}$ if and only if $\theta\circ\Phi$ is a contractive multiplier of $\mathcal{H}$.*

(b) *Let $\Phi: F\to M_{N,1}$ be a function with $\|\Phi(z)\|\leqslant 1$ for all $z\in F$. Then, Φ is a contractive multiplier of $\mathcal{H}$ if and only if $\theta\circ\Phi$ is a contractive multiplier of $\mathcal{H}$.*

We will provide two proofs of Lemma 3.5. The first proof is elementary, but requires some computations. The second proof is shorter, but uses dilation theory.

In the first proof, we require the following analogue of the formula in part (b) of Lemma 3.3. Formulas of this type are certainly known, see for instance [16, Chapter 8]. Since we do not have a reference for the precise formula we need, we provide the proof.

Lemma 3.6. *Let $\theta \in \mathrm{Aut}(\mathbb{B}_N)$ and let $a=\theta(0)\in M_{N,1}(\mathbb{C})$. Then, for all z and w in the closed unit ball of $M_{N,1}$, the identity*

$$I-\theta(z)\theta(w)^* = (I-aa^*)^{1/2}(I-za^*)^{-1}(I-zw^*)(I-aw^*)^{-1}(I-aa^*)^{1/2}$$

holds.

Proof. For $a\in\mathbb{B}_N$ and $z\in\overline{\mathbb{B}}_N$, let

$$\theta_a(z)=\frac{a-P_a z-s_a Q_a z}{1-\langle z,a\rangle},$$

where P_a is the orthogonal projection onto $\mathbb{C}a$, $Q_a=I-P_a$ and $s_a=(1-\|a\|^2)^{1/2}$. Then, $\theta_a\in\mathrm{Aut}(\mathbb{B}_N)$ is an involution that takes zero to a and vice versa; see [32, Theorem 2.2.2]. Moreover, [32, Theorem 2.2.5] shows that the automorphism θ^{-1} is of the form

$$\theta^{-1}=U\circ\theta_a,$$

where $a=\theta(0)$ and U is unitary. Thus,

$$\theta=\theta_a\circ U^*,$$

and it suffices to show the lemma for $\theta=\theta_a$.

We claim that the alternate formula

$$\theta_a(z)=(I-aa^*)^{1/2}(I-za^*)^{-1}(a-z)(1-a^*a)^{-1/2} \tag{3.4}$$

holds. To see this, we will compare inner products with a and with vectors in $(\mathbb{C}a)^\perp$. Let $\tau_a(z)$ denote the right-hand side of (3.4). Using the basic identities

$$(I-aa^*)^{1/2}a=a(1-a^*a)^{1/2} \quad \text{and} \quad (I-az^*)^{-1}a=a(1-z^*a)^{-1},$$

we find that

$$\langle\tau_a(z),a\rangle=\langle(I-za^*)^{-1}(a-z),a\rangle=\frac{\langle a-z,a\rangle}{1-\langle z,a\rangle}=\langle\theta_a(z),a\rangle.$$

On the other hand, let $v \in (\mathbb{C}a)^{\perp}$. Observe that

$$(I-za^*)^{-1}(a-z) = a - z(1-a^*z)^{-1}(1-a^*a),$$

which is easily checked by multiplying both sides with $I-za^*$ on the left. Moreover,

$$(I-aa^*)^{1/2}v = v,$$

for instance by expanding $(1-aa^*)^{1/2}$ in a power series. Thus,

$$\begin{aligned}\langle \tau_a(z), v\rangle &= \langle (I-za^*)^{-1}(a-z)(1-a^*a)^{-1/2}, v\rangle \\ &= -\langle z(1-a^*z)^{-1}(1-a^*a)^{1/2}, v\rangle \\ &= \langle \theta_a(z), v\rangle.\end{aligned}$$

This proves (3.4).

Using (3.4), we now conclude that

$$\begin{aligned}I-\theta_a(z)\theta_a(w)^* = (I-aa^*)^{1/2}&(I-za^*)^{-1} \\ &\times[(I-za^*)(I-aa^*)^{-1}(I-aw^*)-(a-z)(1-a^*a)^{-1}(a^*-w^*)] \\ &\times(I-aw^*)^{-1}(I-aa^*)^{1/2}.\end{aligned}$$

To finish the proof, one checks that the quantity in square brackets equals $I-zw^*$, which is a straightforward computation by repeatedly using the identity

$$(I-aa^*)^{-1}a = a(1-a^*a)^{-1}. \qquad \square$$

We are now ready for the first proof of Lemma 3.5.

First proof of Lemma 3.5. Let K be the reproducing kernel of $\mathcal{H}$.

(a) Suppose that Φ is a contractive row multiplier of $\mathcal{H}$. Then,

$$K(z,w)(1-\Phi(z)\Phi(w)^*) \geqslant 0$$

as a function of (z,w) by Lemma 2.1. Let $a=\theta^{-1}(0) \in M_{1,N}$. Notice that, if $x,y \in M_{1,N}$, then $xy^* = \langle x^T, y^T\rangle$. Thinking of $\mathrm{Aut}(\mathbb{B}_N)$ as acting on rows and applying part (b) of Lemma 3.3, we find that

$$K(z,w)(1-\theta(\Phi(z))\theta(\Phi(w))^*) = K(z,w)\frac{(1-aa^*)(1-\Phi(z)\Phi(w)^*)}{(1-\Phi(z)a^*)(1-a\Phi(w)^*)},$$

which is positive as a function of (z,w) by Lemma 2.2. Hence, $\theta \circ \Phi$ is a contractive multiplier as well. The converse follows by considering θ^{-1}.

(b) We use similar reasoning as in the proof of (a), but applying Lemma 3.6 in place of Lemma 3.3. Explicitly, suppose that Φ is a contractive column multiplier of $\mathcal{H}$. Then,

$$K(z,w)(I-\Phi(z)\Phi(w)^*)\geqslant 0.$$

Let $a=\theta(0)$. We now think of $\mathrm{Aut}(\mathbb{B}_N)$ as acting on columns. By Lemma 3.6, we have

$$\begin{aligned}&K(z,w)(I-\theta(\Phi(z))\theta(\Phi(w))^*)\\&\quad=(I-aa^*)^{1/2}(I-\Phi(z)a^*)^{-1}K(z,w)(I-\Phi(z)\Phi(w)^*)(I-a\Phi(w)^*)^{-1}(I-aa^*)^{1/2},\end{aligned}$$

which is positive by Lemma 2.2. Hence, $\theta\circ\Phi$ is a contractive multiplier of $\mathcal{H}$. The converse again follows by considering θ^{-1}. □

We now provide a second proof of Lemma 3.5, which is dilation theoretic. Recall that a tuple $T=(T_1,...,T_N)$ of operators on a Hilbert space is said to be a row contraction if the row operator

$$[T_1\ \ldots\ T_N]$$

has operator norm at most 1, and a column contraction if the column operator

$$\begin{bmatrix}T_1\\ \vdots\\ T_N\end{bmatrix}$$

has operator norm at most 1. If a tuple $T=(T_1,...,T_N)$ of commuting operators is either a row or a column contraction and if $\theta\in\mathrm{Aut}(\mathbb{B}_N)$ with component functions $\theta_1,...,\theta_N$, then the explicit formula for elements of $\mathrm{Aut}(\mathbb{B}_N)$ given in [32, Theorem 2.2.5] shows that one can define an operator tuple $\theta(T)=(\theta_1(T),...,\theta_N(T))$ by means of a norm convergent power series. (Using more machinery, one could also use the Taylor functional calculus to define $\theta(T)$; the two definitions yield the same result.) Moreover, if T is in fact a tuple of multiplication operators on a reproducing kernel Hilbert space, say $T_j=M_{\varphi_j}$, and $\Phi=(\varphi_1,...,\varphi_N)$, then $\theta_j(T)=M_{\theta_j\circ\Phi}$. Therefore, Lemma 3.5 is also a special case of the following operator theoretic result.

PROPOSITION 3.7. *Let $T=(T_1,...,T_N)$ be a tuple of commuting operators on a Hilbert space and let $\theta\in Aut(\mathbb{B}_N)$.*

(a) *If T is a row contraction, then the tuple $\theta(T)$ is also a row contraction.*

(b) *If T is a column contraction, then the tuple $\theta(T)$ is also a column contraction.*

Proof. (a) Since T is a row contraction, the Drury–Müller–Vasilecu–Arveson dilation theorem (see [9, Theorem 8.1]) shows that

$$\|[\theta_1(T)\ \ldots\ \theta_N(T)]\|\leqslant\|[\theta_1\ \ldots\ \theta_N]\|_{\mathrm{Mult}(H^2_N\otimes\mathbb{C}^N,H^2_N)}.$$

On the other hand, it is well known (and directly follows from Lemma 2.1) that

$$\|[z_1 \ \dots \ z_N]\|_{\mathrm{Mult}(H^2_N\otimes\mathbb{C}^N, H^2_N)} \leqslant 1,$$

and hence Lemma 3.4, applied with $F=\mathbb{B}_d$, implies that

$$\|[\theta_1 \ \dots \ \theta_N]\|_{\mathrm{Mult}(H^2_N\otimes\mathbb{C}^N, H^2_N)} \leqslant 1.$$

Therefore, $\theta(T)$ is a row contraction.

(b) Consider the adjoint tuple $T^*=(T_1^*, \dots, T_N^*)$, which is a row contraction. Define $\tilde{\theta}(z)=\overline{\theta(\bar{z})}$ for $z\in\overline{\mathbb{B}}_N$, where the complex conjugations are defined componentwise. Clearly, $\tilde{\theta}\in\mathrm{Aut}(\mathbb{B}_N)$. Thus, part (a) shows that $\tilde{\theta}(T^*)$ is a row contraction. Moreover, $\tilde{\theta}(T^*)=\theta(T)^*$, so $\theta(T)$ is a column contraction. □

3.5. Factorization

Our next goal is the factorization step in the Schur algorithm. If $\varphi\in H^\infty$ with $\varphi(0)=0$, then we may define $\psi=\varphi/z$, so that $\psi\in H^\infty$ with $\varphi=z\psi$ and $\|\psi\|_\infty=\|\varphi\|_\infty$. A generalization of this fact to the Drury–Arveson space was proved by Greene, Richter and Sundberg in [15, Corollary 4.2]. Using a version of Leech's theorem, they showed that, if $\varphi\in\mathrm{Mult}(H^2_d)$ with $\varphi(0)=0$, then there exist $\psi_1, \dots, \psi_d\in\mathrm{Mult}(H^2_d)$ such that

$$\varphi=\sum_{j=1}^d z_j\psi_j.$$

Moreover, one can achieve that the column norm of $(\psi_1, \dots, \psi_d)$ is at most the multiplier norm of φ. The next proposition is a generalization of this factorization result to columns of multipliers and to restrictions of H^2_d. The proof of [15, Corollary 4.2] carries over with minimal changes.

Let $\mathbf{z}=[z_1 \ \dots \ z_d]$ denote the row vector of coordinate functions.

Proposition 3.8. *Let $F\subset\mathbb{B}_d$ with $0\in F$, and let $\varphi_1, \dots, \varphi_N\in\mathrm{Mult}(H^2_d|_F)$ be such that $\varphi_i(0)=0$ for all i. Then, there exist $\psi_{ij}\in\mathrm{Mult}(H^2_d|_F)$, for $1\leqslant i\leqslant d$ and $1\leqslant j\leqslant N$, such that*

$$\begin{bmatrix}\varphi_1\\ \vdots\\ \varphi_N\end{bmatrix}=\begin{bmatrix}\mathbf{z} & 0 & \dots & 0\\ 0 & \mathbf{z} & \dots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \dots & \mathbf{z}\end{bmatrix}\begin{bmatrix}\psi_{11}\\ \vdots\\ \psi_{d1}\\ \psi_{12}\\ \vdots\\ \psi_{dN}\end{bmatrix}$$

and

$$\left\| \begin{bmatrix} \psi_{11} \\ \vdots \\ \psi_{d1} \\ \psi_{12} \\ \vdots \\ \psi_{dN} \end{bmatrix} \right\|_{\mathrm{Mult}(H^2_d|_F, H^2_d|_F \otimes \mathbb{C}^{dN})} = \left\| \begin{bmatrix} \varphi_1 \\ \vdots \\ \varphi_N \end{bmatrix} \right\|_{\mathrm{Mult}(H^2_d|_F, H^2_d|_F \otimes \mathbb{C}^{N})}.$$

Proof. To shorten notation, set $\mathcal{H}=H^2_d|_F$ and $\mathcal{H}_0=\{f\in\mathcal{H}: f(0)=0\}$. Let

$$\Phi = \begin{bmatrix} \varphi_1 \\ \vdots \\ \varphi_N \end{bmatrix} \in \mathrm{Mult}(\mathcal{H}, \mathcal{H}\otimes\mathbb{C}^N),$$

and assume without loss of generality that Φ has multiplier norm 1. Since $\Phi(0)=0$, we have that Φ is in fact a contractive multiplier from $\mathcal{H}$ to $\mathcal{H}_0\otimes\mathbb{C}^N$. Note that the reproducing kernel of $\mathcal{H}_0$ is

$$\frac{1}{1-\langle z,w\rangle}-1=\frac{\langle z,w\rangle}{1-\langle z,w\rangle}.$$

Therefore, Lemma 2.1 implies that

$$\frac{1}{1-\langle z,w\rangle}(\langle z,w\rangle I_N-\Phi(z)\Phi(w)^*)\geqslant 0 \tag{3.5}$$

as a function of (z,w) on $F\times F$. Consider the $B(\mathbb{C}^{dN},\mathbb{C}^N)$-valued function

$$\Theta(z)=\begin{bmatrix} \mathbf{z} & 0 & \dots & 0 \\ 0 & \mathbf{z} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \mathbf{z} \end{bmatrix}.$$

Then, (3.5) can be equivalently written as

$$\frac{1}{1-\langle z,w\rangle}(\Theta(z)\Theta(w)^*-\Phi(z)\Phi(w)^*)\geqslant 0.$$

In this setting, Leech's theorem for complete Pick spaces (see [2, Theorem 8.57]) yields a contractive multiplier $\Psi\in\mathrm{Mult}(\mathcal{H},\mathcal{H}\otimes\mathbb{C}^{dN})$ such that $\Phi=\Theta\Psi$. Since Θ is a contractive multiplier, we see that Ψ has in fact multiplier norm equal to 1. Writing Ψ as the column of its coordinate functions finishes the proof. $\square$

3.6. From columns to matrices

Suppose that we are given a factorization of a column multiplier as in Proposition 3.8. Then, the corresponding row multiplier is given by

$$[\,\varphi_1\ \dots\ \varphi_N\,] = [\,z_1\ \dots\ z_d\,] \begin{bmatrix} \psi_{11} & \dots & \psi_{1N} \\ \psi_{21} & \dots & \psi_{2N} \\ \vdots & & \vdots \\ \psi_{d1} & \dots & \psi_{dN} \end{bmatrix}.$$

Therefore, we also have to study the process of going from columns of multipliers to rectangular matrices. The Schur algorithm does not seem to be well suited for this problem, as the automorphism groups of the unit balls of $M_{dN,1}(\mathbb{C})$ and $M_{d,N}(\mathbb{C})$ are different.

Instead, we will make use of the following lemma, which crucially uses a result of Jury and Martin [22]; see also [11, Lemma 3.3].

Lemma 3.9. *Let $\mathcal{H}$ be an irreducible complete Pick space and let*

$$\Psi \in \mathrm{Mult}(\mathcal{H}\otimes\mathbb{C}^N, \mathcal{H}\otimes\mathbb{C}^M).$$

Then,

$$\|\Psi\|_{\mathrm{Mult}(\mathcal{H}\otimes\mathbb{C}^N,\mathcal{H}\otimes\mathbb{C}^M)} = \sup\{\|\Psi\Phi\|_{\mathrm{Mult}(\mathcal{H},\mathcal{H}\otimes\mathbb{C}^M)} : \Phi \in \mathrm{Mult}_1(\mathcal{H}, \mathcal{H}\otimes\mathbb{C}^N)\}.$$

Proof. By normalizing the reproducing kernel at a point, we may assume that $\mathcal{H}$ is normalized; see §2.2. The inequality "$\geqslant$" in the lemma is trivial.

To prove the reverse inequality, let $F\in\mathcal{H}\otimes\mathbb{C}^N$ with $\|F\|=1$. Applying [22, Theorem 1.1], we obtain $\Phi\in\mathrm{Mult}_1(\mathcal{H},\mathcal{H}\otimes\mathbb{C}^N)$ and $f\in\mathcal{H}$, with $\|f\|=1$ and $F=\Phi f$. Thus,

$$\|\Psi F\|_{\mathcal{H}\otimes\mathbb{C}^M} = \|\Psi\Phi f\|_{\mathcal{H}\otimes\mathbb{C}^M} \leqslant \|\Psi\Phi\|_{\mathrm{Mult}(\mathcal{H},\mathcal{H}\otimes\mathbb{C}^N)}.$$

Taking the supremum over all F in the unit sphere of $\mathcal{H}\otimes\mathbb{C}^N$ yields the remaining inequality. □

With the help of Lemma 3.9, we can establish the key fact that the column-row property implies the more general "column-matrix property". For the Hardy space H^2, the multiplier norm of any matrix of multipliers is simply the supremum of the pointwise operator norms. In this case, the following result follows from the basic inequality

$$\|A\|_{\mathrm{op}} \leqslant \|A\|_{\mathrm{HS}}$$

between the operator norm and the Hilbert–Schmidt norm of a matrix A.

Proposition 3.10. *Let $\mathcal{H}$ be an irreducible complete Pick space that satisfies the column-row property with constant* 1. *Let $\psi_{ij}\in\mathrm{Mult}(\mathcal{H})$ for $1\leqslant i\leqslant M$ and $1\leqslant j\leqslant N$. Then,*

$$\left\|\begin{bmatrix}\psi_{11} & \dots & \psi_{1N}\\ \psi_{21} & \dots & \psi_{2N}\\ \vdots & & \vdots\\ \psi_{M1} & \dots & \psi_{MN}\end{bmatrix}\right\|_{\mathrm{Mult}(\mathcal{H}\otimes\mathbb{C}^N,\mathcal{H}\otimes\mathbb{C}^M)} \leqslant \left\|\begin{bmatrix}\psi_{11}\\ \vdots\\ \psi_{M1}\\ \psi_{12}\\ \vdots\\ \psi_{MN}\end{bmatrix}\right\|_{\mathrm{Mult}(\mathcal{H},\mathcal{H}\otimes\mathbb{C}^{NM})}.$$

Proof. We will compute the norm of the matrix on the left with the help of Lemma 3.9. To this end, let

$$\Phi=\begin{bmatrix}\varphi_1\\ \vdots\\ \varphi_N\end{bmatrix}\in\mathrm{Mult}_1(\mathcal{H},\mathcal{H}\otimes\mathbb{C}^N).$$

Using commutativity of the multiplication in $\mathrm{Mult}(\mathcal{H})$, we find that

$$\begin{aligned}&\begin{bmatrix}\psi_{11} & \dots & \psi_{1N}\\ \psi_{21} & \dots & \psi_{2N}\\ \vdots & & \vdots\\ \psi_{M1} & \dots & \psi_{MN}\end{bmatrix}\begin{bmatrix}\varphi_1\\ \varphi_2\\ \vdots\\ \varphi_N\end{bmatrix}\\ &\quad=\begin{bmatrix}\varphi_1 & \dots & \varphi_N & 0 & \dots & 0 & \dots & 0 & \dots & 0\\ 0 & \dots & 0 & \varphi_1 & \dots & \varphi_N & \dots & 0 & \dots & 0\\ \vdots & & \vdots & \vdots & & \vdots & \ddots & \vdots & & \vdots\\ 0 & \dots & 0 & 0 & \dots & 0 & \dots & \varphi_1 & \dots & \varphi_N\end{bmatrix}\begin{bmatrix}\psi_{11}\\ \vdots\\ \psi_{1N}\\ \psi_{21}\\ \vdots\\ \psi_{MN}\end{bmatrix}.\end{aligned}$$

Here, the matrix on the left is the $M\times MN$ block diagonal matrix whose diagonal blocks are all equal to the row $[\varphi_1\ \dots\ \varphi_N]$. Since $\mathcal{H}$ satisfies the column-row property with constant 1 by assumption, the matrix on the left, being a direct sum of the rows $[\varphi_1\ \dots\ \varphi_N]$, has multiplier norm at most 1. Thus, Lemma 3.9 implies that

$$\left\| \begin{bmatrix} \psi_{11} & \cdots & \psi_{1N} \\ \psi_{21} & \cdots & \psi_{2N} \\ \vdots & & \vdots \\ \psi_{M1} & \cdots & \psi_{MN} \end{bmatrix} \right\|_{\text{Mult}} \leqslant \left\| \begin{bmatrix} \psi_{11} \\ \vdots \\ \psi_{1N} \\ \psi_{21} \\ \vdots \\ \psi_{MN} \end{bmatrix} \right\|_{\text{Mult}} = \left\| \begin{bmatrix} \psi_{11} \\ \vdots \\ \psi_{M1} \\ \psi_{12} \\ \vdots \\ \psi_{MN} \end{bmatrix} \right\|_{\text{Mult}},$$

where in the last step, we used the elementary fact that the column norm is invariant under permutations. $\square$

The following simple example shows that in general, it is not true that a sequence of operators that forms both a row and a column contraction also forms contractive matrices.

Example 3.11. Consider the scaled 2×2 matrix units

$$E_{11} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad E_{12} = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad E_{21} = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad E_{22} = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

One easily checks that

$$\|[E_{11}\ E_{12}\ E_{21}\ E_{22}]\| = 1 = \left\| \begin{bmatrix} E_{11} \\ E_{12} \\ E_{21} \\ E_{22} \end{bmatrix} \right\|,$$

yet

$$\left\| \begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{bmatrix} \right\| = \sqrt{2}.$$

3.7. Proof of the main result

We are almost ready to put everything together to prove the main result. We isolate one last lemma, which will be useful in the inductive proof. It is an application of the Schur complement technique.

LEMMA 3.12. *Let $F=E\cup\{0\}\subset\mathbb{B}_d$ be a finite set and let $\mathcal{E}$ and $\mathcal{F}$ be Hilbert spaces.*

(a) *A function $\Phi: F\to B(\mathcal{E},\mathcal{F})$ with $\Phi(0)=0$ is a contractive multiplier of $H^2_d|_F$ if and only if*

$$(z,w)\longmapsto\frac{1}{1-\langle z,w\rangle}(I_{\mathcal{F}}-\Phi(z)\Phi(w)^*)-I_{\mathcal{F}}\geqslant 0$$

on $E\times E$.

(b) *Let $\Psi: F\to B(\mathcal{E},\mathbb{C}^d)$ be a function and suppose that $\Psi|_E$ is a contractive multiplier of $H^2_d|_E$. Then, $\mathbf{z}\Psi$ is a contractive row multiplier of $H^2_d|_F$.*

Proof. (a) Let $E=\{\lambda_1,...,\lambda_{n-1}\}$ and set $\lambda_n=0$, so that $F=\{\lambda_1,...,\lambda_n\}$. An application of Lemma 2.1 shows that Φ is a contractive multiplier of $H^2_d|_F$ if and only if

$$\left[\frac{1}{1-\langle\lambda_i,\lambda_j\rangle}(I_{\mathcal{F}}-\Phi(\lambda_i)\Phi(\lambda_j)^*)\right]_{i,j=1}^n\geqslant 0. \tag{3.6}$$

Since $\Phi(0)=0$, each entry in the last row and in the last column of this matrix is equal to $I_{\mathcal{F}}$. Taking the Schur complement of the lower right corner (see, for instance, [2, Lemma 7.27]), we see that (3.6) holds if and only if

$$\left[\frac{1}{1-\langle\lambda_i,\lambda_j\rangle}(I_{\mathcal{F}}-\Phi(\lambda_i)\Phi(\lambda_j)^*)-I_{\mathcal{F}}\right]_{i,j=1}^{n-1}\geqslant 0,$$

which proves part (a).

(b) Since $\Psi|_E$ is a contractive multiplier of $H^2_d|_E$, we find that

$$\frac{1}{1-\langle z,w\rangle}(I_d-\Psi(z)\Psi(w)^*)\geqslant 0$$

on $E\times E$ by Lemma 2.1. Defining $\Phi=\mathbf{z}\Psi$, multiplying this relation with $[z_1\ \dots\ z_d]$ on the left and with $[w_1\ \dots\ w_d]^*$ on the right, and using the identity

$$\frac{\langle z,w\rangle}{1-\langle z,w\rangle}=\frac{1}{1-\langle z,w\rangle}-1,$$

we find, with the help of Lemma 2.2, that

$$\frac{1}{1-\langle z,w\rangle}(1-\Phi(z)(\Phi(w))^*)-1\geqslant 0$$

on $E\times E$. Thus, part (a) shows that Φ is a contractive row multiplier of $H^2_d|_F$. $\square$

Remark 3.13. The complete Pick property of H^2_d can be used to obtain a shorter, but somewhat less explicit proof of part (b) of Lemma 3.12. Indeed, if $\Psi: F\to B(\mathcal{E},\mathbb{C}^d)$ is a function with the property that $\Psi|_E$ is a contractive multiplier of $H^2_d|_E$, then the complete Pick property of H^2_d implies that there exists a contractive multiplier $\hat{\Psi}: \mathbb{B}_d\to B(\mathcal{E},\mathbb{C}^d)$ of H^2_d satisfying $\hat{\Psi}|_E=\Psi|_E$. Moreover, $\mathbf{z}\hat{\Psi}|_F=\mathbf{z}\Psi$, as the two functions agree on E and at the origin. Since $\mathbf{z}$ is a contractive row multiplier of H^2_d, the product $\mathbf{z}\hat{\Psi}$ is contractive row multiplier of H^2_d, hence $\mathbf{z}\Psi=\mathbf{z}\hat{\Psi}|_F$ is a contractive row multiplier of $H^2_d|_F$.

We are now ready to prove Theorem 1.2, which we restate.

THEOREM 3.14. *Every normalized complete Pick space satisfies the column-row property with constant* 1.

Proof. By Lemma 3.2, it suffices to show that $H^2_d|_F$ satisfies the column-row property with constant 1 for all $d\in\mathbb{N}$ and all finite sets $F\subset\mathbb{B}_d$. As remarked at the beginning of §3.2, it is also sufficient to consider columns of a finite length $N\in\mathbb{N}$. We fix $d\in\mathbb{N}$ for the remainder of the proof, and prove the statement by induction on $n=|F|$.

The base case $n=1$ is easy, as the multiplier norm in a reproducing kernel Hilbert space on a singleton is simply the norm at the singleton. More explicitly, suppose that $F=\{\lambda_1\}$ and let Φ be a contractive column multiplier of $H^2_d|_F$, say

$$\Phi\in\operatorname{Mult}_1(H^2_d|_F, H^2_d|_F\otimes\mathbb{C}^N).$$

Then,

$$\frac{1}{1-\|\lambda_1\|^2}(I_N-\Phi(\lambda_1)\Phi(\lambda_1)^*)\geqslant 0$$

by Lemma 2.1, which is equivalent to saying that $\|\Phi(\lambda_1)\|_{B(\mathbb{C},\mathbb{C}^N)}\leqslant 1$. Thus,

$$\|\Phi^T(\lambda_1)\|_{B(\mathbb{C}^N,\mathbb{C})}\leqslant 1,$$

so that

$$\frac{1}{1-\|\lambda_1\|^2}(1-\Phi^T(\lambda_1)(\Phi^T(\lambda_1))^*)\geqslant 0,$$

and hence

$$\Phi^T\in\operatorname{Mult}_1(H^2_d|_F\otimes\mathbb{C}^N, H^2_d|_F),$$

again by Lemma 2.1, i.e. Φ^T is a contractive row multiplier. This finishes the proof in the case $n=1$.

Next, let $n\geqslant 2$ and suppose that we already know that, for each subset $E\subset\mathbb{B}_d$ with $|E|\leqslant n-1$, the space $H^2_d|_E$ has the column-row property with constant 1. Let $F\subset\mathbb{B}_d$ with $|F|=n$, and let Φ be a contractive column multiplier of $H^2_d|_F$, say

$$\Phi\in\operatorname{Mult}_1(H^2_d|_F, H^2_d|_F\otimes\mathbb{C}^N).$$

Our goal is to show that Φ^T is a contractive row multiplier, i.e. that

$$\Phi^T\in\operatorname{Mult}_1(H^2_d|_F\otimes\mathbb{C}^N, H^2_d).$$

First, we apply the conformal automorphism step of the Schur algorithm. Recall that $\operatorname{Aut}(\mathbb{B}_d)$ acts transitively on $\mathbb{B}_d$ (see [32, Theorem 2.2.3]), so there exists $\theta\in\operatorname{Aut}(\mathbb{B}_d)$ with

$0 \in \theta^{-1}(F)$. Lemma 3.4 therefore shows that by replacing F with $\theta^{-1}(F)$ and Φ with $\Phi \circ \theta$, we may assume without loss of generality that $0 \in F$. The fact that Φ is a contractive column multiplier implies that $\|\Phi(\lambda)\| \leqslant 1$ for all $\lambda \in F$, just as in the proof of the base case $n=1$. Thus, Lemma 3.5 shows that by replacing Φ with $\theta \circ \Phi$ for a suitable element of $\mathrm{Aut}(\mathbb{B}_N)$, we may assume that $\Phi(0)=0$.

Next, we apply the factorization step of the Schur algorithm. Let $\varphi_1, ..., \varphi_N$ be the coordinate functions of Φ. In our setting, Proposition 3.8 implies that there exists a contractive column multiplier

$$\Psi = \begin{bmatrix} \psi_{11} \\ \vdots \\ \psi_{d1} \\ \psi_{12} \\ \vdots \\ \psi_{dN} \end{bmatrix} \in \mathrm{Mult}_1(H^2_d|_F, H^2_d|_F \otimes \mathbb{C}^{dN})$$

such that

$$\Phi = \begin{bmatrix} \varphi_1 \\ \vdots \\ \varphi_N \end{bmatrix} = \begin{bmatrix} \mathbf{z} & 0 & \dots & 0 \\ 0 & \mathbf{z} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \mathbf{z} \end{bmatrix} \begin{bmatrix} \psi_{11} \\ \vdots \\ \psi_{d1} \\ \psi_{12} \\ \vdots \\ \psi_{dN} \end{bmatrix}.$$

Let

$$\tilde{\Psi} = \begin{bmatrix} \psi_{11} & \dots & \psi_{1N} \\ \psi_{21} & \dots & \psi_{2N} \\ \vdots & & \vdots \\ \psi_{d1} & \dots & \psi_{dN} \end{bmatrix},$$

so that

$$\Phi^T = [\, \varphi_1 \ \dots \ \varphi_N \,] = \mathbf{z}\tilde{\Psi}.$$

To apply the inductive hypothesis, let us write $F = E \cup \{0\}$, where $|E| = n-1$. Notice that $\Psi|_E$ is in particular a contractive column multiplier of $H^2_d|_E$. By the induction hypothesis, the irreducible complete Pick space $H^2_d|_E$ satisfies the column-row property with constant 1. Proposition 3.10 therefore shows that the matrix $\tilde{\Psi}|_E$ is a contractive $B(\mathbb{C}^N, \mathbb{C}^d)$-valued multiplier of $H^2_d|_E$. Lemma 3.12 (b) now implies that $\Phi^T = \mathbf{z}\tilde{\Psi}$ is a contractive row multiplier of $H^2_d|_F$, as desired. □

Let us illustrate the proof above with a simple example in which the row norm is strictly smaller than the column norm.

Example 3.15. Let $d=2$ and consider

$$\Phi = \frac{1}{\sqrt{2}} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}.$$

Then, $\|\Phi\|_{\mathrm{Mult}(H_2^2, H_d^2 \otimes \mathbb{C}^2)}=1$, whereas $\|\Phi^T\|_{\mathrm{Mult}(H_2^2\otimes\mathbb{C}^2, H_2^2)}=1/\sqrt{2}$. Indeed, it is a fundamental property of the Drury–Arveson space that the coordinate functions form a row contraction, so by the basic $\sqrt{n}$-bound mentioned in the introduction,

$$\|\Phi\|_{\mathrm{Mult}} \leqslant \sqrt{2}\|\Phi^T\|_{\mathrm{Mult}} \leqslant 1,$$

and applying Φ to the constant function $1\in H_2^2$ shows that $\|\Phi\|_{\mathrm{Mult}}\geqslant 1$, so equality holds throughout.

Notice that $\Phi(0)=0$, so we may apply Proposition 3.8. In this case, a factorization is simply

$$\Phi = \begin{bmatrix} z_1 & z_2 & 0 & 0 \\ 0 & 0 & z_1 & z_2 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 0 \\ 1/\sqrt{2} \end{bmatrix}.$$

Proceeding as in the above proof, we see that

$$\Phi^T = [z_1 \;\; z_2] \begin{bmatrix} 1/\sqrt{2} & 0 \\ 0 & 1/\sqrt{2} \end{bmatrix}.$$

Observe that the square matrix in the factorization of Φ^T has norm $1/\sqrt{2}$, whereas the column in the factorization of Φ has norm 1. Thus, we can see the decrease in multiplier norm in the factorization in this case. In this simple example, there is no need to restrict to finite subsets of the ball, as the multiplier becomes constant after one step of the Schur algorithm. Reversing the steps above, we also see where the argument breaks down when going from rows to columns.

Combining Theorem 1.2 with Proposition 3.10, it follows immediately that complete Pick spaces in fact satisfy the "column-matrix property".

Corollary 3.16. *Let $\mathcal{H}$ be a normalized complete Pick space and let $\psi_{ij} \in \mathrm{Mult}(\mathcal{H})$ for $1 \leqslant i \leqslant M$ and $1 \leqslant j \leqslant N$. Then,*

$$\left\| \begin{bmatrix} \psi_{11} & \dots & \psi_{1N} \\ \psi_{21} & \dots & \psi_{2N} \\ \vdots & & \vdots \\ \psi_{M1} & \dots & \psi_{MN} \end{bmatrix} \right\|_{\mathrm{Mult}(\mathcal{H}\otimes\mathbb{C}^N, \mathcal{H}\otimes\mathbb{C}^M)} \leqslant \left\| \begin{bmatrix} \psi_{11} \\ \vdots \\ \psi_{M1} \\ \psi_{12} \\ \vdots \\ \psi_{MN} \end{bmatrix} \right\|_{\mathrm{Mult}(\mathcal{H}, \mathcal{H}\otimes\mathbb{C}^{NM})}.$$

3.8. Spaces with a complete Pick factor

In recent years, several results about complete Pick spaces have been generalized to spaces whose reproducing kernel has a complete Pick factor; see for instance [4], [12] and [5, §4]. We show that the column-row property for complete Pick spaces generalizes in a similar fashion.

Throughout this subsection, we will assume the following setting. Let $\mathcal{H}_K$ and $\mathcal{H}_S$ be two reproducing kernel Hilbert spaces on X with reproducing kernels K and S, respectively. Assume that $\mathcal{H}_S$ is a normalized complete Pick space and that $K/S \geqslant 0$. Then, the positive kernel K/S may be factored as

$$(K/S)(z,w) = G(z)G(w)^*,$$

where $G: X \to B(\mathcal{G}, \mathbb{C})$ for some auxiliary Hilbert space $\mathcal{G}$.

A basic example of this setting occurs when $\mathcal{H}_S$ is the Hardy space and $\mathcal{H}_K$ is the Bergman space on the disc. In the references cited above, results about $\mathrm{Mult}(\mathcal{H}_S)$ were generalized to $\mathrm{Mult}(\mathcal{H}_S, \mathcal{H}_K)$. The following lemma makes it possible to carry out this generalization for the column-row property; it is essentially a vector-valued version of [5, Proposition 4.10].

Lemma 3.17. *Assume the setting above and let $\mathcal{E}$ and $\mathcal{F}$ be Hilbert spaces. For a function $\Phi: X \to \mathcal{B}(\mathcal{E}, \mathcal{F})$, the following are equivalent:*

(i) *the function Φ is a contractive multiplier from $\mathcal{H}_S \otimes \mathcal{E}$ to $\mathcal{H}_K \otimes \mathcal{F}$;*

(ii) *there exists a contractive multiplier Ψ from $\mathcal{H}_S \otimes \mathcal{E}$ to $\mathcal{H}_S \otimes \mathcal{G} \otimes \mathcal{F}$ such that*

$$\Phi = (G \otimes I_{\mathcal{F}})\Psi.$$

Proof. (ii) ⇒ (i) This is immediate from the fact that G is a contractive multiplier from $\mathcal{H}_S \otimes \mathcal{G}$ to $\mathcal{H}_K$, which in turn follows from Lemma 2.1.

(i) ⇒ (ii) Let $\widetilde{G}(z) = G(z) \otimes I_{\mathcal{F}}$. The definition of G and Lemma 2.1 show that

$$S(z,w)(\widetilde{G}(z)\widetilde{G}(w) - \Phi(z)\Phi(w)^*) = K(z,w)I_{\mathcal{F}} - S(z,w)\Phi(z)\Phi(w)^* \geqslant 0$$

as a function of (z,w). As $\mathcal{H}_S$ is a complete Pick space, Leech's theorem [2, Theorem 8.57] yields the desired multiplier Ψ of $\mathcal{H}_S$. □

We are now ready to generalize the column-row property and also the column-matrix property to pairs of spaces. The column-row property corresponds to the case $M=1$ below.

THEOREM 3.18. *Assume the setting of* §3.8. *Let* $\varphi_{ij} \in \mathrm{Mult}(\mathcal{H}_S, \mathcal{H}_K)$ *for* $1 \leqslant i \leqslant M$ *and* $1 \leqslant j \leqslant N$. *Then,*

$$\left\| \begin{bmatrix} \varphi_{11} & \cdots & \varphi_{1N} \\ \varphi_{21} & \cdots & \varphi_{2N} \\ \vdots & & \vdots \\ \varphi_{M1} & \cdots & \varphi_{MN} \end{bmatrix} \right\|_{\mathrm{Mult}(\mathcal{H}_S \otimes \mathbb{C}^N, \mathcal{H}_K \otimes \mathbb{C}^M)} \leqslant \left\| \begin{bmatrix} \varphi_{11} \\ \vdots \\ \varphi_{M1} \\ \varphi_{12} \\ \vdots \\ \varphi_{MN} \end{bmatrix} \right\|_{\mathrm{Mult}(\mathcal{H}_S, \mathcal{H}_K \otimes \mathbb{C}^{NM})}.$$

Proof. Let Φ be the column of the φ_{ij} on the right, and assume that Φ has multiplier norm 1. The implication (i) ⇒ (ii) of Lemma 3.17 yields a contractive multiplier Ψ from $\mathcal{H}_S$ to $\mathcal{H}_S \otimes \mathcal{G} \otimes \mathbb{C}^{MN}$ such that $\Phi = (G \otimes I_{MN})\Psi$. Write Ψ as a column of multipliers Ψ_{ij} from $\mathcal{H}_S$ to $\mathcal{H}_S \otimes \mathcal{G}$, and hence $\varphi_{ij} = G\Psi_{ij}$ for all $1 \leqslant i \leqslant M$ and $1 \leqslant j \leqslant N$. Then,

$$\begin{bmatrix} \varphi_{11} & \cdots & \varphi_{1N} \\ \varphi_{21} & \cdots & \varphi_{2N} \\ \vdots & & \vdots \\ \varphi_{M1} & \cdots & \varphi_{MN} \end{bmatrix} = \begin{bmatrix} G & 0 & \cdots & 0 \\ 0 & G & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & G \end{bmatrix} \begin{bmatrix} \Psi_{11} & \cdots & \Psi_{1N} \\ \Psi_{21} & \cdots & \Psi_{2N} \\ \vdots & & \vdots \\ \Psi_{M1} & \cdots & \Psi_{MN} \end{bmatrix}.$$

Since Ψ is a contractive multiplier of $\mathcal{H}_S$, the column-matrix property of $\mathcal{H}_S$ (Corollary 3.16), combined with an obvious approximation argument to approximate each Ψ_{ij} by a finite column, shows that the matrix on the right is a contractive multiplier of $\mathcal{H}_S$. Thus, the trivial direction (ii) ⇒ (i) of Lemma 3.17 shows that the matrix on the left is a contractive multiplier from $\mathcal{H}_S$ to $\mathcal{H}_K$. □

Note that in the proof above, we used the column-matrix property of $\mathcal{H}_S$ even in the case $M=1$, i.e. when only showing the column-row property for the pair $(\mathcal{H}_S, \mathcal{H}_K)$.

4. Further applications

4.1. Weak product spaces

We already observed in the introduction that combining the main result, Theorem 1.2, with known results in the literature yields several results about weak product spaces. We now collect a few more of these consequences.

The *Smirnov class* of a normalized complete Pick space $\mathcal{H}$ is defined to be

$$N^+(\mathcal{H}) = \left\{ \frac{\varphi}{\eta} : \varphi, \eta \in \mathrm{Mult}(\mathcal{H}), \eta \text{ cyclic} \right\}.$$

Recall that $\eta \in \mathrm{Mult}(\mathcal{H})$ is said to be *cyclic* if the multiplication operator M_η on $\mathcal{H}$ has dense range. (Since the kernel functions of a normalized complete Pick space are multipliers, $\mathrm{Mult}(\mathcal{H})$ is densely contained in $\mathcal{H}$, so η is cyclic if and only if $\eta \cdot \mathrm{Mult}(\mathcal{H})$ is dense in $\mathcal{H}$.) In [3], it was shown that $\mathcal{H} \subset N^+(\mathcal{H})$ for any normalized complete Pick space $\mathcal{H}$. Moreover, [7, Corollary 3.4] shows that, if $\mathcal{H}$ satisfies the column-row property, then $\mathcal{H} \odot \mathcal{H} \subset N^+(\mathcal{H})$. Thus, in combination with Theorem 1.2, we obtain this inclusion for all normalized complete Pick spaces. Notice that since $N^+(\mathcal{H})$ is an algebra, the inclusion $\mathcal{H} \odot \mathcal{H} \subset N^+(\mathcal{H})$ also follows from the description

$$\mathcal{H} \odot \mathcal{H} = \{ f \cdot g : f, g \in \mathcal{H} \}$$

of Theorem 1.3. In fact, [7, Theorem 3.3] yields more precise information. To put the next result into perspective, it is useful to recall that, if $\psi \in \mathrm{Mult}_1(\mathcal{H})$ with $\psi \neq 1$, then $1-\psi$ and $(1-\psi)^2$ are cyclic; see [3, Lemma 2.3].

Theorem 4.1. *Let $\mathcal{H}$ be a complete Pick space that is normalized at z_0 and let $h \in \mathcal{H} \odot \mathcal{H}$ with $\|h\|_{\mathcal{H} \odot \mathcal{H}} \leqslant 1$. Then, there exist $\varphi, \psi \in \mathrm{Mult}(\mathcal{H})$, with*

$$\|\varphi\|_{\mathrm{Mult}(\mathcal{H})} \leqslant 1, \quad \|\psi\|_{\mathrm{Mult}(\mathcal{H})} \leqslant 1 \quad \textit{and} \quad \psi(z_0) = 0$$

such that

$$h = \frac{\varphi}{(1-\psi)^2}.$$

Proof. This is the statement of [7, Theorem 3.3], the only difference being that in [7, Theorem 3.3], one has $\varphi \in \mathrm{Mult}(\mathcal{H} \odot \mathcal{H})$ with $\|\varphi\|_{\mathrm{Mult}(\mathcal{H} \odot \mathcal{H})} \leqslant 1$. Thus, the result follows, for instance, from the equality

$$\mathrm{Mult}(\mathcal{H} \odot \mathcal{H}) = \mathrm{Mult}(\mathcal{H})$$

of Theorem 1.4, which in turn relies on [11].

Alternatively, examination of the proofs of Lemma 3.2 and Theorem 3.3 of [7] show that, if $\mathcal{H}$ satisfies the column-row property with constant 1, then one obtains that $\varphi \in \mathrm{Mult}(\mathcal{H})$, with $\|\varphi\|_{\mathrm{Mult}(\mathcal{H})} \leqslant 1$, so the result also follows directly with the help of Theorem 1.2, independently of the results of [11]. $\square$

Remark 4.2. Using a factorization result of Jury and Martin ([23, Corollary 3.4]) and Theorem 1.3, one obtains for each $h \in \mathcal{H} \odot \mathcal{H}$ an alternate factorization of the form $h = \varphi/\eta^2$, where $\varphi, \eta \in \mathrm{Mult}(\mathcal{H})$ and η is cyclic. This factorization is generally different from the one in Theorem 4.1, even in the case $\mathcal{H} = H^2$.

Nehari's theorem shows that the dual space of H^1 is the space of all symbols of bounded Hankel operators on H^2. This can be generalized to weak product spaces of normalized complete Pick spaces $\mathcal{H}$. In [7, §2.2], a space $\mathrm{Han}(\mathcal{H})$ of symbols of bounded Hankel operators on $\mathcal{H}$ is defined, and it is shown that the dual space of $\mathcal{H} \odot \mathcal{H}$ is isomorphic via a conjugate linear isometry to $\mathrm{Han}(\mathcal{H})$. The definition of $\mathrm{Han}(\mathcal{H})$ is somewhat involved. More concrete is the space $\mathcal{X}(\mathcal{H})$ of all those $b \in \mathcal{H}$ for which the densely defined bilinear form

$$
\begin{aligned}
&\mathcal{H} \times \mathcal{H} \longrightarrow \mathbb{C}, \\
&(\varphi, f) \longmapsto \langle \varphi f, b \rangle, \quad \varphi \in \mathrm{Mult}(\mathcal{H}), f \in \mathcal{H},
\end{aligned}
$$

is bounded, equipped with the norm of the bilinear form. (Recall that the kernel functions of a normalized complete Pick space are multipliers, and hence $\mathrm{Mult}(\mathcal{H})$ is densely contained in $\mathcal{H}$.) Explicitly, $b \in \mathcal{X}(\mathcal{H})$ if and only if there exists $C \geqslant 0$ such that

$$
|\langle \varphi f, b \rangle| \leqslant C \|\varphi\|_{\mathcal{H}} \|f\|_{\mathcal{H}} \quad \text{for all } \varphi \in \mathrm{Mult}(\mathcal{H}) \text{ and all } f \in \mathcal{H}.
$$

In [7, Theorem 2.6], it is shown that, in the presence of the column-row property, one has $\mathrm{Han}(\mathcal{H}) = \mathcal{X}(\mathcal{H})$. Thus, in combination with Theorem 1.2, we obtain the following version of Nehari's theorem.

THEOREM 4.3. *Let $\mathcal{H}$ be a normalized complete Pick space. Then, there is a conjugate linear isometric isomorphism $(\mathcal{H} \odot \mathcal{H})^* = \mathcal{X}(\mathcal{H})$. On the dense subspace $\mathcal{H}$ of $\mathcal{H} \odot \mathcal{H}$, the action of an element $b \in \mathcal{X}(\mathcal{H})$ on $f \in \mathcal{H}$ is given by $\langle f, b \rangle_{\mathcal{H}}$.*

If $b \in \mathrm{Han}(\mathcal{H}) = \mathcal{X}(\mathcal{H})$, then the associated Hankel operator H_b is the unique bounded linear operator $H_b: \mathcal{H} \to \overline{\mathcal{H}}$ satisfying

$$
\langle H_b f, \overline{\varphi} \rangle_{\overline{\mathcal{H}}} = \langle \varphi f, b \rangle_{\mathcal{H}}, \quad \varphi \in \mathrm{Mult}(\mathcal{H}), f \in \mathcal{H};
$$

see [7, §2.2]. Here, $\overline{\mathcal{H}}$ denotes the conjugate Hilbert space. It is easy to see that the kernel of every Hankel operator is multiplier invariant, i.e. invariant under M_φ for all

$\varphi \in \mathrm{Mult}(\mathcal{H})$. In [7, Corollary 3.8], it was shown that, conversely, every closed multiplier invariant subspace of $\mathcal{H}$ is an intersection of kernels of Hankel operators, provided that $\mathcal{H}$ satisfies the column-row property. Thus, we obtain the following consequence.

THEOREM 4.4. *Let $\mathcal{H}$ be a normalized complete Pick space and let $\mathcal{M} \subset \mathcal{H}$ be a closed multiplier invariant subspace. Then, there exists a sequence (b_n) in $\mathcal{X}(\mathcal{H})$ such that*

$$\mathcal{M} = \bigcap_n \ker H_{b_n}.$$

4.2. Interpolating sequences

Let $\mathcal{H}$ be a normalized complete Pick space on X with reproducing kernel K. A sequence (z_n) in X is said to

(IS) be an *interpolating sequence* for $\mathrm{Mult}(\mathcal{H})$ if the evaluation map

$$\begin{aligned} \mathrm{Mult}(\mathcal{H}) &\longrightarrow \ell^\infty, \\ \varphi &\longmapsto (\varphi(z_n)), \end{aligned}$$

is surjective;

(C) satisfy the *Carleson measure condition* if there exists $C \geqslant 0$ such that

$$\sum_n \frac{|f(z_n)|^2}{K(z_n, z_n)} \leqslant C \|f\|_{\mathcal{H}}^2 \quad \text{for all } f \in \mathcal{H};$$

(WS) be *weakly separated* if there exists $\varepsilon > 0$ such that that, for all $n \neq m$, there exists $\varphi \in \mathrm{Mult}_1(\mathcal{H})$ with $\varphi(z_n) = \varepsilon$ and $\varphi(z_m) = 0$.

The weak separation condition can be rephrased in terms of a (pseudo-)metric derived from the reproducing kernel K. For this translation and background on interpolating sequences, see [2, Chapter 9]. Carleson showed that, for $\mathcal{H} = H^2$, a sequence satisfies (IS) if and only if it satisfies (C) and (WS). This result was extended to all normalized complete Pick spaces in [5], using the solution of the Kadison–Singer problem due to Marcus, Spielman and Srivastava [26]. In [5, Remark 3.7], it was observed that, if $\mathcal{H}$ satisfies the column-row property, then a simpler proof is possible. For the convenience of the reader, we present the relevant part of the argument in which the column-row property enters.

THEOREM 4.5. *In every normalized complete Pick space $\mathcal{H}$, the equivalence*

$$\text{(IS)} \iff \text{(C)} + \text{(WS)}$$

holds. In this case, there exists a bounded linear right-inverse of the evaluation map

$$\mathrm{Mult}(\mathcal{H}) \longrightarrow \ell^\infty.$$

Proof. It is well known that every interpolating sequence is weakly separated and satisfies the Carleson measure condition; see [2, Chapter 9].

Conversely, suppose that (z_n) is weakly separated and satisfies the Carleson measure condition. A theorem of Agler and McCarthy [2, Theorem 9.46 (c)] shows that there exists a bounded column multiplier $\Phi\in\mathrm{Mult}(\mathcal{H},\mathcal{H}\otimes\ell^2)$ such that $\Phi(z_n)=e_n$, the nth standard basis vector of ℓ^2, for all $n\in\mathbb{N}$. Theorem 1.2 implies that the transposed multiplier $\Phi^T\in\mathrm{Mult}(\mathcal{H}\otimes\ell^2,\mathcal{H})$ is bounded as well. Let $\Delta\colon\ell^\infty\to B(\ell^2)$ be the embedding via diagonal operators and define

$$\begin{aligned} T\colon \ell^\infty &\longrightarrow \mathrm{Mult}(\mathcal{H}),\\ w &\longmapsto \Phi^T\Delta(w)\Phi. \end{aligned}$$

Observe that, if $w\in\ell^\infty$, then $T(w)$ is indeed a multiplication operator, and that

$$T(w)(z_n)=e_n^T\Delta(w)e_n=w_n$$

for all $n\in\mathbb{N}$. Thus, (z_n) satisfies (IS), and T is the desired right-inverse of the evaluation map. $\square$

4.3. de Branges–Rovnyak spaces and extreme points

In this subsection, we prove Theorem 1.6. We require the following special case of a result of Jury and Martin [18], [19] regarding extreme points of the unit ball of an operator algebra. The proof below is a simplification of their proof.

Lemma 4.6. (Jury–Martin) *Let $\mathcal{H}$ be a Hilbert space, let $A,B\in B(\mathcal{H})$ and suppose that*

$$\left\|\begin{bmatrix} B\\ A\end{bmatrix}\right\|\leqslant 1 \quad \textit{and} \quad \|[\,B\ \ A\,]\|\leqslant 1.$$

Then,

$$\left\|B\pm\frac{1}{2}A^2\right\|\leqslant 1.$$

Proof. Since the column and the row have norm at most 1, so do

$$X=\begin{bmatrix} 1 & 0 & 0\\ 0 & B & A\end{bmatrix} \quad\text{and}\quad Y=\begin{bmatrix} 0 & B\\ 1 & 0\\ 0 & \pm A\end{bmatrix}.$$

Hence,

$$[\,1/\sqrt{2}\quad 1/\sqrt{2}\,]XY\begin{bmatrix} 1/\sqrt{2}\\ 1/\sqrt{2}\end{bmatrix}=\frac{1}{2}[\,1\quad 1\,]\begin{bmatrix} 0 & B\\ B & \pm A^2\end{bmatrix}\begin{bmatrix} 1\\ 1\end{bmatrix}=B\pm\frac{1}{2}A^2$$

has norm at most 1 as well. $\square$

We can now prove Theorem 1.6, which we restate for the reader's convenience. Recall that a multiplier b of a reproducing kernel Hilbert space $\mathcal{H}$ of norm at most 1 is said to be column extreme if there does not exist $a \in \operatorname{Mult}(\mathcal{H}) \setminus \{0\}$ such that

$$\left\| \begin{bmatrix} b \\ a \end{bmatrix} \right\|_{\operatorname{Mult}(\mathcal{H}, \mathcal{H} \otimes \mathbb{C}^2)} \leqslant 1.$$

THEOREM 4.7. *Let $\mathcal{H}$ be a normalized complete Pick space and let b belong to the closed unit ball of* $\operatorname{Mult}(\mathcal{H})$. *Then, b is an extreme point of the closed unit ball of* $\operatorname{Mult}(\mathcal{H})$ *if and only if b is column extreme.*

Proof. The "if" part was already shown by Jury and Martin [21]. We provide a variant of their argument in the spirit of the proof of Lemma 4.6. Suppose that b is not an extreme point of the closed unit ball of $\operatorname{Mult}(\mathcal{H})$. Then, there exists $a \in \operatorname{Mult}(\mathcal{H}) \setminus \{0\}$ such that $\|b \pm a\|_{\operatorname{Mult}(\mathcal{H})} \leqslant 1$. Therefore,

$$\begin{bmatrix} b \\ a \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} b+a & 0 \\ 0 & b-a \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

has multiplier norm at most 1, because the scalar square matrix is unitary. Hence, b is not column extreme.

Conversely, suppose that b is not column extreme. By definition, there exists $a \in \operatorname{Mult}(\mathcal{H}) \setminus \{0\}$ such that

$$\left\| \begin{bmatrix} b \\ a \end{bmatrix} \right\|_{\operatorname{Mult}(\mathcal{H}, \mathcal{H} \otimes \mathbb{C}^2)} \leqslant 1.$$

Theorem 1.2 implies that

$$\| [\, b \quad a \,] \|_{\operatorname{Mult}(\mathcal{H} \otimes \mathbb{C}^2, \mathcal{H})} \leqslant 1,$$

hence $\left\| b \pm \frac{1}{2} a^2 \right\|_{\operatorname{Mult}(\mathcal{H})} \leqslant 1$ by Lemma 4.6. Since $\operatorname{Mult}(\mathcal{H})$ is an algebra of functions, it does not contain any non-zero nilpotent elements, so $a^2 \neq 0$. Thus, b is not an extreme point of the closed unit ball of $\operatorname{Mult}(\mathcal{H})$. $\square$

5. Counterexamples and questions

5.1. Failure of the complete column-row property

We briefly discuss how Theorem 1.2 (the column-row property) and Corollary 3.16 (the column-matrix property) can be interpreted in the theory of operator spaces and how a natural generalization of these results fails.

Let $N \in \mathbb{N}$ and let

$$\mathcal{M}^C = \mathrm{Mult}(\mathcal{H}, \mathcal{H} \otimes \mathbb{C}^N) \quad \text{and} \quad \mathcal{M}^R = \mathrm{Mult}(\mathcal{H} \otimes \mathbb{C}^N, \mathcal{H})$$

be the space of column, respectively row, multipliers with N components, both equipped with the multiplier norm. (One could also allow infinite columns and rows, but we restrict to columns and rows of a fixed finite length for simplicity.) Theorem 1.2 shows that the transpose mapping $T: \mathcal{M}^C \to \mathcal{M}^R$ is contractive.

Identifying a multiplier with its multiplication operator, we see that $\mathcal{M}^C$ and $\mathcal{M}^R$ are in fact concrete operator spaces. In particular, for each $n, m \in \mathbb{N}$, there is a natural norm on $M_{n,m}(\mathcal{M}^C)$ and $M_{n,m}(\mathcal{M}^R)$, via the identifications

$$M_{n,m}(\mathcal{M}^C) = \mathrm{Mult}(\mathcal{H} \otimes \mathbb{C}^m, \mathcal{H} \otimes \mathbb{C}^N \otimes \mathbb{C}^n)$$

and

$$M_{n,m}(\mathcal{M}^R) = \mathrm{Mult}(\mathcal{H} \otimes \mathbb{C}^N \otimes \mathbb{C}^m, \mathcal{H} \otimes \mathbb{C}^n).$$

It is not hard to see that Corollary 3.16 is equivalent to the assertion that, for each $n \in \mathbb{N}$, the induced map

$$T^{(n,1)}: M_{n,1}(\mathcal{M}^C) \longrightarrow M_{n,1}(\mathcal{M}^R),$$

defined by applying T entrywise, is contractive.

But if $N \geqslant 2$, it is very easy to see that T is not completely contractive, i.e. that

$$T^{(n,m)}: M_{n,m}(\mathcal{M}^C) \longrightarrow M_{n,m}(\mathcal{M}^R)$$

is not contractive for all $n, m \in \mathbb{N}$. This is simply because of the failure of complete contractivity of the ordinary transpose map. Indeed, suppose for simplicity that $N=2$ and consider the element

$$\begin{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{bmatrix} \in M_{1,2}(\mathcal{M}^C),$$

which has norm 1, but

$$T^{(1,2)}\left(\begin{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{bmatrix} \right) = [\,[1 \;\; 0]\; [0 \;\; 1]\,] \in M_{1,2}(\mathcal{M}^R),$$

which has norm $\sqrt{2}$.

5.2. Failure of the column-row property in other reproducing kernel Hilbert spaces

While Theorem 1.2 applies to complete Pick spaces, there are other reproducing kernel Hilbert spaces that satisfy the column-row poperty. For instance, subspaces of L^2-spaces, such as (weighted) Bergman spaces or the Hardy space on the ball or the polydisc, trivially satisfy the column-row property with constant 1. Nevertheless, there are counterexamples. We first construct a family of rotationally invariant spaces of holomorphic functions on the unit disc that do not satisfy the column-row property with constant 1.

Let $\alpha>1$ and, for $n\geqslant 0$, define

$$a_n = \begin{cases} 1, & \text{if } n \neq 2, \\ \dfrac{1}{\alpha}, & \text{if } n = 2. \end{cases}$$

Let $\mathcal{H}$ be the reproducing kernel Hilbert space on $\mathbb{D}$ with kernel

$$K(z,w) = \sum_{n=0}^{\infty} a_n (z\overline{w})^n. \tag{5.1}$$

It is well known that the monomials $(z^n)_{n=0}^{\infty}$ form an orthogonal basis of $\mathcal{H}$ with

$$\|z^n\|^2 = \frac{1}{a_n}, \quad n \in \mathbb{N};$$

see, for example, [14, §2.1]. Thus, $\mathcal{H}=H^2$, with equivalent but not equal norms.

We will show that the row multiplier norm of the pair (z, z^2) exceeds the column multiplier norm. Using the fact the monomials form an orthogonal basis, it is easy to check that

$$\left\| \begin{bmatrix} z \\ z^2 \end{bmatrix} \right\|_{\mathrm{Mult}}^2 = \sup_{n\in\mathbb{N}} \frac{1}{\|z^n\|^2} \left\| \begin{bmatrix} z \\ z^2 \end{bmatrix} z^n \right\|^2 = \sup_{n\in\mathbb{N}} \left(\frac{a_n}{a_{n+1}} + \frac{a_n}{a_{n+2}} \right) = 1+\alpha,$$

where we used that $\alpha>1$ in the last step. On the other hand,

$$\| [\, z \ \ z^2 \,] \|_{\mathrm{Mult}}^2 \geqslant \frac{\left\| [\, z \ \ z^2 \,] \begin{bmatrix} z \\ 1 \end{bmatrix} \right\|^2}{\left\| \begin{bmatrix} z \\ 1 \end{bmatrix} \right\|^2} = \frac{1}{2} \|2z^2\|^2 = 2\alpha.$$

Therefore,

$$\frac{\| [\, z \ \ z^2 \,] \|_{\mathrm{Mult}}^2}{\left\| \begin{bmatrix} z \\ z^2 \end{bmatrix} \right\|_{\mathrm{Mult}}^2} \geqslant \frac{2\alpha}{1+\alpha} > 1.$$

Morever, observe that the last ratio tends to 2 as $\alpha\to\infty$, which corresponds to the basic $\sqrt{n}$-bound mentioned in introduction.

It is not hard to check directly that $\mathcal{H}$ is not a complete Pick space. Indeed, if the reproducing kernel K is of the form (5.1), then $\mathcal{H}$ is a complete Pick space if and only if the coefficients $(b_n)_{n=1}^{\infty}$ defined by the power series identity

$$\sum_{n=1}^{\infty} b_n t^n = 1 - \frac{1}{\sum_{n=0}^{\infty} a_n t^n}$$

are all non-negative; see [2, Theorem 7.33]. A small computation shows that

$$b_2 = a_2 - a_1^2 = \frac{1}{\alpha} - 1 < 0,$$

so $\mathcal{H}$ is not a complete Pick space.

This example can be easily modified to obtain, for each constant $c>1$, a reproducing kernel Hilbert space $\mathcal{H}$ on the unit disc that does not satisfy the column-row property with constant c. Explicitly, one replaces the condition $n=2$ in the definition of (a_n) with the condition $n=k$ for some large natural number k, and compares the column and row norms of the tuple $(z, z^2, ..., z^k)$. Moreover, by taking a direct sum of such examples, one obtains a single reproducing kernel Hilbert space (on the disjoint union of countably many copies of the unit disc) that admits sequences of multipliers that induce bounded column, but unbounded row multiplication operators. We omit the details.

With more care, it is possible to obtain a single reproducing kernel Hilbert space on the unit disc that does not satisfy the column-row property with any constant.

PROPOSITION 5.1. *There exists a reproducing kernel Hilbert space $\mathcal{H}$ on the unit disc with the following properties*:

(1) *$\mathcal{H}$ consists of holomorphic functions on the unit disc*;

(2) *the reproducing kernel K of $\mathcal{H}$ has the form*

$$K(z,w) = \sum_{n=0}^{\infty} a_n (z\overline{w})^n,$$

where $a_0=1$ and $a_n>0$ for all $n>0$;

(3) *$\mathcal{H}$ does not satisfy the column-row property with any constant.*

Therefore, there exists a sequence of multipliers of $\mathcal{H}$ that induces a bounded column, but an unbounded row multiplication operator.

Proof. We recursively define a strictly increasing sequence $(p_k)_{k=1}^{\infty}$ of natural numbers by $p_1=1$ and $p_{k+1}=(2k+1)p_k$ for $k\geqslant 2$. These numbers partition the set of non-zero natural numbers into blocks, the kth block being

$$B_k = \{n \in \mathbb{N} : p_k \leqslant n < p_{k+1}\}.$$

On each block B_k, we set a_n to be 1 on the first half of the block, and then decreasing by factors of 2 in k steps of equal length p_k. Explicitly, for each $k \geqslant 1$,

$$a_n = \begin{cases} 1, & \text{if } p_k \leqslant n < (k+1)p_k, \\ 2^{-j}, & \text{if } (k+j)p_k \leqslant n < (k+j+1)p_k, \text{ where } j = 1, ..., k. \end{cases} \tag{5.2}$$

We also set $a_0 = 1$. Clearly, the power series

$$\sum_{n=0}^{\infty} a_n t^n$$

has radius of convergence 1, so defining $\mathcal{H}$ to be the reproducing kernel Hilbert space on $\mathbb{D}$ with kernel

$$K(z,w) = \sum_{n=0}^{\infty} a_n (z\overline{w})^n,$$

we obtain a space satisfying the first two properties of the proposition.

To show that $\mathcal{H}$ does not satisfy the column-row property with any constant, we will compare the row norm and the column norm of the tuple $(z^{p_k}, z^{p_k+1}, ..., z^{p_{k+1}-1})$ for $k \geqslant 1$. As mentioned in the discussion preceding the proposition, the monomials $(z^n)_{n=0}^{\infty}$ form an orthogonal basis of $\mathcal{H}$ with $\|z^n\|^2 = 1/a_n$ for all n. Since

$$\left| \frac{a_n}{a_{n+1}} \right| \leqslant 2 \quad \text{for all } n,$$

it follows that z, and hence all polynomials, are indeed multipliers of $\mathcal{H}$. A routine computation shows that

$$M_{z^j}^* z^n = \frac{a_{n-j}}{a_n} z^{n-j}, \quad j, n \in \mathbb{N},$$

where we use the convention that $a_k = 0$ for $k < 0$. Thus, using the C^*-identity, we find that

$$\|[\, z^{p_k} \;\; z^{p_k+1} \;\; ... \;\; z^{p_{k+1}-1} \,]\|_{\mathrm{Mult}}^2 = \left\| \sum_{j=p_k}^{p_{k+1}-1} M_{z^j} M_{z^j}^* \right\| = \sup_{n \geqslant 0} \sum_{j=p_k}^{p_{k+1}-1} \frac{a_{n-j}}{a_n}, \tag{5.3}$$

as the operator in the middle is diagonal with respect to the orthogonal basis of monomials. Similarly,

$$\left\| \begin{bmatrix} z^{p_k} \\ \vdots \\ z^{p_{k+1}-1} \end{bmatrix} \right\|_{\mathrm{Mult}}^2 = \left\| \sum_{j=p_k}^{p_{k+1}-1} M_{z_j}^* M_{z_j} \right\| = \sup_{n \geqslant 0} \sum_{j=p_k}^{p_{k+1}-1} \frac{a_n}{a_{n+j}}. \tag{5.4}$$

We will now estimate the norm of the row from below and the norm of the column from above. For the row, we set $n=p_{k+1}-1$ in the supremum in (5.3), thus $a_n=2^{-k}$, to find that

$$\|[\, z^{p_k} \;\; z^{p_k+1} \;\; \dots \;\; z^{p_{k+1}-1} \,]\|_{\mathrm{Mult}}^2 \geqslant 2^k \sum_{j=p_k}^{p_{k+1}-1} a_{p_{k+1}-1-j} = 2^k \sum_{r=0}^{2kp_k-1} a_r \geqslant 2^k k p_k,$$

because $a_r=1$ if $p_k \leqslant r < (k+1)p_k$.

For the column, we distinguish two cases when estimating the supremum in (5.4), namely $n \leqslant (k+1)p_{k+1}$ and $n \geqslant p_{k+1}$ (the two cases overlap). In the first case, the trivial estimate $a_n \leqslant 1$ yields

$$\sum_{j=p_k}^{p_{k+1}-1} \frac{a_n}{a_{n+j}} \leqslant \sum_{j=p_k}^{p_{k+1}-1} \frac{1}{a_{n+j}}.$$

This sum has $2kp_k$ terms. For j as in the sum, we have

$$p_k \leqslant n+j < p_{k+1}+n \leqslant (k+2)p_{k+1}.$$

In this range, the only values of a_{n+j} different from 1 occur for

$$(k+1)p_k \leqslant n+j < p_{k+1}$$

and are given by (5.2). Hence,

$$\sum_{j=p_k}^{p_{k+1}-1} \frac{a_n}{a_{n+j}} \leqslant p_k \sum_{j=1}^{k} 2^j + kp_k \leqslant 2^{k+2} p_k.$$

In the second case, $n \geqslant p_{k+1}$, we instead estimate, again using (5.2),

$$\sum_{j=p_k}^{p_{k+1}-1} \frac{a_n}{a_{n+j}} \leqslant 2kp_k \max_{p_k \leqslant j < p_{k+1}} \frac{a_n}{a_{n+j}} \leqslant 4kp_k \leqslant 2^{k+2} p_k.$$

Thus, using (5.4), we obtain the upper bound

$$\left\| \begin{bmatrix} z^{p_k} \\ \vdots \\ z^{p_{k+1}-1} \end{bmatrix} \right\|_{\mathrm{Mult}}^2 \leqslant 2^{k+2} p_k.$$

Combining the bounds for row and column norm, we see that

$$\frac{\|[\, z^{p_k} \;\; z^{p_k+1} \;\; \dots \;\; z^{p_{k+1}-1} \,]\|_{\mathrm{Mult}}^2}{\left\| \begin{bmatrix} z^{p_k} \\ \vdots \\ z^{p_{k+1}-1} \end{bmatrix} \right\|_{\mathrm{Mult}}^2} \geqslant \frac{k}{4}$$

for all $k \geqslant 1$. Hence, $\mathcal{H}$ does not satisfy the column-row property for any constant. Therefore, for each $j \in \mathbb{N}$, there exists a finite tuple Φ_j of multipliers whose column norm is at most 2^{-j}, but whose row norm is at least 2^j. Considering the union of all Φ_j as one sequence, we obtain a sequence of multipliers that yields a bounded column operator, but not a bounded row operator. □

5.3. Questions

While most of the known applications of the column-row property occur within the realm of complete Pick spaces, one may still ask the following natural question.

Question 5.2. Which reproducing kernel Hilbert spaces satisfy the column-row property (or the column-matrix property) with constant 1?

As mentioned in §5.2, weighted Bergman spaces and the Hardy space on the ball or the polydisc satisfy the column-row property with constant 1, but are typically not complete Pick spaces.

For a concrete example, let $\mathcal{H}_a$ be the reproducing kernel Hilbert space on $\mathbb{B}_d$ with kernel $(1-\langle z, w\rangle)^{-a}$, where $a>0$. Then, $\mathcal{H}_a$ is a complete Pick space for $0<a\leqslant 1$, and $\mathrm{Mult}(\mathcal{H}_a) = H^\infty(\mathbb{B}_d)$ completely isometrically for $a \geqslant d$, so $\mathcal{H}_a$ satisfies the column-row property with constant 1 for $a \in (0,1] \cup [d, \infty)$. Moreover, by [7, §4], each space $\mathcal{H}_a$ on $\mathbb{B}_d$ satisfies the column-row property with some finite constant $c_{a,d}$.

Question 5.3. Does $\mathcal{H}_a$ satisfy the column-row property with constant 1 for $a \in (1, d)$?

Naturally, one can define the column-row property not just in the context of multiplier algebras, but more generally for any operator space. By slight abuse of terminology, we therefore ask.

Question 5.4. Which operator spaces satisfy the column-row property with constant 1?

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