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Khintchine's theorem and Diophantine approximation on manifolds

by

Victor Beresnevich

University of York
Heslington, York, U.K.

Lei Yang

Sichuan University
Chengdu, Sichuan, China

Dedicated to G. A. Margulis on the occasion of his 75th birthday.

1. Introduction

1.1. Khintchine's theorem and manifolds

To begin with, let us recall the notion of ψ-approximable points which is convenient for introducing the problems investigated in this paper. Here and elsewhere, $\psi: (0, +\infty) \to (0, 1)$ is a function that will be referred to as an *approximation function*. We will say that the point $\mathbf{y} = (y_1, ..., y_n) \in \mathbb{R}^n$ is *ψ-approximable* if the system

$$\left| y_i - \frac{p_i}{q} \right| < \frac{\psi(q)}{q}, \quad 1 \leqslant i \leqslant n, \tag{1.1}$$

holds for infinitely many $(p_1, ..., p_n, q) \in \mathbb{Z}^n \times \mathbb{N}$. The set of ψ-approximable points in $\mathbb{R}^n$ will be denoted by $\mathcal{S}_n(\psi)$. In the special case of $\psi_\tau(q) := q^{-\tau}$ for some $\tau > 0$, we will also write $\mathcal{S}_n(\tau)$ instead of $\mathcal{S}_n(\psi_\tau)$. Recall that, by Dirichlet's theorem [42], $\mathcal{S}_n(1/n) = \mathbb{R}^n$. For functions ψ that decay faster that $q^{-1/n}$, Khintchine [32], [33] discovered the following simple yet powerful criterion for the proximity of rational points to almost all points $\mathbf{y}$ in $\mathbb{R}^n$. We state it below in a modern (slightly less restrictive) form; see [3], [9], [37] for further details and generalisations. In what follows, $\mathcal{L}_n$ denotes Lebesgue measure on $\mathbb{R}^n$ and $\mathcal{L}_n(X) = \text{Full}$ means that the complement to $X \subset \mathbb{R}^n$ has Lebesgue measure zero.

Khintchine's theorem. *Given any decreasing approximation function ψ,*(1)

$$\mathcal{L}_n(\mathcal{S}_n(\psi)) = \begin{cases} 0, & \text{if } \sum_{q=1}^\infty \psi(q)^n < \infty, \\ \text{Full}, & \text{if } \sum_{q=1}^\infty \psi(q)^n = \infty. \end{cases} \tag{1.2}$$

(1) In the original version of Khintchine's theorem, $q\psi(q)$ is assumed to be decreasing, see [3] for an overview.

The convergence case of Khintchine's theorem is a simple application of the Borel–Cantelli lemma based on the trivial count of rational points of bounded height. However, studying the proximity of rational points to points $\mathbf{y}$ lying on a submanifold $\mathcal{M}\subset\mathbb{R}^n$ gives rise to major challenges. Indeed, extending Khintchine's proof to manifolds requires solving the notoriously difficult problem of counting rational points lying close to $\mathcal{M}$ [§1.6.1.2]MR3618787. This was first observed by Sprindžuk in [45, §2.6]. The main purpose of this paper is to address the following central problem that was initially communicated by Kleinbock and Margulis in their seminal paper on the Baker–Sprinžuk conjecture [36, §6.3] and later stated in a more general form(2) by Kleinbock, Lindestrauss and Weiss [35, Question 10.1].

Problem 1.1. Let $\mathcal{M}\subset\mathbb{R}^n$ be a non-degenerate submanifold. Verify if, for any monotonic function ψ, almost no/every point of $\mathcal{M}$ is ψ-approximable whenever the series

$$\sum_{q=1}^{\infty}\psi(q)^n \tag{1.3}$$

converges/diverges.

In this paper, we use the notion of non-degeneracy introduced in [36]. A map $\mathbf{f}\colon\mathbf{U}\to\mathbb{R}^n$, defined on an open subset $\mathbf{U}\subset\mathbb{R}^d$, is said to be *$l$-non-degenerate* at $\mathbf{x}_0\in\mathbf{U}$ if $\mathbf{f}$ is l times continuously differentiable on a neighborhood of $\mathbf{x}_0$, and the partial derivatives of $\mathbf{f}$ at $\mathbf{x}_0$ of orders up to l span $\mathbb{R}^n$. The map $\mathbf{f}$ is said to be *non-degenerate* at $\mathbf{x}_0$ if it is l-non-degenerate at $\mathbf{x}_0$ for some $l\in\mathbb{N}$. The map $\mathbf{f}$ is said to be *non-degenerate* if it is non-degenerate at $\mathcal{L}_d$-almost every point in $\mathbf{U}$. In turn, the immersed manifold $\mathcal{M}:=\mathbf{f}(\mathbf{U})$ is non-degenerate (at $\mathbf{y}_0=\mathbf{f}(x_0)$) if the immersion $\mathbf{f}\colon\mathbf{U}\to\mathbb{R}^n$ is non-degenerate (at $\mathbf{x}_0$). This readily extends to manifolds $\mathcal{M}$ that do not possess a global parameterisation via local parameterisations. As is well known, any real connected analytic manifold not contained in a hyperplane of $\mathbb{R}^n$ is non-degenerate [36]. In fact, it is non-degenerate at every point.

The special case $\psi(q)=q^{-\tau}$, $\tau>0$, of Problem 1.1 was posed by Sprindžuk for analytic manifolds [46] and famously resolved in full by Kleinbock and Margulis [36]. Note that for these approximation functions the divergence case is trivial thanks to Dirichlet's theorem. In [34] Kleinbock extended [36] to affine subspaces of $\mathbb{R}^n$ satisfying certain Diophantine conditions and to submanifolds of such subspaces that are non-degenerate with respect to them. In another direction, Kleinbock, Lindestrauss and Weiss [35] established the analogue of [36] for the supports of friendly measures. We also refer the reader to [45] and [12] for various preceding results.

(2) The general form incorporates the supports of so-called friendly measures, which essentially generalise the notion of non-degeneracy from manifolds to fractals. See [31] for recent advances on the version of Problem 1.1 for fractals.

For arbitrary monotonic ψ, Problem 1.1 turned out to be far more delicate. Its *divergence case* was settled for C^3 planar curves [4], and then fully resolved for analytic manifolds in arbitrary dimensions [2]. More recently, the latter was also extended to arbitrary non-degenerate curves [8], while for planar curves the non-degeneracy assumption was replaced by weak non-degeneracy in [10].(3)

The *convergence case* of Problem 1.1 for arbitrary ψ is a different story. It was resolved for $n=2$ for all C^3 non-degenerate curves in the breakthrough of Vaughan and Velani [47]. Later, Huang [23] extended this to weakly non-degenerate planar curves. However, known results in higher dimensions require various additional constrains on the geometry and dimension of manifolds, predominantly as a result of the use of tools based on Fourier analysis. A brief account of known results is as follows. Bernik [11] proved it for the manifolds in $\mathbb{R}^{dk}$ defined as the Cartesian products of $d\geqslant k\geqslant 2$ C^{k+1} non-degenerate curves in $\mathbb{R}^k$. Dodson, Rynne and Vickers [20] proved it for the manifolds $\mathcal{M}$ in $\mathbb{R}^n$ having at least two non-zero principle curvatures of the same sign with respect to every normal direction at $\mathbf{y}$ for almost all $\mathbf{y}\in\mathcal{M}$. Note that this geometric condition requires that the dimension $d=\dim\mathcal{M}$ satisfies the inequality $(d+1)d\geqslant 2n$. Vaughan, Velani, Zorin and the first named author of this paper [7] proved the convergence case of Problem 1.1 for 2-non-degenerate manifolds in $\mathbb{R}^n$ of dimension $d\geqslant\frac{1}{2}n+1$, with $n\geqslant 4$. They also proved it for hypersurfaces in $\mathbb{R}^3$ with Gaussian curvature non-vanishing almost everywhere [7, Corollary 5]. Simmons [43] further relaxed the conditions of [20] and [7] imposed on manifolds, albeit the restrictions on their dimension remain broadly the same and, for instance, rule out curves. In a related development Huang and Liu [27] proved a Khintchine-type theorem for affine subspaces satisfying certain Diophantine conditions.

Thus, the convergence case of Problem 1.1 remains fully open for curves in dimensions $n\geqslant 3$. Indeed, it is open for subclasses of non-degenerate manifolds in $\mathbb{R}^n$ of every dimension $d<n$. Even in the case of hypersurfaces, which are most susceptible to the methods used in preceding papers, the problem is not fully resolved, e.g. it is open for hypersurfaces in $\mathbb{R}^3$ of zero Gaussian curvature. In this paper we contrive no additional hypotheses on non-degenerate manifolds, and resolve the convergence case of Problem 1.1 in full. Our main result reads as follows.

THEOREM 1.2. *Let $n\geqslant 2$, a submanifold $\mathcal{M}\subset\mathbb{R}^n$ be non-degenerate, ψ be monotonic, and assume* (1.3) *converges. Then, almost all points on $\mathcal{M}$ are not ψ-approximable.*

Hausdorff measure and Hausdorff dimension are often used to distinguish between sets of Lebesgue measure zero, and thus refine the convergence case of Khintchine's

(3) A curve $\mathcal{C}\subset\mathbb{R}^2$ is *weakly non-degenerate* at a point $\mathbf{p}\in\mathcal{C}$ if there is a neighborhood of $\mathbf{p}$ that can be written as the uniform limit of a sequence of non-degenerate curves whose curvatures are uniformly bounded away from zero and infinity; see [10] for further details.

theorem. In this paper we make an extra step beyond Theorem 1.2 and establish such refinements. The precise statements are provided in §2.2, while §2.1 contains an overview of the preceding results and problems.

1.2. Rational points near manifolds

The proof of Theorem 1.2, and indeed its generalisation to Hausdorff measures stated in §2.2, are underpinned by a new result on rational points near non-degenerate manifolds stated in this section, which is of independent interest. For simplicity and without loss of generality, we will assume that the manifolds $\mathcal{M}$ of interest are immersed by maps $\mathbf{f}\colon\mathbf{U}\to\mathbb{R}^n$, where $\mathbf{U}\subset\mathbb{R}^d$ denotes an open subset. Furthermore, in view of the implicit function theorem, it is non-restrictive to assume that

$$\mathbf{f}(\mathbf{x})=(\mathbf{x},\boldsymbol{f}(\mathbf{x}))=(x_1,...,x_d,f_1(\mathbf{x}),...,f_m(\mathbf{x})), \tag{1.4}$$

where $d=\dim\mathcal{M}$ and $m=\operatorname{codim}\mathcal{M}$, $\mathbf{x}=(x_1,...,x_d)\in\mathbf{U}$ and

$$\boldsymbol{f}=(f_1,...,f_m)\colon\mathbf{U}\longrightarrow\mathbb{R}^m. \tag{1.5}$$

As we are interested in non-degenerate manifolds, the maps $\mathbf{f}$ must necessarily be C^2. Furthermore, when proving Theorem 1.2, we can deal with the manifold locally. Therefore, without loss of generality, we may assume that there is a constant $M\geqslant1$ such that

$$\max_{1\leqslant k\leqslant m}\ \max_{1\leqslant i,j\leqslant d}\ \sup_{\mathbf{x}\in\mathbf{U}}\ \max\left\{\left|\frac{\partial f_k(\mathbf{x})}{\partial x_i}\right|,\left|\frac{\partial^2 f_k(\mathbf{x})}{\partial x_i\partial x_j}\right|\right\}\leqslant M. \tag{1.6}$$

Given $t>0$, $0<\varepsilon<1$ and $\Delta\subset\mathbb{R}^d$, let

$$\mathcal{R}(\Delta;\varepsilon,t)=\left\{(\mathbf{p},q)\in\mathbb{Z}^{n+1}:0<q<e^t\ \text{and}\ \inf_{\mathbf{x}\in\Delta\cap\mathbf{U}}\left\|\mathbf{f}(\mathbf{x})-\frac{\mathbf{p}}{q}\right\|_\infty<\frac{\varepsilon}{e^t}\right\},$$

and let

$$N(\Delta;\varepsilon,t)=\#\mathcal{R}(\Delta;\varepsilon,t).$$

Thus, $N(\Delta;\varepsilon,t)$ counts the rational points $\mathbf{p}/q$ (not necessarily written in the lowest terms) of denominator $0<q<e^t$ lying εe^{-t}-close to $\mathbf{f}(\Delta\cap\mathbf{U})\subset\mathcal{M}$.

Counting rational points on manifolds is usually geared towards establishing estimates of the form

$$N(\mathbf{U};\varepsilon,e^t)\ll\varepsilon^m e^{(d+1)t}+E(\mathbf{U};\varepsilon,t), \tag{1.7}$$

where $E(\mathbf{U};\varepsilon,t)$ is an error term. In general, $E(\mathbf{U};\varepsilon,t)$ cannot be made smaller than e^{dt} for all non-degenerate manifolds, since a d-dimensional non-degenerate manifold may contain a $(d-1)$-dimensional rational subspace. Furthermore, a non-degenerate manifold may accumulate abnormally high number of rational points around points where it has a very high 'contact' with its tangent d-dimensional plane, for example, when this tangent plane is rational. In this case, even the estimate $E(\mathbf{U};\varepsilon,t)\ll e^{(d+1-\delta)t}$ may not be achievable for any $\delta>0$. Hence, establishing (1.7) with a useful error term requires imposing conditions beyond non-degeneracy. Worse still, major limitations on the dimension d and/or ε arise from the tools that are currently in use. In fact, the theory is only reasonably complete for curves in $\mathbb{R}^2$, thanks to the breakthrough of Vaughan and Velani [47], who proved (1.7) with $E(\mathbf{U};\varepsilon,t)=O(e^{t(1+\delta)})$ and arbitrary $\delta>0$ for any compact C^3 curve in $\mathbb{R}^2$ of non-zero curvature, and thus obtained the best possible strengthening of Huxley's earlier result [28]. The complementary lower bound was found in [4] and various improvements for planar curves can be found in [10], [19], [21], [23], [26]. Results in higher dimensions are limited to manifolds with various additional hypothesis that we already discussed in §1.1, and can be found in [7], [27], [43]. More recently, Huang [25] obtained essentially the best possible bound on the error term in (1.7) for a class of hypersurfaces in $\mathbb{R}^n$ with non-zero Gaussian curvature. A further generalisation of Huang was found by Schindler and Yamagishi [39]. It is commonly believed that the biggest challenge is establishing (1.7) is posed by the case of curves in $\mathbb{R}^n$ for which we have virtually no results. In this context, it is worth mentioning the recent work of Huang [24] on rational points near non-degenerate curves in $\mathbb{R}^3$ of fixed denominator and whose main result implies (1.7) with $E(\mathbf{U};\varepsilon,t)=O(t^{4/5}e^{8t/5})$. However, with reference to this bound, the main term in (1.7) becomes dominant only when $\varepsilon\geqslant e^{-t/5}$, and so it cannot be used to resolve Problem 1.1 for curves in $\mathbb{R}^3$, which requires understanding rational points lying much closer to the manifolds in question, namely $\varepsilon=o(e^{-t/3})$.

In this paper, we deal with all non-degenerate manifolds, including non-degenerate curves by introducing a new approach, which involves splitting a manifold into a 'generic part' and a 'special part' using diagonal actions on the space of lattices. In short, we establish a sharp upper bound for the number of rational points lying near the generic part, which agrees with the main term in (1.7); see (1.9) below. Regarding the special part, we establish explicit bounds on the size of the special part which decay exponentially and uniformly for $e^{-3t/2n+\delta}\leqslant\varepsilon<1$ as $t\to\infty$, where $\delta>0$ is arbitrary; see (1.8) below. The key new idea is to apply the so-called quantitative non-divergence estimate on the space of lattices [13, Theorem 6.2] in order to demonstrate that the size of the special part is small, and to use certain tools from homogeneous dynamics and the geometry of numbers to count rational points near the generic part. The previous results that we discussed above have been using tools from analytic number theory (a version of the circle method) relying on Fourier analysis, and Huxley [28] used a version of the determinate method of

Swinnerton–Dyer. Our main result on rational points is as follows.

THEOREM 1.3. *Suppose* $\mathbf{U}\subset\mathbb{R}^d$ *is open, and let* $\mathbf{f}\colon\mathbf{U}\to\mathbb{R}^n$ *be a* C^2 *map satisfying* (1.4) *and* (1.6). *Then, for any* $0<\varepsilon<1$ *and every* $t>0$, *there is a subset* $\mathfrak{M}(\varepsilon,t)\subset\mathbf{U}$, *which can be written as a union of balls in* $\mathbf{U}$ *of radius* $\varepsilon e^{-t/2}$ *of intersection multiplicity* $\leqslant N_d$, *where* N_d *is the Besicovitch constant, satisfying the following properties. For every* $\mathbf{x}_0\in\mathbf{U}$ *such that* $\mathbf{f}$ *is* l*-non-degenerate at* $\mathbf{x}_0$ *there is a ball* $B_0\subset\mathbf{U}$ *centred at* $\mathbf{x}_0$ *and constants* $K_0,t_0>0$, *depending on* B_0 *and* $\mathbf{f}$ *only, such that*

$$\mathcal{L}_d(\mathfrak{M}(\varepsilon,t)\cap B_0)\leqslant K_0(\varepsilon^n e^{3t/2})^{-1/d(2l-1)(n+1)} \quad \text{for } t\geqslant t_0, \tag{1.8}$$

and, for every ball $B\subset\mathbf{U}$ *and for all sufficiently large* t, *we have that*

$$N(B\backslash\mathfrak{M}(\varepsilon,t);\varepsilon,t)\leqslant K_1\varepsilon^m e^{(d+1)t}\mathcal{L}_d(B), \tag{1.9}$$

where K_1 *depends on* n *and* $\mathbf{f}$ *only.*

We now demonstrate how Theorem 1.3 is used to resolve Problem 1.1.

1.3. Proof of Theorem 1.2 modulo Theorem 1.3

To begin with, we give the following two auxiliary statements.

LEMMA 1.4. *If* $\mathbf{f}(\mathbf{x})\in\mathcal{S}_n(\psi)$, *then there are infinitely many* $t\in\mathbb{N}$ *such that*

$$\left\|\mathbf{f}(\mathbf{x})-\frac{\mathbf{p}}{q}\right\|_\infty<\frac{\psi(e^{t-1})}{e^{t-1}} \tag{1.10}$$

for some $(\mathbf{p},q)\in\mathbb{Z}^{n+1}$, *with* $e^{t-1}\leqslant q<e^t$.

Proof. If $\mathbf{f}(\mathbf{x})\in\mathcal{S}_n(\psi)$, then (1.1) holds for infinitely many $(\mathbf{p},q)\in\mathbb{Z}^{n+1}$, with arbitrarily large $q>0$. For each q, we can define the corresponding $t\in\mathbb{N}$ from the inequalities $e^{t-1}\leqslant q<e^t$. There are infinitely many $t\in\mathbb{N}$ arising this way, since q is unbounded. Finally, (1.10) follows from (1.1), since ψ is monotonically decreasing. □

LEMMA 1.5. *Let* $\psi\colon\mathbb{R}\to\mathbb{R}$ *be monotonic. Then,*

$$\sum_{q=1}^{\infty}\psi(q)^n<\infty \quad\Longleftrightarrow\quad \sum_{t=1}^{\infty}\psi(e^t)^n e^t<\infty.$$

Lemma 1.5 is a version of the Cauchy condensation test.

Proof of Theorem 1.2 modulo Theorem 1.3. Without loss of generality, we consider $\mathcal{M}$ of the form $\mathbf{f}(\mathbf{U})$, where $\mathbf{f}\colon\mathbf{U}\to\mathbb{R}^n$ is a non-degenerate immersion on an open subset $\mathbf{U}\subset\mathbb{R}^d$. Since $\mathbf{f}$ is non-degenerate, for almost every $\mathbf{x}_0\in\mathbf{U}$ the map $\mathbf{f}$ is non-degenerate at $\mathbf{x}_0$. Hence, without loss of generality, it suffices to prove that

$$\mathcal{L}_d(\{\mathbf{x}\in B_0:\mathbf{f}(\mathbf{x})\in\mathcal{S}_n(\psi)\})=0 \qquad \text{if (1.3) converges and } \psi \text{ is monotonic}$$

for a sufficiently small ball B_0 centred at $\mathbf{x}_0\in\mathbf{U}$, where $\mathbf{f}$ is l-non-degenerate at $\mathbf{x}_0$ for some $l\in\mathbb{N}$. Fix $\mathbf{x}_0$ and take B_0 as in Theorem 1.2.

Without loss of generality, we will assume that $\psi(q)\geqslant q^{-5/4n}$ for all $q>0$, as otherwise we can replace ψ with $\max\{\psi(q),q^{-5/4n}\}$. By Lemma 1.4, for any $T\geqslant1$,

$$\{\mathbf{x}\in B_0:\mathbf{f}(\mathbf{x})\in\mathcal{S}_n(\psi)\}\subset\bigcup_{t\geqslant T}(A_t\cup B_t), \tag{1.11}$$

where

$$A_t:=\mathfrak{M}(e\psi(e^{t-1}),t)\cap B_0$$

and

$$B_t:=\bigcup_{(\mathbf{p},q)}\left\{\mathbf{x}\in B_0:\left\|\mathbf{x}-\frac{\mathbf{p}'}{q}\right\|_\infty<\frac{\psi(e^{t-1})}{e^{t-1}}\right\},$$

with the union in B_t running over

$$(\mathbf{p},q)\in\mathcal{R}(B_0\setminus\mathfrak{M}(e\psi(e^{t-1}),t);e\psi(e^{t-1}),t).$$

By Theorem 1.3 and the assumption $\psi(q)\geqslant q^{-5/4n}$, we get that

$$\mathcal{L}_d(A_t)\ll(e^{(t-1)/4})^{-1/d(2l-1)(n+1)}$$

and

$$\mathcal{L}_d(B_t)\ll\psi(e^{t-1})^m e^{(d+1)(t-1)}\left(\frac{\psi(e^{t-1})}{e^{t-1}}\right)^d=\psi(e^{t-1})^n e^{t-1}.$$

Hence, by Lemma 1.5, we get that

$$\mathcal{L}_d(\{\mathbf{x}\in B_0:\mathbf{f}(\mathbf{x})\in\mathcal{S}_n(\psi)\})\ll\sum_{t\geqslant T}(e^{(t-1)/4})^{-1/d(2l-1)(n+1)}+\sum_{t\geqslant T}\psi(e^{t-1})^n e^{t-1}$$

which tends to zero as $T\to\infty$, since the series above are convergent. Therefore,

$$\mathcal{L}_d(\{\mathbf{x}\in B_0:\mathbf{f}(\mathbf{x})\in\mathcal{S}_n(\psi)\})=0,$$

and the proof is complete. □

2. Generalisations to Hausdorff measure and dimension

2.1. Problems and known results

To begin with, let us recall two classical results in this area due to Jarník and Besicovitch, which represent the Hausdorff dimension and Hausdorff measure refinements of Khintchine's theorem.

JARNÍK–BESICOVITCH THEOREM. ([14], [29]) *Let* $\tau \geqslant 1/n$. *Then,*

$$\dim \mathcal{S}_n(\tau) = \frac{n+1}{\tau+1}. \tag{2.1}$$

JARNÍK'S THEOREM. ([30]) ([4]) *Given any monotonic function* ψ *and* $0<s<n$,

$$\mathcal{H}^s(\mathcal{S}_n(\psi)) = \begin{cases} 0, & \text{if } \sum_{q=1}^{\infty} q^n(\psi(q)/q)^s < \infty, \\ \infty, & \text{if } \sum_{q=1}^{\infty} q^n(\psi(q)/q)^s = \infty. \end{cases} \tag{2.2}$$

In the above and elsewhere, 'dim' denotes the Hausdorff dimension and $\mathcal{H}^s$ denotes the s-dimensional Hausdorff measure. The following general problem aims at refining the Lebesgue measure theory of ψ-approximable points on manifolds. It is geared towards establishing the analogues of the theorems of Jarník and Besicovitch, and it incorporates Khintchine-type theorems for manifolds as the special case $s = \dim \mathcal{M}$.

Problem 2.1. Given a smooth submanifold $\mathcal{M} \subset \mathbb{R}^n$, determine the Hausdorff dimension s of the set $\mathcal{S}_n(\psi) \cap \mathcal{M}$, and furthermore determine the s-dimensional Hausdorff measure of the set $\mathcal{S}_n(\psi) \cap \mathcal{M}$.

As before, our interest in Problem 2.1 will be focused on non-degenerate manifolds. It is well known that, for approximation functions ψ that decay relatively fast, the problem cannot have the same answer for all such manifolds, even if the manifolds are non-degenerate at every point. This is easily illustrated by the following example, whose details can be found in [2]. Let $\mathcal{C}_r$ be the circle in $\mathbb{R}^2$ defined by the equation

$$x^2 + y^2 = r.$$

Then,

$$\dim \mathcal{S}_2(\tau) \cap \mathcal{C}_1 = \frac{1}{\tau+1} \quad \text{while} \quad \dim \mathcal{S}_2(\tau) \cap \mathcal{C}_3 = 0 \text{ for all } \tau > 1. \tag{2.3}$$

This naturally leads to the following problem.

([4]) The original statement of Jarník's theorem had additional constrains; see [3] for details.

Problem 2.2. (Dimension problem) Let $1\leqslant d<n$ be integers and $\widetilde{\mathcal{M}}$ be a class of submanifolds $\mathcal{M}\subset\mathbb{R}^n$ of dimension d. Find the maximal value $\tau(\widetilde{\mathcal{M}})$ such that

$$\dim \mathcal{S}_n(\tau)\cap\mathcal{M}=\frac{n+1}{\tau+1}-\operatorname{codim}\mathcal{M} \quad \text{whenever } \frac{1}{n}\leqslant\tau<\tau(\widetilde{\mathcal{M}}) \tag{2.4}$$

for every manifold $\mathcal{M}\in\widetilde{\mathcal{M}}$. In particular, find $\tau_{n,d}:=\tau(\widetilde{\mathcal{M}}_{n,d})$ for the class $\widetilde{\mathcal{M}}_{n,d}$ of manifolds in $\mathbb{R}^n$ of dimension d which are non-degenerate at every point.([5])

Formula (2.4) for the dimension is informed by the volume-based expectation for the number of rational points lying close to $\mathcal{M}$; see [2] and [6, §1.6.2]. The dimension problem was resolved for non-degenerate planar curves in [4] and [10] on establishing that $\tau_{2,1}=1$. Furthermore, a Jarník-type theorem was established in [4] and [47] regarding the s-dimensional Hausdorff measure of $\mathcal{S}_2(\psi)\cap\mathcal{C}$ for C^3 non-degenerate planar curves $\mathcal{C}$.

THEOREM 2.3. (See [4], [47]) *Given any monotonic approximation function ψ, any $s\in\left(\frac{1}{2},1\right)$ and any C^3 planar curve $\mathcal{C}$ non-degenerate at every point, we have that*

$$\mathcal{H}^s(\mathcal{S}_2(\psi)\cap\mathcal{C})=\begin{cases}0, & \text{if } \sum_{q=1}^{\infty}q^{1-s}\psi(q)^{1+s}<\infty,\\ \infty, & \text{if } \sum_{q=1}^{\infty}q^{1-s}\psi(q)^{1+s}=\infty.\end{cases} \tag{2.5}$$

The C^3 hypothesis was removed from Theorem 2.3 in the case of divergence [10], and for a subrange of s in the case of convergence [23], where Theorem 2.3 was extended to weakly non-degenerate curves.

In higher dimensions, there are various speculations as to what $\tau_{n,d}$ might be. Let us first discuss the manifolds of dimension $d>1$. Consider the non-degenerate manifold $\mathcal{M}$ in $\mathbb{R}^n$ immersed by the map

$$(x_1,\dots,x_d)\longmapsto(x_1,\dots,x_d,x_d^2,\dots,x_d^{n+1-d}). \tag{2.6}$$

Then, $\mathcal{M}$ contains $\mathbb{R}^{d-1}\times\{\mathbf{0}\}$, and so $\dim\mathcal{S}_n(\tau)\cap\mathcal{M}\geqslant\dim\mathcal{S}_{d-1}(\tau)$. Therefore, by the Jarník–Besicovitch theorem, we have that

$$\dim\mathcal{S}_n(\tau)\cap\mathcal{M}\geqslant\frac{d}{\tau+1}>\frac{n+1}{\tau+1}-\operatorname{codim}\mathcal{M} \quad \text{whenever } \tau>\frac{1}{n-d}.$$

This means that $\tau_{n,d}\leqslant 1/(n-d)$ for $d>1$. Any improvement to this hard bound on $\tau_{n,d}$ would require restricting $\mathcal{M}$ to a smaller subclass of manifolds. Nevertheless, in all likelihood within the class $\widetilde{\mathcal{M}}_{n,d}$ of non-degenerate manifolds defined within Problem 2.2, this upper bound is exact. We state this formally now as a conjecture.

([5]) The hypothesis of non-degeneracy can be asked everywhere except on a set of dimension $\leqslant\dim\mathcal{S}_n(\tau)\cap\mathcal{M}$. However, this relaxation will not make the problem more general.

Conjecture 2.4. Let $1<d<n$. Then,

$$\tau_{n,d}=\frac{1}{n-d}.$$

The following lower bound towards Conjecture 2.4 was established in [5]:

$$\dim\mathcal{S}_n(\tau)\cap\mathcal{M}\geqslant\frac{n+1}{\tau+1}-\operatorname{codim}\mathcal{M}\quad\text{whenever }\ \frac{1}{n}\leqslant\tau<\frac{1}{n-d},\tag{2.7}$$

which is valid literally for every C^2 submanifold $\mathcal{M}\subset\mathbb{R}^n$ of every dimension $1\leqslant d<n$. In particular, it does not require non-degeneracy or any other constrain on $\mathcal{M}$. Furthermore, in the case of analytic non-degenerate submanifolds of $\mathbb{R}^n$, the following more subtle Hausdorff measure version of (2.7) for generic ψ was obtained in [2] generalising the divergence part of Theorem 2.3.

Theorem 2.5. ([2, Theorem 2.5]) *For every analytic non-degenerate submanifold $\mathcal{M}$ of $\mathbb{R}^n$ of dimension d and codimension $m=n-d$, any monotonic ψ such that*

$$q\psi(q)^m\to\infty$$

as $q\to\infty$ and any $s\in(md/(m+1),d)$, we have that

$$\mathcal{H}^s(\mathcal{S}_n(\psi)\cap\mathcal{M})=\infty\tag{2.8}$$

whenever the series

$$\sum_{q=1}^{\infty}q^n\left(\frac{\psi(q)}{q}\right)^{s+m}\tag{2.9}$$

diverges.

The remaining problem in establishing Conjecture 2.4 is to get the upper bound for the dimension. Partial progress was made in [7], [24], [25], [27], [39], [43] as a consequence of results on counting rational points; see §1.2. However, as with Problem 1.1, Problem 2.2 remains open for curves in dimensions $n\geqslant3$ and subclasses of non-degenerate manifolds in $\mathbb{R}^n$ of every dimension $d<n$.

Non-degenerate curves are of special interest for various reasons. First of all, curves cannot contain rational subspaces, and so example (2.6) is not applicable to them. Curves can be used to analyse manifolds of higher dimensions using fibering techniques. In fact, Theorem 2.5, and consequently the lower bound (2.7), hold in the following stronger form for non-degenerate curves.

THEOREM 2.6. (See [2, Theorem 7.2] and [8]) *For every curve $\mathcal{C}$ in $\mathbb{R}^n$ non-degenerate at every point, any monotonic ψ such that $q\psi(q)^{(2n-1)/3}\to\infty$ as $q\to\infty$ and any $s\in\left(\frac{1}{2},1\right)$, we have that*

$$\mathcal{H}^s(\mathcal{S}_n(\psi)\cap\mathcal{C})=\infty \tag{2.10}$$

whenever the series (2.9) with $m=n-1$ diverges. Consequently,

$$\dim\mathcal{S}_n(\tau)\cap\mathcal{C}\geqslant\frac{n+1}{\tau+1}-(n-1)\quad\textit{whenever}\ \frac{1}{n}\leqslant\tau<\frac{3}{2n-1}. \tag{2.11}$$

Conjecture 2.7. (Curves) For every $n\geqslant2$ we have that

$$\tau_{n,1}=\frac{3}{2n-1}.$$

2.2. New results on Hausdorff measure and dimension

Here we provide generalisations of Theorem 1.2 to s-dimensional Hausdorff measures and Hausdorff dimension, which thus contribute to resolving the problems surveyed in §2.1. The following is our key outcome on Hausdorff measures.

THEOREM 2.8. *Let $n\geqslant2$ be an integer, $s>0$ and $\mathcal{M}$ be a submanifold of $\mathbb{R}^n$ such that*

$$\mathcal{H}^s(\{\mathbf{y}\in\mathcal{M}:\mathcal{M}\ \textit{is not } l\textit{-non-degenerate at}\ \mathbf{y}\})=0. \tag{2.12}$$

Let $d=\dim\mathcal{M}$, $m=\operatorname{codim}\mathcal{M}$ and ψ be a monotonic approximation function such that the series (2.9) converges and

$$\sum_{t=1}^{\infty}\left(\frac{\psi(e^t)}{e^{t/2}}\right)^{s-d}(\psi(e^t)^n e^{3t/2})^{-\alpha}<\infty,\quad\textit{where}\ \alpha:=\frac{1}{d(2l-1)(n+1)}. \tag{2.13}$$

Then,

$$\mathcal{H}^s(\mathcal{S}_n(\psi)\cap\mathcal{M})=0. \tag{2.14}$$

The following statement is our key result on the Hausdorff dimension.

COROLLARY 2.9. *Let $n\geqslant2$ be an integer, $\mathcal{M}$ be a submanifold of $\mathbb{R}^n$ of dimension d, which is l-non-degenerate everywhere, except possibly on a set of Hausdorff dimension $\leqslant(n+1)/(\tau+1)-\operatorname{codim}\mathcal{M}$. Let $\tau\geqslant1/n$ satisfy*

$$\frac{n\tau-1}{\tau+1}\leqslant\frac{\alpha(3-2n\tau)}{2\tau+1}, \tag{2.15}$$

where α is the same as in (2.13). Then,

$$\dim(\mathcal{M}\cap\mathcal{S}_n(\tau))=\frac{n+1}{\tau+1}-\operatorname{codim}\mathcal{M}. \tag{2.16}$$

Similarly to Theorem 1.2, the proof of Theorem 2.8 is a rather simple consequence of our main result on rational points. We make no delay in showing its details.

Proof of Theorem 2.8 modulo Theorem 1.3. First of all, note that, since $\psi:\mathbb{R}\to\mathbb{R}$ is monotonic, by Cauchy's condensation test we have that

$$\sum_{q=1}^{\infty} q^n\left(\frac{\psi(q)}{q}\right)^{s+m}<\infty \iff \sum_{t=1}^{\infty} e^{(n+1)t}\left(\frac{\psi(e^t)}{e}^t\right)^{s+m}<\infty. \tag{2.17}$$

As before, without loss of generality, we consider $\mathcal{M}$ of the form $\mathbf{f}(\mathbf{U})$, where $\mathbf{f}:\mathbf{U}\to\mathbb{R}^n$ is a non-degenerate immersion of an open subset $\mathbf{U}\subset\mathbb{R}^d$. By (2.12), it suffices to prove that

$$\mathcal{H}^s(\{\mathbf{x}\in B_0:\mathbf{f}(\mathbf{x})\in\mathcal{S}_n(\psi)\})=0$$

whenever ψ is monotonic, (1.3) converges and (2.13) holds, where B_0 is a sufficiently small ball centred at $\mathbf{x}_0\in\mathbf{U}$, where $\mathbf{f}$ is l-non-degenerate at $\mathbf{x}_0$. Fix B_0 as in Theorem 1.2. By (1.4), for any $T\geqslant 1$ we have inclusion (1.11). By Theorem 1.3 with $\varepsilon=e\psi(e^{t-1})$, the set A_t can be covered by

$$\ll(\psi(e^{t-1})e^{-(t-1)/2})^{-d}(\psi(e^{t-1})^n e^{3(t-1)/2})^{-1/d(2l-1)(n+1)}$$

balls of radius $\psi(e^{t-1})e^{-(t-1)/2}$. Furthermore, by Theorem 1.3, we also have that the set B_t is the union of $\ll\psi(e^{t-1})^m e^{(d+1)(t-1)}$ balls of radius $\ll\psi(e^{t-1})/e^{t-1}$. Hence, by the definition of s-dimensional Hausdorff measure, we get that

$$\begin{aligned}
&\mathcal{H}^s(\{\mathbf{x}\in B_0:\mathbf{f}(\mathbf{x})\in\mathcal{S}_n(\psi)\})\\
&\quad\ll\sum_{t\geqslant T}\psi(e^{t-1})^m e^{(d+1)(t-1)}\left(\frac{\psi(e^{t-1})}{e^{t-1}}\right)^s\\
&\qquad+\sum_{t\geqslant T}(\psi(e^{t-1})e^{-(t-1)/2})^{s-d}(\psi(e^{t-1})^n e^{3(t-1)/2})^{-1/d(2l-1)(n+1)}.
\end{aligned} \tag{2.18}$$

The first sum equals

$$\sum_{t\geqslant T} e^{(n+1)(t-1)}\left(\frac{\psi(e^{t-1})}{e^{t-1}}\right)^{s+m}$$

and, by (2.17), tends to zero. The second sum in (2.18) also tends to zero as a consequence of (2.13). Hence,

$$\mathcal{H}^s(\{\mathbf{x}\in B_0:\mathbf{f}(\mathbf{x})\in\mathcal{S}_n(\psi)\})=0$$

and the proof is complete. □

Proof of Corollary 2.9. Let $L(\tau)$ and $R(\tau)$ denote the left- and right-hand sides of (2.15), respectively. First, observe that $L(\tau)$ is increasing and $R(\tau)$ is decreasing. Next, note that $L(1/n)=0$, while $R(1/n)>0$. Also, observe that $L(1/(n-1))=1/n$, while

$$R\left(\frac{1}{n-1}\right)<\alpha\leqslant\frac{1}{n+1}<L\left(\frac{1}{n-1}\right).$$

Hence, the set of solutions to (2.15) is a closed interval $I_{d,l,n}\subset[1/n,1/(n-1))$. In particular, for any $\tau\in I_{d,l,n}$, estimate (2.7) is applicable and therefore, to prove (2.16), we only need to prove the complementary upper bound. To this end, let

$$s>\frac{n+1}{\tau+1}-\operatorname{codim}\mathcal{M}$$

and $\psi(q)=q^{-\tau}$. Then, it is readily seen that (2.9) is convergent. Furthermore, by (2.15), one easily verifies condition (2.13). Condition (2.12) is also satisfied since $\mathcal{M}$ is l-nondegenerate everywhere except possibly on a set of Hausdorff dimension $<s$. Hence, Theorem 2.8 is applicable and we conclude that $\mathcal{H}^s(\mathcal{S}_n(\tau)\cap\mathcal{M})=0$. By definition, it means that $\dim(\mathcal{S}_n(\tau)\cap\mathcal{M})\leqslant s$. Since

$$s>\frac{n+1}{\tau+1}-\operatorname{codim}\mathcal{M}$$

is arbitrary, we obtain the require upper bound and complete the proof. □

Remark 2.10. It is not difficult to see that the monotonicity of ψ was only used to apply the Cauchy condensation test to establish (2.17) and to replace $\psi(q)/q$ with $\psi(e^{t-1})/e^{t-1}$ in Diophantine inequalities. The requirement that ψ is monotonic within Theorems 1.2 and 2.8 can therefore be replaced by a weaker assumption. For instance, one can replace the monotonicity of ψ with the following requirement: there exist a constant $C>0$ such that

$$\psi(q)\leqslant C\psi(e^{t-1})\qquad\text{for } e^{t-1}\leqslant q<e^t.$$

In fact, the use of the sequence e^t is not critical, and it can be replaced by any other sequence $s_t>0$ such that

$$1<\liminf_{t\to\infty}\frac{s_t}{s_{t-1}}\leqslant\limsup_{t\to\infty}\frac{s_t}{s_{t-1}}<\infty.$$

2.3. Spectrum of Diophantine exponents

Now, let us describe the implications of our results for a problem of Bugeaud and Laurent regarding the spectrum of the following Diophantine exponent introduced in [17]. Given $x\in\mathbb{R}$, let

$$\lambda_n(x):=\sup\{\tau>0:(x,x^2,\dots,x^n)\in\mathcal{S}_n(\tau)\}$$

be the exponent of simultaneous rational approximations to n consecutive powers of a real number x. By Dirichlet's theorem, we have that $\lambda_n(x)\in[1/n,+\infty]$ for any $x\in\mathbb{R}$. The spectrum of λ_n is defined as

$$\operatorname{spec}(\lambda_n):=\lambda_n(\mathbb{R}\setminus\mathbb{Q})=\{\lambda\in[1/n,+\infty]:\lambda=\lambda_n(x)\text{ for some }x\in\mathbb{R}\setminus\mathbb{Q}\}.$$

In 2007 Bugeaud and Laurent posed the following problem.

Problem 2.11. (Bugeaud–Laurent [18, Problem 5.5]) Is $\operatorname{spec}(\lambda_n)=[1/n,+\infty]$?

The following more subtle version of this problem was later raised in [16].

Problem 2.12. (Bugeaud [16, Problem 3.5]) For every $\lambda\geqslant 1/n$, determine

$$\dim\{x\in\mathbb{R}:\lambda_n(x)=\lambda\}\quad\text{and}\quad\dim\{x\in\mathbb{R}:\lambda_n(x)\geqslant\lambda\}.$$

To begin with, note that, by Sprindžuk's theorem [44], $\lambda_n(x)=1/n$ for almost all $x\in\mathbb{R}$. In particular, $1/n\in\operatorname{spec}(\lambda_n)$ for every n. For $n=1$, Problem 2.11 is relatively simple and can be solved, for instance, using continued fractions, while the answer to Problem 2.12 is provided by the Jarník–Besicovitch theorem stated at the start of §2.1. For $n=2$, Problem 2.12, and consequently Problem 2.11, was solved in [4] and [15]. In turn, Bugeaud [16] showed that $[1,+\infty]\subset\operatorname{spec}(\lambda_n)$ for any n using explicit examples, while Schleischitz [40] resolved Problem 2.12 for $\lambda>1$. The most significant challenge within Problems 2.11 and 2.12 is posed by the values of λ in the spectrum of λ_n which are <1. The first step in this direction was made by Schleischitz [41], who proved that $\operatorname{spec}(\lambda_3)$ contains points <1. Most recently, Badziahin and Bugeaud [1] made a major achievement by showing that

$$\left[\frac{n+4}{3n},+\infty\right]\subset\operatorname{spec}(\lambda_n)\quad\text{for every }n\geqslant 3$$

and resolving Problem 2.12 for $\lambda\geqslant(n+4)/3n$. Corollary 2.9 of our paper makes a first step in closing the gap in the spectrum of λ_n from the other end, namely for the values λ close to the Dirichlet exponent $1/n$. To produce an explicit statement, we now specialise Corollary 2.9 to curves. First of all, we state and prove the following proposition which allows us to fix the non-degeneracy parameter l.

PROPOSITION 2.13. *Let* $\mathbf{f}:\mathbf{U}\to\mathbb{R}^n$ *be* l*-non-degenerate at* $x_0\in\mathbf{U}$*, where* $\mathbf{U}$ *is an interval in* $\mathbb{R}$*. Then, there are an interval* B_0 *centred at* x_0 *and a countable subset* $S\subset B_0$ *such that* $\mathbf{f}$ *is* n*-non-degenerate at every point* $x\in B_0\setminus S$*.*

This proposition is a standard exercise in analysis relying on the following lemma.

LEMMA 2.14. *If $\varphi\colon\mathbf{U}\to\mathbb{R}$ is a C^1 function on an interval $\mathbf{U}$ and*

$$\mathcal{N}(\varphi):=\{x\in\mathbf{U}:\varphi(x)=0\},$$

then $\mathcal{N}(\varphi)\setminus\mathcal{N}(\varphi')$ consists on isolated points.

Proof. If $x_0\in\mathcal{N}(\varphi)\setminus\mathcal{N}(\varphi')$ is a limit point of $\mathcal{N}(\varphi)$, then there is a sequence $x_k\in\mathbf{U}\cap\mathcal{N}(\varphi)\setminus\{x_0\}$ converging to x_0. By the mean value theorem,

$$\varphi'(\tilde{x}_k)(x_k-x_0)=\varphi(x_k)-\varphi(x_0)=0,$$

where $\tilde{x}_k$ is between x_k and x_0. Thus, $\varphi'(\tilde{x}_k)=0$. Letting $k\to\infty$ and using the continuity of φ' gives

$$\varphi'(x_0)=\varphi'\Big(\lim_{k\to\infty}\tilde{x}_k\Big)=\lim_{k\to\infty}\varphi'(\tilde{x}_k)=0.$$

However, $x_0\notin\mathcal{N}(\varphi')$. Thus, x_0 cannot be a limit point of $\mathcal{N}(\varphi)$. □

Proof of Proposition 2.13. Since $\mathbf{f}$ is l-non-degenerate at x_0, we have that

$$\operatorname{rank}\{\mathbf{f}^{(i)}(x_0):1\leqslant i\leqslant l\}=n.$$

Since $\mathbf{f}$ is C^l, there is an interval B_0 centred at x_0 such that

$$\operatorname{rank}\{\mathbf{f}^{(i)}(x):1\leqslant i\leqslant l\}=n\quad\text{for all } x\in B_0.$$

If $l=n$, the statement is obvious. Thus, we will assume that $l>n$. Let

$$\varphi(x):=\det(f_j^{(i)}(x))_{1\leqslant i,j\leqslant n}$$

be the Wronskian of $\mathbf{f}'(x)$. Let $S_0=\{x\in B_0:\varphi(x)=0\}$ and $S_i=\{x\in S_{i-1}:\varphi^{(i)}(x)=0\}$ for $i=1,\dots,l-n$. By definition, $S_0\supset S_1\supset\dots\supset S_{l-n}$. By the choice of B_0, we have $S_{l-n}=\varnothing$. By Lemma 2.14, $S_{i-1}\setminus S_i$ is countable for every $1\leqslant i\leqslant l-n$. Hence,

$$S_0=(S_0\setminus S_1)\cup\dots\cup(S_{l-n-1}\setminus S_{l-n})$$

is countable, and the proof is complete. □

In view of Proposition 2.13, we can always apply Corollary 2.9 to non-degenerate curves with $l=n$. This gives the following statement.

COROLLARY 2.15. *Let $n\geqslant 2$ be an integer, $\mathcal{C}$ be a curve in $\mathbb{R}^n$, which is non-degenerate everywhere except possibly on a set of Hausdorff dimension*

$$\leqslant\frac{n+1}{\tau+1}-n+1.$$

Let $\tau\geqslant 1/n$ satisfy

$$\frac{n\tau-1}{\tau+1}\leqslant\frac{3-2n\tau}{(2\tau+1)(2n-1)(n+1)}.\tag{2.19}$$

Then,

$$\dim(\mathcal{C}\cap\mathcal{S}_n(\tau))=\frac{n+1}{\tau+1}-n+1.\tag{2.20}$$

On letting $n\tau=1+\delta$, (2.19) transforms into

$$\frac{\delta}{n+1+\delta}\leqslant\frac{1-2\delta}{(n+2+2\delta)(2n-1)(n+1)},$$

or equivalently

$$\delta^2(4n^2+2n)+\delta(2n^3+5n^2+3n-1)-n-1\leqslant 0. \tag{2.21}$$

Solving (2.21), we get that

$$0\leqslant\delta\leqslant\delta_n:=\frac{\sqrt{D_n}-B_n}{2A_n}, \tag{2.22}$$

where

$$\begin{aligned}
A_n &= 4n^2+2n,\\
B_n &= 2n^3+5n^2+3n-1,\\
D_n &= 4n^6+20n^5+37n^4+42n^3+23n^2+2n+1.
\end{aligned}$$

By (2.21), we also have that

$$\delta_n<\frac{1}{2n^2+5n}.$$

This also means that the first term in (2.21) is <1 for $n\geqslant 3$, and therefore (2.21) will hold whenever

$$\delta(2n^3+5n^2+3n-1)\leqslant n. \tag{2.23}$$

Also, observe that $6n^2\geqslant 5n^2+3n-1$ for $n\geqslant 3$. Hence, (2.23) is implied provided that $\delta(2n^2+6n)<1$. Therefore,

$$\frac{1}{2n^2+6n}<\delta_n<\frac{1}{2n^2+5n}. \tag{2.24}$$

Corollary 2.16. (The spectrum of λ_n) *For every $n\geqslant 3$,*

$$\left[\frac{1}{n},\frac{1}{n}+\frac{\delta_n}{n}\right]\subset\mathrm{spec}(\lambda_n),$$

where δ_n is given by (2.22) and can be estimated by (2.24).

3. Preliminaries

3.1. Notation and conventions

First, let us agree on some notation that we will use throughout the rest of the paper. By $\mathbb{I}_k$ we will denote the identity $k\times k$ matrix. Throughout, $\|\cdot\|$ and $\|\cdot\|_\infty$ will denote the Euclidean and supremum norms on $\mathbb{R}^k$ respectively. Given $r>0$ and $\mathbf{x}\in\mathbb{R}^d$, by $B(\mathbf{x},r)$

we will denote the Euclidean ball in $\mathbb{R}^d$ of radius r centred at $\mathbf{x}$, and respectively, by $\mathcal{B}(\mathbf{x},r)$ we will denote the $\|\cdot\|_\infty$-ball of radius r centred at $\mathbf{x}$, which for obvious reasons will be referred to as a hypercube.

We will use the Vinogradov and Bachmann–Landau notations: for functions f and positive-valued functions g, we write $f\ll g$ or $f=O(g)$ if there exists a constant C such that $|f|\leqslant Cg$ pointwise. We will write $f\asymp g$ if $f\ll g$ and $g\ll f$. Throughout,

$$G=\mathrm{SL}(n+1,\mathbb{R}) \quad \text{and} \quad \Gamma=\mathrm{SL}(n+1,\mathbb{Z}).$$

Then, the homogeneous space

$$X_{n+1}:=\mathrm{SL}(n+1,\mathbb{R})/\,\mathrm{SL}(n+1,\mathbb{Z})$$

can be identified with the set of all unimodular lattices in $\mathbb{R}^{n+1}$, where the coset $g\Gamma$ in X_{n+1} corresponds to the lattice $g\mathbb{Z}^{n+1}$ in $\mathbb{R}^{n+1}$. Note that the column vectors of g form a basis of $g\mathbb{Z}^{n+1}$.

3.2. Preliminaries from the geometry of numbers

Given a lattice $\Lambda\in X_{n+1}$ and an integer $1\leqslant i\leqslant n+1$, let

$$\lambda_i(\Lambda):=\inf\{\lambda>0: B(\mathbf{0},\lambda)\cap\Lambda \text{ contains } i \text{ linearly independent vectors}\}. \tag{3.1}$$

In other words, $\lambda_1(\Lambda)\leqslant\ldots\leqslant\lambda_{n+1}(\Lambda)$ are the *successive minima* of the closed unit ball $B(\mathbf{0},1)$ with respect to the lattice Λ.

Recall that, given a lattice $\Lambda\in X_{n+1}$, its *polar lattice* is defined as follows:

$$\Lambda^*=\{\mathbf{a}\in\mathbb{R}^{n+1}:\mathbf{a}\cdot\mathbf{b}\in\mathbb{Z} \text{ for every } \mathbf{b}\in\Lambda\}. \tag{3.2}$$

The following lemma is well known; see, e.g., [22, Theorem 21.5].

LEMMA 3.1. *Let $g\in G$. Then,*

$$(g\mathbb{Z}^{n+1})^*=(g^{\mathrm{T}})^{-1}\mathbb{Z}^{n+1},$$

where $(g^{\mathrm{T}})^{-1}$ is the inverse of the transpose of g.

Given a convex body $\mathcal{C}$ in $\mathbb{R}^{n+1}$ symmetric about $\mathbf{0}$, one defines the polar body

$$\mathcal{C}^*=\{\mathbf{y}\in\mathbb{R}^{n+1}:\mathbf{x}\cdot\mathbf{y}\leqslant 1 \text{ for all } \mathbf{y}\in\mathcal{C}\}.$$

It is readily seen that

$$B(\mathbf{0},1)^*=B(\mathbf{0},1).$$

Then, the following theorem on successive minima of the polar lattice is a direct consequence of a more general result of Mahler; see [22, Theorem 23.2].

THEOREM 3.2. (Mahler, see [22, Theorem 23.2]) *Let Λ be any lattice in $\mathbb{R}^{n+1}$. Then, for every $1\leqslant i\leqslant n+1$, we have that*

$$1\leqslant \lambda_i(\Lambda)\lambda_{n+2-i}(\Lambda^*)\leqslant ((n+1)!)^2.$$

Given $k\in\mathbb{N}$, define the following square $k\times k$ matrix:

$$\sigma_k=\begin{bmatrix} 0 & 0 & \dots & 0 & 1\\ 0 & 0 & \dots & 1 & 0\\ \vdots & \vdots & \iddots & \vdots & \vdots\\ 0 & 1 & \dots & 0 & 0\\ 1 & 0 & \dots & 0 & 0\end{bmatrix}.$$

In case $k=n+1$, we will simply write σ instead of σ_{n+1}. Note that σ_k is an involution, that is $\sigma_k^{-1}=\sigma_k$. Also note that σ_k acts on row-vectors on the right and on column-vectors on the left by placing their coordinates in the reverse order. Furthermore, we have that $g\mathbb{Z}^{n+1}=g\sigma\mathbb{Z}^{n+1}$ and $\sigma^{-1}(B(\mathbf{0},\lambda))=B(\mathbf{0},\lambda)$ for every $\lambda>0$. Therefore, for every $g\in G$,

$$\lambda_i(g\mathbb{Z}^{n+1})=\lambda_i(\sigma^{-1}g\sigma\mathbb{Z}^{n+1}). \tag{3.3}$$

Given $g\in G$, we will define the *dual* of g, denoted by g^*, by

$$g^*:=\sigma^{-1}(g^{\mathrm{T}})^{-1}\sigma. \tag{3.4}$$

It is readily seen that the dual of the product of matrices equals the product of dual matrices, that is

$$(g_1g_2)^*=g_1^*g_2^* \quad \text{for any } g_1,g_2\in G. \tag{3.5}$$

Further, in view of equation (3.3), Theorem 3.2 implies the following result.

LEMMA 3.3. *For any $g\in G$ and every $1\leqslant i\leqslant n+1$, we have that*

$$1\leqslant \lambda_i(g\mathbb{Z}^{n+1})\lambda_{n+2-i}(g^*\mathbb{Z}^{n+1})\leqslant ((n+1)!)^2.$$

3.3. A quantitative non-divergence estimate

We will make use of a version of the quantitative non-divergence estimate on the space of lattices due to Bernik, Kleinbock and Margulis [13, Theorem 6.2]. To be more precise, we will use a consequence of this non-divergence estimate appearing as Theorem 1.4 in [13]. Below, we state it in a slightly simplified form which fully covers our needs. In what follows, ∇ stands for the gradient of a real-valued function.

THEOREM 3.4. (See [13, Theorem 1.4]) *Let* $\mathbf{U}\subset\mathbb{R}^d$ *be open,* $\mathbf{x}_0\in\mathbf{U}$ *and* $\mathbf{f}:\mathbf{U}\to\mathbb{R}^n$ *be* l*-non-degenerate at* $\mathbf{x}_0$*. Then, there exist a ball* $B_0\subset\mathbf{U}$ *centred at* $\mathbf{x}_0$ *and a constant* $E\geqslant 1$ *such that, for any choice of*

$$0<\delta\leqslant 1,\quad T\geqslant 1\quad\text{and}\quad K>0\quad\text{satisfying}\quad \delta^n<KT^{n-1}, \tag{3.6}$$

the Lebesgue measure of the set

$$\begin{aligned}\mathfrak{S}_{\mathbf{f}}(\delta,K,T):=\{\mathbf{x}\in B_0:\ &\text{there exists } (a_0,\mathbf{a})\in\mathbb{Z}\times\mathbb{Z}^n \text{ such that}\\ &|a_0+\mathbf{f}(\mathbf{x})\mathbf{a}^T|<\delta,\ \|\nabla\mathbf{f}(\mathbf{x})\mathbf{a}^T\|_\infty<K \text{ and } 0<\|\mathbf{a}\|_\infty<T\}\end{aligned} \tag{3.7}$$

satisfies the inequality

$$\mathcal{L}_d(\mathfrak{S}_{\mathbf{f}}(\delta,K,T))\leqslant E(\delta KT^{n-1})^{1/d(2l-1)(n+1)}\mathcal{L}_d(B_0).$$

4. The generic and special parts

4.1. Dynamical reformulation

Recall that

$$\mathcal{R}(\Delta;\varepsilon,t)=\left\{(\mathbf{p},q)\in\mathbb{Z}^{n+1}:0<q<e^t \text{ and there is } \mathbf{x}\in\Delta\cap\mathbf{U} \text{ with } f(\mathbf{x})\in\mathcal{B}\left(\frac{\mathbf{p}}{q},\frac{\varepsilon}{e^t}\right)\right\}.$$

Our goal is to interpret the condition $\mathbf{f}(\mathbf{x})\in\mathcal{B}(\mathbf{p}/q,\varepsilon/e^t)$ in terms of properties of the action of $g_{\varepsilon,t}$ on a certain lattice in $\mathbb{R}^{n+1}$. With this goal in mind, given

$$\mathbf{y}=(y_1,\dots,y_n)\in\mathbb{R}^n,$$

define

$$U(\mathbf{y}):=\begin{bmatrix}\mathbb{I}_n & \sigma_n^{-1}\mathbf{y}^T\\ 0 & 1\end{bmatrix}=\begin{bmatrix}1 & \dots & 0 & y_n\\ & \ddots & \vdots & \vdots\\ & & 1 & y_1\\ & & & 1\end{bmatrix}\in G. \tag{4.1}$$

Also, given an $m\times d$ matrix $\Theta=[\theta_{i,j}]_{\substack{1\leqslant i\leqslant m\\1\leqslant j\leqslant d}}\in\mathbb{R}^{m\times d}$, let

$$Z(\Theta):=\begin{bmatrix}\mathbb{I}_m & \sigma_m^{-1}\Theta\sigma_d & 0\\ 0 & \mathbb{I}_d & 0\\ 0 & 0 & 1\end{bmatrix}=\begin{bmatrix}1 & \dots & 0 & \theta_{m,d} & \dots & \theta_{m,1} & 0\\ & \ddots & \vdots & \vdots & & \vdots & \vdots\\ & & 1 & \theta_{1,d} & \dots & \theta_{1,1} & 0\\ & & & 1 & \dots & 0 & 0\\ & & & & \ddots & \vdots & \vdots\\ & & & & & 1 & 0\\ & & & & & & 1\end{bmatrix}\in G. \tag{4.2}$$

For each $t>0$ and $0<\varepsilon<1$, define the following unimodular diagonal matrix:

$$g_{\varepsilon,t} := \mathrm{diag}\{\underbrace{\phi\varepsilon^{-1},\dots,\phi\varepsilon^{-1}}_{n},\phi e^{-t}\}\in G, \tag{4.3}$$

where

$$\phi := (\varepsilon^n e^t)^{1/(n+1)}. \tag{4.4}$$

Before moving on, we state a couple of conjugation equations involving $g_{\varepsilon,t}$.

LEMMA 4.1. *For any $t>0$, $\Theta\in\mathbb{R}^{m\times d}$ and $\mathbf{y}\in\mathbb{R}^n$, we have that*

$$g_{\varepsilon,t}U(\mathbf{y})g_{\varepsilon,t}^{-1} = U(e^t\varepsilon^{-1}\mathbf{y}), \tag{4.5}$$

$$g_{\varepsilon,t}Z(\Theta)g_{\varepsilon,t}^{-1} = Z(\Theta). \tag{4.6}$$

The proof is elementary and left to the reader.

LEMMA 4.2. *Let $\mathbf{y}\in\mathbb{R}^n$. Then, for any $t>0$ and any $\Theta\in\mathbb{R}^{m\times d}$, if $\mathbf{y}\in\mathcal{B}(\mathbf{p}/q,\varepsilon/e^t)$ for some $(\mathbf{p},q)\in\mathbb{Z}^{n+1}$ with $0<q<e^t$, then*

$$\|g_{\varepsilon,t}Z(\Theta)U(\mathbf{y})(-\mathbf{p}\sigma_n,q)^{\mathrm{T}}\| \leqslant c_0\phi, \tag{4.7}$$

where

$$c_0 = \sqrt{n+1}\max_{1\leqslant i\leqslant m}(1+|\theta_{i,1}|+\dots+|\theta_{i,d}|). \tag{4.8}$$

Proof. To begin with, note that, by $\mathbf{y}\in\mathcal{B}(\mathbf{p}/q,\varepsilon/e^t)$, we trivially have that

$$\|g_{\varepsilon,t}U(\mathbf{y})(-\mathbf{p}\sigma_n,q)^{\mathrm{T}}\|_\infty < \phi. \tag{4.9}$$

Then, using Lemma 4.1 we get that

$$\begin{aligned}\|g_{\varepsilon,t}Z(\Theta)U(\mathbf{y})(-\mathbf{p}\sigma_n,q)^{\mathrm{T}}\|_\infty &\overset{(4.6)}{=} \|Z(\Theta)g_{\varepsilon,t}U(\mathbf{y})(-\mathbf{p}\sigma_n,q)^{\mathrm{T}}\|_\infty\\ &\leqslant \|Z(\Theta)\|_\infty\cdot\|g_{\varepsilon,t}u(\mathbf{x})(-\mathbf{p}\sigma_n,q)^{\mathrm{T}}\|_\infty\\ &\overset{(4.9)}{\leqslant} \|Z(\Theta)\|_\infty\cdot\phi,\end{aligned} \tag{4.10}$$

where $\|Z(\Theta)\|_\infty$ is the operator norm of $Z(\Theta)$ as a linear transformation from $\mathbb{R}^{n+1}$ to itself equipped with the supremum norm. As is well known, $\|Z(\Theta)\|_\infty$ equals the maximum of ℓ_1 norms of its rows, that is,

$$\|Z(\Theta)\|_\infty = \max_{1\leqslant i\leqslant m}(1+|\theta_{i,1}|+\dots+|\theta_{i,d}|).$$

Now, taking into account that

$$\|\mathbf{a}\| \leqslant \sqrt{n+1}\|\mathbf{a}\|_\infty$$

for any $\mathbf{a}\in\mathbb{R}^{n+1}$, we obtain (4.7) immediately from (4.10). $\square$

Our next goal is to produce a similar statement when $\mathbf{y}=\mathbf{f}(\mathbf{x})$, where $\mathbf{f}$ is as in (1.4) and subject to condition (1.6). To this end, for $\mathbf{x}=(x_1,...,x_d)\in\mathbf{U}$, define

$$u(\mathbf{x}) := U(\mathbf{f}(\mathbf{x})), \tag{4.11}$$

where U is given by (4.1), and let

$$\mathbf{J}(\mathbf{x}) := \left[\frac{\partial f_i}{\partial x_j}(\mathbf{x})\right]_{\substack{1\leqslant i\leqslant m\\ 1\leqslant j\leqslant d}} \in \mathbb{R}^{m\times d}$$

denote the Jacobian of the map $\boldsymbol{f}(\mathbf{x})=(f_1(\mathbf{x}),...,f_m(\mathbf{x}))$. Next, for $\mathbf{x}\in\mathbf{U}$, define

$$z(\mathbf{x}) := Z(-\mathbf{J}(\mathbf{x})), \tag{4.12}$$

where Z is given by (4.2), and finally let

$$u_1(\mathbf{x}) := z(\mathbf{x})u(\mathbf{x}). \tag{4.13}$$

Explicitly, by the above definitions, we have that

$$u_1(\mathbf{x}) = \begin{bmatrix} \mathbb{I}_m & -\sigma_m^{-1}\mathbf{J}(\mathbf{x})\sigma_d & \sigma_m^{-1}\boldsymbol{h}(\mathbf{x})^T \\ 0 & \mathbb{I}_d & \sigma_d^{-1}\mathbf{x}^T \\ 0 & 0 & 1 \end{bmatrix}, \tag{4.14}$$

where

$$\boldsymbol{h}(\mathbf{x}) = (h_1(\mathbf{x}),...,h_m(\mathbf{x})) = \boldsymbol{f}(\mathbf{x}) - \mathbf{J}(\mathbf{x})\mathbf{x}^T,$$

that is

$$h_i(\mathbf{x}) = f_i(\mathbf{x}) - \sum_{j=1}^d x_j\frac{\partial f_i(\mathbf{x})}{\partial x_j}, \quad 1\leqslant i\leqslant m.$$

LEMMA 4.3. *Let $\mathbf{x}\in\mathbf{U}$. If $\mathbf{f}(\mathbf{x})\in\mathcal{B}(\mathbf{p}/q,\varepsilon/e^t)$ for some $(\mathbf{p},q)\in\mathbb{Z}^{n+1}$ with $0<q<e^t$, then*

$$\|g_{\varepsilon,t}u_1(\mathbf{x})(-\mathbf{p}\sigma_n,q)\| \leqslant c_1\phi,$$

and, in particular,

$$\lambda_1(g_{\varepsilon,t}u_1(\mathbf{x})\mathbb{Z}^{n+1}) \leqslant c_1\phi, \tag{4.15}$$

where

$$c_1 = \sqrt{n+1}(d+1)M. \tag{4.16}$$

Proof. The proof is rather obvious and requires the following two observations. First, on setting Θ to be $-\mathbf{J}(\mathbf{x})$ and $\mathbf{y}=\mathbf{f}(\mathbf{x})$, by (4.11) and (4.12) we get that

$$Z(\Theta)U(\mathbf{y})=z(\mathbf{x})u(\mathbf{x})=u_1(\mathbf{x}).$$

Second, the quantity

$$\max_{1\leqslant i\leqslant m}(1+|\theta_{i,1}|+\ldots+|\theta_{i,d}|)$$

that appears in (4.8) is bounded by $(d+1)M$, in view of (1.6). The latter means that $c_0\leqslant c_1$, and hence (4.15) follows from (4.7). $\square$

4.2. The generic and special parts of a manifold

Setting up the generic and special parts will require another diagonal action on X_{n+1}. For each $t\in\mathbb{N}$, define the following diagonal matrix:

$$b_t:=\begin{bmatrix} e^{dt/2(n+1)}\mathbb{I}_m & & \\ & e^{-(m+1)t/2(n+1)}\mathbb{I}_d & \\ & & e^{dt/2(n+1)}\end{bmatrix}\in G. \tag{4.17}$$

First, define the 'raw' set of the special part:

$$\mathfrak{M}_0(\varepsilon,t):=\{\mathbf{x}\in\mathbf{U}:\lambda_{n+1}(b_tg_{\varepsilon,t}u_1(\mathbf{x})\mathbb{Z}^{n+1})>\phi e^{dt/2(n+1)}\}. \tag{4.18}$$

Now, define the *special part* as the following enlargement of $\mathfrak{M}_0(\varepsilon,t)$, which will ensure the structural claim about $\mathfrak{M}(\varepsilon,t)$ within Theorem 1.3:

$$\mathfrak{M}(\varepsilon,t):=\bigcup_{\mathbf{x}\in\mathfrak{M}_0(\varepsilon,t)}B(\mathbf{x},\varepsilon e^{-t/2})\cap\mathbf{U}. \tag{4.19}$$

Naturally, the *generic part* is the complement of $\mathfrak{M}(\varepsilon,t)$:

$$\mathfrak{M}'(\varepsilon,t):=\mathbf{U}\setminus\mathfrak{M}(\varepsilon,t). \tag{4.20}$$

Before moving on, we provide two further auxiliary statements. The first presents two conjugation equations involving b_t. For the rest of this paper, given $\mathbf{y}=(y_1,\ldots,y_k)\in\mathbb{R}^k$ for some $1\leqslant k<n$, with reference to (4.1), we define

$$U(\mathbf{y}):=U(\widetilde{\mathbf{y}})\quad\text{with }\ \widetilde{\mathbf{y}}=(y_1,\ldots,y_k,0,\ldots,0)\in\mathbb{R}^n,$$

while, for any $A>0$,

$$U(O(A)):=U(\mathbf{y})\quad\text{for some }\mathbf{y}\in\mathbb{R}^n\text{ such that }\|\mathbf{y}\|\ll A.$$

LEMMA 4.4. *For any* $t>0$, $\Theta\in\mathbb{R}^{m\times d}$ *and* $\mathbf{x}=(x_1,\ldots,x_d)\in\mathbb{R}^d$, *we have that*

$$b_tU(\mathbf{x})b_{-t}=U(e^{-t/2}\mathbf{x}), \tag{4.21}$$

$$b_tZ(\Theta)b_{-t}=Z(e^{t/2}\Theta). \tag{4.22}$$

The proof of these equations is elementary and obtained by inspecting them one by one. The details are left to the reader.

LEMMA 4.5. *For any* $\mathbf{x}\in\mathbf{U}$ *and* $\mathbf{x}'=(x_1',\ldots,x_d')\in\mathbb{R}^d$ *such that the line segment joining* $\mathbf{x}$ *and* $\mathbf{x}+\mathbf{x}'$ *is contained in* $\mathbf{U}$, *we have that*

$$u_1(\mathbf{x}+\mathbf{x}')=Z(O(\|\mathbf{x}'\|))U(O(\|\mathbf{x}'\|^2))U(\mathbf{x}')u_1(\mathbf{x}).$$

The proof is readily obtained on using Taylor's expansion of $\mathbf{f}(\mathbf{x}')$ and (1.6). The details are left to the reader.

5. Proof of Theorem 1.3

5.1. Dealing with the special part

The goal is to prove (1.8), that is to give an explicit exponentially decaying bound for the measure of the special part $\mathfrak{M}(\varepsilon,t)$, and to establish the structural claim about $\mathfrak{M}(\varepsilon,t)$ that it can be written as a union of balls of radius $\varepsilon e^{-t/2}$ of multiplicity $\leqslant N_d$. Specifically, we prove the following statement.

PROPOSITION 5.1. *Suppose* $\mathbf{U}\subset\mathbb{R}^d$ *is open*, $\mathbf{x}_0\in\mathbf{U}$, $\mathbf{f}:\mathbf{U}\to\mathbb{R}^n$ *is given as in* (1.4) *and is l-non-degenerate at* $\mathbf{x}_0$. *Then, there are a ball* $B_0\subset\mathbf{U}$ *centred at* $\mathbf{x}_0$ *and constants* $K_0,t_0>0$, *depending on* $\mathbf{f}$ *and* B_0 *only, with the following properties. For any* $0<\varepsilon\leqslant1$ *and every* $t\geqslant t_0$, *we have that the set defined by* (4.19) *satisfies*

$$\mathcal{L}_d(\mathfrak{M}(\varepsilon,t)\cap B_0)\leqslant K_0(\varepsilon^ne^{3t/2})^{-1/d(2l-1)(n+1)}.$$

Furthermore, $\mathfrak{M}(\varepsilon,t)$ *can be written as a union of balls in* $\mathbf{U}$ *of radius* $\varepsilon e^{-t/2}$ *of intersection multiplicity* $\leqslant N_d$.

Proof. By definition, for any $\mathbf{x}\in\mathfrak{M}_0(\varepsilon,t)$, we have that

$$\lambda_{n+1}(b_tg_{\varepsilon,t}u_1(\mathbf{x})\mathbb{Z}^{n+1})>\phi e^{dt/2(n+1)}.$$

By Theorem 3.2 and property (3.5), we have that

$$\lambda_1(b_t^*g_t^*u_1^*(\mathbf{x})\mathbb{Z}^{n+1})\leqslant c_2\phi^{-1}e^{-dt/2(n+1)}, \tag{5.1}$$

where $c_2=((n+1)!)^2$. It is straightforward to see, using (3.4), (4.3), (4.14) and (4.17), that

$$g^*_{\varepsilon,t}:=\phi^{-1}\operatorname{diag}\{e^t,\underbrace{\varepsilon,\dots,\varepsilon}_{n}\}, \tag{5.2}$$

$$b^*_t:=\begin{bmatrix} e^{-dt/2(n+1)} & & \\ & e^{(m+1)t/2(n+1)}\mathbb{I}_d & \\ & & e^{-dt/2(n+1)}\mathbb{I}_m \end{bmatrix} \tag{5.3}$$

and

$$u^*_1(\mathbf{x})=\begin{bmatrix} 1 & -\mathbf{x} & -\boldsymbol{f}(\mathbf{x}) \\ 0 & \mathbb{I}_d & \mathbf{J}(\mathbf{x}) \\ 0 & 0 & \mathbb{I}_m \end{bmatrix}. \tag{5.4}$$

Therefore, by (5.1), we get that, for any $\mathbf{x}\in\mathfrak{M}_0(\varepsilon,t)$, there exists $(a_0,\mathbf{a})\in\mathbb{Z}\times\mathbb{Z}^n\setminus\{\mathbf{0}\}$ such that

$$|a_0+\mathbf{f}(\mathbf{x})\mathbf{a}^T|<c_2e^{-t}, \tag{5.5}$$

$$\|\nabla\mathbf{f}(\mathbf{x})\mathbf{a}^T\|_\infty<c_2\varepsilon^{-1}e^{-t/2}, \tag{5.6}$$

$$\max\{|a_{d+1}|,\dots,|a_n|\}<c_2\varepsilon^{-1}. \tag{5.7}$$

Using (5.5)–(5.7), (1.4), (1.6) and Taylor's expansion of the function $a_0+\mathbf{f}(\mathbf{x})\mathbf{a}^T$, one has that, for every $\mathbf{x}'\in\mathfrak{M}(\varepsilon,t)$,

$$|a_0+\mathbf{f}(\mathbf{x}')\mathbf{a}^T|<c_2e^{-t}+c_2de^{-t}+\tfrac{1}{2}d^2mMc_2\varepsilon e^{-t}\leqslant c_3e^{-t}, \tag{5.8}$$

where

$$c_3=c_2(1+n+n^3M)=((n+1)!)^2(1+n+n^3M)$$

depends on n and $\mathbf{f}$ only. Similarly, using (5.6), (5.7), (1.4), (1.6) and Taylor's expansion of the gradient $\nabla\mathbf{f}(\mathbf{x})\mathbf{a}^T$, one has that, for every $\mathbf{x}'\in\mathfrak{M}(\varepsilon,t)$,

$$\|\nabla\mathbf{f}(\mathbf{x}')\mathbf{a}^T\|_\infty\leqslant c_2\varepsilon^{-1}e^{-t/2}+dc_2e^{-t/2}\leqslant c_3\varepsilon^{-1}e^{-t/2}. \tag{5.9}$$

Also, by (5.6), (5.7), (1.4) and (1.6), we have that

$$\max\{|a_1|,\dots,|a_n|\}<c_2mM\varepsilon^{-1}\leqslant c_3\varepsilon^{-1}. \tag{5.10}$$

Combining (5.8)–(5.10) gives that

$$\mathfrak{M}(\varepsilon,t)\subset\mathfrak{S}_{\mathbf{f}}(\delta,K,T), \tag{5.11}$$

with

$$\delta = c_3 e^{-t}, \quad K = c_3 \varepsilon^{-1} e^{-t/2} \quad \text{and} \quad T = c_3 \varepsilon^{-1},$$

where $\mathfrak{S}_{\mathbf{f}}(\delta, K, T)$ is defined by (3.7). It is readily seen that conditions (3.6) are satisfied for all t such that $\delta = c_3 e^{-t} \leqslant 1$, that is $t \geqslant \log c_3 =: t_0$. Now, fix any $\mathbf{x}_0 \in \mathbf{U}$ such that $\mathbf{f}$ is l-non-degenerate at $\mathbf{x}_0$, and let B_0 and E be the ball and constant arising from Theorem 3.4. By Theorem 3.4 and (5.11), we obtain that

$$\mathcal{L}_d(\mathfrak{M}(\varepsilon, t) \cap B_0) \leqslant E(c_3^{n+1} \varepsilon^{-n} e^{-3t/2})^{1/d(2l-1)(n+1)} \mathcal{L}_d(B_0),$$

which gives the required bound with $K_0 = E c_3^{1/d(2l-1)} \mathcal{L}_d(B_0)$.

Finally, in view of the definition of $\mathfrak{M}(\varepsilon, t)$, the 'Furthermore' claim trivially follows from Besicovitch's covering theorem (see below) applied to the set $A = \mathfrak{M}(\varepsilon, t)$ and $\mathcal{B}$ being the collection of balls appearing in the right-hand side of (4.19). $\square$

THEOREM 5.2. (Besicovitch's covering theorem [38, Theorem 2.7]) *There is an integer N_d depending only on d with the following property: let A be a bounded subset of $\mathbb{R}^d$ and let $\mathcal{B}$ be a family of non-empty open balls in $\mathbb{R}^d$ such that each $x \in A$ is the center of some ball of $\mathcal{B}$. Then, there exists a finite or countable subfamily $\{B_i\}$ of $\mathcal{B}$ covering A of intersection multiplicity at most N_d, that is, with $1_A \leqslant \sum_i 1_{B_i} \leqslant N_d$.*

5.2. Dealing with the generic part

The goal is to give a sharp counting estimate for the number of rational points of bounded height near the generic part. Indeed, the following statement we prove here completes the proof of Theorem 1.3.

PROPOSITION 5.3. *Let $\mathbf{U} \subset \mathbb{R}^d$ be open and $\mathbf{f}: \mathbf{U} \to \mathbb{R}^n$ be a C^2 maps satisfying (1.4) and (1.6). Then, for any $0 < \varepsilon \leqslant 1$, any ball $B \subset \mathbf{U}$ and all sufficiently large t, we have that*

$$N(B \setminus \mathfrak{M}(\varepsilon, t); \varepsilon, t) \leqslant K_1 \varepsilon^m e^{(d+1)t} \mathcal{L}_d(B), \tag{5.12}$$

where K_1 depends on n and $\mathbf{f}$ only.

We will make use of the following trivial property: for any $\Delta_1, \Delta_2 \subset \mathbb{R}^d$,

$$N(\Delta_1 \cup \Delta_2; \varepsilon, t) \leqslant N(\Delta_1; \varepsilon, t) + N(\Delta_2; \varepsilon, t). \tag{5.13}$$

This allows us to reduce the proof of Proposition 5.3 to considering domains of the form

$$\Delta_t(\mathbf{x}_0) := \{\mathbf{x} \in \mathbb{R}^d : \|\mathbf{x} - \mathbf{x}_0\|_\infty \leqslant (\varepsilon e^{-t})^{1/2}\},$$

where $\mathbf{x}_0 \in \mathfrak{M}'(\varepsilon, t)$. At the heart of the reduction is the following simple statement.

LEMMA 5.4. *For all sufficiently large $t>0$, we have that*

$$N(B\backslash\mathfrak{M}(\varepsilon,t);\varepsilon,t)\leqslant 2(\varepsilon e^{-t})^{-d/2}\mathcal{L}_d(B)\max_{\mathbf{x}_0\in\mathfrak{M}'(\varepsilon,t)\cap B}N(\Delta_t(\mathbf{x}_0)\cap B;\varepsilon,t).$$

Proof. First of all, note that $\varepsilon e^{-t}\to 0$ as $t\to\infty$, since $\varepsilon\leqslant 1$ for all $t>0$. Therefore, for all sufficiently large t, the ball B can be covered by $\leqslant 2(\varepsilon e^{-t})^{-d/2}\mathcal{L}_d(B)$ hypercubes Δ of sidelength $(\varepsilon e^{-t})^{1/2}$. Any of these hypercubes Δ that intersects $\mathfrak{M}'(\varepsilon,t)\cap B$ can be covered by a hypercube $\Delta_t(\mathbf{x}_0)$ with $\mathbf{x}_0\in\mathfrak{M}'(\varepsilon,t)\cap B\cap\Delta$. The collection of the sets $\Delta_t(\mathbf{x}_0)\cap B$ is thus a cover for $\mathfrak{M}'(\varepsilon,t)\cap B=B\backslash\mathfrak{M}(\varepsilon,t)$ of $2(\varepsilon e^{-t})^{-d/2}\mathcal{L}_d(B)$ elements. Applying (5.13) completes the proof. □

In view of Lemma 5.4, the following statement is all we need to complete the proof of Proposition 5.3.

LEMMA 5.5. *Let a ball $B\subset\mathbf{U}$ be given. Then, for all sufficiently large $t>0$ and all $\mathbf{x}_0\in\mathfrak{M}'(\varepsilon,t)\cap B$, we have that*

$$N(\Delta_t(\mathbf{x}_0)\cap B;\varepsilon,t)\ll\varepsilon^n e^t(\varepsilon e^{-t})^{-d/2},$$

where the implied constant depends on n and $\mathbf{f}$ only.

Proof. Let us assume that $N(\Delta_t(\mathbf{x}_0)\cap B;\varepsilon,t)\neq 0$, as otherwise there is nothing to prove. Take any $(\mathbf{p},q)\in\mathcal{R}(\Delta_t(\mathbf{x}_0)\cap B;\varepsilon,t)$. By definition, there exists $\mathbf{x}\in\Delta_t(\mathbf{x}_0)\cap B$ such that

$$\left\|\mathbf{f}(\mathbf{x})-\frac{\mathbf{p}}{q}\right\|_\infty<\frac{\varepsilon}{e^t}.$$

By Lemma 4.3, we have that

$$\|g_{\varepsilon,t}u_1(\mathbf{x})(-\mathbf{p}\sigma_n,q)\|\leqslant c_1\phi. \tag{5.14}$$

Since $\mathbf{x}\in\Delta_t(\mathbf{x}_0)$, we have that

$$\mathbf{x}_0=\mathbf{x}+(\varepsilon e^{-t})^{1/2}\mathbf{x}',$$

with $\|\mathbf{x}'\|\leqslant 1$. Since $\mathbf{x},\mathbf{x}_0\in B\subset\mathbf{U}$, the line segment joining $\mathbf{x}_0$ and $\mathbf{x}$ is contained in $\mathbf{U}$. Then, by Lemma 4.5, we have that

$$u_1(\mathbf{x}_0)=Z(O((\varepsilon e^{-t})^{1/2}))U(O(\varepsilon e^{-t}))U((\varepsilon e^{-t})^{1/2}\mathbf{x}')u_1(\mathbf{x}).$$

By (4.5) and (4.6), we have

$$g_{\varepsilon,t}u_1(\mathbf{x}_0)=Z(O((\varepsilon e^{-t})^{1/2}))U(O(1))U((\varepsilon e^{-t})^{-1/2}\mathbf{x}')g_{\varepsilon,t}u_1(\mathbf{x}).$$

Therefore,

$$g_{\varepsilon,t}u_1(\mathbf{x}_0)(-\mathbf{p}\sigma_n,q)=Z(O((\varepsilon e^{-t})^{1/2}))U(O(1))U((\varepsilon e^{-t})^{-1/2}\mathbf{x}')g_{\varepsilon,t}u_1(\mathbf{x})(-\mathbf{p}\sigma_n,q).$$

Let us denote

$$g_{\varepsilon,t}u_1(\mathbf{x})(-\mathbf{p}\sigma_n,q)=\mathbf{v}=(v_n,...,v_1,v_0). \tag{5.15}$$

Then, by the above calculation, we get that

$$\begin{aligned} g_{\varepsilon,t}u_1(\mathbf{x}_0)(-\mathbf{p}\sigma_n,q) &= Z(O((\varepsilon e^{-t})^{1/2}))U(O(1))U((\varepsilon e^{-t})^{-1/2}\mathbf{x}')\mathbf{v} \\ &= Z(O((\varepsilon e^{-t})^{1/2}))U(O(1))\mathbf{v}', \end{aligned} \tag{5.16}$$

where

$$\mathbf{v}'=(v_n,...,v_{d+1},v_d+(\varepsilon e^{-t})^{-1/2}x'_d v_0,...,v_1+(\varepsilon e^{-t})^{-1/2}x'_1 v_0,v_0).$$

By (5.14) and (5.15), we have that $\|\mathbf{v}\|\leqslant c_1\phi$. Furthermore, since $0<q<e^t$, we get that $|v_0|=\phi e^{-t}q\leqslant\phi$. Therefore, using $\|\mathbf{x}'\|\leqslant 1$, we get that

$$\mathbf{v}'\in[c_1\phi]^m\times[(c_1+1)\phi(\varepsilon e^{-t})^{-1/2}]^d\times[\phi], \tag{5.17}$$

where $[a]$ denotes the closed interval $[-a,a]$. After considering the action of

$$Z(O((\varepsilon e^{-t})^{1/2}))U(O(1))$$

on $\mathbf{v}'$, we get from (5.16) and (5.17) that

$$g_{\varepsilon,t}u_1(\mathbf{x}_0)(-\mathbf{p}\sigma_n,q)\in[c_4\phi]^m\times[c_4\phi(\varepsilon e^{-t})^{-1/2}]^d\times[c_4\phi],$$

for some constant $c_4>0$ depending on n and $\mathbf{f}$ only. Then, it is easy to verify that

$$b_t g_{\varepsilon,t}u_1(\mathbf{x}_0)(-\mathbf{p}\sigma_n,q)\in[c_4\phi e^h]^m\times[c_4\phi\varepsilon^{-1/2}e^h]^d\times[c_4\phi e^h],$$

where $h=dt/2(n+1)$. Let us denote

$$\Omega=[c_4\phi e^h]^m\times[c_4\phi\varepsilon^{-1/2}e^h]^d\times[c_4\phi e^h].$$

Then,

$$(-\mathbf{p}\sigma_n,q)\in\Omega\cap b_t g_{\varepsilon,t}u_1(\mathbf{x}_0)\mathbb{Z}^{n+1}\subset(c_6\Omega)\cap b_t g_{\varepsilon,t}u_1(\mathbf{x}_0)\mathbb{Z}^{n+1}$$

for any $c_6>1$. On the other hand, since $\mathbf{x}_0\in\mathfrak{M}'(\varepsilon,t)$, we have that

$$\lambda_{n+1}(b_t g_{\varepsilon,t}u_1(\mathbf{x}_0)\mathbb{Z}^{n+1})\leqslant\phi e^h.$$

This implies that there exists a constant $c_6>1$ such that $c_6\Omega$ contains a full fundamental domain of $b_t g_{\varepsilon,t}u_1(\mathbf{x}_0)\mathbb{Z}^{n+1}$. Therefore,

$$\#((c_6\Omega)\cap b_t g_{\varepsilon,t}u_1(\mathbf{x}_0)\mathbb{Z}^{n+1})\ll\mathcal{L}_{n+1}(c_6\Omega)\asymp\phi^{n+1}\varepsilon^{-d/2}e^{(n+1)h},$$

which implies the desired estimate. □

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Victor Beresnevich
Department of Mathematics
University of York
Heslington, York, YO10 5DD
U.K.
victor.beresnevich@york.ac.uk

Lei Yang
College of Mathematics
Sichuan University
Chengdu, Sichuan, 610000
China
lyang861028@gmail.com

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