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Non-linear inviscid damping near monotonic shear flows

by

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1. Introduction

In this paper we continue our investigation of asymptotic stability of solutions of the 2-dimensional incompressible Euler equation in a channel. More precisely, we consider solutions $u: [0,\infty)\times\mathbb{T}\times[0,1]\to\mathbb{R}^2$ of the equation

$$\partial_t u+u\cdot\nabla u+\nabla p=0, \quad \operatorname{div} u=0, \tag{1.1}$$

with the boundary condition $u^y|_{y=0,1}\equiv 0$. Letting $\omega:=-\partial_y u^x+\partial_x u^y$ be the vorticity field, the equation (1.1) can be written in vorticity form as

$$\partial_t\omega+u\cdot\nabla\omega=0, \quad u=\nabla^\perp\psi=(-\partial_y\psi,\partial_x\psi), \tag{1.2}$$

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for $(x,y)\in\mathbb{T}\times[0,1]$ and $t\geqslant 0$, where the stream function ψ is determined through

$$\Delta\psi=\omega \text{ on } \mathbb{T}\times[0,1], \quad \psi(x,0)\equiv 0, \quad \psi(x,1)\equiv C_0, \tag{1.3}$$

where C_0 is a constant preserved by the flow. We remark that our domain is a finite, periodic channel: periodicity in x is a key condition for inviscid damping and stability, while compactness in y is a physical choice motivated by finite energy considerations.

The 2-dimensional incompressible Euler equation is globally well posed for smooth initial data, by the classical result of Wolibner [45]. See also [24], [47] for global well-posedness results with rough initial data, such as L^∞ vorticity. The long-time behavior of general solutions is however very difficult to understand, due to the lack of a global relaxation mechanism.

A more realistic goal is to study the global non-linear dynamics of solutions that are close to steady states of the 2D Euler equation. Coherent structures, such as shear flows and vortices, are particularly important in the study of the 2D Euler equation, since precise numerical simulations and physical experiments show that they tend to form dynamically and become the dominant feature of the solution for a long time.

The study of stability properties of these steady states is a classical subject and a fundamental problem in hydrodynamics. Early investigations were started by Kelvin [25], Orr [35], Rayleigh [36], Taylor [40], among many others, with a focus on mode stability. Later, more detailed understanding of the general spectral properties and suitable linear decay estimates were also obtained; see [17], [38]. In the direction of non-linear results, Arnold [1] proved a general stability criteria, using the energy Casimir method, but this method does not give asymptotic information on the global dynamics.

The full non-linear asymptotic stability problem has only been investigated in recent years, starting with the remarkable work of Bedrossian–Masmoudi [8], who proved inviscid damping and non-linear stability in the simplest case of perturbations of the Couette flow on $\mathbb{T}\times\mathbb{R}$.

Motivated by this result, the linearized equations around other stationary solutions were investigated intensely in the last few years, and linear inviscid damping and decay was proved in many cases of physical interest; see for example [4], [15], [19], [23], [42], [43], [44], [49], [48]. However, it also became clear that there are major difficulties in passing from linear to non-linear stability, such as the presence of "resonant times" in the non-linear problem, which require refined Fourier analysis techniques, and the fact that the final state of the flow is determined dynamically by the global evolution and cannot be described in terms of the initial data.

In this paper we close this gap and establish inviscid damping and full non-linear asymptotic stability for a general class of monotone shear flows, which are not close to the Couette flow. We hope that the general framework we develop here can be adapted to establish non-linear asymptotic stability in other outstanding open problems involving 2D or 3D Euler and Navier–Stokes equations, such as the stability of smooth radially decreasing vortices in 2D.

1.1. The main theorem

We consider a perturbative regime for the Euler equation (1.1), with velocity field given by $(b(y),0))+u(x,y)$ and vorticity given by $-b'(y)+\omega$.

To state our main theorem we define the Gevrey spaces $\mathcal{G}^{\lambda,s}(\mathbb{T}\times\mathbb{R})$ as the spaces of L^2 functions f on $\mathbb{T}\times\mathbb{R}$ defined by the norm

$$\|f\|_{\mathcal{G}^{\lambda,s}(\mathbb{T}\times\mathbb{R})} := \|e^{\lambda\langle k,\xi\rangle^s}\tilde{f}(k,\xi)\|_{L^2_{k,\xi}} < \infty \quad \text{for } s\in(0,1] \text{ and } \lambda>0. \tag{1.4}$$

In the above, $(k,\xi)\in\mathbb{Z}\times\mathbb{R}$ and $\tilde{f}$ denotes the Fourier transform of f in (x,y). More generally, for any interval $I\subseteq\mathbb{R}$, we define the Gevrey spaces $\mathcal{G}^{\lambda,s}(\mathbb{T}\times I)$ by

$$\|f\|_{\mathcal{G}^{\lambda,s}(\mathbb{T}\times I)} := \|Ef\|_{\mathcal{G}^{\lambda,s}(\mathbb{T}\times\mathbb{R})}, \tag{1.5}$$

where

$$Ef(x) := \begin{cases} f(x), & \text{if } x\in I, \\ 0, & \text{if } x\notin I. \end{cases}$$

We refer to §3.1 below for more details as well as further references on Gevrey spaces.

The use of Gevrey spaces is necessary in the context of inviscid damping, mainly due to loss of regularity during the flow. In contrast, Sobolev spaces provide control only on finitely many derivatives, which is not sufficient in our case, while the classical C^∞ spaces do not provide adequate quantitative bounds on the growth of the high-order derivatives. Analytic functions have also been used in certain cases, but analyticity is a very rigid condition which is not compatible with the type of localization arguments we need in our problem (such as the main assumption (1.6) below).

Concerning the background shear flow $b\in C^\infty(\mathbb{R})$, our main assumptions are the following:

(A) For some $\vartheta_0\in\big(0,\frac{1}{10}\big]$ and $\beta_0>0$,

$$\vartheta_0 \leqslant b'(y) \leqslant \frac{1}{\vartheta_0} \text{ for } y\in[0,1], \quad b''(y)\equiv 0 \text{ for } y\notin[2\vartheta_0, 1-2\vartheta_0], \tag{1.6}$$

and

$$\|b\|_{L^\infty(0,1)} + \|b''\|_{\mathcal{G}^{\beta_0,1/2}} \leqslant \frac{1}{\vartheta_0}. \tag{1.7}$$

(B) The associated linearized operator $L_k: L^2(0,1)\to L^2(0,1)$, $k\in\mathbb{Z}\setminus\{0\}$, given by

$$L_k f = b(y)f - b''(y)\varphi_k, \quad \text{where } \partial_y^2\varphi_k - k^2\varphi_k = f \text{ and } \varphi_k(0)=\varphi_k(1)=0, \tag{1.8}$$

has no discrete eigenvalues and therefore, by the general theory of Fredholm operators, the spectrum of L_k is purely continuous spectrum $[b(0), b(1)]$ for all $k\in\mathbb{Z}\setminus\{0\}$.

The spectral condition (B) is a qualitative condition, and we need to make it quantitative in order to link it to the perturbation theory. For this we define, for any $k\in\mathbb{Z}\setminus\{0\}$,

$$\|f\|_{H^1_k(\mathbb{R})} := \|f\|_{L^2(\mathbb{R})} + |k|^{-1}\|f'\|_{L^2(\mathbb{R})}. \tag{1.9}$$

The following quantitative bounds were proved in [22, Lemmas 3.1 and 3.2].

Lemma 1.1. *Assume that $\varphi\in H^{10}$ is supported in $\left[\frac14\vartheta_0, 1-\frac14\vartheta_0\right]$. For $k\in\mathbb{Z}\setminus\{0\}$, $y_0\in[0,1]$, $\varepsilon\in\left[-\frac14,\frac14\right]\setminus\{0\}$, and any $f\in L^2(0,1)$, we define the operator*

$$T_{k,y_0,\varepsilon} f(y) := \int_{\mathbb{R}} \varphi(y) G_k(y,z) \frac{b''(z)f(z)}{b(z)-b(y_0)+i\varepsilon}\, dz, \tag{1.10}$$

where G_k is the Green function associated with the operator $-\partial_y^2+k^2$ on $[0,1]$ (see (4.24) for explicit formulas). Then, there is $\kappa>0$ such that, for any $f\in H^1_k(\mathbb{R})$,

$$\|T_{k,y_0,\varepsilon} f\|_{H^1_k(\mathbb{R})} \lesssim |k|^{-1/3}\|f\|_{H^1_k(\mathbb{R})} \quad \text{and} \quad \|f+T_{k,y_0,\varepsilon} f\|_{H^1_k(\mathbb{R})} \geqslant \kappa \|f\|_{H^1_k(\mathbb{R})}, \tag{1.11}$$

uniformly in $y_0\in[0,1]$, $k\in\mathbb{Z}\setminus\{0\}$, and ε sufficiently small.

In our case, the function φ will be a fixed Gevrey cutoff function, $\varphi(y)=\Psi(b(y))$, where Ψ is defined in (2.42). The parameter $\kappa>0$ in (1.11) will be one of the parameters that determine the smallness of the perturbation in our main theorem.

For any function $H(x,y)$, let $\langle H\rangle(y)$ denote the average of H in x. Our main result in this paper is the following theorem.

Theorem 1.2. *Assume that $\beta_0, \vartheta_0, \kappa>0$ are constants as defined in (1.6), (1.7), and (1.11). Then, there are constants $\beta_1=\beta_1(\beta_0,\vartheta_0,\kappa)>0$ and $\bar\varepsilon=\bar\varepsilon(\beta_0,\vartheta_0,\kappa)>0$ such that the following statement is true:*

Assume that the initial data ω_0 has compact support in $\mathbb{T}\times[2\vartheta_0, 1-2\vartheta_0]$, and satisfies

$$\|\omega_0\|_{\mathcal{G}^{\beta_0,1/2}(\mathbb{T}\times\mathbb{R})} = \varepsilon \leqslant \bar\varepsilon \quad \text{and} \quad \int_{\mathbb{T}} \omega_0(x,y)\, dx = 0 \ \text{ for any } y\in[0,1]. \tag{1.12}$$

Let $\omega: [0,\infty)\times\mathbb{T}\times[0,1]\to\mathbb{R}$ denote the global smooth solution to the Euler equation

$$\begin{cases} \partial_t\omega + b(y)\partial_x\omega - b''(y)\partial_x\psi + u\cdot\nabla\omega = 0, \\ u=(u^x,u^y)=(-\partial_y\psi, \partial_x\psi), \quad \Delta\psi=\omega, \quad \psi(t,x,0)=\psi(t,x,1)=0. \end{cases} \tag{1.13}$$

Then, we have the following conclusions:

(i) *For all* $t\geqslant 0$, $\operatorname{supp}\omega(t)\subseteq\mathbb{T}\times[\vartheta_0,1-\vartheta_0]$.

(ii) *There exists* $F_\infty(x,y)\in\mathcal{G}^{\beta_1,1/2}$, *with* $\operatorname{supp}F_\infty\subseteq\mathbb{T}\times[\vartheta_0,1-\vartheta_0]$, *such that, for all* $t\geqslant 0$,

$$\|\omega(t,x+tb(y)+\Phi(t,y),y)-F_\infty(x,y)\|_{\mathcal{G}^{\beta_1,1/2}(\mathbb{T}\times[0,1])}\lesssim_{\beta_0,\vartheta_0,\kappa}\frac{\varepsilon}{\langle t\rangle},\tag{1.14}$$

where

$$\Phi(t,y):=\int_0^t\langle u^x\rangle(\tau,y)\,d\tau.\tag{1.15}$$

(iii) *We define the smooth functions* $\psi_\infty,u_\infty\colon[0,1]\to\mathbb{R}$ *by*

$$\partial_y^2\psi_\infty=\langle F_\infty\rangle,\quad \psi_\infty(0)=\psi_\infty(1)=1,\quad u_\infty(y):=-\partial_y\psi_\infty.\tag{1.16}$$

Then, the velocity field $u=(u^x,u^y)$ *satisfies*

$$\|\langle u^x\rangle(t,y)-u_\infty(y)\|_{\mathcal{G}^{\beta_1,1/2}(\mathbb{T}\times[0,1])}\lesssim_{\beta_0,\vartheta_0,\kappa}\frac{\varepsilon}{\langle t\rangle^2},\tag{1.17}$$

$$\|u^x(t,x,y)-\langle u^x\rangle(t,y)\|_{L^\infty(\mathbb{T}\times[0,1])}\lesssim_{\beta_0,\vartheta_0,\kappa}\frac{\varepsilon}{\langle t\rangle},\tag{1.18}$$

$$\|u^y(t,x,y)\|_{L^\infty(\mathbb{T}\times[0,1])}\lesssim_{\beta_0,\vartheta_0,\kappa}\frac{\varepsilon}{\langle t\rangle^2}.\tag{1.19}$$

A similar theorem was proved slightly later and independently by Masmoudi–Zhao [31].

1.2. Remarks

We discuss now some of the assumptions and the conclusions of Theorem 1.2.

(1) The equation (1.13) for the vorticity deviation is equivalent to the original Euler equations (1.1)–(1.3). The condition

$$\int_{\mathbb{T}}\omega_0(x,y)\,dx=0$$

can be imposed without loss of generality, because we may replace the shear flow $b(y)$ by the nearby shear flow $b(y)+\langle u_0^x\rangle(y)$. In fact, since

$$\partial_y\langle\partial_y\psi\rangle=\langle\omega\rangle,$$

this condition is equivalent to

$$\langle u_0^x\rangle(y)=0\quad\text{for any } y\in[0,1].\tag{1.20}$$

These identities only hold for the initial data, and are not propagated by the flow (1.13). However, as we show in (2.3) below, we have

$$\langle u^x\rangle(t,y)\equiv 0 \quad \text{for } y\in[0,1]\setminus[\vartheta_0,1-\vartheta_0] \text{ and } t\in[0,T],$$

as long as the vorticity ω is supported in $[0,T]\times\mathbb{T}\times[\vartheta_0,1-\vartheta_0]$. In particular,

$$\langle u^x\rangle(t,y)-u_\infty(y)$$

is compactly supported in $[\vartheta_0,1-\vartheta_0]$.

(2) The assumption on the compact support of ω_0 is likely necessary to prove scattering in Gevrey spaces. Indeed, Zillinger [48] showed that scattering does not hold in high Sobolev spaces unless one assumes that the vorticity vanishes at high order at the boundary. This is due to what is called "boundary effect", which is not consistent with inviscid damping. This boundary effect can also be seen clearly in [23] as the main asymptotic term for the stream function.

Understanding quantitatively the boundary effect in the context of asymptotic stability of Euler or Navier–Stokes equations is an interesting topic by itself, but we will not address it here.

The assumption on the support of b'' is necessary to preserve the compact support of $\omega(t)$ in $\mathbb{T}\times[\vartheta_0,1-\vartheta_0]$, due to the non-local term $b''(y)\partial_x\psi$ in (1.13). In principle, one could hope to remove this strong assumption (and replace it with a milder decay assumption) by working in the infinite cylinder $\mathbb{T}\times\mathbb{R}$ domain instead of the finite channel $\mathbb{T}\times[0,1]$, but this would be at the expense of considering solutions of infinite energy.

We also assume that $b(y)$ is strictly monotone in y. This assumption is important for our proof, to ensure a uniform rate of inviscid damping. It is an important question to investigate what happens to non-monotone shear flows which are linearly stable, such as Kolmogorov flow on a torus with unequal sides (see e.g. [44] for linear stability results).

(3) There is a large class of shear flows b satisfying our assumptions. For instance, if $b(y)$ satisfies $|b'|\geqslant 1$ and $|b'''|<1$, then the spectrum of the operators L_k consist entirely of the continuous spectrum $[b(0),b(1)]$ for $k\in\mathbb{Z}\setminus\{0\}$.

(4) The Gevrey regularity assumption (1.12) on the initial data ω_0 is likely sharp. See the recent construction of non-linear instability of Deng–Masmoudi [16] for the Couette flow in slightly larger Gevrey spaces, and the more definitive counter-examples to inviscid damping in low Sobolev spaces by Lin–Zeng [29].

(5) The most important statement in Theorem 1.2 is (1.14), which provides strong control on the "profile" of the vorticity and from which the other statements follow easily. We note that the convergence (1.14) of the profile for vorticity holds in a slightly

weaker Gevrey space ($\beta_1<\beta_0$). This is connected with the use of energy functionals with decreasing time-dependent weights to control the profile, and is a reflection of the phenomenon that "decay costs regularity" in inviscid damping.

We also remark that in (1.18)–(1.19) we used the L^∞ norm instead of L^2-based norms, which are used for measuring most other quantities in the paper. In our compact channel case, the L^∞ norm provides the strongest control on u^x and u^y without sacrificing the rates of convergence in time in (1.18)–(1.19) (compare with the explicit formulas (1.22) in the Couette case).

(6) At the qualitative level, our main conclusion (1.14) shows that the vorticity ω converges weakly to the function $\langle F_\infty\rangle(y)$. This is consistent with a far-reaching conjecture regarding the long-time behavior of the 2D Euler equation, see [39], which predicts that for general generic solutions the vorticity field converges, as $t\to\infty$, *weakly but not strongly* in L^2_{loc} to a steady state. Proving such a conjecture for general solutions is, of course, well beyond the current PDE techniques, but the non-linear asymptotic stability results we have so far in [8], [20], [21] are consistent with this conjecture.

(7) There are several parameters in our proof, and we summarize their roles here. The parameters $\beta_0, \vartheta_0, \kappa>0$ (the structural constants of the problem) are assumed fixed, and implicit constants in inequalities like $A\lesssim B$ are allowed to depend on these parameters. We will later fix a constant $\delta_0>0$ sufficiently small depending on these parameters, as part of the construction of our main weights; see (2.36).

The weights will also depend on a small parameter $\delta>0$, much smaller than δ_0, which is needed at many places, such as in commutator estimates using inequalities like (1.40). We will use the general notation $A\lesssim_\delta B$ to indicate inequalities where the implicit constants may depend on δ. Finally, the parameters ε and $\varepsilon_1=\varepsilon^{2/3}$, which bound the size of the perturbation, are assumed to be much smaller than δ.

1.2.1. Linear inviscid damping and the Orr mechanism

One can gain some intuition and explain the conclusions in Theorem 1.2 by examining a simple explicit case, corresponding to the Couette flow $b(y)=y$. In this case, $b''(y)=0$ and the linearization of the main equation (1.13) is

$$\partial_t\omega+y\partial_x\omega=0, \tag{1.21}$$

which was studied by Orr in a pioneering work [35]. To simplify the discussion, we assume $x\in\mathbb{T}$ and $y\in\mathbb{R}$ (to avoid the boundary issue which is not our main concern here).

By direct calculation, we have $\omega(t,x,y)=\omega_0(x-yt,y)$. The stream function is given by

$$\Delta\psi(t,x,y)=\omega(t,x,y)$$

for $(x,y)\in\mathbb{T}\times\mathbb{R}$, so in the Fourier space we have the formulas

$$\widetilde{\omega}(t,k,\xi)=\widetilde{\omega}_0(k,\xi+kt) \quad \text{and} \quad \tilde{\psi}(t,k,\xi)=-\frac{\widetilde{\omega}_0(k,\xi+kt)}{k^2+|\xi|^2}. \tag{1.22}$$

We remark that the conclusions in the full non-linear Theorem 1.2 are consistent with these explicit formulas. Indeed, assume that ω_0 is smooth, so $\widetilde{\omega}_0(k,\xi)$ decays fast in k and ξ. Then, the following holds.

(1) The main contribution comes from the frequencies $\xi=-kt+O(1)$, therefore $\tilde{\psi}(t,k,\xi)$ decays like $|k|^{-2}\langle t\rangle^{-2}$ if $k\neq 0$. Similarly, the relations $u^x=-\partial_y\psi$ and $u^y=\partial_x\psi$ show that $\widetilde{u^x}$ decays like $|k|^{-1}\langle t\rangle^{-1}$ and $\widetilde{u^y}$ like $|k|^{-1}\langle t\rangle^{-2}$, as claimed in (1.18)–(1.19).

(2) It can be seen from (1.22) that the functions $\omega(t,x,y)$ and $\psi(t,x,y)$ are not uniformly smooth as $t\to\infty$, in the original coordinates x and y. To obtain smooth "profiles" we define

$$z=x-tv, \quad v=y, \quad F(t,z,v)=\omega(t,x,y), \quad \phi(t,z,v)=\psi(t,x,y). \tag{1.23}$$

Notice that $F(t,z,v)=\omega_0(z,v)$ (independent of t), while $\phi(t,z,v)$ is uniformly smooth for all t provided that ω_0 is smooth. Taking the Fourier transform in z and v, we have the formula

$$\tilde{\phi}(t,k,\xi)=-\frac{\widetilde{\omega}_0(k,\xi)}{k^2+|\xi-kt|^2}. \tag{1.24}$$

(3) An important observation by Orr is that for $k\neq 0$ and large ξ, the normalized stream function ϕ (as well as the velocity field) may experience a *transient growth* as t approaches the "critical time" $t_c=\xi/k$ before decaying to zero. This can be easily seen from the formula (1.24). This transient growth on the linearized level turns out to be crucial for the non-linear analysis as well, and leads to the high-regularity assumptions (Gevrey spaces) that are required for the non-linear perturbation theory; see [16].

1.3. Previous work and related results

The study of stability properties of shear flows and vortices is one of the most important problems in hydrodynamics, and has a long history starting with work of Kelvin [25], Rayleigh [36], and Orr [35]. The problem is well motivated physically, since numerical simulations and physical experiments, such as those of [2], [3], [11], [12], [13], [32], [33], show that coherent structures tend to form and become the dominant feature of incompressible 2D Euler evolutions. This indicates a reverse cascade of energy from high to low frequencies, which is in sharp contrast to the 3D situation, where it is expected that energy flows from small frequencies to high frequencies until the dissipation scale.

We refer also to the recent papers [26], [30] for other interesting results concerning the dynamics of solutions of the 2D Euler equations.

Our main topic in this paper is asymptotic stability. Non-linear asymptotic stability results are difficult for the 2D incompressible Euler equation, because the rate of stabilization is slow, the convergence of the vorticity field holds only in the weak sense, and the non-linear effect is strong. In a recent remarkable paper Bedrossian–Masmoudi [8] proved the first non-linear asymptotic stability result, showing that small perturbations of the Couette flow on the infinite cylinder $\mathbb{T}\times\mathbb{R}$ converge weakly to nearby shear flows. This result was extended by the authors [20] to the finite channel $\mathbb{T}\times[0,1]$, in order to be able to consider solutions with finite energy. In [21] the authors also proved asymptotic stability of point vortex solutions in $\mathbb{R}^2$, showing that small and Gevrey smooth perturbations converge to a smooth radial profile, and the position of the point vortex stabilizes rapidly and forms the center of the final radial profile. These three results appear to be the only known results on non-linear asymptotic stability of stationary solutions for the Euler equations.

A key common feature of these stability results is that the steady states are simple explicit functions, and, more importantly, the associated linearized flow can be solved explicitly.

To expand the stability theory to more general steady states, one can first consider the linearized equation and prove inviscid damping of linear solutions. The linear evolution problem has been investigated intensely in the last few years, in particular around general shear flows and vortices; see for example [19], [23], [42], [48], [49]. In particular, Wei–Zhang–Zhao [42] proved optimal decay rate of the stream function for the linearized problem near monotone shear flows, and Bedrossian–Coti Zelati–Vicol [4] obtained sharp decay estimates for general vortices with decreasing profile. We also refer the reader to important developments for the linear inviscid damping in the case of non-monotone shear flows [43], [44] and circular flows [4], [15].

There is a large gap, however, between linear and non-linear theory. As we know, even in the simplest case of the Couette flow, to prove non-linear stability one needs to bound the contribution of the so-called "resonant times", which can only be detected by working in the Fourier space, in a specific coordinate system. This requires refined Fourier analysis techniques, which are not compatible with the natural spectral theory of the variable-coefficient linearized problems associated with general shear flows and vortices. In addition, non-linear decay comes at the expense of loss of regularity, and one needs a subtle interplay of energy functionals with suitable weights (in the Fourier space) to successfully close the argument.

This gap was bridged in part by the second author in [22], who proved a precise linear result, which combined Fourier analysis and spectral analysis, and provided accurate estimates that are compatible with non-linear analysis. In this paper, we close this gap completely in one important case, namely the case of monotone shear flows satisfying a suitable spectral assumption.

The problem of non-linear inviscid damping we consider here is connected to the well-known Landau damping effect for Vlasov–Poisson equations, and we refer to the celebrated work of Mouhot–Villani [34] for the physical background and more references. Inviscid damping is a very subtle mechanism of stability, and has only been proved rigorously in 2D for Euler-type equations. It can also be viewed as the limiting case of the Navier–Stokes equation with small viscosity $\nu>0$. In the presence of viscosity, one can have more robust stability results for initial data that is sufficiently small relative to ν, which exploit the enhanced dissipation due to the mixing of the fluid. See [6], [9], [10], [41] and references therein. Moreover, in the limit $\nu\to 0$ and if there is boundary, then the boundary layer becomes an important issue, and there are significant additional difficulties. We refer the interested reader to [7], [14] for more details and further references.

1.4. Main ideas

We describe now some of the main ideas involved in the proof.

1.4.1. Renormalization and time-dependent energy functionals

These are two key ideas introduced by Bedrossian–Masmoudi [8] in the case of Couette flow, and which are important in this work as well. We refer to [8], [20] and the recent excellent survey [5] for longer discussions on this topic and its connection with Landau damping of Vlasov–Poisson equations.

As in [8] and [20], [21], we make a non-linear change of variable, and define v and z by

$$v(t,y):=b(y)+\frac{1}{t}\int_0^t \langle u^x\rangle(s,y)\,ds \quad \text{and} \quad z(t,y):=x-tv(t,y). \tag{1.25}$$

The main point is to remove the terms containing the non-decaying components $b(y)\partial_x\omega$ and $\langle u^x\rangle\partial_x\omega$ from the evolution equation satisfied by the renormalized vorticity. We also remark that the change of variable $y\mapsto v$ is crucial since, roughly speaking, it linearizes an oscillatory factor of the type $e^{-iktb(y)}$ to e^{-iktv}, which allows us to precisely capture the main decay factor. Compare e.g. (1.24) for Couette flow and the elliptic equation (4.27) for general shear flows.

Denote

$$F(t,z,v) := \omega(t,x,y) \quad \text{and} \quad \phi(t,z,v) := \psi(t,x,y). \tag{1.26}$$

Under this change of variable, the equation (1.13) becomes

$$\partial_t F - B''\partial_z\phi - V'\partial_v P_{\neq 0}\phi\,\partial_z F + (\dot V + V'\partial_z\phi)\,\partial_v F = 0, \tag{1.27}$$

where $P_{\neq 0}$ is projection off the zero mode. The coefficients B'', V', $\dot V$, and V' are suitable coordinate functions, connected to the change of variable (1.25), see (2.5)–(2.7) for the precise definitions.

The main idea is to control the regularity of F for all $t \geqslant 0$, as well as other quantities such as V', V'', B'', $\dot V$, and ϕ, using a bootstrap argument involving nine time-dependent energy functionals and space-time norms. These norms depend on a family of weights

$$A_k(t,\xi), \quad A_{NR}(t,\xi), \quad \text{and} \quad A_R(t,\xi), \tag{1.28}$$

for $k \in \mathbb{Z}$ and $\xi \in \mathbb{R}$, which have to be designed carefully to control the non-linearities.

To motivate the choice of weights, assume that F and ϕ satisfy the simplified system

$$\partial_t F - \partial_v P_{\neq 0}\phi\,\partial_z F + \partial_z\phi\,\partial_v F = 0, \quad \partial_z^2\phi + (\partial_v - t\partial_z)^2\phi = F, \tag{1.29}$$

for $(z,v,t) \in \mathbb{T}\times\mathbb{R}\times[0,\infty)$. Compared to the original system, we assume that $b'' \equiv 0$ (the Couette flow) and keep only one non-linear term, the "reaction term" $\partial_v \mathbb{P}_{\neq 0}\phi \cdot \partial_z F$. We would like to control, uniformly in time, an energy functional of the form

$$\mathcal{E}(t) := \sum_{k\in\mathbb{Z}} \int_{\mathbb{R}} A_k^2(t,\xi)|\widetilde F(t,k,\xi)|^2\,d\xi, \tag{1.30}$$

as well as a similar energy functional for the function ϕ, for a suitable weight $A_k(t,\xi)$ which decreases in t. The main observation is that

$$(\widetilde{\partial_v P_{\neq 0}\phi})(t,k,\xi) = -\frac{i\xi}{k^2}\frac{\widetilde F(t,k,\xi)}{1+|t-\xi/k|^2}\mathbf{1}_{k\neq 0}. \tag{1.31}$$

When $|\xi| \gg k^2$, the factor ξ/k^2 in (1.31) indicates a loss of one full derivative in v, which occurs in the resonant region $\{(t,k,\xi) : |t-\xi/k| \ll |\xi|/k^2 \text{ and } k^2+1 \ll |\xi|\}$. This is a major obstruction to proving stability, which cannot be removed by standard symmetrization techniques.

The key original idea of [8] is to use *imbalanced weights* $A_k(t,\xi)$ to absorb this derivative loss, taking advantage of the favorable structure of the non-linearity that does not allow for contributions to the resonant region to come from bilinear interactions of small frequencies and frequencies in the resonant region (due to the factor $\partial_z F$ in the reaction term). More precisely, the weights A satisfy the property

$$\frac{A_\ell(t,\eta)}{A_k(t,\xi)} \approx \left|\frac{\eta}{\ell^2}\right| \frac{1}{1+|t-\eta/\ell|}, \tag{1.32}$$

when $k\neq\ell$, $\ell\neq 0$, $\xi=\eta+O(1)$, $k=\ell+O(1)$, and $1+|t-\eta/\ell|\ll|\eta|/\ell^2$.

The weights $A_k(t,\xi)$ decrease in time, in the quantitative form,

$$-\frac{\partial_t A_k(t,\xi)}{A_k(t,\xi)} \approx \frac{1}{\langle t-\xi/k\rangle}, \tag{1.33}$$

if $k\in\mathbb{Z}\setminus\{0\}$, $k^2\lesssim|\xi|$, and $|t-\xi/k|\lesssim|\xi|/k^2$, which is needed in order to be able to control some of the non-linear terms using the Cauchy–Kowalevski terms coming from time differentiation of the energy functional $\mathcal{E}$ in (1.30). This leads to loss of regularity of the profile F during the evolution, which is the price to pay to prove non-linear decay of the stream function ϕ.

Finally, to prove commutator estimates in the context of our problem, we need to know that the weights vary sufficiently slowly in ξ, ideally something like

$$|A_k(t,\xi)-A_k(t,\eta)| \lesssim \langle k,\xi\rangle^{-1/2}[A_k(t,\xi)+A_k(t,\eta)]$$

if $\langle\xi-\eta\rangle\lesssim 1$. This is not possible, however, in the framework of imbalanced weights as defined above. Our solution to this problem is to allow the weights to depend on another parameter $\delta\ll 1$, and prove weaker estimates of the form

$$|A_k(t,\xi)-A_k(t,\eta)| \lesssim \left[\frac{C(\delta)}{\langle k,\xi\rangle^{1/2}}+\sqrt{\delta}\right] \max\{A_k(t,\xi), A_k(t,\eta)\} \tag{1.34}$$

if $\langle\xi-\eta\rangle\lesssim 1\ll\min\{\langle k,\xi\rangle,\langle k,\eta\rangle\}$. Such bounds are still suitable to control the commutators, due to the gain of $\sqrt{\delta}$ for large frequencies.

The resonant and non-resonant weights A_R and A_{NR} are used to control quantities that do not depend on z, like the change-of-coordinates functions V, $\dot{V}$, etc. The special weights we use here are the same as the weights we used in our earlier work [20], [21], and we rely on many estimates proved in these papers. Our weights are refinements of the weights of [8], but depend on an additional small parameter δ which gives critical flexibility at several stages of the argument.

1.4.2. The auxiliary non-linear profile

In the case of general shear flows, an essential new difficulty that is not present in the Couette case, is the additional linear term $B''(t,v)\partial_z\phi$ in (1.27). This extra linear term cannot be treated as a perturbation if b'' is not assumed small. On the linearized level, one can understand the evolution by using spectral analysis, especially the regularity analysis of generalized eigenfunctions corresponding to the continuous spectrum. However, it is still a challenge to combine the linear spectral analysis with the more sophisticated Fourier analysis tools needed for controlling the non-linearity. We deal with this basic issue in two steps: first we define an auxiliary non-linear profile $F^*(t)$ given by

$$F^*(t,z,v)=F(t,z,v)-\int_0^t B''(0,v)\partial_z\phi'(s,z,v)\,ds. \tag{1.35}$$

Thus F^* takes into account the linear effect accumulated up to time t and can be bounded perturbatively, using the method in [20], [21] (outlined in §1.4.1 above). The function ϕ' (not to be confused with the derivative of ϕ) is a small but crucial modification of ϕ, obtained by freezing the coefficients of the elliptic equation defining stream functions at time $t=0$, in order to keep these coefficients very smooth. See (2.39)–(2.40) for the precise definitions.

On a heuristic level, we expect that the full evolution of F consists of two contributions: the main, linear evolution that changes the size of the profile most significantly, and a small but rough (compared with the linear evolution) non-linear correction. We can view (1.35) as a bounded linear transformation in both space and time from F to F^* which takes into account the bulk linear evolution. Remarkably, the transformation (1.35) can be chosen independently of the non-linear evolution, once the non-linear change of coordinates is fixed, and can be studied using just linear analysis. The key point is that this transformation can be inverted to get bounds on the full profile F from bounds on F^*; see §1.4.3 for details.

The modified profile F^* now evolves in a perturbative fashion, and can be bounded using the method in [20], [21]. However, this construction leads to loss of symmetry in the transport terms $V'\partial_v P_{\neq 0}\phi\,\partial_z F$ and $(\dot{V}+V'\partial_z\phi)\,\partial_v F$, since the main perturbative variable is now F^*. This loss of symmetry causes a derivative loss, so we need to prove stronger bounds on $F-F^*$ than on the variables F and F^*, as described in (2.44).

1.4.3. Control of the full profile

We still need to recover the bounds on F and the improved bounds on $F-F^*$. Since the bounds on F^* are already proved, it suffices to prove the improved bounds (2.50) for $F-F^*$.

This is a critical step where we need to use our main spectral assumption and the precise estimates on the linearized flow. To link $F-F^*$ with the linearized flow, we define an auxiliary function ϕ^*, which can be approximately viewed as a stream function associated with F^*, see (7.7) for the precise definitions. Now, setting $g:=F-F^*$ and $\varphi:=\phi'-\phi^*$, the functions g and φ satisfy the inhomogeneous linear system with trivial initial data

$$\begin{gathered}\partial_t g - B_0''(v)\partial_z\varphi = H, \quad g(0,z,v)=0,\\ B_0'(v)^2(\partial_v - t\partial_z)^2\varphi + B_0''(v)(\partial_v - t\partial_z)\varphi + \partial_z^2\varphi = g(t,z,v),\end{gathered} \tag{1.36}$$

where $(t,z,v)\in[0,\infty)\times\mathbb{T}\times[b(0),b(1)]$. The functions

$$B_0'(v)=B'(0,v) \quad \text{and} \quad B_0''(v)=B''(0,v)$$

are time-independent, very smooth, and can be expressed in terms of the original shear flow b. The source term H is given by $H=B_0''(v)\partial_z\phi^*$.

The function ϕ^* is determined by the auxiliary profile F^*. Since we have already proved quadratic bounds on the profile F^*, we can use elliptic estimates to prove quadratic bounds on ϕ^*, and then on the source term H. Therefore, we can think of (1.36) as a linear inhomogeneous system with trivial initial data, and attempt to adapt the linear theory to our situation.

Decomposing in modes, conjugating by e^{-ikvt}, and using Duhamel's formula, we can further reduce to the study of the homogeneous initial-value problem

$$\begin{gathered}\partial_t g_k + ikvg_k - ikB_0''\varphi_k = 0, \quad g_k(0,v) = X_k(v)e^{-ikav},\\ (B_0')^2\partial_v^2\varphi_k + B_0''(v)\partial_v\varphi_k - k^2\varphi_k = g_k, \quad \varphi_k(b(0)) = \varphi_k(b(1)) = 0.\end{gathered} \tag{1.37}$$

for $(t,v)\in[0,\infty)\times[b(0),b(1)]$, where $k\in\mathbb{Z}\setminus\{0\}$ and $a\in\mathbb{R}$.

1.4.4. Analysis of the linearized flow

The equation (1.37) was analyzed, at least when $a=0$, by Wei–Zhang–Zhao in [42] and by the second author in [22]. We follow the approach in [22]. The main idea is to use the spectral representation formula and reduce the analysis of the linearized flow to the analysis of generalized eigenfunctions corresponding to the continuous spectrum. More precisely, using general spectral theory, we can express the stream function as an oscillatory integral of the spectral density function (which depends both on the physical and the spectral variables); see Proposition 8.2. An important new feature in the analysis of the linearized equation here, in comparison with [22] is that we have to consider initial data with an oscillatory factor, see (1.37), and the norms we use to measure the spectral density function are adapted to the oscillatory factor. It is well known that the spectral density function, see e.g. (8.22), contains singularities. To obtain precise characterization of these singularities, we make suitable re-normalizations (8.33)–(8.34) and estimate the resulting functions in Gevrey spaces.

As a result, given data X_k smooth and satisfying supp $X_k \subseteq [b(\vartheta_0), b(1-\vartheta_0)]$, we find a representation formula (see Lemma 8.1 and Proposition 7.1 for the precise formulations)

$$\tilde{g}_k(t,\xi) = \widetilde{X}_k(\xi+kt+ka) + ik\int_0^t\int_{\mathbb{R}} \widetilde{B}_0''(\zeta)\widetilde{\Pi}_k'(\xi+kt-\zeta-k\tau, \xi+kt-\zeta, a)\,d\zeta\,d\tau \tag{1.38}$$

for the solution g_k of the linear evolution equation (1.37), where $\Pi_k'(\xi,\eta,a)$ can be expressed in terms of a family of generalized eigenfunctions. As proved in [22], these eigenfunctions cannot be calculated explicitly, but can be estimated very precisely in the Fourier space,

$$\|(|k|+|\xi|)W_k(\eta+ka)\widetilde{\Pi}_k'(\xi,\eta,a)\|_{L^2_{\xi,\eta}} \lesssim_\delta \|W_k(\eta)\widetilde{X}_k(\eta)\|_{L^2_\eta}, \tag{1.39}$$

for any $a\in\mathbb{R}$, where W_k is a family of weights satisfying smoothness properties of the type

$$|W_k(\xi)-W_k(\eta)| \lesssim e^{2\delta_0\langle\xi-\eta\rangle^{1/2}} W_k(\eta)\left[\frac{C(\delta)}{\langle k,\eta\rangle^{1/8}}+\sqrt{\delta}\right] \quad \text{for any } \xi,\eta\in\mathbb{R}. \tag{1.40}$$

The inequality (1.40) holds for standard weights, like polynomial weights

$$W_k(\xi) = (1+|\xi|^2)^{N/2},$$

which correspond to Sobolev spaces, or exponential weights

$$W_k(\xi) = e^{\lambda\langle\xi\rangle^s}, \quad s<\tfrac{1}{2},$$

which correspond to Gevrey spaces. More importantly, it also holds for our carefully designed weights $A_k(t,\xi)$, as we have already seen in [21]. This allows us to adapt and incorporate the linear theory, and close the argument.

1.5. Organization

The rest of the paper is organized as follows. In §2 we renormalize the variables using a non-linear change of coordinates and set up the main bootstrap Proposition 2.2. In §3 we collect some lemmas concerning Gevrey spaces and describe in detail our main weights A_k, A_R, and A_{NR}. In §4 we prove several bilinear estimates and, more importantly, an elliptic estimate that can be applied many times to control stream-like functions. In §§5–7 we prove the main bootstrap Proposition 2.2. In §8 we prove the main estimates on the linear flow, by adapting the analysis in [22]. Finally, in §9, we use the main bootstrap proposition to complete the proof of Theorem 1.2.

2. The main bootstrap proposition

2.1. Renormalization and the new equations

In this subsection we introduce the non-linear change of variables and define the main quantities we need to control uniformly over time. As illustrated in §1.2.1, to obtain uniform control we need to un-wind the transportation in x; see (2.4) for the precise definitions. Since the coordinate system is time and solution dependent, we need to derive the equations of not only the profile of vorticity and the renormalized stream function, but also of the coordinate system itself. The calculations are all summarized in Proposition 2.1.

Assume that $\omega\colon[0,T]\times\mathbb{T}\times[0,1]$ is a sufficiently smooth solution of the system

$$\begin{gathered}\partial_t\omega+b(y)\partial_x\omega-b''(y)\partial_x\psi+u\cdot\nabla\omega=0,\\ (u^x,u^y)=(-\partial_y\psi,\partial_x\psi),\quad \Delta\psi=\omega,\quad \psi(t,x,1)=\psi(t,x,0)=0,\end{gathered}\tag{2.1}$$

which is supported in $\mathbb{T}\times[\vartheta_0,1-\vartheta_0]$ at all times $t\in[0,T]$, satisfying $\|\langle\omega\rangle(t)\|_{H^{10}}\ll 1$ and

$$\int_{\mathbb{T}}u^x(0,x,y)\,dx=0\quad\text{for any } y\in[0,1].\tag{2.2}$$

Using (2.1) and (2.2), it is easy to show that

$$\int_{\mathbb{T}}u^x(t,x,y)\,dx\equiv 0\quad\text{for any } t\in[0,T] \text{ and } y\in[0,\vartheta_0]\cup[1-\vartheta_0,1].\tag{2.3}$$

Indeed, as $u^x=-\partial_y\psi$ and $\Delta\psi=\omega$, we have $\partial_y\langle u^x\rangle=-\langle\omega\rangle$. We also have $\partial_t\langle u^x\rangle=\langle\omega\partial_x\psi\rangle$ (see the proof of (9.23) below), and the desired identities (2.3) follow using the support assumption on ω.

As in [8], [20], [21], we make the non-linear change of variables

$$v=b(y)+\frac{1}{t}\int_0^t\langle u^x\rangle(\tau,y)\,d\tau,\quad z=x-tv.\tag{2.4}$$

The point of this change of variables is to eliminate two of the non-decaying terms in the evolution equation in (2.1), namely the terms

$$b(y)\,\partial_x\omega\quad\text{and}\quad\langle u^x\rangle\,\partial_x\omega.$$

The change of variable $y\mapsto v$ is crucial for our analysis, and it allows us to link the renormalized stream function ϕ to the profile F using the elliptic equation (2.32). The point is that this equation has constant coefficients at the linear level, so it is compatible with Fourier analysis.

Then, we define the functions

$$F(t,z,v):=\omega(t,x,y),\quad \phi(t,z,v):=\psi(t,x,y), \tag{2.5}$$

$$V'(t,v):=\partial_y v(t,y),\quad V''(t,v):=\partial_{yy}v(t,y),\quad \dot V(t,v):=\partial_t v(t,y), \tag{2.6}$$

$$B'(t,v):=\partial_y b(y),\quad B''(t,v):=\partial_{yy}b(y). \tag{2.7}$$

Using (2.3), we have

$$v\in[b(0),b(1)]\quad\text{and}\quad \operatorname{supp}F(t)\subset\mathbb{T}\times[b(\vartheta_0),b(1-\vartheta_0)]\ \text{ for any } t\in[0,T]. \tag{2.8}$$

The evolution equation in (2.1) becomes

$$\partial_t F-B''\partial_z\phi-V'\partial_v P_{\neq 0}\phi\,\partial_z F+(\dot V+V'\partial_z\phi)\,\partial_v F=0, \tag{2.9}$$

where $P_{\neq 0}$ is projection off the zero mode, i.e., for any function $H(t,z,v)$,

$$P_{\neq 0}H(t,z,v)=H(t,z,v)-\langle H\rangle(t,v). \tag{2.10}$$

Moreover, we have

$$\partial_x\psi=\partial_z\phi\quad\text{and}\quad \partial_y\psi=V'(\partial_v\phi-t\partial_z\phi)=V'(\partial_v-t\partial_z)\phi, \tag{2.11}$$

and therefore

$$\partial_{xx}\psi=\partial_{zz}\phi\quad\text{and}\quad \partial_{yy}\psi=(V')^2(\partial_v-t\partial_z)^2\phi+V''(\partial_v-t\partial_z)\phi. \tag{2.12}$$

Recalling the equation $\Delta\psi=\omega$, we see that ϕ satisfies

$$\partial_z^2\phi+(V')^2(\partial_v-t\partial_z)^2\phi+V''(\partial_v-t\partial_z)\phi=F, \tag{2.13}$$

with $\phi(t,x,b(0))=\phi(t,x,b(1))=0$ for any $t\in[0,T]$ and $x\in\mathbb{T}$.

We also need to establish equations for the functions V', V'', $\dot V$, B', and B'' associated with the change of variables. Using (2.4) and the observation $-\partial_y\langle u^x\rangle=\langle\omega\rangle$, we have

$$\begin{aligned}
\partial_y v(t,y)&=b'(y)-\frac{1}{t}\int_0^t\langle\omega\rangle(\tau,y)\,d\tau,\\
\partial_t v(t,y)&=\frac{1}{t}\left[-\frac{1}{t}\int_0^t\langle u^x\rangle(\tau,y)\,d\tau+\langle u^x\rangle(t,y)\right],\\
\partial_y\partial_t v(t,y)&=\frac{1}{t}\left[\frac{1}{t}\int_0^t\langle\omega\rangle(\tau,y)\,d\tau-\langle\omega\rangle(t,y)\right].
\end{aligned} \tag{2.14}$$

Thus,

$$-\frac{1}{t}\int_0^t \langle\omega\rangle(\tau,y)d\tau = V'(t,v(t,y)) - b'(y). \tag{2.15}$$

By the chain rule, it follows that

$$\begin{aligned}\partial_t[t(V'(t,v)-B'(t,v))] + t\dot{V}(t,v)\,\partial_v[V'(t,v)-B'(t,v)] &= -\langle F\rangle(t,v)\\ &:= -\frac{1}{2\pi}\int_{\mathbb{T}} F(t,z,v)\,dz.\end{aligned} \tag{2.16}$$

We notice that

$$\partial_y(\partial_t v(t,y)) = \partial_y[\dot{V}(t,v(t,y))] = V'(t,v(t,y))\,\partial_v\dot{V}(t,v(t,y)). \tag{2.17}$$

Hence, using the last identity in (2.14) and the identities (2.15) and (2.17), we have

$$tV'(t,v)\,\partial_v\dot{V}(t,v) = B'(t,v) - V'(t,v) - \langle F\rangle(t,v). \tag{2.18}$$

We derive now our main evolution equations. It follows from (2.16) and (2.18) that

$$\partial_t(V'-B') = V'\partial_v\dot{V} - \dot{V}\partial_v(V'-B'). \tag{2.19}$$

Set

$$\mathcal{H} := tV'\partial_v\dot{V} = B' - V' - \langle F\rangle. \tag{2.20}$$

Using (2.19) and (2.9), we calculate

$$\begin{aligned}\partial_t\mathcal{H} &= -\partial_t(V'-B') - \langle\partial_t F\rangle\\ &= -V'\partial_v\dot{V} + \dot{V}\partial_v(V'-B') - V'\langle\partial_v P_{\neq 0}\phi\,\partial_z F\rangle + \langle(\dot{V}+V'\partial_z\phi)\,\partial_v F\rangle.\end{aligned}$$

Using again (2.20) and simplifying, we get

$$\partial_t\mathcal{H} = -\frac{\mathcal{H}}{t} - \dot{V}\,\partial_v\mathcal{H} - V'\langle\partial_v P_{\neq 0}\phi\,\partial_z F\rangle + V'\langle\partial_z\phi\,\partial_v F\rangle.$$

Finally, using (2.7), we have

$$\partial_t B'(t,v) + \dot{V}\partial_v B'(t,v) = \partial_t B''(t,v) + \dot{V}\partial_v B''(t,v) = 0. \tag{2.21}$$

We summarize our calculations so far in the following result.

PROPOSITION 2.1. *Let $\omega:[0,T]\times\mathbb{T}\times[0,1]\to\mathbb{R}$ be a sufficiently smooth solution of the system (2.1)–(2.2) on some time interval $[0,T]$. Assume that $\omega(t)$ is supported in $\mathbb{T}\times[\vartheta_0,1-\vartheta_0]$ and that $\|\langle\omega\rangle(t)\|_{H^{10}}\ll 1$ for all $t\in[0,T]$. Then,*

$$\langle u^x\rangle(t,y)=0 \quad \text{for any } t\in[0,T] \text{ and } y\in[0,\vartheta_0]\cup[1-\vartheta_0,1]. \tag{2.22}$$

We define the change-of-coordinates functions $(z,v):\mathbb{T}\times[0,1]\to\mathbb{T}\times[b(0),b(1)]$ by

$$v:=b(y)+\frac{1}{t}\int_0^t\langle u^x\rangle(\tau,y)\,d\tau \quad \text{and} \quad z:=x-tv, \tag{2.23}$$

and the new variables

$$F,\phi:[0,T]\times\mathbb{T}\times[b(0),b(1)]\longrightarrow\mathbb{R}$$

and

$$V',V'',\dot V,B',B'',\mathcal{H}:[0,T]\times[b(0),b(1)]\longrightarrow\mathbb{R}$$

by

$$F(t,z,v):=\omega(t,x,y), \quad \phi(t,z,v):=\psi(t,x,y), \tag{2.24}$$

$$V'(t,v):=\partial_y v(t,y), \quad V''(t,v)=\partial_{yy}v(t,y), \quad \dot V(t,v)=\partial_t v(t,y), \tag{2.25}$$

$$B'(t,v):=\partial_y b(y), \quad B''(t,v):=\partial_{yy}b(y), \tag{2.26}$$

$$\mathcal{H}(t,v):=tV'(t,v)\,\partial_v\dot V(t,v)=B'(t,v)-V'(t,v)-\langle F\rangle(t,v). \tag{2.27}$$

Then, $V'(t,v)\geqslant\frac{1}{2}\vartheta_0$. Moreover, the new variables F, $V'-B'$, $\dot V$, and $\mathcal{H}$ are supported in $[0,T]\times\mathbb{T}\times[b(\vartheta_0),b(1-\vartheta_0)]$ and satisfy the evolution equations

$$\partial_t F-B''\partial_z\phi=V'\partial_v P_{\neq 0}\phi\,\partial_z F-(\dot V+V'\partial_z\phi)\,\partial_v F, \tag{2.28}$$

$$\partial_t B'(t,v)+\dot V\partial_v B'(t,v)=\partial_t B''(t,v)+\dot V\,\partial_v B''(t,v)=0, \tag{2.29}$$

$$\partial_t(V'-B')+\dot V\,\partial_v(V'-B')=\frac{\mathcal{H}}{t}, \tag{2.30}$$

$$\partial_t\mathcal{H}+\dot V\,\partial_v\mathcal{H}=-\frac{\mathcal{H}}{t}-V'\langle\partial_v P_{\neq 0}\phi\,\partial_z F\rangle+V'\langle\partial_z\phi\,\partial_v F\rangle. \tag{2.31}$$

The variables ϕ, V'', and $\dot V$ satisfy the elliptic-type identities

$$\partial_z^2\phi+(V')^2(\partial_v-t\partial_z)^2\phi+V''(\partial_v-t\partial_z)\phi=F, \tag{2.32}$$

$$\partial_v\dot V=\frac{\mathcal{H}}{tV'}, \quad \dot V(t,b(0))=\dot V(t,b(1))=0, \quad V''=V'\partial_v V'. \tag{2.33}$$

We remark that the main variable we need to control is F, which is the profile for the vorticity ω. The variable $\mathcal{H}=tV'\partial_v\dot V$ is constructed from $\dot V(t,v)=\partial_t v(t,y)$, and encodes the convergence of the coordinate system. In addition, $\mathcal{H}$ satisfies a more favorable equation than $\dot V$. We refer to §2.3 for further discussion on these variables, as well as the other variables.

2.2. Energy functionals and the bootstrap proposition

The main idea of the proof is to estimate the increment of suitable energy functionals, which are defined using special weights. For simplicity, we use exactly the same weights A_{NR}, A_R, and A_k, as in our earlier papers [20] and [21], so we can use some of their properties proved there. These weights are defined by

$$A_{NR}(t,\xi):=\frac{e^{\lambda(t)\langle\xi\rangle^{1/2}}}{b_{NR}(t,\xi)}e^{\sqrt{\delta}\langle\xi\rangle^{1/2}},\quad A_R(t,\xi):=\frac{e^{\lambda(t)\langle\xi\rangle^{1/2}}}{b_R(t,\xi)}e^{\sqrt{\delta}\langle\xi\rangle^{1/2}}, \tag{2.34}$$

and

$$A_k(t,\xi):=e^{\lambda(t)\langle k,\xi\rangle^{1/2}}\bigg(\frac{e^{\sqrt{\delta}\langle\xi\rangle^{1/2}}}{b_k(t,\xi)}+e^{\sqrt{\delta}|k|^{1/2}}\bigg), \tag{2.35}$$

where $k\in\mathbb{Z}$, $t\in[0,\infty)$, and $\xi\in\mathbb{R}$. The function $\lambda:[0,\infty)\to\big[\delta_0,\frac{3}{2}\delta_0\big]$ is defined by

$$\lambda(0)=\frac{3}{2}\delta_0\quad\text{and}\quad\lambda'(t)=-\frac{\delta_0\sigma_0^2}{\langle t\rangle^{1+\sigma_0}}, \tag{2.36}$$

where $\delta_0>0$ is a fixed parameter and $\sigma_0=0.01$. In particular, λ is decreasing on $[0,\infty)$, and the functions A_{NR}, A_R, and A_k are also decreasing in t. The parameter $\delta>0$, which appears also in the weights b_R, b_{NR}, and b_k, is to be taken sufficiently small, depending only on the structural parameters δ_0, ϑ_0, and κ.

The precise definitions of the weights b_{NR}, b_R, and b_k are very important; all the technical details are provided in §3.2 (see also §1.4.1 for some motivation). For now, we note that these functions are essentially increasing in t and satisfy

$$e^{-\delta\sqrt{|\xi|}}\leqslant b_R(t,\xi)\leqslant b_k(t,\xi)\leqslant b_{NR}(t,\xi)\leqslant 1\quad\text{for any } t,\ \xi,\text{ and } k. \tag{2.37}$$

In other words, the weights $1/b_{NR}$, $1/b_R$, and $1/b_k$ are small when compared to the main factors $e^{\lambda(t)\langle\xi\rangle^{1/2}}$ and $e^{\lambda(t)\langle k,\xi\rangle^{1/2}}$ in (2.34)–(2.35). However, their relative contributions are important as they are used to distinguish between resonant and non-resonant times.

Assume that $\omega:[0,T]\times\mathbb{T}\times[0,1]\to\mathbb{R}$ is as in Proposition 2.1 and define the functions F, ϕ, V', V'', $\dot V$, B', B'', and $\mathcal{H}$ as in (2.24)–(2.27). To construct useful energy functionals we need to modify the functions V', B', and B'' which are not "small", so we define the new variables

$$\begin{gathered}B_0'(v):=B'(0,v)=(\partial_y b)(b^{-1}(v)),\quad B_0''(v):=B''(0,v)=(\partial_y^2 b)(b^{-1}(v)),\\ V_*':=V'-B_0',\quad B_*':=B'-B_0',\quad B_*'':=B''-B_0''.\end{gathered} \tag{2.38}$$

Our main goal is to control the functions F and ϕ. For this, we need to consider two auxiliary functions F^* and ϕ'. We define first the function

$$\phi'(t,z,v):[0,T]\times\mathbb{T}\times[b(0),b(1)]\longrightarrow\mathbb{R}$$

as the unique solution to the equation (see Lemma 4.5 for existence and uniqueness)

$$\partial_z^2\phi'+(B_0')^2(\partial_v-t\partial_z)^2\phi'+B_0''(\partial_v-t\partial_z)\phi'=F,\quad \phi'(t,b(0))=\phi'(t,b(1))=0, \tag{2.39}$$

on $\mathbb{T}\times[b(0),b(1)]$. Then, we define the modified profile

$$F^*(t,z,v):=F(t,z,v)-B_0''(v)\int_0^t \partial_z\phi'(\tau,z,v)\,d\tau, \tag{2.40}$$

and the renormalized elliptic profiles

$$\begin{aligned}\Theta(t,z,v)&:=(\partial_z^2+(\partial_v-t\partial_z)^2)\,(\Psi(v)\,\phi(t,z,v))\,,\\ \Theta^*(t,z,v)&:=(\partial_z^2+(\partial_v-t\partial_z)^2)\,(\Psi(v)\,(\phi(t,z,v)-\phi'(t,z,v)))\,,\end{aligned} \tag{2.41}$$

where $\Psi:\mathbb{R}\to[0,1]$ is a Gevrey class cut-off function, satisfying

$$\begin{aligned}&\|e^{\langle\xi\rangle^{3/4}}\widetilde{\Psi}(\xi)\|_{L^\infty}\lesssim 1,\\ &\operatorname{supp}\Psi\subseteq\big[b\big(\tfrac14\vartheta_0\big),b\big(1-\tfrac14\vartheta_0\big)\big],\\ &\Psi\equiv 1\quad\text{in }\big[b\big(\tfrac13\vartheta_0\big),b\big(1-\tfrac13\vartheta_0\big)\big].\end{aligned} \tag{2.42}$$

Our bootstrap argument is based on controlling simultaneously energy functionals and space-time integrals. Let $\dot A_Y(t,\xi):=(\partial_t A_Y)(t,\xi)\leqslant 0$ for $Y\in\{NR,R,k\}$, and define, for any $t\in[0,T]$,

$$\begin{aligned}\mathcal{E}_f(t)&:=\sum_{k\in\mathbb{Z}}\int_{\mathbb{R}}A_k^2(t,\xi)|\tilde f(t,k,\xi)|^2\,d\xi,\quad f\in\{F,F^*\},\\ \mathcal{B}_f(t)&:=\int_1^t\sum_{k\in\mathbb{Z}}\int_{\mathbb{R}}|\dot A_k(s,\xi)|\,A_k(s,\xi)\,|\tilde f(s,k,\xi)|^2\,d\xi\,ds,\end{aligned} \tag{2.43}$$

$$\begin{aligned}\mathcal{E}_{F-F^*}(t)&:=\sum_{k\in\mathbb{Z}^*}\int_{\mathbb{R}}\bigg(1+\frac{\langle k,\xi\rangle}{\langle t\rangle}\bigg)A_k^2(t,\xi)|\,\widetilde{(F-F^*)}(t,k,\xi)|^2\,d\xi,\\ \mathcal{B}_{F-F^*}(t)&:=\int_1^t\sum_{k\in\mathbb{Z}^*}\int_{\mathbb{R}}\bigg(1+\frac{\langle k,\xi\rangle}{\langle s\rangle}\bigg)|\dot A_k(s,\xi)|A_k(s,\xi)\,|\widetilde{(F-F^*)}(s,k,\xi)|^2\,d\xi\,ds,\end{aligned} \tag{2.44}$$

$$\begin{aligned}\mathcal{E}_\Phi(t)&:=\sum_{k\in\mathbb{Z}^*}\int_{\mathbb{R}}A_k^2(t,\xi)\frac{|k|^2\langle t\rangle^2}{|\xi|^2+|k|^2\langle t\rangle^2}|\widetilde\Phi(t,k,\xi)|^2\,d\xi,\quad \Phi\in\{\Theta,\Theta^*\},\\ \mathcal{B}_\Phi(t)&:=\int_1^t\sum_{k\in\mathbb{Z}^*}\int_{\mathbb{R}}|\dot A_k(s,\xi)|A_k(s,\xi)\frac{|k|^2\langle s\rangle^2}{|\xi|^2+|k|^2\langle s\rangle^2}|\widetilde\Phi(s,k,\xi)|^2\,d\xi\,ds,\end{aligned} \tag{2.45}$$

$$\begin{aligned}\mathcal{E}_g(t)&:=\int_{\mathbb{R}}A_R^2(t,\xi)|\tilde g(t,\xi)|^2\,d\xi,\quad g\in\{V_*',B_*',B_*''\},\\ \mathcal{B}_g(t)&:=\int_1^t\int_{\mathbb{R}}|\dot A_R(s,\xi)|\,A_R(s,\xi)\,|\tilde g(s,\xi)|^2\,d\xi\,ds,\end{aligned} \tag{2.46}$$

$$\begin{aligned}\mathcal{E}_{\mathcal{H}}(t) &:= \mathcal{K}^2 \int_{\mathbb{R}} A_{NR}^2(t,\xi)\Big(\frac{\langle t\rangle}{\langle \xi\rangle}\Big)^{3/2} |\widetilde{\mathcal{H}}(t,\xi)|^2\,d\xi,\\ \mathcal{B}_{\mathcal{H}}(t) &:= \mathcal{K}^2 \int_1^t\int_{\mathbb{R}} |\dot{A}_{NR}(s,\xi)|\,A_{NR}(s,\xi)\Big(\frac{\langle s\rangle}{\langle \xi\rangle}\Big)^{3/2} |\widetilde{\mathcal{H}}(s,\xi)|^2\,d\xi\,ds,\end{aligned} \tag{2.47}$$

where $\mathbb{Z}^*:=\mathbb{Z}\setminus\{0\}$ and $\mathcal{K}\geqslant 1$ is a large constant that depends only on δ.

Our main bootstrap proposition is the following.

PROPOSITION 2.2. *Assume that $T\geqslant 1$ and let $\omega\in C([0,T]:\mathcal{G}^{2\delta_0,1/2})$ be a sufficiently smooth solution of the system* (2.1)–(2.2), *with the property that $\omega(t)$ is supported in $\mathbb{T}\times[\vartheta_0,1-\vartheta_0]$ and that $\|\langle\omega\rangle(t)\|_{H^{10}}\ll 1$ for all $t\in[0,T]$. Define F, F^*, Θ, Θ^*, B_*', B_*'', V_*', $\mathcal{H}$ as above. Assume that ε_1 is sufficiently small* (*depending on δ*),

$$\sum_{g\in\{F,F^*,F-F^*,\Theta,\Theta^*,V_*',B_*',B_*'',\mathcal{H}\}} \mathcal{E}_g(t)\leqslant \varepsilon_1^3 \quad \text{for any } t\in[0,1], \tag{2.48}$$

and

$$\sum_{g\in\{F,F^*,F-F^*,\Theta,\Theta^*,V_*',B_*',B_*'',\mathcal{H}\}} [\mathcal{E}_g(t)+\mathcal{B}_g(t)]\leqslant \varepsilon_1^2 \quad \text{for any } t\in[1,T]. \tag{2.49}$$

Then, for any $t\in[1,T]$, we have the improved bounds

$$\sum_{g\in\{F,F^*,F-F^*,\Theta,\Theta^*,V_*',B_*',B_*'',\mathcal{H}\}} [\mathcal{E}_g(t)+\mathcal{B}_g(t)]\leqslant \frac{\varepsilon_1^2}{2}. \tag{2.50}$$

Moreover, for $t\in[1,T]$, we also have the stronger bounds

$$\sum_{g\in\{F,\Theta\}} [\mathcal{E}_g(t)+\mathcal{B}_g(t)]\lesssim_\delta \varepsilon_1^3. \tag{2.51}$$

The proof of Proposition 2.2 is the main part of this paper, and covers §§3–8. In §9 we then show how to use this proposition to prove the main theorem.

The use of bootstrap arguments in perturbation analysis is, of course, well established. The key point is an improvement in the bounds over the ones assumed at the beginning (in our case from ε_1^2 to $\frac{1}{2}\varepsilon_1^2$, compare (2.49) and (2.50)), using the evolution equations. This allows us to use continuity in time of the various quantities to increase the time interval from $[1,T]$ to $[1,T']$, $T'>T$, so that we still have the weaker bounds (2.49) on the longer time interval $[1,T']$. Then, we apply the bootstrap proposition on the longer time interval $[1,T']$ again to get the improved bounds (2.50). This allows us to extend the time on which we have control of the solution indefinitely. We refer to §9 for details.

We also remark that our main bootstrap bounds (2.49) are assumed on $[1,T]$ rather than the more often used $[0,T]$. This is only for convenience of arguments in later sections. Since the main task is to control large time behavior of our solutions, the removal of the time interval $[0,1]$ saves us from having to distinguish the cases $t\in[0,1]$ and $t\in[1,\infty)$, for example when treating equations (2.4) or (2.31), which contain a $1/t$ factor. The desired bounds on the time interval $[0,1]$ are consequences of local well-posedness theory; see Lemma 9.1.

2.3. The variables of the bootstrap argument

Our argument outlined in Proposition 2.2 involves control of nine quantities. We explain now the roles of these quantities:

(1) The main variables are the vorticity profile F and the renormalized elliptic profile Θ. Our primary goal is to prove global bounds on these quantities.

(2) The functions F^* and Θ^* are auxiliary variables, and we analyze them as an intermediate step to controlling the main variables F and Θ. The function F^* satisfies a better transport equation than F, without any other linear terms, while the function Θ^* satisfies a better elliptic equation than Θ, again without linear terms in the right-hand side.

(3) A significant component of the proof is to control the function $F-F^*$, which allows us to pass from the modified profile F^* to the true profile F. This is based on the theory of the linearized equation in Gevrey spaces, as developed by the second author in [22], and requires the spectral assumption (B) on the shear flow. We remark that the bootstrap control on the variable $F-F^*$ is slightly stronger than on the variables F and F^* separately, which is needed to compensate for the lack of symmetry in some of the transport terms.

(4) The functions V'_*, B'_*, and B''_* are connected to the change of variables $y\mapsto v$. These functions appear in many of the non-linear terms in the equations, so it is important to control their smoothness precisely, as part of a combined bootstrap argument, in a way that is consistent with the smoothness of the functions F and Θ.

(5) Finally, the function $\mathcal{H}$, which decays in time, encodes the convergence of the system as $t\to\infty$. This function decays at a rate of $\langle t\rangle^{-3/4}$, in a weaker topology, which shows that the function $\partial_v\dot{V}$ decays fast at an integrable rate of $\langle t\rangle^{-7/4}$, again in a weaker topology.

2.4. Compact support and localization

We note that the variables F, F^*, V'_*, B'_*, B''_*, and $\mathcal{H}$ are all supported in

$$\mathbb{T}\times[b(\vartheta_0), b(1-\vartheta_0)].$$

This compact support property allows us to freely insert a Gevrey regular cutoff function which equals 1 on $[b(\vartheta_0), b(1-\vartheta_0)]$ (such as $\Psi(y)$ from (2.42)), in front of these variables. For example, in some cases we need to bound $V'F$, where we note that V' is not compactly supported inside $\mathbb{T}\times(b(0), b(1))$. By the support property of F, it then suffices to bound $(\Psi V')F$, which can be treated by using the bootstrap bounds (2.49). This localization argument plays an important role at various stages of the proof.

3. Gevrey spaces and the weights A_k, A_R, and A_{NR}

In this section we collect some results on Gevrey spaces and on the weights A_k, A_R, and A_{NR}, that are useful below. Most of the results were proved in [20] and [21].

3.1. Gevrey spaces

We summarize here some general properties of the Gevrey spaces of functions. See [37], [46] for more discussion and further references on Gevrey spaces.

To perform certain algebraic operations, it is very useful to have a related definition in the physical space. For any domain $D\subseteq\mathbb{T}\times\mathbb{R}$ (or $D\subseteq\mathbb{R}$) and parameters $s\in(0,1)$ and $M\geqslant 1$, we define the spaces

$$\widetilde{\mathcal{G}}^s_M(D) := \{f\colon D\to\mathbb{C} : \|f\|_{\widetilde{\mathcal{G}}^s_M(D)} < \infty\}, \tag{3.1}$$

where

$$\|f\|_{\widetilde{\mathcal{G}}^s_M(D)} := \sup_{\substack{x\in D\\ m\geqslant 0\\ |\alpha|\leqslant m}} |D^\alpha f(x)| M^{-m}(m+1)^{-m/s}.$$

We start with a lemma connecting the spaces $\mathcal{G}^{\mu,s}$ and $\widetilde{\mathcal{G}}^s_M$.

LEMMA 3.1. ([20, §A.1]) (i) *Suppose that* $s\in(0,1)$, $K>1$, *and* $f\in C^\infty(\mathbb{T}\times\mathbb{R})$, *with* $\operatorname{supp} f\subseteq\mathbb{T}\times[-L,L]$, *satisfies the bounds* $\|f\|_{\widetilde{\mathcal{G}}^s_K(\mathbb{T}\times\mathbb{R})}\leqslant 1$. *Then, there is* $\mu=\mu(K,s)>0$ *such that*

$$|\tilde{f}(k,\xi)| \lesssim_{K,s} Le^{-\mu|k,\xi|^s} \quad \textit{for all } k\in\mathbb{Z} \textit{ and } \xi\in\mathbb{R}. \tag{3.2}$$

(ii) *Conversely, if* $\mu>0$ *and* $s\in(0,1)$, *then there is* $K=K(s,\mu)>1$ *such that*

$$\|f\|_{\widetilde{\mathcal{G}}^s_K(\mathbb{T}\times\mathbb{R})} \lesssim_{\mu,s} \|f\|_{\mathcal{G}^{\mu,s}(\mathbb{T}\times\mathbb{R})}. \tag{3.3}$$

Using this lemma one can construct cutoff functions in Gevrey spaces: for any points $a'<a\leqslant b<b'\in\mathbb{R}$ and any $s\in(0,1)$, there are functions Ψ supported in $[a',b']$, equal to 1 in $[a,b]$, and satisfying $|\widetilde{\Psi}(\xi)|\lesssim e^{-\langle\xi\rangle^s}$ for any $\xi\in\mathbb{R}$. See [20, §A.1] for an explicit construction of such functions, as well as an elementary proof of Lemma 3.1. We use several functions of this type in the proof of our main theorem.

The physical space characterization of Gevrey functions is useful when studying compositions and algebraic operations of functions.

LEMMA 3.2. (i) *Assume* $s\in(0,1)$, $M\geqslant 1$, *and* $f_1,f_2\in\widetilde{\mathcal{G}}^s_M(D)$. *Then,* $f_1f_2\in\widetilde{\mathcal{G}}^s_{M'}(D)$ *and*

$$\|f_1f_2\|_{\widetilde{\mathcal{G}}^s_{M'}(D)}\lesssim\|f_1\|_{\widetilde{\mathcal{G}}^s_M(D)}\|f_2\|_{\widetilde{\mathcal{G}}^s_M(D)}$$

for some $M'=M'(s,M)\geqslant M$. *Similarly, if* $f_1\geqslant 1$ *in* D, *then*

$$\left\|\frac{1}{f_1}\right\|_{\widetilde{\mathcal{G}}^s_{M'}(D)}\lesssim 1.$$

(ii) *Suppose* $s\in(0,1)$, $M\geqslant 1$, $I_1\subseteq\mathbb{R}$ *is an interval, and* $g\colon\mathbb{T}\times I_1\to\mathbb{T}\times I_2$ *satisfies*

$$|D^\alpha g(x)|\leqslant M^m(m+1)^{m/s}\quad\text{for any } x\in\mathbb{T}\times I_1,\ m\geqslant 1,\ \text{and } |\alpha|\in[1,m]. \tag{3.4}$$

If $K\geqslant 1$ *and* $f\in\widetilde{G}^s_K(\mathbb{T}\times I_2)$, *then* $f\circ g\in\widetilde{G}^s_L(\mathbb{T}\times I_1)$ *for some* $L=L(s,K,M)\geqslant 1$ *and*

$$\|f\circ g\|_{\widetilde{G}^s_L(\mathbb{T}\times I_1)}\lesssim_{s,K,M}\|f\|_{\widetilde{G}^s_K(\mathbb{T}\times I_2)}. \tag{3.5}$$

(iii) *Assume* $s\in(0,1)$, $L\in[1,\infty)$, $I,J\subseteq\mathbb{R}$ *are open intervals, and* $g\colon I\to J$ *is a smooth bijective map satisfying, for any* $m\geqslant 1$,

$$|D^\alpha g(x)|\leqslant L^m(m+1)^{m/s}\quad\text{for any } x\in I\ \text{and } |\alpha|\in[1,m]. \tag{3.6}$$

If $|g'(x)|\geqslant\rho>0$ *for any* $x\in I$, *then the inverse function* $g^{-1}\colon J\to I$ *satisfies the bounds*

$$|D^\alpha(g^{-1})(x)|\leqslant M^m(m+1)^{m/s}\quad\text{for any } x\in J\ \text{and } |\alpha|\in[1,m], \tag{3.7}$$

for some constant $M=M(s,L,\rho)\geqslant L$.

Lemma 3.2 can be proved by elementary means using just the definition (3.1). See also [46, Theorems 6.1 and 3.2] for more general estimates on functions in Gevrey spaces.

3.2. The weights A_{NR}, A_R, and A_k

We summarize here the construction of our main imbalanced weights A_R, A_{NR}, and A_k in [20]. We start by defining the functions $w_{NR},w_R\colon[0,\infty)\times\mathbb{R}\to[0,1]$, which model the non-resonant and resonant growth. The main point of these definitions is to distinguish between resonant and non-resonant regions, which plays a key role in the analysis. Resonance is measured in terms of the size of the denominators $\langle t-\xi/k\rangle$, which appear in the formula (1.31) expressing the normalized stream function in terms of the vorticity profile. The intervals $I_{k,\eta}$ defined below, where this factor is small are called "resonant" intervals. Notice the imbalance in (3.12) between the weights $w_R(t,\eta)$ and $w_{NR}(t,\eta)$, especially around the center of the resonant intervals, consistent with the loss of derivative discussed in §1.4.1.

Assume that $\delta>0$ is small, $\delta\ll\delta_0$. For $|\eta|\leqslant\delta^{-10}$ we simply define

$$w_{NR}(t,\eta):=1 \quad \text{and} \quad w_R(t,\eta):=1. \tag{3.8}$$

For $\eta>\delta^{-10}$ we define $k_0(\eta):=\lfloor\sqrt{\delta^3\eta}\rfloor$. For $l\in\{1,...,k_0(\eta)\}$, we define

$$t_{l,\eta}:=\frac{1}{2}\left(\frac{\eta}{l+1}+\frac{\eta}{l}\right), \quad t_{0,\eta}:=2\eta, \quad I_{l,\eta}:=[t_{l,\eta},t_{l-1,\eta}]. \tag{3.9}$$

Notice that $|I_{l,\eta}|\approx\eta/l^2$ and

$$\frac{\delta^{-3/2}\sqrt{\eta}}{2}\leqslant t_{k_0(\eta),\eta}\leqslant...\leqslant t_{l,\eta}\leqslant\frac{\eta}{l}\leqslant t_{l-1,\eta}\leqslant...\leqslant t_{0,\eta}=2\eta.$$

We define

$$w_{NR}(t,\eta):=1 \quad \text{and} \quad w_R(t,\eta):=1 \quad \text{if } t\geqslant t_{0,\eta}=2\eta. \tag{3.10}$$

Then we define, for $k\in\{1,...,k_0(\eta)\}$,

$$w_{NR}(t,\eta):=\begin{cases}\left(\dfrac{1+\delta^2|t-\eta/k|}{1+\delta^2|t_{k-1,\eta}-\eta/k|}\right)^{\delta_0}w_{NR}(t_{k-1,\eta},\eta), & \text{if } t\in\left[\dfrac{\eta}{k},t_{k-1,\eta}\right],\\ \left(\dfrac{1}{1+\delta^2|t-\eta/k|}\right)^{1+\delta_0}w_{NR}\left(\dfrac{\eta}{k},\eta\right), & \text{if } t\in\left[t_{k,\eta},\dfrac{\eta}{k}\right].\end{cases} \tag{3.11}$$

We define also the weight w_R by the formula

$$w_R(t,\eta):=\begin{cases}w_{NR}(t,\eta)\dfrac{1+\delta^2|t-\eta/k|}{1+\delta^2\eta/(8k^2)}, & \text{if } \left|t-\dfrac{\eta}{k}\right|\leqslant\dfrac{\eta}{8k^2},\\ w_{NR}(t,\eta), & \text{if } t\in I_{k,\eta} \text{ and } \left|t-\dfrac{\eta}{k}\right|\geqslant\dfrac{\eta}{8k^2},\end{cases} \tag{3.12}$$

for any $k\in\{1,...,k_0(\eta)\}$, and notice that, for $t\in I_{k,\eta}$,

$$\frac{\partial_t w_{NR}(t,\eta)}{w_{NR}(t,\eta)}\approx\frac{\partial_t w_R(t,\eta)}{w_R(t,\eta)}\approx\frac{\delta^2}{1+\delta^2\,|t-\eta/k|}. \tag{3.13}$$

It is easy to see that, for $\eta>\delta^{-10}$,

$$w_{NR}(t_{k_0(\eta),\eta},\eta)=w_R(t_{k_0(\eta),\eta},\eta)\in[X_\delta(\eta)^4,X_\delta(\eta)^{1/4}], \tag{3.14}$$

where

$$X_\delta(\eta):=e^{-\delta^{3/2}\ln(\delta^{-1})\sqrt{\eta}}.$$

For small values of $t=(1-\beta)t_{k_0(\eta),\eta}$, $\beta\in[0,1]$, we define w_{NR} and w_R by the formulas

$$w_{NR}(t,\eta)=w_R(t,\eta):=(e^{-\delta\sqrt{\eta}})^\beta w_{NR}(t_{k_0(\eta),\eta},\eta)^{1-\beta}. \tag{3.15}$$

If $\eta<-\delta^{-10}$, then we define

$$w_R(t,\eta):=w_R(t,|\eta|),\quad w_{NR}(t,\eta):=w_{NR}(t,|\eta|),\quad \text{and}\quad I_{k,\eta}:=I_{-k,-\eta}.$$

To summarize, the resonant intervals $I_{k,\eta}$ are defined for $(k,\eta)\in\mathbb{Z}\times\mathbb{R}$ satisfying $|\eta|>\delta^{-10}$, $1\leqslant|k|\leqslant\sqrt{\delta^3|\eta|}$, and $\eta/k>0$.

Finally, we define the weights $w_k(t,\eta)$ by the formula

$$w_k(t,\eta):=\begin{cases} w_{NR}(t,\eta), & \text{if } t\notin I_{k,\eta},\\ w_R(t,\eta), & \text{if } t\in I_{k,\eta}.\end{cases} \tag{3.16}$$

In particular, $w_k(t,\eta)=w_{NR}(t,\eta)$ unless $|\eta|>\delta^{-10}$, $1\leqslant|k|\leqslant\sqrt{\delta^3|\eta|}$, $\eta/k>0$, and $t\in I_{k,\eta}$.

The functions w_{NR}, w_R, and w_k have the right size but lack optimal smoothness in the frequency parameter η, mainly due to the jump discontinuities of the function $k_0(\eta)$. This smoothness is important in symmetrization arguments (energy control of the transport terms) and in commutator arguments. To correct this problem, we fix $\varphi:\mathbb{R}\to[0,1]$, an even smooth function supported in $\left[-\frac{8}{5},\frac{8}{5}\right]$ and equal to 1 in $\left[-\frac{5}{4},\frac{5}{4}\right]$, and let $d_0:=\int_{\mathbb{R}}\varphi(x)\,dx$. For $k\in\mathbb{Z}$ and $Y\in\{NR,R,k\}$, let

$$\begin{aligned} b_Y(t,\xi)&:=\int_{\mathbb{R}} w_Y(t,\rho)\varphi\left(\frac{\xi-\rho}{L_{\delta'}(t,\xi)}\right)\frac{1}{d_0L_{\delta'}(t,\xi)}\,d\rho,\\ L_{\delta'}(t,\xi)&:=1+\frac{\delta'\langle\xi\rangle}{\langle\xi\rangle^{1/2}+\delta't},\quad \delta'\in[0,1].\end{aligned} \tag{3.17}$$

The length $L_{\delta'}(t,\xi)$ in (3.17) is chosen to optimize the smoothness in ξ of the functions $b_Y(t,\cdot)$, while not changing significantly the size of the weights. The parameter δ' is fixed sufficiently small, depending only on δ.

We can now finally define our main weights A_{NR}, A_R, and A_k. We define first the decreasing function $\lambda:[0,\infty)\to\left[\delta_0,\frac{3}{2}\delta_0\right]$ by

$$\lambda(0)=\frac{3}{2}\delta_0\quad\text{and}\quad\lambda'(t)=-\frac{\delta_0\sigma_0^2}{\langle t\rangle^{1+\sigma_0}}, \tag{3.18}$$

for small positive constant σ_0 (say $\sigma_0=0.01$). Then, we define

$$A_R(t,\xi):=\frac{e^{\lambda(t)\langle\xi\rangle^{1/2}}}{b_R(t,\xi)}e^{\sqrt{\delta}\langle\xi\rangle^{1/2}} \quad\text{and}\quad A_{NR}(t,\xi):=\frac{e^{\lambda(t)\langle\xi\rangle^{1/2}}}{b_{NR}(t,\xi)}e^{\sqrt{\delta}\langle\xi\rangle^{1/2}}, \tag{3.19}$$

and, for any $k\in\mathbb{Z}$,

$$A_k(t,\xi):=e^{\lambda(t)\langle k,\xi\rangle^{1/2}}\bigg(\frac{e^{\sqrt{\delta}\langle\xi\rangle^{1/2}}}{b_k(t,\xi)}+e^{\sqrt{\delta}|k|^{1/2}}\bigg). \tag{3.20}$$

We record the simple inequalities

$$\begin{gathered} e^{\lambda(t)\langle\xi\rangle^{1/2}}\leqslant A_{NR}(t,\xi)\leqslant A_R(t,\xi)\leqslant e^{\lambda(t)\langle\xi\rangle^{1/2}}e^{2\sqrt{\delta}\langle\xi\rangle^{1/2}},\\ e^{\lambda(t)\langle k,\xi\rangle^{1/2}}\leqslant A_k(t,\xi)\leqslant 2e^{\lambda(t)\langle k,\xi\rangle^{1/2}}e^{2\sqrt{\delta}\langle k,\xi\rangle^{1/2}}, \end{gathered} \tag{3.21}$$

for any $k\in\mathbb{Z}$, $t\geqslant 0$, and $\xi\in\mathbb{R}$.

3.2.1. Properties of the weights

We collect now several bounds on these weights, which are proved either in [20] or in [21]. In these papers we prove many more properties of the weights, but we summarize here only the ones that we need explicitly in this paper.

We start with some bounds on w_Y and b_Y, see [20, Lemmas 7.1 and 7.2] for the proof.

Lemma 3.3. ([20, Lemmas 7.1 and 7.2]) *For $t\geqslant 0$, $\xi,\eta\in\mathbb{R}$, $k\in\mathbb{Z}$, and $Y\in\{NR,R,k\}$, we have*

$$\frac{w_Y(t,\xi)}{w_Y(t,\eta)}\lesssim_\delta e^{\sqrt{\delta}|\eta-\xi|^{1/2}}, \tag{3.22}$$

$$b_Y(t,\xi)\approx_\delta w_Y(t,\xi),\quad |\partial_\xi b_Y(t,\xi)|\lesssim_\delta b_Y(t,\xi)\frac{1}{L_{\delta'}(t,\xi)}. \tag{3.23}$$

We recall several bounds on the main weights A_{NR}, A_R, and A_k; see [20, Lemma 7.3].

Lemma 3.4. ([20, Lemma 7.3]) (i) *Assume $t\in[0,\infty)$, $k\in\mathbb{Z}$, and $Y\in\{NR,R,k\}$. If $\xi,\eta\in\mathbb{R}$ satisfy $|\eta|\geqslant\frac14|\xi|$ (or $|(k,\eta)|\geqslant\frac14|(k,\xi)|$ if $Y=k$), then*

$$\frac{A_Y(t,\xi)}{A_Y(t,\eta)}\lesssim_\delta e^{0.9\lambda(t)|\xi-\eta|^{1/2}}. \tag{3.24}$$

(ii) *Assume $t\in[0,\infty)$, and let $k,\ell\in\mathbb{Z}$ and $\xi,\eta\in\mathbb{R}$ satisfy $|(\ell,\eta)|\geqslant\frac14|(k,\xi)|$. If $t\notin I_{k,\xi}$ or $t\in I_{k,\xi}\cap I_{\ell,\eta}$, then*

$$\frac{A_k(t,\xi)}{A_\ell(t,\eta)}\lesssim_\delta e^{0.9\lambda(t)|(k-\ell,\xi-\eta)|^{1/2}}. \tag{3.25}$$

If $t\in I_{k,\xi}$ *and* $t\notin I_{\ell,\eta}$, *then*

$$\frac{A_k(t,\xi)}{A_\ell(t,\eta)}\lesssim_\delta \frac{|\xi|}{k^2}\frac{1}{1+|t-\xi/k|}e^{0.9\lambda(t)|(k-\ell,\xi-\eta)|^{1/2}}. \tag{3.26}$$

In some commutator estimates we need an additional property of the weights A_k.

LEMMA 3.5. ([21, Lemma 7.5]) *There is a constant constant* $C_0(\delta)\gg 1$ *such that, if* $\xi,\eta\in\mathbb{R}$, $t\geqslant 0$, $k\in\mathbb{Z}$, *and* $\langle\xi-\eta\rangle\leqslant\frac{1}{8}(\langle k,\xi\rangle+\langle k,\eta\rangle)$, *then*

$$|A_k(t,\xi)-A_k(t,\eta)|\lesssim A_R(t,\xi-\eta)A_k(t,\eta)e^{-(\lambda(t)/40)\langle\xi-\eta\rangle^{1/2}}\left[\frac{C_0(\delta)}{\langle k,\xi\rangle^{1/8}}+\sqrt{\delta}\right]. \tag{3.27}$$

To control the space-time integrals defined in (2.43)–(2.47) we also need estimates on the time derivatives of the weights A_Y.

LEMMA 3.6. (i) *For all* $t\geqslant 0$, $\rho\in\mathbb{R}$, *and* $Y\in\{NR,R\}$, *we have*

$$\frac{-\dot{A}_Y(t,\rho)}{A_Y(t,\rho)}\approx_\delta\left[\frac{\langle\rho\rangle^{1/2}}{\langle t\rangle^{1+\sigma_0}}+\frac{\partial_t w_Y(t,\rho)}{w_Y(t,\rho)}\right]. \tag{3.28}$$

and, for any $k\in\mathbb{Z}$,

$$\frac{-\dot{A}_k(t,\rho)}{A_k(t,\rho)}\approx_\delta\left[\frac{\langle k,\rho\rangle^{1/2}}{\langle t\rangle^{1+\sigma_0}}+\frac{\partial_t w_k(t,\rho)}{w_k(t,\rho)}\frac{1}{1+e^{\sqrt{\delta}\,(|k|^{1/2}-\langle\rho\rangle^{1/2})}w_k(t,\rho)}\right]. \tag{3.29}$$

In particular, if $k\in\mathbb{Z}^*$, $t\geqslant 0$ *and* $\rho\in\mathbb{R}$, *then*

$$\left|\left(\frac{\dot{A}_k}{A_k}\right)(t,\rho)\right|\gtrsim_\delta\left\langle t-\frac{\rho}{k}\right\rangle^{-1-\sigma_0}. \tag{3.30}$$

(ii) *For all* $t\geqslant 0$, $\xi,\eta\in\mathbb{R}$, *and* $Y\in\{NR,R\}$, *we have*

$$\left|\left(\frac{\dot{A}_Y}{A_Y}\right)(t,\xi)\right|\lesssim_\delta\left|\left(\frac{\dot{A}_Y}{A_Y}\right)(t,\eta)\right|e^{4\sqrt{\delta}\,|\xi-\eta|^{1/2}}. \tag{3.31}$$

Moreover, if $k,\ell\in\mathbb{Z}$, *then*

$$\left|\left(\frac{\dot{A}_k}{A_k}\right)(t,\xi)\right|\lesssim_\delta\left|\left(\frac{\dot{A}_\ell}{A_\ell}\right)(t,\eta)\right|e^{4\sqrt{\delta}\,|k-\ell,\xi-\eta|^{1/2}}. \tag{3.32}$$

Finally, if $\rho\in\mathbb{R}$ *and* $k\in\mathbb{Z}$ *satisfy* $|k|\leqslant\langle\rho\rangle+10$, *then*

$$\left|\left(\frac{\dot{A}_k}{A_k}\right)(t,\rho)\right|\approx_\delta\left|\left(\frac{\dot{A}_{NR}}{A_{NR}}\right)(t,\rho)\right|\approx_\delta\left|\left(\frac{\dot{A}_R}{A_R}\right)(t,\rho)\right|. \tag{3.33}$$

Proof. All the estimates except for (3.30) are proved in [20, Lemma 7.4]. To prove (3.30), we use first (3.29), thus

$$\left|\left(\frac{\dot{A}_k}{A_k}\right)(t,\rho)\right| \gtrsim_\delta \langle k,\rho\rangle^{1/2}\langle t\rangle^{-1-\sigma_0},$$

and the bounds (3.30) follow unless

$$\left|\frac{\rho}{k}\right| \geqslant \delta^{-12} \quad \text{and} \quad \left|t-\frac{\rho}{k}\right| \leqslant \frac{|\rho|}{10|k|}.$$

In this case, we use the second term in (3.29), thus

$$\left|\left(\frac{\dot{A}_k}{A_k}\right)(t,\rho)\right| \gtrsim_\delta \frac{\partial_t w_k(t,\rho)}{w_k(t,\rho)} \gtrsim_\delta \frac{\partial_t w_{NR}(t,\rho)}{w_{NR}(t,\rho)}.$$

In view of (3.13) and (3.15), this suffices to prove (3.30) in the remaining range. □

To control commutators in the space-time integrals, we need to also regularize the weights $|\dot{A}_Y/A_Y|$. We start by defining

$$\mu^{\#}(t,\xi) := \begin{cases} 0, & \text{if } |\xi| \leqslant \delta^{-10} \text{ or if } |\xi| > \delta^{-10} \text{ and } t > 2|\xi|, \\ \delta^2, & \text{if } |\xi| > \delta^{-10} \text{ and } t < t_{k_0(\xi),\xi}, \\ \dfrac{\delta^2}{1+\delta^2|t-\xi/k|}, & \text{if } |\xi| > \delta^{-10} \text{ and } t \in I_{k,\xi},\ k \in \{1,2,\ldots,k_0(\xi)\}, \end{cases} \tag{3.34}$$

for $t \geqslant 0$ and $\xi \geqslant 0$. Compare with the formulas (3.13). Then, we define $\mu^{\#}(t,\xi) := \mu^{\#}(t,|\xi|)$ if $\xi \leqslant 0$ and regularize the weight, as in (3.17),

$$\mu^*(t,\xi) := \int_{\mathbb{R}} \mu^{\#}(t,\rho)\frac{1}{d_0 L_{\delta'}(t,\xi)}\varphi\left(\frac{\xi-\rho}{L_{\delta'}(t,\xi)}\right)d\rho, \quad L_{\delta'}(t,\xi) := 1+\frac{\delta'\langle\xi\rangle}{\langle\xi\rangle^{1/2}+\delta' t}. \tag{3.35}$$

Finally, we define, motivated by the formulas (3.28) and (3.29),

$$\begin{aligned} \mu_k(t,\xi) &:= \frac{\langle k,\xi\rangle^{1/2}}{\langle t\rangle^{1+\sigma_0}} + \frac{\mu^*(t,\xi)}{1+e^{\sqrt{\delta}\,(|k|^{1/2}-\langle\xi\rangle^{1/2})}b_k(t,\xi)}, \\ \mu_R(t,\xi) &:= \frac{\langle\xi\rangle^{1/2}}{\langle t\rangle^{1+\sigma_0}} + \mu^*(t,\xi). \end{aligned} \tag{3.36}$$

We record below the main properties of the weights μ_R and μ_k.

LEMMA 3.7. ([21, Lemmas 7.6 and 7.7]) (i) *For $t \geqslant 0$, $\xi \in \mathbb{R}$, $k \in \mathbb{Z}$, we have*

$$\mu_k(t,\xi) \approx_\delta \left|\left(\frac{\dot{A}_k}{A_k}\right)(t,\xi)\right| \quad \textit{and} \quad \mu_R(t,\xi) \approx_\delta \left|\left(\frac{\dot{A}_R}{A_R}\right)(t,\xi)\right|. \tag{3.37}$$

(ii) *Assume that* $\xi,\eta\in\mathbb{R}$, $k\in\mathbb{Z}$, *and* $t\geqslant 0$. *Then,*

$$\mu_k(t,\xi)\lesssim_\delta \mu_k(t,\eta)e^{6\sqrt{\delta}\,|\xi-\eta|^{1/2}}. \tag{3.38}$$

Moreover, if $\langle\xi-\eta\rangle\leqslant\frac{1}{8}(\langle k,\xi\rangle+\langle k,\eta\rangle)$, *then there is* $C_1(\delta)\gg 1$ *such that*

$$|\mu_k(t,\xi)-\mu_k(t,\eta)|\lesssim\langle\xi-\eta\rangle\mu_k(t,\eta)e^{4\sqrt{\delta}\,|\xi-\eta|^{1/2}}\biggl[\frac{C_1(\delta)}{\langle k,\xi\rangle^{1/8}}+\sqrt{\delta}\biggr]. \tag{3.39}$$

In other words, the weights μ_Y are proportional to the weights $|\dot{A}_Y/A_Y|$, but have better smoothness properties. See [21, Lemmas 7.6 and 7.7] for the proofs.

3.3. Bilinear estimates

To bound non-linear terms we need bilinear estimates involving the weights. Many such estimates are proved in [20, §8]. We use all of these bilinear estimates in this paper as well, since our proof here contains all the difficulties of the proof for the Couette flow treated in [20]. In addition, we need four more bilinear estimates to deal with the new terms in the equation (2.28) for F, which we prove in this section.

We start with a lemma that is used many times in this paper. See [20, Lemmas 8.2 and 8.3] for the proofs.

LEMMA 3.8. ([20, Lemmas 8.2 and 8.3]) (i) *For any* $t\in[0,\infty)$, $\alpha\in[0,4]$, $\xi,\eta\in\mathbb{R}$, *and* $Y\in\{NR,R\}$, *we have*

$$\langle\xi\rangle^{-\alpha}A_Y(t,\xi)\lesssim_\delta\langle\xi-\eta\rangle^{-\alpha}A_Y(t,\xi-\eta)\langle\eta\rangle^{-\alpha}A_Y(t,\eta)e^{-(\delta_0/20)\min(\langle\xi-\eta\rangle,\langle\eta\rangle)^{1/2}} \tag{3.40}$$

and

$$\biggl|\biggl(\frac{\dot{A}_Y}{A_Y}\biggr)(t,\xi)\biggr|\lesssim_\delta\biggl\{\biggl|\biggl(\frac{\dot{A}_Y}{A_Y}\biggr)(t,\xi-\eta)\biggr|+\biggl|\biggl(\frac{\dot{A}_Y}{A_Y}\biggr)(t,\eta)\biggr|\biggr\}e^{4\sqrt{\delta}\min(\langle\xi-\eta\rangle,\langle\eta\rangle)^{1/2}}. \tag{3.41}$$

(ii) *For any* $t\in[0,\infty)$, $\xi,\eta\in\mathbb{R}$, *and* $k\in\mathbb{Z}$, *we have*

$$A_k(t,\xi)\lesssim_\delta A_R(t,\xi-\eta)A_k(t,\eta)e^{-(\delta_0/20)\min(\langle\xi-\eta\rangle,\langle k,\eta\rangle)^{1/2}} \tag{3.42}$$

and

$$\biggl|\biggl(\frac{\dot{A}_k}{A_k}\biggr)(t,\xi)\biggr|\lesssim_\delta\biggl\{\biggl|\biggl(\frac{\dot{A}_R}{A_R}\biggr)(t,\xi-\eta)\biggr|+\biggl|\biggl(\frac{\dot{A}_k}{A_k}\biggr)(t,\eta)\biggr|\biggr\}e^{12\sqrt{\delta}\min(\langle\xi-\eta\rangle,\langle k,\eta\rangle)^{1/2}}. \tag{3.43}$$

To state our new estimates, we let $\delta_0' := \frac{1}{200}\delta_0$ and define the sets

$$R_0 := \big\{((k,\xi),(\ell,\eta))\in(\mathbb{Z}\times\mathbb{R})^2 : \min(\langle k,\xi\rangle,\langle \ell,\eta\rangle,\langle k-\ell,\xi-\eta\rangle) \geqslant \tfrac{1}{20}(\langle k,\xi\rangle+\langle \ell,\eta\rangle+\langle k-\ell,\xi-\eta\rangle)\big\}, \tag{3.44}$$

$$R_1 := \big\{((k,\xi),(\ell,\eta))\in(\mathbb{Z}\times\mathbb{R})^2 : \langle k-\ell,\xi-\eta\rangle \leqslant \tfrac{1}{10}(\langle k,\xi\rangle+\langle \ell,\eta\rangle+\langle k-\ell,\xi-\eta\rangle)\big\}, \tag{3.45}$$

$$R_2 := \big\{((k,\xi),(\ell,\eta))\in(\mathbb{Z}\times\mathbb{R})^2 : \langle \ell,\eta\rangle \leqslant \tfrac{1}{10}(\langle k,\xi\rangle+\langle \ell,\eta\rangle+\langle k-\ell,\xi-\eta\rangle)\big\}, \tag{3.46}$$

$$R_3 := \big\{((k,\xi),(\ell,\eta))\in(\mathbb{Z}\times\mathbb{R})^2 : \langle k,\xi\rangle \leqslant \tfrac{1}{10}(\langle k,\xi\rangle+\langle \ell,\eta\rangle+\langle k-\ell,\xi-\eta\rangle)\big\}. \tag{3.47}$$

LEMMA 3.9. *Assume that* $t\geqslant 1$, $k,\ell\in\mathbb{Z}$, $\xi,\eta\in\mathbb{R}$, *let* $(m,\rho):=(k-\ell,\xi-\eta)$, *and assume that* $m\neq 0$.

(i) *If* $((k,\xi),(\ell,\eta))\in R_0\cup R_1$, *then*

$$\frac{(|\rho/m|+\langle t\rangle)\langle\rho\rangle}{\langle t\rangle m^2\langle t-\rho/m\rangle^2}\,|\ell A_k^2(t,\xi)-kA_\ell^2(t,\eta)| \lesssim_\delta \sqrt{|(A_k\dot A_k)(t,\xi)|}\,\sqrt{|(A_\ell\dot A_\ell)(t,\eta)|}\,A_m(t,\rho)e^{-\delta_0'\langle m,\rho\rangle^{1/2}} \tag{3.48}$$

and

$$\frac{(|\rho/m|+\langle t\rangle)\langle\rho\rangle}{\langle t\rangle m^2\langle t-\rho/m\rangle^2}\,\frac{|\ell A_k^2(t,\xi)|+|kA_\ell^2(t,\eta)|}{(1+\langle k,\xi\rangle/\langle t\rangle)^{1/2}} \lesssim_\delta \sqrt{|(A_k\dot A_k)(t,\xi)|}\,\sqrt{|(A_\ell\dot A_\ell)(t,\eta)|}\,A_m(t,\rho)e^{-\delta_0'\langle m,\rho\rangle^{1/2}}. \tag{3.49}$$

(ii) *If* $((k,\xi),(\ell,\eta))\in R_2$, *then*

$$\frac{(|\rho/m|+\langle t\rangle)\langle\rho\rangle}{\langle t\rangle m^2\langle t-\rho/m\rangle^2}\{|\ell A_k^2(t,\xi)|+|kA_\ell^2(t,\eta)|\} \lesssim_\delta \sqrt{|(A_k\dot A_k)(t,\xi)|}\,\sqrt{|(A_m\dot A_m)(t,\rho)|}\,A_\ell(t,\eta)e^{-\delta_0'\langle \ell,\eta\rangle^{1/2}}. \tag{3.50}$$

Proof. The bounds (3.48) and (3.50) are proved in [20, Lemma 8.4]. The statement of (3.50) is slightly weaker in [20, Lemma 8.4], in the sense that the quantity

$$|\ell A_k^2(t,\xi)|+|kA_\ell^2(t,\eta)|$$

in the left-hand side is replaced by the smaller quantity $|\ell A_k^2(t,\xi)-kA_\ell^2(t,\eta)|$, but the proof itself does not use the symmetrization and applies to the larger quantity as well.

We now prove the new bounds (3.49). Notice that

$$\langle t\rangle^2\frac{(|\rho/m|+\langle t\rangle)\langle\rho\rangle}{\langle t\rangle m^2\langle t-\rho/m\rangle^2}+\frac{1+\langle \ell,\eta\rangle/\langle t\rangle}{1+\langle k,\xi\rangle/\langle t\rangle}+\frac{1+\langle k,\xi\rangle/\langle t\rangle}{1+\langle \ell,\eta\rangle/\langle t\rangle}\lesssim_\delta e^{\delta\langle m,\rho\rangle^{1/2}}.$$

By symmetry, for (3.49) it suffices to prove that

$$|\ell A_k^2(t,\xi)| \lesssim_\delta \left(1+\frac{\langle k,\xi\rangle}{\langle t\rangle}\right)^{1/2} \langle t\rangle^2 \sqrt{|(A_k\dot{A}_k)(t,\xi)|}\,\sqrt{|(A_\ell\dot{A}_\ell)(t,\eta)|} \times A_m(t,\rho)e^{-(3\delta_0'/2)\langle m,\rho\rangle^{1/2}}.$$

This is equivalent to proving that

$$|\ell A_k(t,\xi)| \lesssim_\delta \left(1+\frac{\langle k,\xi\rangle}{\langle t\rangle}\right)^{1/2} \langle t\rangle^2 \sqrt{\left|\left(\frac{\dot{A}_k}{A_k}\right)(t,\xi)\right|}\,\sqrt{\left|\left(\frac{\dot{A}_\ell}{A_\ell}\right)(t,\eta)\right|} \times A_\ell(t,\eta)A_m(t,\rho)e^{-(3\delta_0'/2)\langle m,\rho\rangle^{1/2}}. \tag{3.51}$$

In view of (3.32) we may replace $|(\dot{A}_\ell/A_\ell)(t,\eta)|$ with $|(\dot{A}_k/A_k)(t,\xi)|$ at the expense of an acceptable factor. The desired bounds (3.51) follow from the lemma below. □

LEMMA 3.10. *Assume that $t\geqslant 1$, $k,\ell\in\mathbb{Z}$, $\xi,\eta\in\mathbb{R}$, let $(m,\rho):=(k-\ell,\xi-\eta)$, and assume that $m\neq 0$. Then,*

$$\langle k,\xi\rangle \frac{A_k(t,\xi)}{A_\ell(t,\eta)} \lesssim_\delta \left(1+\frac{\langle k,\xi\rangle}{\langle t\rangle}\right)^{1/2} \langle t\rangle^2 \left|\left(\frac{\dot{A}_k}{A_k}\right)(t,\xi)\right| A_m(t,\rho)e^{-2\delta_0'\langle m,\rho\rangle^{1/2}}, \tag{3.52}$$

provided that $((k,\xi),(\ell,\eta))\in R_0\cup R_1$ and $t\geqslant 1$.

Proof. For this, we first use the following elementary observation: if $a,b\in\mathbb{R}^d$ and $\beta\in[0,1]$, then

$$\langle b\rangle \geqslant \beta\langle a\rangle \quad\Longrightarrow\quad \langle a+b\rangle^{1/2} \leqslant \langle b\rangle^{1/2} + \big(1-\tfrac{1}{2}\sqrt{\beta}\big)\langle a\rangle^{1/2}. \tag{3.53}$$

Notice that $20\langle l,\eta\rangle \geqslant \langle m,\rho\rangle$ (since $((k,\xi),(\ell,\eta))\in R_0\cup R_1$), and $|(\dot{A}_k/A_k)(t,\xi)| \gtrsim \langle t\rangle^{-1-\sigma_0}$ (see (3.29)). Using also (3.21) and (3.53), the bounds (3.52) follow if $\langle k,\xi\rangle \leqslant 100\langle m,\rho\rangle$.

On the other hand, if $\langle k,\xi\rangle \geqslant 100\langle m,\rho\rangle$, then we consider two cases. If $t\notin I_{k,\xi}$, then we simply use (3.29) to bound $|(\dot{A}_k/A_k)(t,\xi)| \gtrsim_\delta \langle k,\xi\rangle^{1/2}\langle t\rangle^{-1-\sigma_0}$. The desired bounds (3.52) follow using also (3.25). If $t\in I_{k,\xi}$ (in particular $1\leqslant |k|\leqslant \delta|\xi|$ and $t\approx \xi/k$), then

$$\langle k,\xi\rangle \frac{A_k(t,\xi)}{A_\ell(t,\eta)} \lesssim_\delta \frac{|\xi|^2/k^2}{\langle t-\xi/k\rangle} e^{0.9\lambda(t)\langle m,\rho\rangle^{1/2}} \lesssim_\delta \frac{\langle t\rangle^2}{\langle t-\xi/k\rangle} A_m(t,\rho)e^{-(\delta_0/20)\langle m,\rho\rangle^{1/2}}$$

using (3.26). The bounds (3.52) follow since

$$\frac{1}{\langle t-\xi/k\rangle} \lesssim_\delta \frac{\partial_t w_k(t,\xi)}{w_k(t,\xi)} \lesssim_\delta \frac{|\dot{A}_k(t,\xi)|}{A_k(t,\xi)},$$

as a consequence of (3.13) and (3.29). □

Lemma 3.11. *Assume that* $t\geqslant 1$, $k,\ell\in\mathbb{Z}$, $\xi,\eta\in\mathbb{R}$, *let* $(m,\rho):=(k-\ell,\xi-\eta)$, *and assume that* $m\neq 0$.

(i) *If* $((k,\xi),(\ell,\eta))\in R_0\cup R_1$, *then*

$$\begin{aligned}&\frac{|\rho/m|^2+\langle t\rangle^2}{|m|\langle t\rangle^2\langle t-\rho/m\rangle^2}|\eta A_k^2(t,\xi)-\xi A_\ell^2(t,\eta)|\\&\qquad\lesssim_\delta\sqrt{|(A_k\dot{A}_k)(t,\xi)|}\sqrt{|(A_\ell\dot{A}_\ell)(t,\eta)|}\,A_m(t,\rho)e^{-\delta_0'\langle m,\rho\rangle^{1/2}}\end{aligned}\tag{3.54}$$

and

$$\begin{aligned}&\frac{|\rho/m|^2+\langle t\rangle^2}{|m|\langle t\rangle^2\langle t-\rho/m\rangle^2}\frac{|\eta A_k^2(t,\xi)|+|\xi A_\ell^2(t,\eta)|}{(1+\langle k,\xi\rangle/\langle t\rangle)^{1/2}}\\&\qquad\lesssim_\delta\sqrt{|(A_k\dot{A}_k)(t,\xi)|}\sqrt{|(A_\ell\dot{A}_\ell)(t,\eta)|}\,A_m(t,\rho)e^{-\delta_0'\langle m,\rho\rangle^{1/2}}.\end{aligned}\tag{3.55}$$

(ii) *If* $((k,\xi),(\ell,\eta))\in R_2$, *then*

$$\begin{aligned}&\frac{|\rho/m|^2+\langle t\rangle^2}{|m|\langle t\rangle^2\langle t-\rho/m\rangle^2}\{|\eta A_k^2(t,\xi)|+|\xi A_\ell^2(t,\eta)|\}\\&\qquad\lesssim_\delta\sqrt{|(A_k\dot{A}_k)(t,\xi)|}\sqrt{|(A_m\dot{A}_m)(t,\rho)|}\,A_\ell(t,\eta)e^{-\delta_0'\langle \ell,\eta\rangle^{1/2}}.\end{aligned}\tag{3.56}$$

Proof. The bounds (3.54) and (3.56) are proved in [20, Lemma 8.5], with the same remark as before that the inequality (3.56) is slightly weaker in [20, Lemma 8.5], but its proof does not use the symmetrization. To prove (3.55), we notice again that

$$\langle t\rangle^2\frac{|\rho/m|^2+\langle t\rangle^2}{|m|\langle t\rangle^2\langle t-\rho/m\rangle^2}+\frac{1+\langle \ell,\eta\rangle/\langle t\rangle}{1+\langle k,\xi\rangle/\langle t\rangle}+\frac{1+\langle k,\xi\rangle/\langle t\rangle}{1+\langle \ell,\eta\rangle/\langle t\rangle}\lesssim_\delta e^{\delta\langle m,\rho\rangle^{1/2}}.$$

Therefore, for (3.55), it suffices to prove that, if $((k,\xi),(\ell,\eta))\in R_0\cup R_1$, then

$$\begin{aligned}|\eta|A_k^2(t,\xi)\lesssim_\delta\left(1+\frac{\langle k,\xi\rangle}{\langle t\rangle}\right)^{1/2}\langle t\rangle^2\sqrt{|(A_k\dot{A}_k)(t,\xi)|}\sqrt{|(A_\ell\dot{A}_\ell)(t,\eta)|}\\\times A_m(t,\rho)e^{-(3\delta_0'/2)\langle m,\rho\rangle^{1/2}}.\end{aligned}$$

As in the proof of Lemma 3.9, this follows from (3.52). □

Lemma 3.12. *Define the sets*

$$R_n^*:=\{((k,\xi),(\ell,\eta))\in R_n:k=l\}.$$

Assume that $t\geqslant 1$, $k\in\mathbb{Z}$, $\xi,\eta\in\mathbb{R}$, *and let* $\rho:=\xi-\eta$.

(i) *If* $((k,\xi),(k,\eta))\in R_0^*\cup R_1^*$, *then*

$$\frac{|\eta A_k^2(t,\xi)-\xi A_k^2(t,\eta)|}{\langle\rho\rangle\langle t\rangle+\langle\rho\rangle^{1/4}\langle t\rangle^{7/4}}\lesssim_\delta\sqrt{|(A_k\dot{A}_k)(t,\xi)|}\sqrt{|(A_k\dot{A}_k)(t,\eta)|}\,A_{NR}(t,\rho)e^{-\delta_0'\langle\rho\rangle^{1/2}}.\tag{3.57}$$

and

$$\frac{|\eta A_k^2(t,\xi)|+|\xi A_k^2(t,\eta)|}{(\langle\rho\rangle\langle t\rangle+\langle\rho\rangle^{1/4}\langle t\rangle^{7/4})(1+\langle k,\xi\rangle/\langle t\rangle)^{1/2}} \lesssim_\delta \sqrt{|(A_k\dot{A}_k)(t,\xi)|}\sqrt{|(A_k\dot{A}_k)(t,\eta)|}\,A_{NR}(t,\rho)e^{-\delta_0'\langle\rho\rangle^{1/2}}. \tag{3.58}$$

(ii) *If* $((k,\xi),(k,\eta))\in R_2^*$, *then*

$$\frac{|\eta A_k^2(t,\xi)|+|\xi A_k^2(t,\eta)|}{\langle\rho\rangle\langle t\rangle+\langle\rho\rangle^{1/4}\langle t\rangle^{7/4}} \lesssim_\delta \sqrt{|(A_k\dot{A}_k)(t,\xi)|}\sqrt{|(A_{NR}\dot{A}_{NR})(t,\rho)|}\,A_k(t,\eta)e^{-\delta_0'\langle k,\eta\rangle^{1/2}}. \tag{3.59}$$

Proof. The bounds (3.57) and (3.59) are proved in [20, Lemma 8.6], with the same remark as before that the inequality (3.59) is slightly weaker in [20, Lemma 8.6], but its proof does not use the symmetrization. For the bound (3.58), it suffices to prove that, if $((k,\xi),(k,\eta))\in R_0^*\cup R_1^*$, then

$$|\eta|A_k^2(t,\xi) \lesssim_\delta \left(1+\frac{\langle k,\xi\rangle}{\langle t\rangle}\right)^{1/2}\langle t\rangle^{7/4}\sqrt{|(A_k\dot{A}_k)(t,\xi)|}\sqrt{|(A_k\dot{A}_k)(t,\eta)|}\,A_{NR}(t,\rho)e^{-(3\delta_0'/2)\langle\rho\rangle^{1/2}}.$$

As in the proof of Lemma 3.9, using (3.32) it suffices to prove that

$$\langle k,\xi\rangle\frac{A_k(t,\xi)}{A_k(t,\eta)} \lesssim_\delta \left(1+\frac{\langle k,\xi\rangle}{\langle t\rangle}\right)^{1/2}\langle t\rangle^{7/4}\left|\left(\frac{\dot{A}_k}{A_k}\right)(t,\xi)\right|A_{NR}(t,\rho)e^{-2\delta_0'\langle\rho\rangle^{1/2}}. \tag{3.60}$$

Since

$$\left|\left(\frac{\dot{A}_k}{A_k}\right)(t,\xi)\right| \gtrsim_\delta \langle k,\xi\rangle^{1/2}\langle t\rangle^{-1-\sigma_0}$$

(see (3.29)), for (3.60) it suffices to prove that

$$A_k(t,\xi) \lesssim_\delta A_k(t,\eta)A_{NR}(t,\rho)e^{-2\delta_0'\langle\rho\rangle^{1/2}}. \tag{3.61}$$

This follows from (3.24) if $((k,\xi),(k,\eta))\in R_1^*$, or from the bounds (3.53),

$$A_k(t,\eta) \geqslant e^{\lambda(t)\langle k,\eta\rangle^{1/2}},$$
$$A_{NR}(t,\rho) \geqslant e^{\lambda(t)\langle\rho\rangle^{1/2}},$$
$$A_k(t,\xi) \leqslant 2e^{\lambda(t)\langle k,\xi\rangle^{1/2}}e^{2\sqrt{\delta}\langle k,\xi\rangle^{1/2}}$$

(see (3.21)), if $((k,\xi),(k,\eta))\in R_0^*$. □

LEMMA 3.13. *Assume that* $t\geqslant 1$, $k\in\mathbb{Z}$, $\xi,\eta\in\mathbb{R}$, *and let* $\rho:=\xi-\eta$.

(i) *If* $((k,\xi),(k,\eta))\in R_0^*\cup R_1^*$ *and* $k\neq 0$, *then*

$$A_k^2(t,\xi)\lesssim_\delta \sqrt{|(A_k\dot{A}_k)(t,\xi)|}\,\sqrt{|(A_k\dot{A}_k)(t,\eta)|}\,\frac{|k|\langle t\rangle\langle t-\eta/k\rangle^2}{\langle t\rangle+|\eta/k|}A_R(t,\rho)e^{-\delta_0'\langle\rho\rangle^{1/2}}. \tag{3.62}$$

(ii) *If* $((k,\xi),(k,\eta))\in R_2^*$ *and* $k\neq 0$, *then*

$$A_k^2(t,\xi)\lesssim_\delta \sqrt{|(A_k\dot{A}_k)(t,\xi)|}\,\sqrt{|(A_R\dot{A}_R)(t,\rho)|}\,\frac{|k|\langle t\rangle\langle t-\eta/k\rangle^2}{\langle t\rangle+|\eta/k|}A_k(t,\eta)e^{-\delta_0'\langle k,\eta\rangle^{1/2}}. \tag{3.63}$$

(iii) *If* $((k,\xi),(k,\eta))\in R_3^*$ *and* $k\neq 0$, *then*

$$A_k^2(t,\xi)\lesssim_\delta \sqrt{|(A_k\dot{A}_k)(t,\eta)|}\,\sqrt{|(A_R\dot{A}_R)(t,\rho)|}\,\frac{|k|\langle t\rangle\langle t-\eta/k\rangle^2}{\langle t\rangle+|\eta/k|}A_k(t,\xi)e^{-\delta_0'\langle k,\xi\rangle^{1/2}}. \tag{3.64}$$

Proof. (i) Using (3.32), it suffices to prove that

$$A_k(t,\xi)\lesssim_\delta A_k(t,\eta)\left|\left(\frac{\dot{A}_k}{A_k}\right)(t,\eta)\right|\frac{|k|\langle t\rangle\langle t-\eta/k\rangle^2}{\langle t\rangle+|\eta/k|}A_R(t,\rho)e^{-(3\delta_0'/2)\langle\rho\rangle^{1/2}}.$$

This follows from (3.42) and (3.29)–(3.30).

(ii) Since $4\langle k,\eta\rangle\leqslant\min(\langle k,\xi\rangle,\langle\rho\rangle)$, we can apply (3.32)–(3.33). Notice also that

$$\frac{|k|\langle t\rangle\langle t-\eta/k\rangle^2}{\langle t\rangle+|\eta/k|}\gtrsim_\delta \langle t\rangle^2e^{-\delta\langle k,\eta\rangle^{1/2}}.$$

For (3.63), it suffices to prove that

$$A_k(t,\xi)\lesssim_\delta \left|\left(\frac{\dot{A}_k}{A_k}\right)(t,\xi)\right|\langle t\rangle^2A_R(t,\rho)A_k(t,\eta)e^{-(3\delta_0'/2)\langle k,\eta\rangle^{1/2}}.$$

Since

$$\left|\left(\frac{\dot{A}_k}{A_k}\right)(t,\xi)\right|\gtrsim_\delta \langle t\rangle^{-1-\sigma_0},$$

this follows from (3.42).

(iii) Since $4\langle k,\xi\rangle\leqslant\min(\langle k,\eta\rangle,\langle\rho\rangle)$, the desired bounds (3.64) follow easily from (3.21) and (3.28)–(3.29). □

4. Non-linear bounds and the main elliptic estimate

We prove now estimates on some of the functions defined in Proposition 2.1. In most cases we apply the definitions, the bootstrap assumptions (2.49), and the following general lemma (see [20, Lemma 8.1] for the proof).

LEMMA 4.1. ([20, Lemma 8.1]) (i) *Assume that* $m, m_1, m_2: \mathbb{R} \to \mathbb{C}$ *are symbols satisfying*

$$|m(\xi)| \leqslant |m_1(\xi-\eta)|\,|m_2(\eta)|\{\langle\xi-\eta\rangle^{-2}+\langle\eta\rangle^{-2}\} \tag{4.1}$$

for any $\xi, \eta \in \mathbb{R}$. *If* M, M_1, *and* M_2 *are the operators defined by these symbols, then*

$$\|M(gh)\|_{L^2(\mathbb{R})} \lesssim \|M_1 g\|_{L^2(\mathbb{R})}\,\|M_2 h\|_{L^2(\mathbb{R})}. \tag{4.2}$$

(ii) *Similarly, if* $m, m_2: \mathbb{Z}\times\mathbb{R} \to \mathbb{C}$ *and* $m_1: \mathbb{R} \to \mathbb{C}$ *are symbols satisfying*

$$|m(k,\xi)| \leqslant |m_1(\xi-\eta)|\,|m_2(k,\eta)|\{\langle\xi-\eta\rangle^{-2}+\langle k,\eta\rangle^{-2}\} \tag{4.3}$$

for any $\xi, \eta \in \mathbb{R}$, $k \in \mathbb{Z}$, *and* M, M_1, *and* M_2 *are the operators defined by these symbols, then*

$$\|M(gh)\|_{L^2(\mathbb{T}\times\mathbb{R})} \lesssim \|M_1 g\|_{L^2(\mathbb{R})}\,\|M_2 h\|_{L^2(\mathbb{T}\times\mathbb{R})}. \tag{4.4}$$

(iii) *Finally, assume that* $m, m_1, m_2: \mathbb{Z}\times\mathbb{R} \to \mathbb{C}$ *are symbols satisfying*

$$|m(k,\xi)| \leqslant |m_1(k-\ell,\xi-\eta)|\,|m_2(\ell,\eta)|\{\langle k-\ell,\xi-\eta\rangle^{-2}+\langle \ell,\eta\rangle^{-2}\} \tag{4.5}$$

for any $\xi, \eta \in \mathbb{R}$, $k, \ell \in \mathbb{Z}$. *If* M, M_1, *and* M_2 *are the operators defined by these symbols, then*

$$\|M(gh)\|_{L^2(\mathbb{T}\times\mathbb{R})} \lesssim \|M_1 g\|_{L^2(\mathbb{T}\times\mathbb{R})}\,\|M_2 h\|_{L^2(\mathbb{T}\times\mathbb{R})}. \tag{4.6}$$

For simplicity of notation, we introduce the following definitions.

Definition 4.2. For $f \in C([0,T]:H^4(\mathbb{R}))$, $g \in C([0,T]:H^4(\mathbb{T}\times\mathbb{R}))$, $Y \in \{R, NR\}$, and $t_1, t_2 \in [0,T]$, we define

$$\begin{aligned} \|f\|^2_{Y[t_1,t_2]} := \sup_{t\in[t_1,t_2]} &\int_{\mathbb{R}} A_Y^2(t,\xi)|\tilde f(t,\xi)|^2\,d\xi \\ &+ \int_{t_1}^{t_2}\int_{\mathbb{R}} |\dot A_Y(s,\xi)|\,A_Y(s,\xi)\,|\tilde f(s,\xi)|^2\,d\xi\,ds, \end{aligned} \tag{4.7}$$

$$\begin{aligned} \|g\|^2_{W[t_1,t_2]} := \sup_{t\in[t_1,t_2]} &\biggl\{\sum_{k\in\mathbb{Z}}\int_{\mathbb{R}} A_k^2(t,\xi)|\tilde g(t,k,\xi)|^2\,d\xi\biggr\} \\ &+ \int_{t_1}^{t_2}\sum_{k\in\mathbb{Z}}\int_{\mathbb{R}} |(A_k\dot A_k)(s,\xi)|\,|\tilde g(s,k,\xi)|^2\,d\xi\,ds, \end{aligned} \tag{4.8}$$

$$\begin{aligned} \|g\|^2_{\widetilde W[t_1,t_2]} := \sup_{t\in[t_1,t_2]} &\biggl\{\sum_{k\in\mathbb{Z}^*}\int_{\mathbb{R}} A_k^2(t,\xi)\frac{k^2\langle t\rangle^2}{|\xi|^2+k^2\langle t\rangle^2}|\tilde g(t,k,\xi)|^2\,d\xi\biggr\} \\ &+ \int_{t_1}^{t_2}\sum_{k\in\mathbb{Z}^*}\int_{\mathbb{R}} |(A_k\dot A_k)(s,\xi)|\frac{k^2\langle s\rangle^2}{|\xi|^2+k^2\langle s\rangle^2}|\tilde g(s,k,\xi)|^2\,d\xi\,ds. \end{aligned} \tag{4.9}$$

For simplicity of notation, let $Y := Y[1,T]$, $W := W[1,T]$, and $\widetilde W := \widetilde W[1,T]$.

The resonant weights A_R are our strongest weights. We first show that R is an algebra, and, in fact, multiplication by functions in R preserves the norms W and $\widetilde{W}$. More precisely, we have the following.

LEMMA 4.3. (i) *If* $f,g\in C([1,T]:H^4(\mathbb{R}))$ *and* $H\in C([1,T]:H^4(\mathbb{T}\times\mathbb{R}))$, *then*

$$\|fg\|_R \lesssim_\delta \|f\|_R\|g\|_R \tag{4.10}$$

and

$$\|fH\|_W \lesssim_\delta \|f\|_R\|H\|_W, \quad \|fH\|_{\widetilde{W}} \lesssim_\delta \|f\|_R\|H\|_{\widetilde{W}}. \tag{4.11}$$

(ii) *As a consequence, if* Ψ_1 *is a Gevrey cutoff function supported in*

$$\big[b\big(\tfrac{1}{20}\vartheta_0\big), b\big(1-\tfrac{1}{20}\vartheta_0\big)\big]$$

and satisfying $\|e^{\langle\xi\rangle^{3/4}}\widetilde{\Psi}_1(\xi)\|_{L^\infty}\lesssim 1$, *and*

$$h\in\{\Psi_1(V')^a, \Psi_1(B')^a, \langle\partial_v\rangle^{-1}V'', B'' : a\in[-2,2]\cap\mathbb{Z}\}, \tag{4.12}$$

then

$$\|h\|_R \lesssim_\delta 1. \tag{4.13}$$

Moreover, the functions B_0' *and* B_0'' *do not depend on* t *and satisfy the stronger bounds*

$$\|\Psi_1 B_0'\|_{\mathcal{G}^{4\delta_0,1/2}} + \left\|\Psi_1\left(\frac{1}{B_0'}\right)\right\|_{\mathcal{G}^{4\delta_0,1/2}} + \|B_0''\|_{\mathcal{G}^{4\delta_0,1/2}} \lesssim 1. \tag{4.14}$$

(iii) *With* $\mathcal{K}$ *as in* (2.47) *and* h *satisfying* $\|h\|_R+\|\partial_v h\|_R\leqslant 1$, *for any* $t\in[1,T]$ *we have*

$$\begin{aligned}\int_{\mathbb{R}} A_{NR}^2(t,\xi)(\langle\xi\rangle^2\langle t\rangle^2+\mathcal{K}^2\langle\xi\rangle^{1/2}\langle t\rangle^{7/2})|\widetilde{(h\dot{V})}(t,\xi)|^2\,d\xi \lesssim_\delta \varepsilon_1^2,\\ \int_1^t\int_{\mathbb{R}} |\dot{A}_{NR}(s,\xi)|A_{NR}(s,\xi)(\langle\xi\rangle^2\langle s\rangle^2+\mathcal{K}^2\langle\xi\rangle^{1/2}\langle s\rangle^{7/2})|\widetilde{(h\dot{V})}(s,\xi)|^2\,d\xi\,ds \lesssim_\delta \varepsilon_1^2.\end{aligned} \tag{4.15}$$

The implicit constants in (4.15) *may depend on* δ, *and* $\mathcal{K}$ *is assumed large enough compared to these constants.*

Proof. (i) The bounds (4.10) follow using Lemma 4.1 (i) and the bilinear estimates (3.40)–(3.41) with $Y=R$ and $\alpha=0$ (see [21, Lemma 4.2] for complete details). To prove the bounds (4.11), we use the bilinear estimates (3.42)–(3.43). Moreover, if $k\neq 0$, it is easy to see that

$$\frac{|k|\langle t\rangle}{|\xi|+|k|\langle t\rangle} \lesssim_\delta \frac{|k|\langle t\rangle}{|\eta|+|k|\langle t\rangle} e^{\delta\min(\langle\xi-\eta\rangle,\langle k,\eta\rangle)^{1/2}}, \tag{4.16}$$

and the desired bounds (4.11) follow using also Lemma 4.1 (ii).

(ii) To prove the bounds (4.13), we write

$$\begin{aligned} B'(t,v) &= B'_*(t,v)+B'_0(v), \\ V'(t,v) &= V'_*(t,v)+B'_0(v), \\ B''(t,v) &= B''_*(t,v)+B''_0(v), \end{aligned} \tag{4.17}$$

and recall also that $V''=\frac{1}{2}\partial_v(V')^2$; see (2.33). The functions B'_0, $1/B'_0$, and B''_0 do not depend on t and satisfy the bounds (4.14), as a consequence of Lemmas 3.1 and 3.2 and the assumptions (1.7)–(1.8). The desired bounds (4.13) follow using the algebra property (4.10), the bootstrap assumptions (2.49) on V'_*, B'_*, and B''_*, and the identities (4.17), as long as ε_1 is sufficiently small depending on δ (see [21, Lemma 4.2] for complete details).

(iii) To prove (4.15), we use the formula $\partial_v\dot{V}=\mathcal{H}/(tV')$ (see (2.33)) and the bootstrap assumptions (2.49). Since $\dot{V}$ and $\mathcal{H}$ are supported in $[b(\vartheta_0), b(1-\vartheta_0)]$, we have

$$tV'\partial_v\dot{V}=\Psi\mathcal{H}=\Psi(B'_*-V'_*-\langle F\rangle),$$

see (2.27), where Ψ is as in (2.42). The bootstrap assumptions (2.49) show that

$$\|V'_*\|_R+\|B'_*\|_R+\|\langle F\rangle\|_{NR}\lesssim\varepsilon_1. \tag{4.18}$$

The desired bounds (4.15) follow using (2.49), the bilinear estimates (3.40)–(3.41) with $Y=NR$, and the bounds $\|\Psi(V')^{-1}\|_R\lesssim 1$ (see also [21, Lemma 4.5]). □

We record now bounds on some of the functions that apear in the right-hand sides of the equations (2.28) and (2.31).

LEMMA 4.4. (i) *For any* $t\in[1,T]$ *and* $h_1\in\{(V')^a\partial_z(\Psi\phi):a\in[-2,2]\}$, *we have*

$$\begin{aligned} \sum_{k\in\mathbb{Z}\setminus\{0\}}\int_{\mathbb{R}} A_k^2(t,\xi)\frac{k^2\langle t\rangle^4\langle t-\xi/k\rangle^4}{(|\xi/k|^2+\langle t\rangle^2)^2}|\tilde{h}_1(t,k,\xi)|^2\,d\xi\lesssim_\delta\varepsilon_1^2, \\ \int_1^t\sum_{k\in\mathbb{Z}\setminus\{0\}}\int_{\mathbb{R}}|\dot{A}_k(s,\xi)|A_k(s,\xi)\frac{k^2\langle s\rangle^4\langle s-\xi/k\rangle^4}{(|\xi/k|^2+\langle s\rangle^2)^2}|\tilde{h}_1(s,k,\xi)|^2\,d\xi\,ds\lesssim_\delta\varepsilon_1^2. \end{aligned} \tag{4.19}$$

(ii) *For any* $t\in[1,T]$ *and* $h_2\in\{(V')^a\partial_v\mathbb{P}_{\neq 0}(\Psi\phi):a\in[-2,2]\}$, *we have*

$$\begin{aligned} \sum_{k\in\mathbb{Z}\setminus\{0\}}\int_{\mathbb{R}} A_k^2(t,\xi)\frac{k^4\langle t\rangle^2\langle t-\xi/k\rangle^4}{(|\xi/k|^2+\langle t\rangle^2)\langle\xi\rangle^2}|\tilde{h}_2(t,k,\xi)|^2\,d\xi\lesssim_\delta\varepsilon_1^2, \\ \int_1^t\sum_{k\in\mathbb{Z}\setminus\{0\}}\int_{\mathbb{R}}|\dot{A}_k(s,\xi)|A_k(s,\xi)\frac{k^4\langle s\rangle^2\langle s-\xi/k\rangle^4}{(|\xi/k|^2+\langle s\rangle^2)\langle\xi\rangle^2}|\tilde{h}_2(s,k,\xi)|^2\,d\xi\,ds\lesssim_\delta\varepsilon_1^2. \end{aligned} \tag{4.20}$$

(iii) *If* $g\in\{(V')^a\langle\partial_z\phi\partial_v F\rangle, (V')^a\langle\partial_v\mathbb{P}_{\neq 0}\phi\partial_z F\rangle : a\in[-2,2]\cap\mathbb{Z}\}$ *then, for any* $t\in[1,T]$,

$$\begin{aligned}\int_{\mathbb{R}} |\dot{A}_{NR}(t,\xi)|^{-2} A^4_{NR}(t,\xi)(\langle t\rangle^{3/2}\langle\xi\rangle^{-3/2})|\tilde{g}(t,\xi)|^2\, d\xi \lesssim_\delta \varepsilon_1^4,\\ \int_1^t\int_{\mathbb{R}} |\dot{A}_{NR}(s,\xi)|^{-1} A^3_{NR}(s,\xi)(\langle s\rangle^{3/2}\langle\xi\rangle^{-3/2})|\tilde{g}(s,\xi)|^2\, d\xi\, ds \lesssim_\delta \varepsilon_1^4.\end{aligned} \tag{4.21}$$

Proof. See [21, Lemma 4.3] for statements (i) and (ii), and [21, Lemma 4.6] for the proof of (iii). ☐

4.1. Green's functions and elliptic estimates

Assume that $\varphi', f' : \mathbb{T}\times[0,1]\to\mathbb{C}$ are C^2 functions satisfying

$$(\partial_x^2+\partial_y^2)\varphi' = f' \quad \text{and} \quad \varphi'(x,0)=\varphi'(x,1)=0. \tag{4.22}$$

Then, φ' can be determined explicitly through an integral operator. Indeed, we can write

$$\varphi'_k(y) = -\int_0^1 f'_k(y')G_k(y,y')\, dy', \tag{4.23}$$

where $G_k(y,z)$, defined by

$$G_k(y,z) := \frac{1}{k\sinh k}\begin{cases} \sinh(k(1-z))\sinh(ky), & \text{if } y\leqslant z,\\ \sinh(kz)\sinh(k(1-y)), & \text{if } y\geqslant z,\end{cases} \tag{4.24}$$

for $k\in\mathbb{Z}\setminus\{0\}$ and

$$G_0(y,z) := \begin{cases} (1-z)y, & \text{if } y\leqslant z,\\ z(1-y), & \text{if } y\geqslant z,\end{cases} \tag{4.25}$$

is the Green function associated with the equation (4.22), and

$$\begin{aligned}\varphi'_k(y) &:= \frac{1}{2\pi}\int_{\mathbb{T}} \varphi'(x,y)e^{-ikx}\, dx\\ f'_k(y) &:= \frac{1}{2\pi}\int_{\mathbb{T}} f'(x,y)e^{-ikx}\, dx,\end{aligned} \tag{4.26}$$

denote the kth Fourier coefficient of the functions φ' and f', respectively.

We prove now an important lemma concerning elliptic estimates adapted to our situation.

LEMMA 4.5. *Assume that* $f\in C([0,T]:H^4(\mathbb{T}\times[b(0),b(1)]))$ *is supported in*

$$\mathbb{T}\times[b(\vartheta_0),b(1-\vartheta_0)].$$

Then, there is a unique solution $\varphi\in C([0,T]:H^4(\mathbb{T}\times[b(0),b(1)]))$ *of the problem*

$$\partial_z^2\varphi+(B_0')^2(\partial_v-t\partial_z)^2\varphi+B_0''(\partial_v-t\partial_z)\varphi=f(t,z,v), \tag{4.27}$$

with Dirichlet boundary conditions

$$\varphi(t,z,b(0))=\varphi(t,z,b(1))=0.$$

Moreover, if $t_1,t_2\in[0,T]$ *then, recalling the definitions* (4.7)–(4.9),

$$\begin{aligned}\|P_{\neq 0}[\partial_z^2+(\partial_v-t\partial_z)^2](\Psi\varphi)\|_{W[t_1,t_2]}&\lesssim_\delta \|f\|_{W[t_1,t_2]},\\ \|P_{\neq 0}[\partial_z^2+(\partial_v-t\partial_z)^2](\Psi\varphi)\|_{\widetilde{W}[t_1,t_2]}&\lesssim_\delta \|f\|_{\widetilde{W}[t_1,t_2]}.\end{aligned} \tag{4.28}$$

Proof. We reverse the change of variables (2.4), so we define

$$\begin{aligned}\varphi'(t,x,y)&:=\varphi(t,x-tb(y),b(y)),\\ f'(t,x,y)&:=f(t,x-tb(y),b(y)).\end{aligned} \tag{4.29}$$

The functions φ' and f' satisfy equation (4.22) for any $t\in[0,T]$, therefore, using (4.29),

$$\varphi_k(t,b(y))=-\int_0^1 f_k(t,b(y'))G_k(y,y')e^{ikt((b(y)-b(y'))}\,dy'.$$

Thus, letting $\mathcal{G}_k(b(y),b(y')):=G_k(y,y')$, we have

$$\varphi_k(t,v)=-\int_{b(0)}^{b(1)} f_k(t,w)\mathcal{G}_k(v,w)e^{ikt(v-w)}\left(\frac{1}{B_0'(w)}\right)dw, \tag{4.30}$$

Recall that $f(t)$ is supported in $\mathbb{T}\times[b(\vartheta_0),b(1-\vartheta_0)]$. We multiply (4.30) by $\Psi(v)\Psi(w)$, and take the Fourier transform in v and w. Thus,

$$(\widetilde{\Psi\varphi})(t,k,\xi)=C\int_{\mathbb{R}}\tilde{f}(t,k,\eta)K(\xi-kt,kt-\eta)\,d\eta, \tag{4.31}$$

where

$$K(\mu,\nu):=\int_{\mathbb{R}^2}\Psi(v)\Psi(w)\mathcal{G}_k(v,w)\left(\frac{1}{B_0'(w)}\right)e^{-iv\mu}e^{-iw\nu}\,dv\,dw.$$

The kernel K satisfies the bounds, for $k\neq 0$,

$$|K(\mu,\nu)|\lesssim\frac{e^{-4\delta_0\langle\mu+\nu\rangle^{1/2}}}{k^2+|\mu|^2}. \tag{4.32}$$

This is proved in [22, Lemma A3], using the explicit formula (4.24). Thus, using (4.31),

$$\begin{aligned}(k^2+|\xi-kt|^2)|(\widetilde{\Psi\varphi})(t,k,\xi)| &\lesssim \int_{\mathbb{R}} |\tilde{f}(t,k,\eta)|e^{-4\delta_0\langle\xi-\eta\rangle^{1/2}}\,d\eta \\ &= \int_{\mathbb{R}} |\tilde{f}(t,k,\xi-\eta)|e^{-4\delta_0\langle\eta\rangle^{1/2}}\,d\eta\end{aligned} \tag{4.33}$$

for $k\neq 0$. It follows from (3.42) that, for any $\xi,\eta\in\mathbb{R}$, $k\in\mathbb{Z}\setminus\{0\}$, and $t\geqslant 0$,

$$A_k(t,\xi)\lesssim_\delta A_k(t,\xi-\eta)e^{2\delta_0\langle\eta\rangle^{1/2}}. \tag{4.34}$$

The inequalities in (4.28) follow from (4.33)–(4.34) and the definitions (4.8)–(4.9), using also (3.32) (for the space-time bound). To illustrate the idea, we sketch the proof for the second inequality in (4.28). Using (3.24) and (3.32), we obtain that, for $k\in\mathbb{Z}\setminus\{0\},\xi,\eta\in\mathbb{R}$,

$$\begin{aligned}\frac{|k|\langle t\rangle}{|\xi|+|k|\langle t\rangle}A_k(t,\xi) &\lesssim_\delta \frac{|k|\langle t\rangle}{|\xi-\eta|+|k|\langle t\rangle}A_k(t,\xi-\eta)e^{3\delta_0\langle\eta\rangle^{1/2}}, \\ \frac{|k|\langle t\rangle}{|\xi|+|k|\langle t\rangle}|A_k\dot{A}_k(t,\xi)|^{1/2} &\lesssim_\delta \frac{|k|\langle t\rangle}{|\xi-\eta|+|k|\langle t\rangle}|A_k\dot{A}_k(t,\xi-\eta)|^{1/2}e^{3\delta_0\langle\eta\rangle^{1/2}}.\end{aligned} \tag{4.35}$$

Therefore, using (4.33), (4.35), and Minkowski inequality, we can bound

$$\|P_{\neq 0}[\partial_z^2+(\partial_v-t\partial_z)^2](\Psi\varphi)\|_{\widetilde{W}[t_1,t_2]}\lesssim_\delta \int_{\mathbb{R}}\|f\|_{\widetilde{W}[t_1,t_2]}e^{-\delta_0\langle\eta\rangle^{1/2}}d\eta, \tag{4.36}$$

from which the second inequality in (4.28) follows. □

5. Improved control on the coordinate functions V_*', B_*', B_*'', and $\mathcal{H}$

In this subsection, we prove the following bounds.

Proposition 5.1. *With the definitions and assumptions in Proposition 2.2, we have*

$$\sum_{g\in\{V_*',B_*',B_*'',\mathcal{H}\}}[\mathcal{E}_g(t)+\mathcal{B}_g(t)]\leqslant\frac{\varepsilon_1^2}{2}\quad \textit{for any } t\in[1,T]. \tag{5.1}$$

The rest of the subsection is concerned with the proof of this proposition. The arguments are similar to the arguments in [20, §6], and we will be somewhat brief.

Using definitions (2.46) and (2.47), we calculate

$$\begin{aligned}\frac{d}{dt}\sum_{g\in\{\mathcal{H},V_*',B_*',B_*''\}}\mathcal{E}_g(t) &= 2\mathcal{K}^2\int_{\mathbb{R}}\dot{A}_{NR}(t,\xi)A_{NR}(t,\xi)(\langle t\rangle^{3/2}\langle\xi\rangle^{-3/2})|\widetilde{\mathcal{H}}(t,\xi)|^2\,d\xi\\ &+\mathcal{K}^2 2\mathrm{Re}\int_{\mathbb{R}}A_{NR}^2(t,\xi)(\langle t\rangle^{3/2}\langle\xi\rangle^{-3/2})\partial_t\widetilde{\mathcal{H}}(t,\xi)\overline{\widetilde{\mathcal{H}}(t,\xi)}\,d\xi\\ &+\mathcal{K}^2\int_{\mathbb{R}}A_{NR}^2(t,\xi)\frac{3}{2}(t\langle t\rangle^{-1/2}\langle\xi\rangle^{-3/2})|\widetilde{\mathcal{H}}(t,\xi)|^2\,d\xi\\ &+2\sum_{U\in\{V_*',B_*',B_*''\}}\int_{\mathbb{R}}\dot{A}_R(t,\xi)A_R(t,\xi)|\widetilde{U}(t,\xi)|^2\,d\xi\\ &+2\sum_{U\in\{V_*',B_*',B_*''\}}\mathrm{Re}\int_{\mathbb{R}}A_R^2(t,\xi)\partial_t\widetilde{U}(t,\xi)\overline{\widetilde{U}(t,\xi)}\,d\xi.\end{aligned}$$

Therefore, since $\partial_t A_R\leqslant 0$ and $\partial_t A_{NR}\leqslant 0$, for any $t\in[1,T]$ we have

$$\begin{aligned}&\sum_{g\in\{\mathcal{H},V_*',B_*',B_*''\}}[\mathcal{E}_g(t)+\mathcal{B}_g(t)]\\ &\quad=\sum_{g\in\{\mathcal{H},V_*',B_*',B_*''\}}\mathcal{E}_g(1)-\bigg[\sum_{g\in\{\mathcal{H},V_*',B_*',B_*''\}}\mathcal{B}_g(t)\bigg]+\mathcal{L}_1(t)+\mathcal{L}_2(t),\end{aligned}\tag{5.2}$$

where

$$\mathcal{L}_1(t):=2\mathrm{Re}\sum_{U\in\{V_*',B_*',B_*''\}}\int_1^t\int_{\mathbb{R}}A_R^2(s,\xi)\,\partial_s\widetilde{U}(s,\xi)\overline{\widetilde{U}(s,\xi)}\,d\xi\,ds,\tag{5.3}$$

$$\begin{aligned}\mathcal{L}_2(t):=\mathcal{K}^2 2\mathrm{Re}&\int_1^t\int_{\mathbb{R}}A_{NR}^2(s,\xi)(\langle s\rangle^{3/2}\langle\xi\rangle^{-3/2})\,\partial_s\widetilde{\mathcal{H}}(s,\xi)\overline{\widetilde{\mathcal{H}}(s,\xi)}\,d\xi\,ds\\ +\mathcal{K}^2&\int_1^t\int_{\mathbb{R}}A_{NR}^2(s,\xi)\frac{3}{2}(s\langle s\rangle^{-1/2}\langle\xi\rangle^{-3/2})|\widetilde{\mathcal{H}}(s,\xi)|^2\,d\xi\,ds.\end{aligned}\tag{5.4}$$

Since

$$\sum_{g\in\{\mathcal{H},V_*',B_*',B_*''\}}\mathcal{E}_g(1)\lesssim\varepsilon_1^3,$$

for (5.1) it suffices to prove that, for any $t\in[1,T]$,

$$-\bigg[\sum_{g\in\{\mathcal{H},V_*',B_*',B_*''\}}\mathcal{B}_g(t)\bigg]+\mathcal{L}_1(t)+\mathcal{L}_2(t)\leqslant\frac{\varepsilon_1^2}{4}.\tag{5.5}$$

To prove (5.5), we rewrite equations (2.29) and (2.30) in the form

$$\begin{aligned}\partial_t B_*' &= -\dot{V}\partial_v B_*'-\dot{V}\partial_v B_0',\\ \partial_t B_*'' &= -\dot{V}\partial_v B_*''-\dot{V}\partial_v B_0'',\\ \partial_t V_*' &= -\dot{V}\partial_v V_*'-\dot{V}\partial_v B_0'+\frac{\mathcal{H}}{t},\end{aligned}\tag{5.6}$$

We extract the quadratic components of $\mathcal{L}_1$ and $\mathcal{L}_2$ (corresponding to the linear terms in the right-hand sides of (5.6) and (2.31), so we define

$$\begin{aligned}\mathcal{L}_{1,2}(t):=2\mathrm{Re}\int_1^t\int_{\mathbb{R}}A_R^2(s,\xi)\\ \times\Big\{\Big[\frac{\widetilde{\mathcal{H}}(s,\xi)}{s}-\widetilde{\dot{V}_1'}(s,\xi)\Big]\overline{\widetilde{V_*'}(s,\xi)}-\sum_{a\in\{1,2\}}\widetilde{\dot{V}_a'}(s,\xi)\overline{\widetilde{U}_a(s,\xi)}\Big\}\,d\xi\,ds,\end{aligned}\tag{5.7}$$

where

$$\dot{V}_1':=\dot{V}\partial_v B_0',\quad \dot{V}_2':=\dot{V}\partial_v B_0'',\quad U_1:=B_*',\quad U_2:=B_*'',\tag{5.8}$$

and

$$\begin{aligned}\mathcal{L}_{2,2}(t)&:=\mathcal{K}^2\int_1^t\int_{\mathbb{R}}A_{NR}^2(s,\xi)\Big\{-\frac{2\langle s\rangle^{3/2}}{s\langle\xi\rangle^{3/2}}|\widetilde{\mathcal{H}}(s,\xi)|^2+\frac{3s/2}{\langle s\rangle^{1/2}\langle\xi\rangle^{3/2}}|\widetilde{\mathcal{H}}(s,\xi)|^2\Big\}\,d\xi\,ds\\ &=-\mathcal{K}^2\int_1^t\int_{\mathbb{R}}A_{NR}^2(s,\xi)\frac{2+s^2/2}{s\langle\xi\rangle^{3/2}\langle s\rangle^{1/2}}|\widetilde{\mathcal{H}}(s,\xi)|^2\,d\xi\,ds.\end{aligned}\tag{5.9}$$

We examine the identities (5.6) and (2.31) and let

$$\begin{aligned}&f_1:=-\dot{V}\partial_v B_*',\qquad f_2:=-\dot{V}\partial_v B_*'',\quad f_3:=-\dot{V}\partial_v V_*',\\ &g_1:=-\dot{V}\partial_v\mathcal{H},\qquad g_2:=V'[\langle\partial_z\phi\,\partial_v F\rangle-\langle\partial_v P_{\neq 0}\phi\,\partial_z F\rangle].\end{aligned}\tag{5.10}$$

Notice that

$$\begin{aligned}\mathcal{L}_1(t)&=\mathcal{L}_{1,2}(t)+2\mathrm{Re}\int_1^t\int_{\mathbb{R}}A_R^2(s,\xi)\Big\{\sum_{a\in\{1,2\}}\tilde{f}_a(s,\xi)\overline{\widetilde{U}_a(s,\xi)}+\tilde{f}_3(s,\xi)\overline{\widetilde{V}_*'(s,\xi)}\Big\}\,d\xi\,ds,\\ \mathcal{L}_2(t)&=\mathcal{L}_{2,2}(t)+\sum_{a\in\{1,2\}}\mathcal{K}^2 2\mathrm{Re}\int_1^t\int_{\mathbb{R}}A_{NR}^2(s,\xi)(\langle s\rangle^{3/2}\langle\xi\rangle^{-3/2})\tilde{g}_a(s,\xi)\overline{\widetilde{\mathcal{H}}(s,\xi)}\,d\xi\,ds.\end{aligned}\tag{5.11}$$

The desired bounds (5.5) follow from Lemmas 5.2 and 5.3 below.

LEMMA 5.2. *For any $t\in[1,T]$ we have*

$$-\Big[\sum_{g\in\{\mathcal{H},V_*',B_*',B_*''\}}\mathcal{B}_g(t)\Big]+\mathcal{L}_{1,2}(t)+\mathcal{L}_{2,2}(t)\leqslant\frac{\varepsilon_1^2}{8}.\tag{5.12}$$

Proof. Since $\mathcal{L}_{2,2}(t)\leqslant 0$, it suffices to prove that, for any $t\in[1,T]$,

$$\mathcal{L}_{1,2}(t)\leqslant\Big[\sum_{g\in\{\mathcal{H},V_*',B_*',B_*''\}}\mathcal{B}_g(t)\Big]+\frac{\varepsilon_1^2}{8}.\tag{5.13}$$

Using Cauchy–Schwarz and the definitions, we have

$$\begin{aligned}\mathcal{L}_{1,2}(t) \leqslant \frac{1}{2}\mathcal{B}_{V_*'}(t)+32\int_1^t\int_{\mathbb{R}}\frac{A_R^3(s,\xi)}{|\dot{A}_R(s,\xi)|}\frac{|\widetilde{\mathcal{H}}(s,\xi)|^2}{s^2}\,d\xi\,ds\\ +32\int_1^t\int_{\mathbb{R}}\frac{A_R^3(s,\xi)}{|\dot{A}_R(s,\xi)|}|\dot{V}_1'(s,\xi)|^2\,d\xi\,ds\\ +\frac{1}{2}\mathcal{B}_{B_*'}(t)+8\int_1^t\int_{\mathbb{R}}\frac{A_R^3(s,\xi)}{|\dot{A}_R(s,\xi)|}|\dot{V}_1'(s,\xi)|^2\,d\xi\,ds\\ +\frac{1}{2}\mathcal{B}_{B_*''}(t)+8\int_1^t\int_{\mathbb{R}}\frac{A_R^3(s,\xi)}{|\dot{A}_R(s,\xi)|}|\dot{V}_2'(s,\xi)|^2\,d\xi\,ds.\end{aligned}$$

The functions $\dot{V}_a'$, $a\in\{1,2\}$, satisfy the bounds (4.15). Notice also that, for any $C_\delta\geqslant 1$, there is $\mathcal{K}(\delta)$ large enough such that

$$\frac{A_R^3(s,\xi)}{s^2|\dot{A}_R(s,\xi)|}\leqslant A_{NR}(s,\xi)|\dot{A}_{NR}(s,\xi)|(C_\delta^{-1}+\mathcal{K}(\delta)^2\langle s\rangle^{3/2}\langle\xi\rangle^{-3/2}).$$

This inequality is proved in [20, Lemma 6.2]. The desired bounds (5.13) follow by letting $\mathcal{K}$ large enough, using also the estimates (4.15). □

We now prove estimates on the cubic terms.

LEMMA 5.3. *For any $t\in[1,T]$ and $a\in\{1,2\}$, we have*

$$\left|2\mathrm{Re}\int_1^t\int_{\mathbb{R}}A_R^2(s,\xi)\tilde{f}_a(s,\xi)\overline{\widetilde{U}_a(s,\xi)}\,d\xi\,ds\right|\lesssim_\delta\varepsilon_1^3, \tag{5.14}$$

$$\left|2\mathrm{Re}\int_1^t\int_{\mathbb{R}}A_R^2(s,\xi)\tilde{f}_3(s,\xi)\overline{\widetilde{V}_*'(s,\xi)}\,d\xi\,ds\right|\lesssim_\delta\varepsilon_1^3, \tag{5.15}$$

$$\left|2\mathrm{Re}\int_1^t\int_{\mathbb{R}}A_{NR}^2(s,\xi)(\langle s\rangle^{3/2}\langle\xi\rangle^{-3/2})\widetilde{g}_a(s,\xi)\overline{\widetilde{\mathcal{H}}(s,\xi)}\,d\xi\,ds\right|\lesssim_\delta\varepsilon_1^3. \tag{5.16}$$

Proof. Step 1. We start with (5.14) and (5.15). The two bounds are similar, so we only provide all the details for the estimate (5.15). See also [20, Lemma 6.5] for a similar argument.

We write the left-hand side of (5.15) in the form

$$\begin{aligned}C&\left|2\mathrm{Re}\int_1^t\int_{\mathbb{R}}\int_{\mathbb{R}}A_R^2(s,\xi)\tilde{\dot{V}}(s,\xi-\eta)(i\eta)\widetilde{V}_*'(s,\eta)\overline{\widetilde{V}_*'(s,\xi)}\,d\xi\,d\eta\,ds\right|\\ &=C\left|\int_1^t\int_{\mathbb{R}}\int_{\mathbb{R}}[\eta A_R^2(s,\xi)-\xi A_R^2(s,\eta)]\tilde{\dot{V}}(s,\xi-\eta)\widetilde{V}_*'(s,\eta)\overline{\widetilde{V}_*'(s,\xi)}\,d\xi\,d\eta\,ds\right|,\end{aligned}$$

using symmetrization and the fact that $\dot V$ is real-valued. We define the sets

$$\begin{aligned} S_0 &:= \big\{(\xi,\eta)\in\mathbb{R}^2 : \min(\langle\xi\rangle,\langle\eta\rangle,\langle\xi-\eta\rangle)\geqslant \tfrac{1}{20}(\langle\xi\rangle+\langle\eta\rangle+\langle\xi-\eta\rangle)\big\},\\ S_1 &:= \big\{(\xi,\eta)\in\mathbb{R}^2 : \langle\xi-\eta\rangle\leqslant \tfrac{1}{10}(\langle\xi\rangle+\langle\eta\rangle+\langle\xi-\eta\rangle)\big\},\\ S_2 &:= \big\{(\xi,\eta)\in\mathbb{R}^2 : \langle\eta\rangle\leqslant \tfrac{1}{10}(\langle\xi\rangle+\langle\eta\rangle+\langle\xi-\eta\rangle)\big\},\\ S_3 &:= \big\{(\xi,\eta)\in\mathbb{R}^2 : \langle\xi\rangle\leqslant \tfrac{1}{10}(\langle\xi\rangle+\langle\eta\rangle+\langle\xi-\eta\rangle)\big\}, \end{aligned} \tag{5.17}$$

and the corresponding integrals

$$\begin{aligned} \mathcal{I}_n := \int_1^t\int_{\mathbb{R}}\int_{\mathbb{R}} &\mathbf{1}_{S_n}(\xi,\eta)\\ &\times|\eta A_R^2(s,\xi)-\xi A_R^2(s,\eta)|\,|\widetilde{\dot V}(s,\xi-\eta)|\,|\widetilde{V}'_*(s,\eta)|\,|\widetilde{V}'_*(s,\xi)|\,d\xi\,d\eta\,ds. \end{aligned} \tag{5.18}$$

For (5.15), it suffices to prove that

$$\mathcal{I}_n \lesssim_\delta \varepsilon_1^3 \quad \text{for } n\in\{0,1,2,3\}. \tag{5.19}$$

We use the following bilinear estimates for the weights, proved in [20, Lemma 8.9]. Letting $\delta_0'=\frac{1}{200}\delta_0$, we have

- If $(\xi,\eta)\in S_0\cup S_1$, $\rho=\xi-\eta$, $s\geqslant 1$, $\alpha\in[0,4]$, and $Y\in\{NR,R\}$, then

$$\begin{aligned} &|\eta A_Y^2(s,\xi)\langle\xi\rangle^{-\alpha}-\xi A_Y^2(s,\eta)\langle\eta\rangle^{-\alpha}|\\ &\quad\lesssim_\delta s^{1.6}\frac{\sqrt{|(A_Y\dot A_Y)(s,\xi)|}}{\langle\xi\rangle^{\alpha/2}}\frac{\sqrt{|(A_Y\dot A_Y)(s,\eta)|}}{\langle\eta\rangle^{\alpha/2}}\cdot A_{NR}(s,\rho)e^{-\delta_0'\langle\rho\rangle^{1/2}}. \end{aligned} \tag{5.20}$$

- If $(\xi,\eta)\in S_2$, $\rho=\xi-\eta$, and $s\geqslant 1$, then

$$\begin{aligned} &\langle\eta\rangle A_R^2(s,\xi)\\ &\quad\lesssim_\delta s^{1.1}\langle\xi\rangle^{0.6}\sqrt{|(A_R\dot A_R)(s,\xi)|}\sqrt{|(A_{NR}\dot A_{NR})(s,\rho)|}\cdot A_R(s,\eta)e^{-\delta_0'\langle\eta\rangle^{1/2}} \end{aligned} \tag{5.21}$$

and

$$\begin{aligned} &\langle\eta\rangle A_{NR}^2(s,\xi)\\ &\quad\lesssim_\delta s^{1.1}\langle\xi\rangle^{-0.4}\sqrt{|(A_{NR}\dot A_{NR})(s,\xi)|}\sqrt{|(A_{NR}\dot A_{NR})(s,\rho)|}\cdot A_{NR}(s,\eta)e^{-\delta_0'\langle\eta\rangle^{1/2}}. \end{aligned} \tag{5.22}$$

We remark that there is some room in the choice of exponents of s and $\langle\xi\rangle$ in (5.20)–(5.22). For instance, in (5.20) we can choose the exponent to be any number between $1+\sigma_0$ and $\frac{7}{4}$, where the range is determined by the requirement that the inequality holds and that the resulting weight can be absorbed by $\dot V$.

For $n\in\{0,1\}$, we can now estimate, using (5.20),

$$\mathcal{I}_n \lesssim_\delta \Big\|\sqrt{|(A_R\dot{A}_R)(s,\xi)|}\,\widetilde{V}'_*(s,\xi)\Big\|_{L^2_sL^2_\xi}\Big\|\sqrt{|(A_R\dot{A}_R)(s,\eta)|}\,\widetilde{V}'_*(s,\eta)\Big\|_{L^2_sL^2_\eta} \times\|s^{1.6}A_{NR}(s,\rho)\langle\rho\rangle^2e^{-\delta'_0\langle\rho\rangle^{1/2}}\widetilde{\dot{V}}(s,\rho)\|_{L^\infty_sL^2_\rho},$$

and the bounds (5.19) follow for $n\in\{0,1\}$ from (2.49) and (4.15). Similarly, for $n=2$, we use (5.21) and (3.22) to estimate

$$\mathcal{I}_2 \lesssim_\delta \Big\|\sqrt{|(A_R\dot{A}_R)(s,\xi)|}\,\widetilde{V}'_*(s,\xi)\Big\|_{L^2_sL^2_\xi}\Big\|s^{1.1}\langle\rho\rangle^{0.6}\sqrt{|(A_{NR}\dot{A}_{NR})(s,\rho)|}\,\widetilde{\dot{V}}(s,\rho)\Big\|_{L^2_sL^2_\rho} \times\|A_R(s,\eta)\langle\eta\rangle e^{-\delta'_0\langle\eta\rangle^{1/2}}\widetilde{V}'_*(s,\eta)\|_{L^\infty_sL^2_\eta},$$

and the desired bounds follow from (2.49) and (4.15). The case $n=3$ is similar, by changes of variables, which completes the proof of (5.15).

Step 2. The bounds (5.16) for $a=1$ are similar, using symmetrization, the bounds (5.20) with $Y=NR$, and the bounds (5.22). See also [20, Lemma 6.6] for a similar argument. Finally, the bounds (5.16) for $a=2$ follow from (4.21), (2.49), and the Cauchy inequality (see also [20, Lemma 6.4] for a similar proof). □

6. Improved control on the auxiliary variables Θ^* and F^*

In this section we prove the main bootstrap bounds (2.50) for the functions Θ^* and F^*.

PROPOSITION 6.1. *With the definitions and assumptions in Proposition 2.2, we have*

$$\mathcal{E}_{\Theta^*}(t)+\mathcal{B}_{\Theta^*}(t)\lesssim_\delta \varepsilon_1^4 \quad \textit{for any } t\in[1,T]. \tag{6.1}$$

Proof. We use the equations (2.32) and (2.39), and thus

$$\partial_z^2(\phi-\phi')+(B'_0)^2(\partial_v-t\partial_z)^2(\phi-\phi')+B''_0(\partial_v-t\partial_z)(\phi-\phi')=\mathcal{G}_1+\mathcal{G}_2, \tag{6.2}$$

where

$$\mathcal{G}_1:=[(B'_0)^2-(V')^2](\partial_v-t\partial_z)^2\phi \quad\text{and}\quad \mathcal{G}_2:=(B''_0-V'')(\partial_v-t\partial_z)\phi.$$

In view of Lemma 4.5, it suffices to prove that

$$\|\mathcal{G}_1\|_{\widetilde{W}}+\|\mathcal{G}_2\|_{\widetilde{W}}\lesssim_\delta \varepsilon_1^2. \tag{6.3}$$

Since V'_* is supported in $[b(\vartheta_0),b(1-\vartheta_0)]$, we can write

$$\mathcal{G}_1=-V'_*\cdot\Psi(B'_0+V')\cdot(\partial_v-t\partial_z)^2(\Psi\phi),$$

where Ψ is the Gevrey cut-off function in (2.42). Using Lemma 4.3 (i) and (ii), and the bootstrap assumptions (2.49) for V'_* and Θ, we can estimate

$$\|\mathcal{G}_1\|_{\widetilde{W}} \lesssim_\delta \|V'_*\|_R \, \|\Psi(B'_0+V')\|_R \, \|(\partial_v - t\partial_z)^2(\Psi\phi)\|_{\widetilde{W}} \lesssim_\delta \varepsilon_1^2,$$

as claimed in (6.3).

Similarly, since $V''=V'\partial_v V'$ and $B''_0=B'_0\,\partial_v B'_0$, we can write

$$\mathcal{G}_2 = -\tfrac{1}{2}\partial_v[V'_*\cdot\Psi(B'_0+V')]\cdot(\partial_v - t\partial_z)(\Psi\phi). \tag{6.4}$$

Moreover,

$$\frac{|k|\langle t\rangle}{|\xi|+|k|\langle t\rangle} \lesssim_\delta \frac{\langle \eta - tk\rangle}{\langle \xi-\eta\rangle}\,\frac{|k|\langle t\rangle}{|\eta|+|k|\langle t\rangle}\, e^{\delta \min(\langle \xi-\eta\rangle,\langle k,\eta\rangle)^{1/2}}, \tag{6.5}$$

if $k\in\mathbb{Z}^*$, $t\geqslant 1$, and $\xi,\eta\in\mathbb{R}$, as one can check easily by considering the cases $|\xi-\eta|\leqslant 10|k,\eta|$ and $|\xi-\eta|\geqslant 10|k,\eta|$. Therefore, using also (3.42)–(3.43),

$$\frac{A_k(t,\xi)|k|\langle t\rangle}{|\xi|+|k|\langle t\rangle} \lesssim_\delta \frac{A_R(t,\xi-\eta)}{\langle \xi-\eta\rangle}\,\frac{A_k(t,\eta)|k|\langle t\rangle}{|\eta|+|k|\langle t\rangle}\langle \eta - tk\rangle e^{-(\delta_0/30)\min(\langle \xi-\eta\rangle,\langle k,\eta\rangle)^{1/2}}$$

and

$$\begin{aligned}\frac{|(A_k\dot{A}_k)(t,\xi)|^{1/2}|k|\langle t\rangle}{|\xi|+|k|\langle t\rangle} &\lesssim_\delta e^{-(\delta_0/30)\min(\langle \xi-\eta\rangle,\langle k,\eta\rangle)^{1/2}} \\ &\times\bigg\{\frac{|(A_R\dot{A}_R)(t,\xi-\eta)|^{1/2}}{\langle \xi-\eta\rangle}\,\frac{A_k(t,\eta)|k|\langle t\rangle}{|\eta|+|k|\langle t\rangle}\langle \eta-tk\rangle \\ &\quad + \frac{A_R(t,\xi-\eta)}{\langle \xi-\eta\rangle}\,\frac{|(A_k\dot{A}_k)(t,\eta)|^{1/2}|k|\langle t\rangle}{|\eta|+|k|\langle t\rangle}\langle \eta-tk\rangle\bigg\}.\end{aligned}$$

We examine the formula (6.4) and notice that

$$\|V'_*\cdot\Psi(B'_0+V')\|_R \lesssim_\delta \varepsilon_1$$

(due to Lemma 4.3 (i) and (ii)) and

$$\|\langle \partial_v - t\partial_z\rangle(\partial_v - t\partial_z)(\Psi\phi)\|_{\widetilde{W}} \lesssim_\delta \varepsilon_1$$

(due to the bootstrap assumption (2.49)). The desired conclusion $\|\mathcal{G}_2\|_{\widetilde{W}} \lesssim_\delta \varepsilon_1^2$ in (6.3) follows using Lemma 4.1 (ii) and the two weighted estimates above. □

We prove now bootstrap bounds on the function F^*.

PROPOSITION 6.2. *With the definitions and assumptions in Proposition 2.2, we have*

$$\mathcal{E}_{F^*}(t)+\mathcal{B}_{F^*}(t) \lesssim_\delta \varepsilon_1^3 \quad \textit{for any } t\in[1,T]. \tag{6.6}$$

Proof. The function F^* satisfies the evolution equation

$$\partial_t F^* = V'\partial_v P_{\neq 0}\phi\,\partial_z F - (\dot{V} + V'\partial_z\phi)\,\partial_v F + (B''\partial_z\phi - B_0''\,\partial_z\phi'), \tag{6.7}$$

which follows from (2.28) and (2.40). Recalling the definition (2.43), we calculate

$$\begin{aligned}\frac{d}{dt}\mathcal{E}_{F^*}(t) = &\sum_{k\in\mathbb{Z}}\int_{\mathbb{R}} 2\dot{A}_k(t,\xi)A_k(t,\xi)|\widetilde{F^*}(t,k,\xi)|^2\,d\xi \\ &+ 2\mathrm{Re}\sum_{k\in\mathbb{Z}}\int_{\mathbb{R}} A_k^2(t,\xi)\,\partial_t\widetilde{F^*}(t,k,\xi)\overline{\widetilde{F^*}(t,k,\xi)}\,d\xi.\end{aligned} \tag{6.8}$$

Therefore, since $\partial_t A_k \leqslant 0$, for any $t\in[1,T]$ we have

$$\begin{aligned}&\mathcal{E}_f(t) + \int_1^t \sum_{k\in\mathbb{Z}}\int_{\mathbb{R}} 2|\dot{A}_k(s,\xi)|\,A_k(s,\xi)\,|\widetilde{F^*}(s,k,\xi)|^2\,d\xi\,ds \\ &\quad= \mathcal{E}_f(1) + \int_1^t \left\{2\mathrm{Re}\sum_{k\in\mathbb{Z}}\int_{\mathbb{R}} A_k^2(s,\xi)\partial_s\widetilde{F^*}(s,k,\xi)\overline{\widetilde{F^*}(s,k,\xi)}\,d\xi\right\}ds.\end{aligned}$$

We examine the equation (6.7) and decompose the non-linearity in the right-hand side. Let

$$\begin{gathered}\mathcal{N}_1 := V'\partial_v P_{\neq 0}\phi\,\partial_z F^*, \quad \mathcal{N}_2 := -V'\partial_z\phi\partial_v F^*, \quad \mathcal{N}_3 := -\dot{V}\partial_v F^*, \\ \mathcal{N}_4 := V'\partial_v P_{\neq 0}\phi\,\partial_z(F - F^*), \quad \mathcal{N}_5 := -V'\partial_z\phi\partial_v(F - F^*), \\ \mathcal{N}_6 := -\dot{V}\partial_v(F - F^*), \quad \mathcal{N}_7 := B''\partial_z\phi - B_0''\partial_z\phi'.\end{gathered} \tag{6.9}$$

Since $\mathcal{E}_f(1) \lesssim \varepsilon_1^3$ (see (2.48)), for (6.6) it suffices to prove that, for any $t\in[1,T]$,

$$\left|2\mathrm{Re}\int_1^t \sum_{k\in\mathbb{Z}}\int_{\mathbb{R}} A_k^2(s,\xi)\widetilde{\mathcal{N}_a}(s,k,\xi) - \overline{\widetilde{F^*}(s,k,\xi)}\,d\xi\,ds\right| \lesssim_\delta \varepsilon_1^3, \tag{6.10}$$

for $a\in\{1,\dots,7\}$. We prove these bounds in Lemmas 6.3–6.5 below. □

LEMMA 6.3. *The bounds* (6.10) *hold for* $a\in\{1,2,3\}$.

Proof. This is similar to the proofs of [20, Lemmas 4.4, 4.6, and 4.8], and we will be somewhat brief. The common point is that one can symmetrize the integrals to avoid loss of derivatives.

Step 1. We consider first the non-linearity $\mathcal{N}_1$. Letting $H_1 := V'\partial_v P_{\neq 0}(\Psi\phi)$, we write

$$\begin{aligned}&\left|2\mathrm{Re}\int_1^t \sum_{k\in\mathbb{Z}}\int_{\mathbb{R}} A_k^2(s,\xi)\widetilde{\mathcal{N}_1}(s,k,\xi)\overline{\widetilde{F^*}(s,k,\xi)}\,d\xi\,ds\right| \\ &\quad= C\left|2\mathrm{Re}\left\{\sum_{k,\ell\in\mathbb{Z}}\int_1^t\int_{\mathbb{R}^2} A_k^2(s,\xi)\widetilde{H}_1(s,k-\ell,\xi-\eta)i\ell\widetilde{F^*}(s,\ell,\eta)\overline{\widetilde{F^*}(s,k,\xi)}\,d\xi\,d\eta\,ds\right\}\right| \\ &\quad= C\left|\int_1^t \sum_{k,\ell\in\mathbb{Z}}\int_{\mathbb{R}^2} [\ell A_k^2(s,\xi) - kA_\ell^2(s,\eta)]\widetilde{H}_1(s,k-\ell,\xi-\eta)\widetilde{F^*}(s,\ell,\eta)\overline{\widetilde{F^*}(s,k,\xi)}\,d\xi\,d\eta\,ds\right|,\end{aligned}$$

where the second identity uses symmetrization based on the fact that H_1 is real-valued. The cutoff function Ψ can be inserted in the definition of H_1, because F and F^* are supported in $[0,T]\times\mathbb{T}\times[b(\vartheta_0),b(1-\vartheta_0)]$. With R_n as in equations (3.44)–(3.47), we define the integrals

$$\begin{aligned}\mathcal{U}_1^n := \int_1^t \sum_{k,\ell\in\mathbb{Z}} \int_{\mathbb{R}^2} \mathbf{1}_{R_n}((k,\xi),(\ell,\eta))|\ell A_k^2(s,\xi)-kA_\ell^2(s,\eta)|\,|\widetilde{H}_1(s,k-\ell,\xi-\eta)| \\ \times|\widetilde{F^*}(s,\ell,\eta)|\,|\widetilde{F^*}(s,k,\xi)|\,d\xi\,d\eta\,ds.\end{aligned} \tag{6.11}$$

We use Lemma 3.9, and remark that $\widetilde{H}_1(t,0,\rho)=0$ for $\rho\in\mathbb{R}$, due to the definition

$$H_1 = V'\partial_v P_{\neq 0}(\Psi\phi).$$

Denote $(m,\rho)=(k-\ell,\xi-\eta)$. Using (3.48), (4.20), and (2.49), for $n\in\{0,1\}$ we can bound

$$\begin{aligned}\mathcal{U}_1^n &\lesssim_\delta \int_1^t \sum_{k,\ell\in\mathbb{Z}} \int_{\mathbb{R}^2} \sqrt{|(A_k\dot{A}_k)(s,\xi)|}\,|\widetilde{F^*}(s,k,\xi)|\sqrt{|(A_\ell\dot{A}_\ell)(s,\eta)|}\,|\widetilde{F^*}(s,\ell,\eta)| \\ &\qquad\times \mathbf{1}_{\mathbb{Z}^*}(m)\frac{\langle s\rangle\langle s-\rho/m\rangle^2 m^2}{(|\rho/m|+\langle s\rangle)\langle\rho\rangle}A_m(s,\rho)|\widetilde{H}_1(s,m,\rho)|e^{-\delta_0'\langle m,\rho\rangle^{1/2}}\,d\xi\,d\eta\,ds \\ &\lesssim_\delta \Big\|\sqrt{|(A_k\dot{A}_k)(s,\xi)|}\,\widetilde{F^*}(s,k,\xi)\Big\|_{L^2_sL^2_{k,\xi}}\Big\|\sqrt{|(A_\ell\dot{A}_\ell)(s,\eta)|}\,\widetilde{F^*}(s,\ell,\eta)\Big\|_{L^2_sL^2_{\ell,\eta}} \\ &\qquad\times\Big\|\mathbf{1}_{\mathbb{Z}^*}(m)A_m(s,\rho)\frac{\langle s\rangle\langle s-\rho/m\rangle^2 m^2}{(|\rho/m|+\langle s\rangle)\langle\rho\rangle}e^{-(\delta_0'/2)\langle m,\rho\rangle^{1/2}}\widetilde{H}_1(s,m,\rho)\Big\|_{L^\infty_sL^2_{m,\rho}} \\ &\lesssim_\delta \varepsilon_1^3.\end{aligned}$$

Similarly, for $n=2$, we use (3.50), (4.20), and (2.49) to bound

$$\begin{aligned}\mathcal{U}_1^2 &\lesssim_\delta \int_1^t \sum_{k,\ell\in\mathbb{Z}} \int_{\mathbb{R}^2} \mathbf{1}_{\mathbb{Z}^*}(m)\sqrt{|(A_m\dot{A}_m)(s,\rho)|}\,\frac{\langle s\rangle\langle s-\rho/m\rangle^2 m^2}{(|\rho/m|+\langle s\rangle)\langle\rho\rangle}|\widetilde{H}_1(s,m,\rho)| \\ &\qquad\times\sqrt{|(A_k\dot{A}_k)(s,\xi)|}\,|\widetilde{F^*}(s,k,\xi)|A_\ell(s,\eta)e^{-\delta_0'\langle\ell,\eta\rangle^{1/2}}|\widetilde{F^*}(s,\ell,\eta)|\,d\xi\,d\eta\,ds \\ &\lesssim_\delta \Big\|\sqrt{|(A_k\dot{A}_k)(s,\xi)|}\,\widetilde{F^*}(s,k,\xi)\Big\|_{L^2_sL^2_{k,\xi}}\Big\|A_\ell(s,\eta)e^{-(\delta_0'/2)\langle\ell,\eta\rangle^{1/2}}\widetilde{F^*}(s,\ell,\eta)\Big\|_{L^\infty_sL^2_{\ell,\eta}} \\ &\qquad\times\Big\|\mathbf{1}_{\mathbb{Z}^*}(m)\sqrt{|(A_m\dot{A}_m)(s,\rho)|}\,\frac{\langle s\rangle\langle s-\rho/m\rangle^2 m^2}{(|\rho/m|+\langle s\rangle)\langle\rho\rangle}\widetilde{H}_1(s,m,\rho)\Big\|_{L^2_sL^2_{m,\rho}} \\ &\lesssim_\delta \varepsilon_1^3.\end{aligned} \tag{6.12}$$

The case $n=3$ is identical to the case $n=2$, by symmetry, so $\mathcal{U}_1^n\lesssim_\delta\varepsilon_1^3$ for all $n\in\{0,1,2,3\}$. The desired bounds (6.10) follow for $a=1$.

Step 2. The bounds for the non-linearity $\mathcal{N}_2$ follow in the same way, using the estimates (3.54)–(3.56) and (4.19) (see [20, Lemma 4.6] for complete details). The bounds for the non-linearity $\mathcal{N}_3$ also follow in the same way, using the estimates (3.57)–(3.59) and (4.15) (see [20, Lemma 4.8] for complete details). □

LEMMA 6.4. *The bounds* (6.10) *hold for* $a\in\{4,5,6\}$.

Proof. Step 1. We first consider the non-linearity $\mathcal{N}_4$, and estimate

$$\begin{aligned}
&\Big|2\mathrm{Re}\int_1^t\sum_{k\in\mathbb{Z}}\int_{\mathbb{R}}A_k^2(s,\xi)\widetilde{\mathcal{N}_4}(s,k,\xi)\overline{\widetilde{F^*}(s,k,\xi)}\,d\xi\,ds\Big|\\
&\quad\lesssim\Big|\sum_{k,\ell\in\mathbb{Z}}\int_1^t\int_{\mathbb{R}^2}A_k^2(s,\xi)\widetilde{H}_1(s,k-\ell,\xi-\eta)\ell\widetilde{(F-F^*)}(s,\ell,\eta)\overline{\widetilde{F^*}(s,k,\xi)}\,d\xi\,d\eta\,ds\Big|\\
&\quad\lesssim\sum_{n\in\{0,1,2,3\}}\mathcal{U}_4^n,
\end{aligned}$$

where

$$H_1=V'\partial_vP_{\neq0}(\Psi\phi),$$

as in the proof of Lemma 6.3, and, recall the definitions (3.44)–(3.47),

$$\begin{aligned}
\mathcal{U}_4^n:=\int_1^t\sum_{k,\ell\in\mathbb{Z}}\int_{\mathbb{R}^2}&\mathbf{1}_{R_n}((k,\xi),(\ell,\eta))|\ell A_k^2(s,\xi)|\,|\widetilde{H}_1(s,k-\ell,\xi-\eta)|\\
&\times|\widetilde{(F-F^*)}(s,\ell,\eta)|\,|\widetilde{F^*}(s,k,\xi)|\,d\xi\,d\eta\,ds.
\end{aligned}$$

Recall that $\widetilde{H}_1(t,0,\rho)=0$ for $\rho\in\mathbb{R}$. Letting $(m,\rho)=(k-\ell,\xi-\eta)$ and using (3.49), (4.20), and (2.49), for $n\in\{0,1\}$ we can bound

$$\begin{aligned}
\mathcal{U}_4^n&\lesssim_\delta\int_1^t\sum_{k,\ell\in\mathbb{Z}}\int_{\mathbb{R}^2}\sqrt{|(A_\ell\dot{A}_\ell)(s,\eta)|}\Big(1+\frac{\langle\ell,\eta\rangle}{\langle s\rangle}\Big)^{1/2}|\widetilde{(F-F^*)}(s,\ell,\eta)|\cdot\sqrt{|(A_k\dot{A}_k)(s,\xi)|}\\
&\qquad\times|\widetilde{F^*}(s,k,\xi)|\cdot\mathbf{1}_{\mathbb{Z}^*}(m)\frac{\langle s\rangle\langle s-\rho/m\rangle^2m^2}{(|\rho/m|+\langle s\rangle)\langle\rho\rangle}A_m(s,\rho)|\widetilde{H}_1(s,m,\rho)|e^{-\delta_0'\langle m,\rho\rangle^{1/2}}\,d\xi\,d\eta\,ds\\
&\lesssim_\delta\Big\|\sqrt{|(A_k\dot{A}_k)(s,\xi)|}\,\widetilde{F^*}(s,k,\xi)\Big\|_{L^2_sL^2_{k,\xi}}\\
&\qquad\times\Big\|\sqrt{|(A_\ell\dot{A}_\ell)(s,\eta)|\Big(1+\frac{\langle\ell,\eta\rangle}{\langle s\rangle}\Big)}\widetilde{(F-F^*)}(s,\ell,\eta)\Big\|_{L^2_sL^2_{\ell,\eta}}\\
&\qquad\times\Big\|\mathbf{1}_{\mathbb{Z}^*}(m)A_m(s,\rho)\frac{\langle s\rangle\langle s-\rho/m\rangle^2m^2}{(|\rho/m|+\langle s\rangle)\langle\rho\rangle}e^{-(\delta_0'/2)\langle m,\rho\rangle^{1/2}}\widetilde{H}_1(s,m,\rho)\Big\|_{L^\infty_sL^2_{m,\rho}}\\
&\lesssim_\delta\varepsilon_1^3.
\end{aligned}$$

Moreover, we can also estimate $\mathcal{U}_4^2\lesssim_\delta\varepsilon_1^3$, using (3.50), (4.20), and (2.49) as in (6.12). Then, we can estimate $\mathcal{U}_4^3\lesssim_\delta\varepsilon_1^3$ by symmetry. The desired bounds (6.10) follow for $a{=}4$.

Step 2. We consider now the non-linearity $\mathcal{N}_5$, and estimate

$$\Big|2\mathrm{Re}\int_1^t\sum_{k\in\mathbb{Z}}\int_{\mathbb{R}}A_k^2(s,\xi)\widetilde{\mathcal{N}}_5(s,k,\xi)\overline{\widetilde{F^*}(s,k,\xi)}\,d\xi\,ds\Big|\lesssim\sum_{n\in\{0,1,2,3\}}\mathcal{U}_5^n,$$

where $H_2{:=}V'\partial_z(\Psi\phi)$, and

$$\mathcal{U}_5^n:=\int_1^t\sum_{k,\ell\in\mathbb{Z}}\int_{\mathbb{R}^2}\mathbf{1}_{R_n}((k,\xi),(\ell,\eta))|\eta A_k^2(s,\xi)|\,|\widetilde{H}_2(s,k-\ell,\xi-\eta)|\\ \times|\widetilde{(F-F^*)}(s,\ell,\eta)|\,|\widetilde{F^*}(s,k,\xi)|\,d\xi\,d\eta\,ds.$$

Notice that $\widetilde{H}_2(t,0,\rho){=}0$ for $\rho{\in}\mathbb{R}$. Letting $(m,\rho){=}(k{-}\ell,\xi{-}\eta)$ and using (3.55), (4.19), and (2.49), for $n{\in}\{0,1\}$ we can bound, as before,

$$\begin{aligned}\mathcal{U}_5^n&\lesssim_\delta\int_1^t\sum_{k,\ell\in\mathbb{Z}}\int_{\mathbb{R}^2}\sqrt{|(A_\ell\dot{A}_\ell)(s,\eta)|}\Big(1+\frac{\langle\ell,\eta\rangle}{\langle s\rangle}\Big)^{1/2}|\widetilde{(F-F^*)}(s,\ell,\eta)|\cdot\sqrt{|(A_k\dot{A}_k)(s,\xi)|}\\ &\quad\times|\widetilde{F^*}(s,k,\xi)|\cdot\mathbf{1}_{\mathbb{Z}^*}(m)\frac{|m|\langle s\rangle^2\langle s-\rho/m\rangle^2}{|\rho/m|^2+\langle s\rangle^2}A_m(s,\rho)|\widetilde{H}_2(s,m,\rho)|e^{-\delta_0'\langle m,\rho\rangle^{1/2}}\,d\xi\,d\eta\,ds\\ &\lesssim_\delta\Big\|\sqrt{|(A_k\dot{A}_k)(s,\xi)|}\,\widetilde{F^*}(s,k,\xi)\Big\|_{L^2_sL^2_{k,\xi}}\\ &\quad\times\Big\|\sqrt{|(A_\ell\dot{A}_\ell)(s,\eta)|\Big(1+\frac{\langle\ell,\eta\rangle}{\langle s\rangle}\Big)}\widetilde{(F-F^*)}(s,\ell,\eta)\Big\|_{L^2_sL^2_{\ell,\eta}}\\ &\quad\times\Big\|\mathbf{1}_{\mathbb{Z}^*}(m)\frac{|m|\langle s\rangle^2\langle s-\rho/m\rangle^2}{|\rho/m|^2+\langle s\rangle^2}A_m(s,\rho)e^{-(\delta_0'/2)\langle m,\rho\rangle^{1/2}}\widetilde{H}_2(s,m,\rho)\Big\|_{L^\infty_sL^2_{m,\rho}}\\ &\lesssim_\delta\varepsilon_1^3.\end{aligned}$$

The term $\mathcal{U}_5^2$ can be bounded in the same way, using (3.56), (4.19), and (2.49), while the term $\mathcal{U}_5^3$ can be bounded by symmetry. The desired bounds (6.10) follow for $a{=}5$.

Step 3. Similarly, for $a{=}6$, we estimate

$$\Big|2\mathrm{Re}\int_1^t\sum_{k\in\mathbb{Z}}\int_{\mathbb{R}}A_k^2(s,\xi)\widetilde{\mathcal{N}}_6(s,k,\xi)\overline{\widetilde{F^*}(s,k,\xi)}\,d\xi\,ds\Big|\lesssim\sum_{n\in\{0,1,2,3\}}\mathcal{U}_6^n,$$

where $R_n^*{:=}\{((k,\xi),(\ell,\eta)){\in}R_n{:}k{=}l\}$ and

$$\mathcal{U}_6^n:=\int_1^t\sum_{k\in\mathbb{Z}}\int_{\mathbb{R}^2}\mathbf{1}_{R_n^*}((k,\xi),(k,\eta))|\eta A_k^2(s,\xi)|\,|\widetilde{\dot{V}}(s,\xi-\eta)|\\ \times|\widetilde{(F-F^*)}(s,k,\eta)|\,|\widetilde{F^*}(s,k,\xi)|\,d\xi\,d\eta\,ds.$$

Letting $\rho=\xi-\eta$ and using (3.58), (4.15), and (2.49), for $n\in\{0,1\}$ we can bound

$$\begin{aligned}\mathcal{U}_6^n &\lesssim_\delta \int_1^t \sum_{k\in\mathbb{Z}} \int_{\mathbb{R}^2} \sqrt{|(A_k\dot{A}_k)(s,\eta)|}\Big(1+\frac{\langle k,\eta\rangle}{\langle s\rangle}\Big)^{1/2} |(\widetilde{F-F^*})(s,\ell,\eta)|\cdot\sqrt{|(A_k\dot{A}_k)(s,\xi)|}\\ &\qquad\times|\widetilde{F^*}(s,k,\xi)|\cdot(\langle\rho\rangle\langle s\rangle+\langle\rho\rangle^{1/4}\langle s\rangle^{7/4})A_{NR}(s,\rho)|\widetilde{\dot{V}}(s,\rho)|e^{-\delta_0'\langle\rho\rangle^{1/2}}\,d\xi\,d\eta\,ds\\ &\lesssim_\delta \Big\|\sqrt{|(A_k\dot{A}_k)(s,\xi)|}\,\widetilde{F^*}(s,k,\xi)\Big\|_{L^2_sL^2_{k,\xi}}\\ &\qquad\times\Big\|\sqrt{|(A_k\dot{A}_k)(s,\eta)|\Big(1+\frac{\langle k,\eta\rangle}{\langle s\rangle}\Big)}(\widetilde{F-F^*})(s,k,\eta)\Big\|_{L^2_sL^2_{k,\eta}}\\ &\qquad\times\|(\langle\rho\rangle\langle s\rangle+\langle\rho\rangle^{1/4}\langle s\rangle^{7/4})A_{NR}(s,\rho)e^{-(\delta_0'/2)\langle\rho\rangle^{1/2}}\widetilde{\dot{V}}(s,\rho)\|_{L^\infty_sL^2_\rho}\\ &\lesssim_\delta \varepsilon_1^3.\end{aligned}$$

The term $\mathcal{U}_6^2$ can be bounded in the same way, using (3.59), (4.15), and (2.49), while the term $\mathcal{U}_6^3$ can be bounded by symmetry. The desired bounds (6.10) follow for $a=6$. □

LEMMA 6.5. *The bounds* (6.10) *hold for* $a=7$.

Proof. Since B'' and B''_0 are supported in $[0,T]\times\mathbb{T}\times[b(\vartheta_0),b(1-\vartheta_0)]$, we can write

$$\mathcal{N}_7=B''_*\partial_z(\Psi\phi)+B''_0\partial_z(\Psi(\phi-\phi')).$$

In view of (4.13), (2.49), and (6.1), and recalling the definitions (4.7)–(4.9), we have

$$\begin{aligned}&\|B''_*\|_R\lesssim_\delta\varepsilon_1, &&\|(\partial_z^2+(\partial_v-t\partial_z)^2)(\Psi\phi)\|_{\widetilde{W}}\lesssim_\delta\varepsilon_1,\\ &\|B''_0\|_R\lesssim_\delta 1, &&\|(\partial_z^2+(\partial_v-t\partial_z)^2)(\Psi(\phi-\phi'))\|_{\widetilde{W}}\lesssim_\delta\varepsilon_1^2.\end{aligned}\tag{6.13}$$

Therefore, to prove (6.10) for $a=7$, it suffices to show that

$$\Big|\sum_{k\in\mathbb{Z}}\int_1^t\int_{\mathbb{R}^2} kA_k^2(s,\xi)\tilde{h}(s,\xi-\eta)\widetilde{\varphi}(s,k,\eta)\overline{\widetilde{F^*}(s,k,\xi)}\,d\xi\,d\eta\,ds\Big|\lesssim_\delta\varepsilon_1,\tag{6.14}$$

for any functions h and φ satisfying $\|h\|_R\leqslant 1$ and $\|(\partial_z^2+(\partial_v-t\partial_z)^2)\varphi\|_{\widetilde{W}}\leqslant 1$. With R_n^* defined as before, for $n\in\{0,1,2,3\}$ we let

$$\mathcal{U}_7^n:=\int_1^t\sum_{k\in\mathbb{Z}}\int_{\mathbb{R}^2}\mathbf{1}_{R_n^*}((k,\xi),(k,\eta))|k|\,A_k^2(s,\xi)\,|\tilde{h}(s,\xi-\eta)|\,|\widetilde{\varphi}(s,k,\eta)|\,|\widetilde{F^*}(s,k,\xi)|\,d\xi\,d\eta\,ds.$$

Letting $\rho=\xi-\eta$ and using (3.62), for $n\in\{0,1\}$ we can estimate

$$\begin{aligned}\mathcal{U}_7^n &\lesssim_\delta \int_1^t \sum_{k\in\mathbb{Z}^*}\int_{\mathbb{R}^2}\sqrt{|(A_k\dot{A}_k)(s,\eta)|}\frac{|k|^2\langle s\rangle\langle s-\eta/k\rangle^2}{\langle s\rangle+|\eta/k|}|\widetilde{\varphi}(s,k,\eta)| \\ &\qquad\times\sqrt{|(A_k\dot{A}_k)(s,\xi)|}\,|\widetilde{F^*}(s,k,\xi)|\cdot A_R(s,\rho)|\tilde{h}(s,\rho)|e^{-\delta_0'\langle\rho\rangle^{1/2}}\,d\xi\,d\eta\,ds \\ &\lesssim_\delta \Big\|\sqrt{|(A_k\dot{A}_k)(s,\xi)|}\,\widetilde{F^*}(s,k,\xi)\Big\|_{L^2_sL^2_{k,\xi}}\|A_R(s,\rho)e^{-(\delta_0'/2)\langle\rho\rangle^{1/2}}\tilde{h}(s,\rho)\|_{L^\infty_sL^2_\rho} \\ &\qquad\times\Big\|\mathbf{1}_{\mathbb{Z}^*}(k)\sqrt{|(A_k\dot{A}_k)(s,\eta)|}\frac{|k|^2\langle s\rangle\langle s-\eta/k\rangle^2}{\langle s\rangle+|\eta/k|}\widetilde{\varphi}(s,k,\eta)\Big\|_{L^2_sL^2_{k,\eta}} \\ &\lesssim_\delta \varepsilon_1.\end{aligned}$$

Similarly, we can use (3.63) to estimate $\mathcal{U}_7^2\lesssim_\delta\varepsilon_1$, and then use (3.64) to estimate $\mathcal{U}_7^3\lesssim_\delta\varepsilon_1$. This completes the proof of the lemma. □

7. Improved control on $F-F^*$ and the main variables F and Θ

In this section we improve the remaining bootstrap bounds.

Proposition 7.1. *With the definitions and assumptions in Proposition 2.2, we have*

$$\sum_{g\in\{F-F^*,F,\Theta\}}[\mathcal{E}_g(t)+\mathcal{B}_g(t)]\lesssim_\delta\varepsilon_1^3\quad\textit{for any } t\in[1,T]. \tag{7.1}$$

The key issue is to prove the bounds (7.1) for the variable $F-F^*$, from which the other bounds follow easily. Our main tool is the following precise estimates on the linearized flow.

Proposition 7.2. *Given $k\in\mathbb{Z}^*$, assume that f_k is a smooth solution to the equation*

$$\partial_t f_k-ikB_0''\psi_k=X_k(t,v), \tag{7.2}$$

$$(B_0')^2(\partial_v-itk)^2\psi_k+B_0''(\partial_v-itk)\psi_k-k^2\psi_k=f_k,\qquad \psi_k(b(0))=\psi_k(b(1))=0, \tag{7.3}$$

for $t\in[0,T]$ and $v\in[b(0),b(1)]$, with vanishing initial data $f_k(0,v)\equiv0$. Assume that X_k is supported in $[0,T]\times[b(\vartheta_0),b(1-\vartheta_0)]$. Then,

$$\tilde{f}_k(t,\xi)=\int_0^t\widetilde{X}_k(s,\xi)\,ds+ik\int_0^t\int_s^t\int_{\mathbb{R}}\widetilde{B}_0''(\zeta)\widetilde{\Pi}_k(s,\xi-\zeta-k\tau,\xi-\zeta)\,d\zeta\,d\tau\,ds, \tag{7.4}$$

for some functions $\Pi_k\colon[0,T]\times\mathbb{R}^2\to\mathbb{C}$. Moreover, the functions Π_k satisfy the bounds

$$\|(|k|+|\xi|)W_k(\eta)\widetilde{\Pi}_k(t,\xi,\eta)\|_{L^2_{\xi,\eta}}\lesssim_\delta\|W_k(\xi)\widetilde{X}_k(t,\xi)\|_{L^2_\xi}, \tag{7.5}$$

for any $t\in[0,T]$ *and* δ *sufficiently small. Here* $W_k\geqslant 1$ *is a family of weights which depend on a small parameter* $\delta\in(0,1]$, *and satisfy, for any* $k\in\mathbb{Z}^*$ *and* $\xi,\eta\in\mathbb{R}$,

$$|W_k(\xi)-W_k(\eta)|\lesssim e^{2\delta_0\langle\xi-\eta\rangle^{1/2}}W_k(\eta)\left[\frac{C(\delta)}{\langle k,\eta\rangle^{1/8}}+\sqrt{\delta}\right], \tag{7.6}$$

where $C(\delta)\gg 1$ *is a large constant, and the implied constant in* (7.6) *does not depend on* k *and* δ.

The weights W_k we use for our application are connected to the main weights A_k; see (7.15) and (7.17). They are allowed to depend on t as well, as long as the bounds (7.6) hold uniformly.

We remark that the condition (7.6) on the rate of change of the weights $W_k(\xi)$ is crucial for the commutator arguments below. Such a property clearly holds in a strong sense for standard Sobolev weights and Gevrey weights, with derivative gains at large frequencies, but only holds in the weak sense stated in (7.6) for our special weights A_k (compare with Lemma 3.5)

Roughly speaking, Proposition 7.2 allows us to invert the linear transformation defined in (2.40) taking the full profile F to the auxilliary profile F^*, with the right bounds.

The proof of Proposition 7.2 is based on the ideas introduced in [22]. For our purposes here, we need to consider the linearized flow with an inhomogeneous term and to obtain more precise estimates. We provide the detailed proof of this proposition in the next section.

7.1. Proof of Proposition 7.1

In the rest of this section, we assume Proposition 7.2 and prove Proposition 7.1. For $k\in\mathbb{Z}^*$, we define the function $\phi_k^*(t,v)$ as the solution to

$$(B_0')^2(\partial_v-itk)^2\phi_k^*+B_0''(\partial_v-itk)\phi_k^*-k^2\phi_k^*=F_k^* \quad \text{for } v\in(b(0),b(1)), \tag{7.7}$$

with boundary value $\phi_k^*(t,b(0))=\phi_k^*(t,b(1))=0$. Notice that

$$\begin{aligned}&\sup_{t\in[0,T]}\left\{\sum_{k\in\mathbb{Z}^*}\int_{\mathbb{R}}A_k^2(t,\xi)(|k|^2+|\xi-kt|^2)^2\,|\tilde{h}_k(t,\xi)|^2\,d\xi\right\}\lesssim_\delta \varepsilon_1^3,\\ &\sum_{k\in\mathbb{Z}^*}\int_0^T\int_{\mathbb{R}}|\dot{A}_kA_k(s,\xi)|(|k|^2+|\xi-ks|^2)^2\,|\tilde{h}_k(s,\xi)|^2\,d\xi\,ds\lesssim_\delta \varepsilon_1^3,\end{aligned} \tag{7.8}$$

where $h_k=\Psi\phi_k^*$ or $h_k=B_0''\phi_k^*$. Indeed, since $\|F^*\|_{W[0,T]}\lesssim_\delta\varepsilon_1^{3/2}$ (see (6.6)), the bounds (7.8) follow from the elliptic bounds in Lemma 4.5 if $h_k=\Psi\phi_k^*$. The bounds for $B_0''\phi_k^*=B_0''\Psi\phi_k^*$ then follow using Lemma 3.8 (ii) and the bounds (4.14) on the function B_0''.

We now write

$$F_k(t,v)-F_k^*(t,v)-ik\int_0^t B_0''(v)(\phi_k'-\phi_k^*)(\tau,v)\,d\tau=ik\int_0^t B_0''(v)\phi_k^*(\tau,v)\,d\tau. \tag{7.9}$$

Setting

$$g_k(t,v):=F_k(t,v)-F_k^*(t,v)\quad\text{and}\quad \psi_k(t,v):=\phi_k'(t,v)-\phi_k^*(t,v), \tag{7.10}$$

then g_k satisfies the equation

$$\partial_t g_k-ikB_0''(v)\psi_k=ik(B_0''\phi_k^*)(t,v), \tag{7.11}$$

with initial data $g_k(0,v)\equiv 0$, while ψ_k solves the elliptic equation

$$(B_0')^2(\partial_v-itk)^2\psi_k+B_0''(\partial_v-itk)\psi_k-k^2\psi_k=g_k,\quad \psi_k(t,b(0))=\psi_k(t,b(1))=0, \tag{7.12}$$

in $[0,T]\times[b(0),b(1)]$. Using Proposition 7.2, we obtain that

$$\tilde g_k(t,\xi)=ik\int_0^t(\widetilde{B_0''\phi_k^*})(s,\xi)\,ds+ik\int_0^t\int_s^t\int_{\mathbb{R}}\widetilde{B}_0''(\zeta)\,\widetilde{\Pi}_k(s,\xi-\zeta-k\tau,\xi-\zeta)\,d\zeta\,d\tau\,ds. \tag{7.13}$$

This is the main formula we need to estimate the functions $g_k=F_k-F_k^*$. To use it effectively, we need bounds on the functions Π_k, which we prove below.

LEMMA 7.3. *The functions $\widetilde{\Pi}_k$ satisfy the bounds*

$$\begin{gathered}\sup_{t\in[0,T]}\left\{\sum_{k\in\mathbb{Z}^*}\int_{\mathbb{R}^2}\left(1+\left|\frac{\xi}{k}\right|\right)^2(|k|^2+|\eta-kt|^2)^2A_k^2(t,\eta)|\widetilde{\Pi}_k(t,\xi,\eta)|^2\,d\xi\,d\eta\right\}\lesssim_\delta\varepsilon_1^3,\\ \int_0^T\sum_{k\in\mathbb{Z}^*}\int_{\mathbb{R}^2}\left(1+\left|\frac{\xi}{k}\right|\right)^2(|k|^2+|\eta-ks|^2)^2|\dot A_kA_k(s,\eta)|\,|\widetilde{\Pi}_k(s,\xi,\eta)|^2\,d\xi\,d\eta\,ds\lesssim_\delta\varepsilon_1^3.\end{gathered} \tag{7.14}$$

Proof. We would like to use the bounds (7.5) and (7.8), but we need to be careful because our weights have to satisfy condition (7.6). We first use the weights

$$W_k(\eta):=A_k(t,\eta)(|k|^2+\delta^2|\eta-kt|^2). \tag{7.15}$$

We verify now the estimates (7.6). If $\langle k,\xi\rangle+\langle k,\eta\rangle\leqslant 8\langle\xi-\eta\rangle$ then, in view of (3.21),

$$A_k(t,\xi)\leqslant 2A_k(t,\eta)e^{\lambda(t)\langle\xi-\eta\rangle^{1/2}}e^{2\sqrt{\delta}\langle k,\xi\rangle^{1/2}}, \tag{7.16}$$

which gives (7.6) in the stronger form

$$W_k(\xi)+W_k(\eta)\lesssim_\delta e^{2\delta_0\langle\xi-\eta\rangle^{1/2}}W_k(\eta)\langle k,\eta\rangle^{-1/8}.$$

On the other hand, if $\langle\xi-\eta\rangle\leqslant\frac{1}{8}(\langle k,\xi\rangle+\langle k,\eta\rangle)$, then we write $|W_k(\xi)-W_k(\eta)|\leqslant \mathrm{I}+\mathrm{II}$, where

$$\mathrm{I}:=|A_k(t,\xi)-A_k(t,\eta)|(|k|^2+\delta^2|\eta-kt|^2),$$
$$\mathrm{II}:=\delta^2A_k(t,\xi)\big||\xi-kt|^2-|\eta-kt|^2\big|.$$

The desired estimates (7.6) easily follow using (3.27).

We can therefore use (7.5) to estimate, for any $t\in[0,T]$,

$$\int_{\mathbb{R}^2}(|k|+|\xi|)^2(|k|^2+\delta^2|\eta-kt|^2)^2A_k^2(t,\eta)|\widetilde{\Pi}_k(t,\xi,\eta)|^2\,d\xi\,d\eta \\ \lesssim_\delta \int_{\mathbb{R}}A_k^2(t,\xi)(|k|^2+\delta^2|\xi-kt|^2)^2\,k^2|\widetilde{(B_0''\phi_k^*)}(t,\xi)|^2\,d\xi,$$

and the desired bounds in the first line of (7.14) follow from (7.8), after dividing by k^2 and summing over $k\in\mathbb{Z}^*$.

The bounds in the second line of (7.14) are similar, using (7.5) with the different weights

$$W_k(\eta):=\sqrt{\mu_k(t,\eta)}A_k(t,\eta)(|k|^2+\delta^2|\eta-kt|^2), \tag{7.17}$$

where the functions μ_k are defined in equations (3.36). These weights satisfy the bounds (7.6) as well, using (7.16) and (3.38) if $\langle k,\xi\rangle+\langle k,\eta\rangle\leqslant 8\langle\xi-\eta\rangle$, or (3.27) and (3.39) if $\langle\xi-\eta\rangle\leqslant\frac{1}{8}(\langle k,\xi\rangle+\langle k,\eta\rangle)$. We can therefore use (7.5) to estimate, for any $t\in[0,T]$,

$$\int_{\mathbb{R}^2}(|k|+|\xi|)^2(|k|^2+\delta^2|\eta-kt|^2)^2\mu_k(t,\eta)A_k^2(t,\eta)|\widetilde{\Pi}_k(t,\xi,\eta)|^2\,d\xi\,d\eta \\ \lesssim_\delta \int_{\mathbb{R}}\mu_k(t,\xi)A_k^2(t,\xi)(|k|^2+\delta^2|\xi-kt|^2)^2\,k^2|\widetilde{(B_0''\phi_k^*)}(t,\xi)|^2\,d\xi,$$

and the desired bounds in the second line of (7.14) follow from (7.8) and (3.37), after dividing by k^2, summing over $k\in\mathbb{Z}^*$ and integrating in $t\in[0,T]$. □

We are now ready to bound the functions g_k.

LEMMA 7.4. *For any $t\in[1,T]$, we have*

$$\sum_{k\in\mathbb{Z}^*}\int_{\mathbb{R}}\left(1+\frac{\langle k,\xi\rangle}{\langle t\rangle}\right)A_k^2(t,\xi)|\tilde g_k(t,\xi)|^2\,d\xi\lesssim_\delta\varepsilon_1^3 \tag{7.18}$$

$$\sum_{k\in\mathbb{Z}^*}\int_1^t\int_{\mathbb{R}}\left(1+\frac{\langle k,\xi\rangle}{\langle s\rangle}\right)|\dot A_kA_k(s,\xi)|\,|\tilde g_k(s,\xi)|^2\,d\xi\,ds\lesssim_\delta\varepsilon_1^3. \tag{7.19}$$

Proof. Using the identity (7.13), we have

$$|\tilde g_k(t,\xi)|\leqslant|k|\gamma_{k,1}(t,\xi)+|k|\gamma_{k,2}(t,\xi),$$

where

$$\begin{aligned}\gamma_{k,1}(t,\xi) &:= \int_0^t |(\widetilde{B_0''\phi_k^*})(s,\xi)|\,ds,\\ \gamma_{k,2}(t,\xi) &:= \int_0^t\int_s^t\int_{\mathbb{R}} |\widetilde{B}_0''(\zeta)|\,|\widetilde{\Pi}_k(s,\xi-\zeta-k\tau,\xi-\zeta)|\,d\zeta\,d\tau\,ds.\end{aligned} \tag{7.20}$$

To simplify the notation we define, for any $k\in\mathbb{Z}^*$, $t\geqslant 0$, and $\xi,\eta\in\mathbb{R}$,

$$\begin{aligned}\alpha_k(t,\xi) &:= (k^2+|\xi-kt|^2)|(\widetilde{B_0''\phi_k^*})(t,\xi)|,\\ \beta_k(t,\xi,\eta) &:= \left(1+\left|\frac{\xi}{k}\right|\right)(|k|^2+|\eta-kt|^2)\int_{\mathbb{R}} |\widetilde{B}_0''(\zeta)|\,|\widetilde{\Pi}_k(t,\xi-\zeta,\eta-\zeta)|\,d\zeta.\end{aligned} \tag{7.21}$$

Using (7.8), we have

$$\begin{aligned}&\sup_{t\in[0,T]}\left\{\sum_{k\in\mathbb{Z}^*}\int_{\mathbb{R}} A_k^2(t,\xi)\alpha_k^2(t,\xi)\,d\xi\right\}\\ &\qquad+\sum_{k\in\mathbb{Z}^*}\int_0^T\int_{\mathbb{R}} |\dot{A}_kA_k(s,\xi)|\alpha_k^2(s,\xi)\,d\xi\,ds \lesssim_\delta \varepsilon_1^3.\end{aligned} \tag{7.22}$$

Also, using (7.14), the strong smoothness bounds (4.14) on B_0'', and the bilinear estimates (3.42)–(3.43), we have

$$\begin{aligned}&\sup_{t\in[0,T]}\left\{\sum_{k\in\mathbb{Z}^*}\int_{\mathbb{R}^2} A_k^2(t,\eta)\beta_k^2(t,\xi,\eta)\,d\xi\,d\eta\right\}\\ &\qquad+\int_0^T\sum_{k\in\mathbb{Z}^*}\int_{\mathbb{R}^2} |\dot{A}_kA_k(s,\eta)|\beta_k^2(s,\xi,\eta)\,d\xi\,d\eta\,ds \lesssim_\delta \varepsilon_1^3.\end{aligned} \tag{7.23}$$

Step 1. We first prove the bounds (7.18). Using the definitions (7.20)–(7.21), we estimate, for any $k\in\mathbb{Z}^*$ and $t\in[1,T]$,

$$\begin{aligned}&\left\|\left(1+\frac{\langle k,\xi\rangle}{\langle t\rangle}\right)^{1/2} A_k(t,\xi)|k|\gamma_{k,1}(t,\xi)\right\|_{L^2_\xi}\\ &\quad= \sup_{\|P\|_{L^2_\xi}\leqslant 1}\int_{\mathbb{R}}\int_0^t |P(\xi)|\left(1+\frac{\langle k,\xi\rangle}{\langle t\rangle}\right)^{1/2} A_k(t,\xi)|k|\frac{\alpha_k(s,\xi)}{k^2+|\xi-ks|^2}\,ds\,d\xi\\ &\quad\lesssim \frac{1}{|k|}\big\|\alpha_k(s,\xi)|(A_k\dot{A}_k)(s,\xi)|^{1/2}\big\|_{L^2_{s,\xi}}\\ &\qquad\times\sup_{\|P\|_{L^2_\xi}\leqslant 1}\left\|\frac{(1+\langle k,\xi\rangle/\langle s\rangle)^{1/2}}{1+|\xi/k-s|^2}\frac{P(\xi)A_k(s,\xi)}{|(A_k\dot{A}_k)(s,\xi)|^{1/2}}\right\|_{L^2_{s,\xi}},\end{aligned} \tag{7.24}$$

using also the fact that $A_k(t,\xi)\leqslant A_k(s,\xi)$ if $s\in[0,t]$. Using (3.30), we have

$$\frac{1}{|k|^{1/2}}\bigg\|P(\xi)\frac{(1+\langle k,\xi\rangle/\langle s\rangle)^{1/2}}{1+|\xi/k-s|^2}\frac{A_k(s,\xi)}{|(A_k\dot{A}_k)(s,\xi)|^{1/2}}\bigg\|_{L^2_{s,\xi}}\lesssim_\delta \|P\|_{L^2}. \tag{7.25}$$

Therefore,

$$\bigg\|\Big(1+\frac{\langle k,\xi\rangle}{\langle t\rangle}\Big)^{1/2}A_k(t,\xi)|k|\gamma_{k,1}(t,\xi)\bigg\|^2_{L^2_\xi}\lesssim_\delta \big\|\alpha_k(s,\xi)|(A_k\dot{A}_k)(s,\xi)|^{1/2}\big\|^2_{L^2_{s,\xi}}. \tag{7.26}$$

Similarly, to bound the contribution of $\gamma_{k,2}$, we write

$$\begin{aligned}
&\bigg\|\Big(1+\frac{\langle k,\xi\rangle}{\langle t\rangle}\Big)^{1/2}A_k(t,\xi)|k|\gamma_{k,2}(t,\xi)\bigg\|_{L^2_\xi}\\
&\quad=\sup_{\|P\|_{L^2_\xi}\leqslant 1}\int_{\mathbb{R}}\int_0^t\int_s^t |P(\xi)|\Big(1+\frac{\langle k,\xi\rangle}{\langle t\rangle}\Big)^{1/2}\frac{A_k(t,\xi)}{|k|\langle \xi/k-s\rangle^2}\frac{\beta_k(s,\xi-k\tau,\xi)}{1+|\xi-k\tau|/|k|}\,d\tau\,ds\,d\xi\\
&\quad\lesssim \sup_{\|P\|_{L^2_\xi}\leqslant 1}\frac{1}{k^2}\int_{\mathbb{R}^2}\int_0^t |P(\xi)|\Big(1+\frac{\langle k,\xi\rangle}{\langle s\rangle}\Big)^{1/2}\frac{A_k(s,\xi)}{\langle \xi/k-s\rangle^2}\frac{\beta_k(s,\eta,\xi)}{1+|\eta|/|k|}\,ds\,d\eta\,d\xi\\
&\quad\lesssim \frac{1}{k^2}\big\|\beta_k(s,\eta,\xi)|(A_k\dot{A}_k)(s,\xi)|^{1/2}\big\|_{L^2_{s,\xi,\eta}}\\
&\qquad\qquad\times\sup_{\|P\|_{L^2_\xi}\leqslant 1}\bigg\|\frac{(1+\langle k,\xi\rangle/\langle s\rangle)^{1/2}}{\langle \xi/k-s\rangle^2\langle \eta/k\rangle}\frac{P(\xi)A_k(s,\xi)}{|(A_k\dot{A}_k)(s,\xi)|^{1/2}}\bigg\|_{L^2_{s,\xi,\eta}}.
\end{aligned}\tag{7.27}$$

We can use again (7.25), and note that bounding the L^2 norm in η requires an additional factor $|k|^{1/2}$. It follows that

$$\bigg\|\Big(1+\frac{\langle k,\xi\rangle}{\langle t\rangle}\Big)^{1/2}A_k(t,\xi)|k|\gamma_{k,2}(t,\xi)\bigg\|^2_{L^2_\xi}\lesssim_\delta \big\|\beta_k(s,\eta,\xi)|(A_k\dot{A}_k)(s,\xi)|^{1/2}\big\|^2_{L^2_{s,\xi,\eta}}. \tag{7.28}$$

The bounds (7.18) follow from (7.26)–(7.28) and (7.22)–(7.23), by summation over $k\in\mathbb{Z}^*$.

Step 2. We now prove the bounds (7.19). Using the definitions (7.20)–(7.21), we estimate, for any $k\in\mathbb{Z}^*$ and $t\in[1,T]$,

$$\begin{aligned}
&\bigg\|\Big(1+\frac{\langle k,\xi\rangle}{\langle s\rangle}\Big)^{1/2}|(A_k\dot{A}_k)(s,\xi)|^{1/2}\,|k|\gamma_{k,1}(s,\xi)\bigg\|_{L^2_{s,\xi}}\\
&\quad=\sup_{\|P\|_{L^2_{s,\xi}}\leqslant 1}\int_{\mathbb{R}}\int_1^t\int_0^s |P(s,\xi)|\Big(1+\frac{\langle k,\xi\rangle}{\langle s\rangle}\Big)^{1/2}|(A_k\dot{A}_k)(s,\xi)|^{1/2}\frac{|k|\alpha_k(\tau,\xi)}{k^2+|\xi-k\tau|^2}\,d\tau\,ds\,d\xi\\
&\quad\leqslant \sup_{\|P\|_{L^2_{s,\xi}}\leqslant 1}\int_{\mathbb{R}}\int_0^t\int_\tau^t \{|P(s,\xi)|\,|(A_k\dot{A}_k)(s,\xi)|^{1/2}\}\\
&\qquad\qquad\qquad\qquad\times\Big(1+\frac{\langle k,\xi\rangle}{\langle \tau\rangle}\Big)^{1/2}\frac{\alpha_k(\tau,\xi)}{|k|\langle \xi/k-\tau\rangle^2}\,ds\,d\tau\,d\xi.
\end{aligned}$$

The integral in $s\in[\tau,t]$ can be estimated using the Cauchy inequality and the observation

$$2\int_\tau^t |\dot{A}_k A_k(s,\xi)|\,ds = A_k^2(\tau,\xi)-A_k^2(t,\xi)\leqslant A_k^2(\tau,\xi), \tag{7.29}$$

since the functions A_k are decreasing in s. Therefore, the right-hand side of the expression above is bounded by

$$\sup_{\|P'\|_{L^2_\xi}\leqslant 1}\int_{\mathbb{R}}\int_0^t |P'(\xi)|A_k(\tau,\xi)\left(1+\frac{\langle k,\xi\rangle}{\langle\tau\rangle}\right)^{1/2}\frac{\alpha_k(\tau,\xi)}{|k|\langle\xi/k-\tau\rangle^2}\,d\tau\,d\xi.$$

This is similar to the expression in the second line of (7.24), so it can be estimated in the same way to give

$$\left\|\left(1+\frac{\langle k,\xi\rangle}{\langle s\rangle}\right)^{1/2}|(A_k\dot{A}_k)(s,\xi)|^{1/2}\,|k|\gamma_{k,1}(s,\xi)\right\|^2_{L^2_{s,\xi}}\lesssim_\delta \|\alpha_k(s,\xi)|(A_k\dot{A}_k)(s,\xi)|^{1/2}\|^2_{L^2_{s,\xi}}. \tag{7.30}$$

Similarly, to bound the contribution of $\gamma_{k,2}$ we write

$$\begin{aligned}
&\left\|\left(1+\frac{\langle k,\xi\rangle}{\langle s\rangle}\right)^{1/2}|(A_k\dot{A}_k)(s,\xi)|^{1/2}\,|k|\gamma_{k,2}(s,\xi)\right\|_{L^2_{s,\xi}}\\
&\quad=\sup_{\|P\|_{L^2_{s,\xi}}\leqslant 1}\int_{\mathbb{R}}\int_1^t\int_0^s\int_u^s |P(s,\xi)|\left(1+\frac{\langle k,\xi\rangle}{\langle s\rangle}\right)^{1/2}\\
&\qquad\qquad\times\frac{|(A_k\dot{A}_k)(s,\xi)|^{1/2}}{|k|\langle\xi/k-u\rangle^2}\frac{\beta_k(u,\xi-k\tau,\xi)}{1+|\xi-k\tau|/|k|}\,d\tau\,du\,ds\,d\xi\\
&\quad\leqslant\sup_{\|P\|_{L^2_{s,\xi}}\leqslant 1}\frac{1}{k^2}\int_{\mathbb{R}^2}\int_0^t\int_u^t |P(s,\xi)|\left(1+\frac{\langle k,\xi\rangle}{\langle s\rangle}\right)^{1/2}\\
&\qquad\qquad\times\frac{|(A_k\dot{A}_k)(s,\xi)|^{1/2}}{\langle\xi/k-u\rangle^2}\frac{\beta_k(u,\eta,\xi)}{1+|\eta|/|k|}\,ds\,du\,d\xi\,d\eta.
\end{aligned}$$

We use (7.29) and the Cauchy inequality to estimate first the integral in $s\in[u,t]$, so the right-hand side of the expression above is bounded by

$$\sup_{\|P'\|_{L^2_\xi}\leqslant 1}\frac{1}{k^2}\int_{\mathbb{R}^2}\int_0^t P'(\xi)\left(1+\frac{\langle k,\xi\rangle}{\langle u\rangle}\right)^{1/2}\frac{A_k(u,\xi)}{\langle\xi/k-u\rangle^2}\frac{\beta_k(u,\eta,\xi)}{1+|\eta|/|k|}\,du\,d\xi\,d\eta.$$

This is similar to the expression in the third line of (7.27), therefore

$$\left\|\left(1+\frac{\langle k,\xi\rangle}{\langle s\rangle}\right)^{1/2}|(A_k\dot{A}_k)(s,\xi)|^{1/2}\,|k|\gamma_{k,2}(s,\xi)\right\|^2_{L^2_{s,\xi}}\lesssim_\delta \|\beta_k(s,\eta,\xi)|(A_k\dot{A}_k)(s,\xi)|^{1/2}\|^2_{L^2_{s,\xi,\eta}}. \tag{7.31}$$

The bounds (7.19) follow from inequalities (7.30)–(7.31) and (7.22)–(7.23), by summation over $k\in\mathbb{Z}^*$. □

We can now complete the proof of Proposition 7.1. The bounds for the function $F-F^*$ follow from Lemma 7.4, once we recall that $g_k=F_k-F_k^*$ (compare with the definitions in (2.44)). The bounds for the main variable F then follow using also Proposition 6.2. Finally, to prove the bounds for Θ we start from the main elliptic equation (2.32), and rewrite it in the form

$$\partial_z^2\phi+(B_0')^2(\partial_v-t\partial_z)^2\phi+B_0''(\partial_v-t\partial_z)\phi=F+\mathcal{G}_1+\mathcal{G}_2,$$

where

$$\mathcal{G}_1=[(B_0')^2-(V')^2](\partial_v-t\partial_z)^2\phi \quad\text{and}\quad \mathcal{G}_2=[B_0''-V''](\partial_v-t\partial_z)\phi$$

are as in (6.2). In view of (6.3), we have $\|\mathcal{G}_1\|_{\widetilde{W}[1,T]}+\|\mathcal{G}_2\|_{\widetilde{W}[1,T]}\lesssim_\delta\varepsilon_1^2$, while the bounds $\mathcal{E}_F+\mathcal{B}_F\lesssim_\delta\varepsilon_1^3$ we have just proved show that $\|F\|_{W[1,T]}\lesssim_\delta\varepsilon_1^{3/2}$. The desired bounds $\|\Theta\|_{\widetilde{W}[1,T]}\lesssim_\delta\varepsilon_1^{3/2}$ follow from Lemma 4.5. This completes the proof of Proposition 7.1.

8. Analysis of the linearized operator: proof of Proposition 7.2

In this section, we provide the proof of the key Proposition 7.2, which is the only place where the spectral assumption on the linearized operator L_k is used. As we have seen before, the linear estimates we prove here are essential to link the non-linear profile F^*, which evolves perturbatively, with the full profile F. The proof of Proposition 7.2 relies on the following homogeneous bounds on the linearized flow.

LEMMA 8.1. *Assume $k\in\mathbb{Z}^*$ and $a\in\mathbb{R}$, and consider the initial value problem*

$$\partial_t g_k+ikvg_k-ikB_0''\varphi_k=0,\quad g_k(0,v)=X_k(v)e^{-ikav}, \tag{8.1}$$

for $(v,t)\in[b(0),b(1)]\times[0,\infty)$, where φ_k is determined through the elliptic equation

$$(B_0')^2\partial_v^2\varphi_k+B_0''(v)\partial_v\varphi_k-k^2\varphi_k=g_k,\quad \varphi_k(b(0))=\varphi_k(b(1))=0. \tag{8.2}$$

Assume that $X_k\in L^2[b(0),b(1)]$ and $\operatorname{supp} X_k\subseteq[b(\vartheta_0),b(1-\vartheta_0)]$. Then, there is a unique global solution $g_k\in C^1([0,\infty):L^2[b(0),b(1)])$ of the initial-value problem (8.1) with

$$\operatorname{supp} g_k(t)\subseteq[b(\vartheta_0),b(1-\vartheta_0)]$$

for any $t\geqslant 0$. Moreover, there is a function $\Pi_k'=\Pi_k'(\xi,\eta,a)$ such that

$$\tilde{g}_k(t,\xi)=\widetilde{X}_k(\xi+kt+ka)+ik\int_0^t\int_{\mathbb{R}}\widetilde{B}_0''(\zeta)\widetilde{\Pi}_k'(\xi+kt-\zeta-k\tau,\xi+kt-\zeta,a)\,d\zeta\,d\tau. \tag{8.3}$$

Finally, if the weights W_k satisfy the bounds (7.6), then

$$\big\|(|k|+|\xi|)W_k(\eta+ka)\widetilde{\Pi}_k'(\xi,\eta,a)\big\|_{L^2_{\xi,\eta}}\lesssim_\delta\|W_k(\eta)\widetilde{X}_k(\eta)\|_{L^2_\eta}. \tag{8.4}$$

The equation (8.1) is of the form

$$\partial_t G - iT_k(G) = 0, \quad T_k(G) := -kvG + kB_0''\Phi, \quad G(0) = G_0, \tag{8.5}$$

where Φ is the solution of the elliptic equation

$$(B_0')^2\partial_v^2\Phi + B_0''(v)\partial_v\Phi - k^2\Phi = G$$

with Dirichlet boundary conditions $\Phi(b(0))=\Phi(b(1))=0$. This elliptic equation can be solved explicitly using the change of variables $v=b(y)$ (see (8.18) below), thus T_k is a bounded operator on $L^2[b(0), b(1)]$. Therefore, equation (8.5) can be solved explicitly

$$G(t) = e^{itT_k}G_0 = \sum_{n\geqslant 0} \frac{(itT_k)^n G_0}{n!}, \tag{8.6}$$

and the solution $G(t)$ is unique, by energy estimates. The main point of the lemma is to derive the representation formula (8.3) and the strong bounds (8.4).

We first show that Lemma 8.1 implies Proposition 7.2.

Proof of Proposition 7.2. With f_k and ψ_k as in Proposition 7.2 let

$$g_k(t,v) := f_k(t,v)e^{-ikvt} \quad \text{and} \quad \varphi_k(t,v) := \psi_k(t,v)e^{-ikvt}. \tag{8.7}$$

The functions g_k and φ_k satisfy, for $(t,v)\in[0,T]\times[b(0),b(1)]$,

$$\partial_t g_k + ikvg_k - ikB_0''(v)\varphi_k = X_k(t,v)e^{-iktv}, \tag{8.8}$$

$$(B_0')^2\partial_v^2\varphi_k + B_0''\partial_v\varphi_k - k^2\varphi_k = g_k, \tag{8.9}$$

with initial data $g_k(0,v)=0$. By Duhamel's formula, we obtain the representation formula

$$g_k(t,v) := \int_0^t \{e^{i(t-a)T_k}[X_k(a,\cdot)e^{-ika\cdot}]\}(v)\,da, \tag{8.10}$$

where e^{itT_k} is the evolution operator defined in (8.6). Notice that

$$h_k(t-a,v) := e^{i(t-a)T_k}[X_k(a,\cdot)e^{-ika\cdot}](v)$$

is the solution to (8.1)–(8.2) at time $t-a$ with initial data $X_k(a,v)e^{-ikav}$. In view of formula (8.3), we have

$$\begin{aligned}\tilde{h}_k(t-a,\xi) = {}& \widetilde{X}_k(a,\xi+kt) \\ & + ik\int_0^{t-a}\int_{\mathbb{R}} \widetilde{B}_0''(\zeta)\widetilde{\Pi}_k'(\xi+k(t-a)-\zeta-k\tau, \xi+k(t-a)-\zeta, a)\,d\zeta\,d\tau,\end{aligned}$$

where the functions Π'_k satisfy the bounds

$$\big\|(|k|+|\xi|)W_k(\eta+ka)\widetilde{\Pi}'_k(\xi,\eta,a)\big\|_{L^2_{\xi,\eta}} \lesssim_\delta \|W_k(\xi)\widetilde{X}_k(a,\xi)\|_{L^2_\xi}. \tag{8.11}$$

Therefore,

$$\begin{aligned}\tilde{g}_k(t,\xi) = &\int_0^t \widetilde{X}_k(a,\xi+kt)\,da \\ &+ik\int_0^t\int_0^{t-a}\int_{\mathbb{R}} \widetilde{B}''_0(\zeta)\widetilde{\Pi}'_k(\xi+k(t-a)-\zeta-k\tau,\xi+k(t-a)-\zeta,a)\,d\zeta\,d\tau\,da.\end{aligned} \tag{8.12}$$

Define, for $\xi,\eta\in\mathbb{R}$ and $a\in[0,T]$,

$$\widetilde{\Pi}_k(a,\xi,\eta) := \widetilde{\Pi}'_k(\xi,\eta-ka,a), \tag{8.13}$$

The desired bounds (7.5) follow from (8.11). Using (8.7), (8.12), and (8.13), we also have

$$\begin{aligned}\tilde{f}_k(t,\xi) &= \int_0^t \widetilde{X}_k(a,\xi)\,da + ik\int_0^t\int_0^{t-a}\int_{\mathbb{R}} \widetilde{B}''_0(\zeta)\,\widetilde{\Pi}_k(a,\xi-ka-\zeta-k\tau,\xi-\zeta)\,d\zeta\,d\tau\,da \\ &= \int_0^t \widetilde{X}_k(a,\xi)\,da + ik\int_0^t\int_a^t\int_{\mathbb{R}} \widetilde{B}''_0(\zeta)\,\widetilde{\Pi}_k(a,\xi-\zeta-k\tau,\xi-\zeta)\,d\zeta\,d\tau\,da.\end{aligned} \tag{8.14}$$

The proposition is now proved. □

In the rest of this section, we provide the proof of Lemma 8.1. The main idea is the same as in [22]. However, we need to consider more general initial data with the additional modulation factor e^{-ikav}, in order to analyze the inhomogeneous linear evolution, and we need to prove stronger estimates. We divide the proof into several steps, organized in subsections.

8.1. The representation formula and limiting absorption principle

In this subsection, we recall some important properties of the linear evolution operator from [22]. Throughout this section, we use the change of variables

$$v = b(y) \quad \text{for } y\in[0,1]. \tag{8.15}$$

The change of variable (8.15) is just the non-linear change of variable (2.4) at $t=0$. Define

$$g^*_k(t,y) := g_k(t,v), \quad \varphi^*_k(t,y) := \varphi_k(t,v), \quad X^*_k(y) := X_k(v), \tag{8.16}$$

where $v=b(y)$ for $y\in[0,1]$. Then, g_k^* and φ_k^* satisfy

$$\partial_t g_k^*(t,y)+ikb(y)g_k^*(t,y)-ikb''(y)\varphi_k^*(t,y)=0, \tag{8.17}$$

$$-k^2\varphi_k^*(t,y)+\partial_y^2\varphi_k^*(t,y)=g_k^*(t,y), \quad \varphi_k^*(t,0)=\varphi^*(t,1)=0, \tag{8.18}$$

for $(y,t)\in[0,1]\times[0,\infty)$, with initial data

$$g_k^*(0,y)=X_k^*(y)e^{-ikab(y)}.$$

For each $k\in\mathbb{Z}\setminus\{0\}$, we set, for any $f\in L^2[0,1]$,

$$L_kf(y):=b(y)f(y)+b''(y)\int_0^1 G_k(y,z)f(z)\,dz, \tag{8.19}$$

where G_k is the Green function for the operator $k^2-\partial_y^2$ on $[0,1]$ with zero Dirichlet boundary condition defined in (4.24). Then the system (8.17)–(8.18) can be reformulated as

$$\partial_t g_k^*(t,y)+ikL_kg_k^*(t,y)=0. \tag{8.20}$$

We first record an important representation formula, see [22, Proposition 2.1].

Proposition 8.2. *For $k\in\mathbb{Z}^*$, we have the following representation formula for φ_k^*:*

$$\varphi_k^*(t,y)=-\frac{1}{2\pi i}\lim_{\varepsilon\to0^+}\int_0^1 e^{-ikb(y_0)t}b'(y_0)[\psi_{k,\varepsilon}^-(y,y_0)-\psi_{k,\varepsilon}^+(y,y_0)]\,dy_0, \tag{8.21}$$

where $\psi_{k,\varepsilon}^\iota\colon[0,1]^2\to\mathbb{C}$ are defined, for $\iota\in\{+,-\}$ and $\varepsilon\in\left[-\frac14,\frac14\right]\setminus\{0\}$, by

$$\psi_{k,\varepsilon}^\pm(y,y_0):=\int_0^1 G_k(y,z)[(-b(y_0)+L_k\pm i\varepsilon)^{-1}(X_k(\cdot)e^{-ikab(\cdot)})](z)\,dz, \tag{8.22}$$

where G_k are the Green functions defined in (4.24). Also, the generalized eigenfunctions $\psi_{k,\varepsilon}^\pm$ are solutions of the equation

$$-k^2\psi_{k,\varepsilon}^\iota(y,y_0)+\frac{d^2}{dy^2}\psi_{k,\varepsilon}^\iota(y,y_0)-\frac{b''(y)}{b(y)-b(y_0)+i\iota\varepsilon}\psi_{k,\varepsilon}^\iota(y,y_0)=\frac{-X_k^*(y)e^{-ikab(y)}}{b(y)-b(y_0)+i\iota\varepsilon}. \tag{8.23}$$

Remark 8.3. The existence of the functions $\psi_{k,\varepsilon}^\iota$ for $\varepsilon\in\left[-\frac14,\frac14\right]\setminus\{0\}$ follows from our spectral assumptions. These functions depend on the parameter a as well, but we suppress this dependence for simplicity of notation.

We transfer now the results of Lemma 1.1 to the new variables.

LEMMA 8.4. *For any* $f\in H^1_k(\mathbb{R})$, $\varepsilon\in[-\frac14,\frac14]\setminus\{0\}, k\in\mathbb{Z}^*$, *and* $w\in[b(0),b(1)]$, *let*

$$S_{k,w,\varepsilon}f(v):=\int_{\mathbb{R}}\Psi(v)\mathcal{G}_k(v,v')(\partial_{v'}B_0')(v')\frac{f(v')}{v'-w+i\varepsilon}\,dv', \tag{8.24}$$

where $\mathcal{G}_k(v,v')=G_k(b^{-1}(v),b^{-1}(v'))$ *are the renormalized Green functions defined in the proof of Lemma 4.5. Then, for all* $k\in\mathbb{Z}^*$, $w\in[b(0),b(1)]$, $f\in H^1_k(\mathbb{R})$, *and sufficiently small* $\varepsilon\neq0$,

$$\|S_{k,w,\varepsilon}f\|_{H^1_k(\mathbb{R})}\lesssim|k|^{-1/3}\|f\|_{H^1_k(\mathbb{R})}\quad\text{and}\quad\|f\|_{H^1_k(\mathbb{R})}\lesssim\|f+S_{k,w,\varepsilon}f\|_{H^1_k(\mathbb{R})}. \tag{8.25}$$

Define also

$$S'_{k,w,\varepsilon}f(v):=\int_{\mathbb{R}}\Psi(v+w)\mathcal{G}_k(v+w,v'+w)(\partial_{v'}B_0')(v'+w)\frac{f(v')}{v'+i\varepsilon}\,dv'. \tag{8.26}$$

Then, for all $k\in\mathbb{Z}^*$, $w\in[b(0),b(1)]$, $f\in H^1_k(\mathbb{R})$, *and sufficiently small* $\varepsilon\neq0$,

$$\|S'_{k,w,\varepsilon}f\|_{H^1_k(\mathbb{R})}\lesssim|k|^{-1/3}\,\|f\|_{H^1_k(\mathbb{R})}\quad\text{and}\quad\|f\|_{H^1_k(\mathbb{R})}\lesssim\|f+S'_{k,w,\varepsilon}f\|_{H^1_k(\mathbb{R})}. \tag{8.27}$$

Proof. The bounds (8.25) follow from Lemma 1.1, using the change of variable formula (8.15). The bounds (8.27) follow by a shift of variables $v\mapsto v-w$. □

8.2. Gevrey bounds for generalized eigenfunctions

In this section we study the regularity of the generalized eigenfunctions $\psi^\iota_{k,\varepsilon}(y,y_0)$, with $y,y_0\in[0,1]$ and $\iota\in\{+,-\}$. The starting point is equation (8.23), which can be reformulated as

$$\psi^\iota_{k,\varepsilon}(y,y_0)+\int_0^1G_k(y,z)\frac{b''(z)\psi^\iota_{k,\varepsilon}(z,y_0)}{b(z)-b(y_0)+i\iota\varepsilon}\,dz=\int_0^1G_k(y,z)\frac{X^*_k(z)e^{-ikab(z)}}{b(z)-b(y_0)+i\iota\varepsilon}\,dz. \tag{8.28}$$

Denote, for $y,y_0\in[0,1]$,

$$h(y,y_0):=\int_0^1G_k(y,z)\frac{X^*_k(z)e^{-ikab(z)}}{b(z)-b(y_0)+i\iota\varepsilon}\,dz. \tag{8.29}$$

We can now prove bounds on the low frequencies of the generalized eigenfunctions.

LEMMA 8.5. (i) *We have*

$$\|h(y,y_0)\|_{L^2_{y,y_0}}+|k|^{-1}\|\partial_yh(y,y_0)\|_{L^2_{y,y_0}}\lesssim|k|^{-1}\|X^*_k\|_{L^2_v}. \tag{8.30}$$

(ii) *For* $\iota\in\{+,-\}$, $k\in\mathbb{Z}^*$, *and* $\varepsilon\in[-\frac14,\frac14]\setminus\{0\}$ *sufficiently small, we have*

$$|k|\,\|\Psi(b(y))\psi^\iota_{k,\varepsilon}(y,y_0)\|_{L^2_{y,y_0}}+\|\partial_y(\Psi(b(y))\psi^\iota_{k,\varepsilon})(y,y_0)\|_{L^2_{y,y_0}}\lesssim\|X_k\|_{L^2}. \tag{8.31}$$

Moreover,

$$\lim_{\varepsilon\to0+}[\psi^-_{k,\varepsilon}(y,y_0)-\psi^+_{k,\varepsilon}(y,y_0)]\equiv0\quad\text{for } y_0\in\left[0,\tfrac12\vartheta_0\right]\cup\left[1-\tfrac12\vartheta_0,1\right]. \tag{8.32}$$

Proof. (i) Using L^2 boundedness of the Hilbert transform, we estimate

$$\begin{aligned}\|h(y,y_0)\|_{L^2_{y,y_0}} &= \sup_{\|P\|_{L^2}\leqslant 1}\Big|\int_{[0,1]^3} P(y,y_0)G_k(y,z)\frac{X_k^*(z)e^{-ikab(z)}}{b(z)-b(y_0)+i\iota\varepsilon}\,dz\,dy\,dy_0\Big|\\ &\lesssim \sup_{\|P'\|_{L^2}\leqslant 1}\int_{[0,1]^2}|P'(y,z)G_k(y,z)X_k^*(z)|\,dz\,dy\\ &\lesssim |k|^{-3/2}\|X_k^*\|_{L^2},\end{aligned}$$

where in the last inequality we used the bounds $\|G_k(y,z)\|_{L^2_y}\lesssim|k|^{-3/2}$ for any $z\in[0,1]$ (compare with (4.24)). The estimate on the second term in the left-hand side of (8.30) is similar, since $\|(\partial_y G_k)(y,z)\|_{L^2_y}\lesssim|k|^{-1/2}$.

(ii) The bounds (8.31) follow from (1.11) and equations (8.28)–(8.29) (the functions $\Psi(b(\cdot))\psi^{\pm}_{k,\varepsilon}(\cdot,y_0)$ are in $H^1_k(\mathbb{R})$, due to (8.22)). The identities (8.32) were proved in [22, Lemma 4.1]. □

We now turn to the main case when $y_0\in\big[\frac{1}{2}\vartheta_0,1-\frac{1}{2}\vartheta_0\big]$. Recall the change of variables (8.15), and set, for $k\in\mathbb{Z}\setminus\{0\}$, $\iota\in\{+,-\}$, and sufficiently small $\varepsilon\neq 0$,

$$\phi^{\iota}_{k,\varepsilon}(v,w):=\psi^{\iota}_{k,\varepsilon}(y,y_0),\quad \text{with } v=b(y) \text{ and } w=b(y_0). \tag{8.33}$$

The following lemma contains the main estimates for the generalized eigenfunctions.

LEMMA 8.6. *Define, for* $\iota\in\{+,-\}$, $k\in\mathbb{Z}\setminus\{0\}$, *and sufficiently small* $\varepsilon>0$,

$$\Pi^{\iota}_{k,\varepsilon}(v,w):=\Psi(v+w)\phi^{\iota}_{k,\varepsilon}(v+w,w)\Psi(w)\quad \textit{for } v,w\in\mathbb{R}. \tag{8.34}$$

If W_k *are weights satisfying* (7.6) *then, for* $\delta,\varepsilon>0$ *sufficiently small,*

$$\big\|(|k|+|\xi|)W_k(\eta+ka)\widetilde{\Pi}^{\iota}_{k,\varepsilon}(\xi,\eta)\big\|_{L^2_{\xi,\eta}}\lesssim_\delta \|W_k(\eta)\,\widetilde{X}_k(\eta)\|_{L^2_\eta}. \tag{8.35}$$

Proof. Using (8.31) and the definitions (8.34), we have the bounds

$$\big\|(|k|+|\partial_v|)\Pi^{\iota}_{k,\varepsilon}\big\|_{L^2_{v,w}}\lesssim\|X_k\|_{L^2}, \tag{8.36}$$

which are useful to control the low-frequency components of $\Pi^{\iota}_{k,\varepsilon}$.

For the commutator argument below, to begin with, we need the qualitative bounds

$$\big\|(|k|+|\partial_v|)W_k^a\Pi^{\iota}_{k,\varepsilon}\big\|_{L^2_{v,w}}<\infty. \tag{8.37}$$

We can arrange this by working first with the weights

$$W_{k,\rho}(\xi)=\frac{W_k(\xi)}{1+\rho W_k(\xi)},\quad \rho>0,$$

which still satisfy the main bounds (7.6) uniformly in ρ. The qualitative bounds (8.37) are satisfied for these weights, due to (8.36). We prove the bounds (8.35) for the weights $W_{k,\rho}$ uniformly in ρ, and then let $\rho\to 0$. We will therefore assume (8.37) in the rest of the proof.

We divide the rest of the proof into several steps.

Step 1. We first derive the main equations for $\Pi^{\iota}_{k,\varepsilon}(v,w)$. Using the definitions (8.33), we can reformulate equation (8.28) as

$$\begin{aligned}\Psi(v)\phi^{\iota}_{k,\varepsilon}(v,w)+\int_{\mathbb{R}}\Psi(v)\mathcal{G}_k(v,v')(\partial_{v'}B_0')(v')\frac{\Psi(v')\phi^{\iota}_{k,\varepsilon}(v',w)}{v'-w+i\iota\varepsilon}\,dv'\\ =\int_{\mathbb{R}}\Psi(v)\mathcal{G}_k(v,v')\frac{1}{B_0'(v')}\frac{X_k(v')e^{-ikav'}}{v'-w+i\iota\varepsilon}\,dv',\end{aligned}\tag{8.38}$$

for $v\in\mathbb{R}$ and $w\in[b(0),b(1)]$, since $\Psi\equiv 1$ on the support of $\partial_v B_0'$. Recall also that $\Psi\equiv 1$ on the support of X_k, and let

$$\mathcal{G}_k'(v,w):=\Psi(v)\mathcal{G}_k(v,w)\Psi(w).$$

It follows that the function $\Pi^{\iota}_{k,\varepsilon}(v,w)$ satisfies the more regular (in w) equation

$$\begin{aligned}\Pi^{\iota}_{k,\varepsilon}(v,w)+\int_{\mathbb{R}}\mathcal{G}_k'(v+w,v'+w)(\partial_{v'}B_0')(v'+w)\frac{\Pi^{\iota}_{k,\varepsilon}(v',w)}{v'+i\iota\varepsilon}\,dv'\\ =\int_{\mathbb{R}}\mathcal{G}_k'(v+w,v'+w)\frac{\Psi(w)}{B_0'(v'+w)}\frac{X_k(v'+w)e^{-ika(v'+w)}}{v'+i\iota\varepsilon}\,dv'.\end{aligned}\tag{8.39}$$

Step 2. We now study the regularity of the functions $\Pi^{\iota}_{k,\varepsilon}$ using equation (8.39). Define the operator W_k^a by the Fourier multiplier

$$(\widetilde{W_k^a h})(\eta):=W_k(\eta+ka)\tilde{h}(\eta)\quad\text{for any } h\in L^2(\mathbb{R}).\tag{8.40}$$

The basic idea is to use the limiting absorption principle in Lemma 8.4 to bound $\Pi^{\iota}_{k,\varepsilon}$. We note that $\Pi^{\iota}_{k,\varepsilon}$ is very smooth in w, but not so smooth in v, due to the presence of the singular factor $1/(v'+i\iota\varepsilon)$. In order to prove Gevrey regularity of $\Pi^{\iota}_{k,\varepsilon}$ in w, we apply the operator W_k^a, which acts on the variable w, to equation (8.39) and obtain

$$\begin{aligned}W_k^a\Pi^{\iota}_{k,\varepsilon}(v,w)+\int_{\mathbb{R}}\mathcal{G}_k'(v+w,v'+w)(\partial_{v'}B_0')(v'+w)\frac{W_k^a\Pi^{\iota}_{k,\varepsilon}(v',w)}{v'+i\iota\varepsilon}\,dv'\\ =W_k^a\bigg[\int_{\mathbb{R}}\mathcal{G}_k'(v+\cdot\,,v'+\cdot\,)\frac{\Psi(\cdot)}{B_0'(v'+\cdot)}\frac{X_k(v'+\cdot)e^{-ika(v'+\cdot)}}{v'+i\iota\varepsilon}\,dv'\bigg](w)+\mathcal{C}^{\iota}_{k,\varepsilon}(v,w)\\ =:F^{\iota}_{k,\varepsilon}(v,w)+\mathcal{C}^{\iota}_{k,\varepsilon}(v,w),\end{aligned}\tag{8.41}$$

for $v,w\in\mathbb{R}$, where the commutator term $\mathcal{C}^{\iota}_{k,\varepsilon}(v,w)$ is defined as

$$\begin{aligned}\mathcal{C}^{\iota}_{k,\varepsilon}(v,w):=&\int_{\mathbb{R}}\mathcal{G}'_k(v+w,v'+w)(\partial_{v'}B'_0)(v'+w)\frac{W^a_k\Pi^{\iota}_{k,\varepsilon}(v',w)}{v'+i\iota\varepsilon}\,dv'\\&-W^a_k\bigg[\int_{\mathbb{R}}\mathcal{G}'_k(v+\cdot,v'+\cdot)(\partial_{v'}B'_0)(v'+\cdot)\frac{\Pi^{\iota}_{k,\varepsilon}(v',\cdot)}{v'+i\iota\varepsilon}\,dv'\bigg](w).\end{aligned}\tag{8.42}$$

We now fix a cutoff function Ψ_0 supported in $\big[b\big(\tfrac18\vartheta_0\big),b\big(1-\tfrac18\vartheta_0\big)\big]$, equal to 1 in $\big[b\big(\tfrac14\vartheta_0\big),b\big(1-\tfrac14\vartheta_0\big)\big]$, and satisfying $\|e^{\langle\xi\rangle^{3/4}}\tilde{\Psi}_0(\xi)\|_{L^\infty}\lesssim 1$. Applying (8.27) for each w, and taking L^2 in w, we obtain from (8.41)

$$\big\|\Psi_0(w)(|k|+|\partial_v|)W^a_k\Pi^{\iota}_{k,\varepsilon}\big\|_{L^2_{v,w}}\lesssim\big\|(|k|+|\partial_v|)F^{\iota}_{k,\varepsilon}\big\|_{L^2_{v,w}}+\big\|(|k|+|\partial_v|)\mathcal{C}^{\iota}_{k,\varepsilon}\big\|_{L^2_{v,w}}.\tag{8.43}$$

Step 3. We now bound the terms in the right-hand side of (8.43). We first show that

$$\big\|(|k|+|\partial_v|)F^{\iota}_{k,\varepsilon}\big\|_{L^2_{v,w}}\lesssim_\delta\|W_k(\eta)\widetilde{X}_k(\eta)\|_{L^2_\eta}.\tag{8.44}$$

Using (8.41) and taking Fourier transform in v and w, we obtain that

$$\begin{aligned}\widetilde{F}^{\iota}_{k,\varepsilon}(\xi,\eta)=&\,CW_k(\eta+ka)\int_{\mathbb{R}^4}\widetilde{\mathcal{G}}'_k(\xi,\zeta)e^{-iw\eta+i\xi w+i\zeta(v'+w)}\Psi(w)\\&\times\frac{X_k(v'+w)e^{-ika(v'+w)}}{B'_0(v'+w)}e^{i(v'+i\iota\varepsilon)\gamma}\mathbf{1}_+(\iota\gamma)\,dv'\,dw\,d\zeta\,d\gamma\\=&\,CW_k(\eta+ka)\int_{\mathbb{R}^2}\widetilde{\mathcal{G}}'_k(\xi,\zeta)\tilde{h}^a_k(-\zeta-\gamma,\eta-\xi-\zeta)e^{-\varepsilon\iota\gamma}\mathbf{1}_+(\iota\gamma)\,d\zeta\,d\gamma,\end{aligned}\tag{8.45}$$

for some constant C, where $\mathbf{1}_+$ denotes the characteristic function of the interval $[0,\infty)$ and

$$h^a_k(v,w):=\frac{\Psi(w)X_k(v+w)e^{-ika(v+w)}}{B'_0(v+w)}\quad\text{for } v,w\in\mathbb{R}.\tag{8.46}$$

We have also used the fact that the Fourier transform of $(v'+i\iota\varepsilon)^{-1}$ in v' is $e^{-\iota\gamma\varepsilon}\mathbf{1}_+(\iota\gamma)$ (up to a constant), with γ being the Fourier variable.

Since $\Psi\equiv 1$ on the support of X_k, we can write

$$h^a_k(v,w)=\Upsilon(v,w)X_k(v+w)e^{-ika(v+w)},\quad\text{where }\Upsilon(v,w):=\frac{\Psi(w)\Psi(v+w)}{B'_0(v+w)}.\tag{8.47}$$

Using general properties of Gevrey spaces (Lemmas 3.1 and 3.2), and the regularity of b, see (1.6)–(1.7), we obtain that $|\widetilde{\Upsilon}(\xi,\eta)|\lesssim e^{-4\delta_0\langle\xi,\eta\rangle^{1/2}}$ for any $\xi,\eta\in\mathbb{R}$. Therefore,

$$|\tilde{h}^a_k(\xi,\eta)|\lesssim\int_{\mathbb{R}}e^{-4\delta_0\langle\xi-\alpha,\eta-\alpha\rangle^{1/2}}|\widetilde{X}_k(\alpha+ka)|\,d\alpha.\tag{8.48}$$

As in (4.32), in view of [22, Lemma A3], we have

$$|\widetilde{\mathcal{G}}'_k(\xi,\zeta)| \lesssim e^{-4\delta_0\langle\xi+\zeta\rangle^{1/2}}(k^2+|\xi|^2)^{-1}.$$

Using now (8.45), it follows that

$$\begin{aligned}
&(k^2+\xi^2)|\widetilde{F}^\iota_{k,\varepsilon}(\xi,\eta)| \\
&\quad\lesssim \int_{\mathbb{R}^3} W_k(\eta+ka)e^{-4\delta_0\langle\xi+\zeta\rangle^{1/2}}e^{-4\delta_0\langle-\zeta-\gamma-\alpha,\eta-\xi-\zeta-\alpha\rangle^{1/2}}|\widetilde{X}_k(\alpha+ka)|\,d\zeta\,d\gamma\,d\alpha \\
&\quad\lesssim \int_{\mathbb{R}} W_k(\eta+ka)e^{-3\delta_0\langle\eta-\alpha\rangle^{1/2}}|\widetilde{X}_k(\alpha+ka)|\,d\alpha.
\end{aligned}$$

The desired bounds (8.44) then follow, since

$$W_k(\eta+ka) \lesssim_\delta W_k(\alpha+ka)e^{2\delta_0\langle\eta-\alpha\rangle^{1/2}}.$$

Step 4. We now show that the term $\mathcal{C}^\iota_{k,\varepsilon}$ satisfies the bounds

$$\big\|(|k|+|\partial_v|)\mathcal{C}^\iota_{k,\varepsilon}\big\|_{L^2_{v,w}} \lesssim \delta^{1/2}\big\|(|k|+|\partial_v|)W^a_k\Pi^\iota_{k,\varepsilon}\big\|_{L^2_{v,w}}+C_\delta\big\|(|k|+|\partial_v|)\Pi^\iota_{k,\varepsilon}\big\|_{L^2_{v,w}}. \tag{8.49}$$

Indeed, using the definition (8.42) and expanding as in (8.45), we have

$$\begin{aligned}
\widetilde{\mathcal{C}}^\iota_{k,\varepsilon}(\xi,\eta) = C\int_{\mathbb{R}^3} &\widetilde{\mathcal{G}}'_k(\xi,\zeta)(\widetilde{\partial_v B'_0})(\alpha)[W^a_k(\eta-\xi-\zeta-\alpha)-W^a_k(\eta)] \\
&\times\widetilde{\Pi}^\iota_{k,\varepsilon}(-\zeta-\gamma-\alpha,\eta-\xi-\zeta-\alpha)\mathbf{1}_+(\iota\gamma)e^{-\varepsilon\iota\gamma}\,d\alpha\,d\zeta\,d\gamma.
\end{aligned}$$

Since

$$|\widetilde{\mathcal{G}}'_k(\xi,\zeta)| \lesssim e^{-4\delta_0\langle\xi+\zeta\rangle^{1/2}}(k^2+|\xi|^2)^{-1},$$

and using also (7.6), we can estimate

$$\begin{aligned}
(k^2+\xi^2)|\widetilde{\mathcal{C}}^\iota_{k,\varepsilon}(\xi,\eta)| &\lesssim \int_{\mathbb{R}^3} e^{-2\delta_0\langle\xi+\zeta\rangle^{1/2}}e^{2\delta_0\langle\alpha\rangle^{1/2}}|(\widetilde{\partial_v B'_0})(\alpha)|\left[\sqrt{\delta}+\frac{C_\delta}{\langle ka+\eta-\xi-\zeta-\alpha\rangle^{1/8}}\right] \\
&\qquad\times W_k(ka+\eta-\xi-\zeta-\alpha)|\widetilde{\Pi}^\iota_{k,\varepsilon}(-\zeta-\gamma-\alpha,\eta-\xi-\zeta-\alpha)|\,d\alpha\,d\zeta\,d\gamma \\
&\lesssim \int_{\mathbb{R}^3} e^{-2\delta_0\langle\alpha,\zeta\rangle^{1/2}}\left[\sqrt{\delta}+\frac{C_\delta}{\langle ka+\eta-\zeta-\alpha\rangle^{1/8}}\right] \\
&\qquad\times W_k(ka+\eta-\zeta-\alpha)|\widetilde{\Pi}^\iota_{k,\varepsilon}(\gamma,\eta-\zeta-\alpha)|\,d\alpha\,d\zeta\,d\gamma,
\end{aligned}$$

from which (8.49) follows.

Step 5. We now show that

$$\begin{aligned}
\big\|(|k|+|\partial_v|)W^a_k\Pi^\iota_{k,\varepsilon}\big\|_{L^2_{v,w}} \lesssim &\big\|\Psi_0(w)(|k|+|\partial_v|)W^a_k\Pi^\iota_{k,\varepsilon}\big\|_{L^2_{v,w}} \\
&+C_\delta\big\|(|k|+|\partial_v|)\Pi^\iota_{k,\varepsilon}\big\|_{L^2_{v,w}}.
\end{aligned} \tag{8.50}$$

Indeed, for $v,w\in\mathbb{R}$, let

$$\begin{aligned}H(v,w)&:=(|k|+|\partial_v|)W_k^a\Pi_{k,\varepsilon}^{\iota}(v,w)-(|k|+|\partial_v|)\Psi_0(w)W_k^a\Pi_{k,\varepsilon}^{\iota}(v,w)\\&=(|k|+|\partial_v|)W_k^a\,(\Psi_0\Pi_{k,\varepsilon}^{\iota})(v,w)-(|k|+|\partial_v|)\Psi_0(w)W_k^a\Pi_{k,\varepsilon}^{\iota}(v,w).\end{aligned}\tag{8.51}$$

For simplicity of notation, we suppressed the dependence of H on ι, ε, and k in the above definition. By the support property of Ψ_0 and the bounds (7.6), we have

$$\begin{aligned}|\widetilde{H}(\xi,\eta)|&=(|k|+|\xi|)\left|\int_{\mathbb{R}}\widetilde{\Psi}_0(\zeta)\widetilde{\Pi}_{k,\varepsilon}^{\iota}(\xi,\eta-\zeta)[W_k(\eta+ka)-W_k(\eta+ka-\zeta)]\,d\zeta\right|\\&\lesssim\int_{\mathbb{R}}e^{-2\delta_0\langle\zeta\rangle^{1/2}}\,[\sqrt{\delta}+C_\delta\langle ka+\eta-\zeta\rangle^{-1/8}](|k|+|\xi|)W_k(ka+\eta-\zeta)|\widetilde{\Pi}_{k,\varepsilon}^{\iota}(\xi,\eta-\zeta)|\,d\zeta.\end{aligned}$$

Therefore,

$$\begin{aligned}\|\widetilde{H}(\xi,\eta)\|_{L^2_{\xi,\eta}}\lesssim\sqrt{\delta}\,\big\|(|k|+|\partial_v|)W_k^a\Pi_{k,\varepsilon}^{\iota}(v,w)\big\|_{L^2_{v,w}}&\\+C_\delta\,\big\|(|k|+|\partial_v|)\Pi_{k,\varepsilon}^{\iota}(v,w)\big\|_{L^2_{v,w}}&.\end{aligned}\tag{8.52}$$

It follows from (8.51) that

$$\big\|(|k|+|\partial_v|)W_k^a\Pi_{k,\varepsilon}^{\iota}\big\|_{L^2_{v,w}}\lesssim\|\widetilde{H}(\xi,\eta)\|_{L^2_{\xi,\eta}}+\big\|\Psi_0(w)(|k|+|\partial_v|)W_k^a\Pi_{k,\varepsilon}^{\iota}\big\|_{L^2_{v,w}}.\tag{8.53}$$

The bounds (8.50) then follow from (8.52)–(8.53), provided that $\delta>0$ is sufficiently small.

Step 6. We now complete the proof of (8.35). Using the bounds (8.50), (8.43), (8.44), and (8.49), we have

$$\begin{aligned}&\big\|(|k|+|\partial_v|)W_k^a\,\Pi_{k,\varepsilon}^{\iota}\big\|_{L^2_{v,w}}\\&\quad\lesssim\big\|\Psi_0(w)(|k|+|\partial_v|)W_k^a\Pi_{k,\varepsilon}^{\iota}\big\|_{L^2_{v,w}}+C_\delta\big\|(|k|+|\partial_v|)\Pi_{k,\varepsilon}^{\iota}\big\|_{L^2_{v,w}}\\&\quad\lesssim\big\|(|k|+|\partial_v|)F_{k,\varepsilon}^{\iota}\big\|_{L^2_{v,w}}+\big\|(|k|+|\partial_v|)\mathcal{C}_{k,\varepsilon}^{\iota}\big\|_{L^2_{v,w}}+C_\delta\big\|(|k|+|\partial_v|)\Pi_{k,\varepsilon}^{\iota}\big\|_{L^2_{v,w}}\\&\quad\lesssim C_\delta\|W_k(\eta)\widetilde{X}_k(\eta)\|_{L^2_\eta}+\delta^{1/2}\big\|(|k|+|\partial_v|)W_k^a\Pi_{k,\varepsilon}^{\iota}\big\|_{L^2_{v,w}}\\&\qquad\quad+C_\delta\big\|(|k|+|\partial_v|)\Pi_{k,\varepsilon}^{\iota}\big\|_{L^2_{v,w}}.\end{aligned}\tag{8.54}$$

We can absorb the term

$$\delta^{1/2}\big\|(|k|+|\partial_v|)W_k^a\Pi_{k,\varepsilon}^{\iota}\big\|_{L^2_{v,w}}$$

into the left-hand side, and use (8.36) to conclude the proof of the lemma. □

8.3. Proof of Lemma 8.1

We can now complete the proof of Lemma 8.1. We define

$$\Pi'_k(v,w,a) := \frac{i}{4\pi^2} \lim_{\varepsilon_n\to 0^+} [\Pi^-_{k,\varepsilon_n}(v,w) - \Pi^+_{k,\varepsilon_n}(v,w)], \tag{8.55}$$

as a weak limit along along a subsequence (in fact, the limit above exists in the strong sense, see [23, Lemma 4.3], but this is not needed here). The desired bounds (8.4) follow from (8.35). To prove the representation formula (8.3), we start from the identities (8.21). We make the change of variables $v=b(y)$ and $w=b(y_0)$, and use (8.32) and (8.34) to get

$$\begin{aligned}\Psi(v)\varphi_k(t,v) &= -\frac{1}{2\pi i} \lim_{\varepsilon_n\to 0^+} \int_{\mathbb{R}} e^{-ikwt}\Psi(v)[\phi^-_{k,\varepsilon_n}(v,w) - \phi^+_{k,\varepsilon_n}(v,w)]\Psi(w)\,dw \\ &= \frac{i}{2\pi}\int_{\mathbb{R}} e^{-ikwt}(-4i\pi^2)\Pi'_k(v-w,w,a)\,dw.\end{aligned} \tag{8.56}$$

Hence,

$$(\widetilde{\Psi\varphi_k})(t,\xi) = 2\pi\widetilde{\Pi}'_k(\xi,\xi+kt,a). \tag{8.57}$$

In view of equation (8.1) and definition (8.55), we obtain that

$$\partial_t[e^{ikvt}g_k(t,v)] = ikB''_0(v)\varphi_k(t,v)e^{ikvt} \quad \text{for } v\in[b(0),b(1)]. \tag{8.58}$$

We notice that $\Psi\equiv 1$ on the support of B''_0. Therefore,

$$e^{ikvt}g_k(t,v) - g_k(0,v) = ik\int_0^t B''_0(v)\Psi(v)\varphi_k(\tau,v)e^{ikv\tau}\,d\tau. \tag{8.59}$$

Using (8.57), we obtain

$$\tilde{g}_k(t,\xi-kt) - \widetilde{X}_k(\xi+ak) = ik\int_0^t\int_{\mathbb{R}} \widetilde{B}''_0(\zeta)\widetilde{\Pi}'_k(\xi-\zeta-k\tau,\xi-\zeta,a)\,d\zeta\,d\tau, \tag{8.60}$$

which gives (8.3). This completes the proof of Lemma 8.1.

9. Proof of the main theorem

In this section, we complete the proof of Theorem 1.2. We start with a local regularity lemma (see [20, Lemma 3.1] for a simple proof adapted to our situation, or more general results on the Gevrey regularity of Euler flows in [18], [27], and [28]).

LEMMA 9.1. *Assume that* $s\in\left[\frac{1}{4},\frac{3}{4}\right]$, $\lambda_0\in(0,1)$, *and* $\operatorname{supp}\omega_0\subseteq\mathbb{T}\times[2\vartheta_0,1-2\vartheta_0]$. *Assume also that*

$$A:=\|\langle\nabla\rangle^3\,\omega_0\|_{\mathcal{G}^{\lambda_0,s}}<\infty \quad \textit{and} \quad \int_{\mathbb{T}\times[0,1]}\omega_0(x,y)\,dx\,dy=0. \tag{9.1}$$

Let $\omega\in C([0,\infty):H^{10})$ *denote the unique smooth solution of the system* (2.1). *Assume that, for some* $T>0$ *and all* $t\in[0,T]$,

$$\operatorname{supp}\omega(t)\subseteq\mathbb{T}\times[\vartheta_0,1-\vartheta_0]. \tag{9.2}$$

Then, for any smooth cutoff function $\Upsilon\in\mathcal{G}^{1,3/4}$ *with* $\operatorname{supp}\Upsilon\subseteq\left[\frac{1}{20}\vartheta_0,1-\frac{1}{20}\vartheta_0\right]$, *and any* $t\in[0,T]$, *we have*

$$\begin{aligned}\|\langle\nabla\rangle^5\,(\Upsilon\psi)(t)\|_{\mathcal{G}^{\lambda(t),s}}&+\|\langle\nabla\rangle^3\,\omega(t)\|_{\mathcal{G}^{\lambda(t),s}}\\ &\leqslant\exp\left[C_*\int_0^t(\|\omega(s)\|_{H^6}+1)ds\right]\|\langle\nabla\rangle^3\omega_0\|_{\mathcal{G}^{\lambda_0,s}},\end{aligned} \tag{9.3}$$

if we choose, for some large constant $C_*=C_*(\vartheta_0)>1$,

$$\lambda(t):=\lambda_0\exp\left\{-C_*At\exp\left[C_*\int_0^t(\|\omega(s)\|_{H^6}+1)ds\right]-C_*t\right\}. \tag{9.4}$$

We note that an important aspect of the regularity theory for Euler equations in Gevrey spaces is the shrinking in time, at a fast rate, of the radius of convergence (the function $\lambda(t)$ in Lemma 9.1). In our case, the support assumption (9.2) on $\omega(t)$ is satisfied if $T=2$, as a consequence of the smallness and the support assumptions on ω_0, and the standard local well-posedness theory in Sobolev spaces of the Euler equation (2.1). In fact, as we show below, it is satisfied as part of the bootstrap argument for all $t\in[0,\infty)$.

Lemma 9.1 is used in our problem in two ways. First, the local Gevrey regularity estimates (9.3) applied for $T=2$ allow us to assume that our solutions satisfy (2.48), thus we can avoid dealing with the apparent singularities at $t=0$ in some of our definitions. Second, the Gevrey regularity (9.3) ensures the continuity in time of various variables, which is required for the bootstrap argument.

9.1. Proof of Theorem 1.2

For the purpose of proving continuity in time of the energy functionals $\mathcal{E}_g$ and $\mathcal{B}_g$, we make the a-priori assumption that $\omega_0\in\mathcal{G}^{1,2/3}$. The argument is similar to the argument in [20, §3], and we will be somewhat brief. We divide the proof in several steps.

Step 1. Given small data ω_0 satisfying (1.12), we first apply Lemma 9.1. Therefore, $\omega\in C\big([0,2]:\mathcal{G}^{\lambda_1,2/3}\big)$, with $\lambda_1>0$, satisfies the quantitative estimates

$$\sup_{t\in[0,2]}\|e^{\beta_0'\langle k,\xi\rangle^{1/2}}\widetilde{\omega}(t,k,\xi)\|_{L^2_{k,\xi}}\lesssim\varepsilon, \tag{9.5}$$

for some $\beta_0'=\beta_0'(\beta_0,\vartheta_0)>0$. In addition, letting $\Psi'\in\mathcal{G}^{1,3/4}$ denote a cutoff function supported in $\big[\frac{1}{8}\vartheta_0,1-\frac{1}{8}\vartheta_0\big]$ and equal to 1 in $\big[\frac{1}{4}\vartheta_0,1-\frac{1}{4}\vartheta_0\big]$, the localized stream function $\Psi'\psi$ satisfies similar bounds:

$$\sup_{t\in[0,2]}\|\langle k,\xi\rangle^2e^{\beta_0'\langle k,\xi\rangle^{1/2}}(\widetilde{\Psi'\psi})(t,k,\xi)\|_{L^2_{k,\xi}}\lesssim\varepsilon. \tag{9.6}$$

Recalling definition (3.1), and using formula (2.14) and Lemmas 3.1–3.2, it follows that there is a constant $K_1=K_1(\beta_0,\vartheta_0)$ such that

$$\|v(t,\cdot)\|_{\widetilde{G}^{1/2}_{K_1}[0,1]}\lesssim1\quad\text{and}\quad\|\mathcal{Y}(t,\cdot)\|_{\widetilde{G}^{1/2}_{K_1}[b(0),b(1)]}\lesssim1 \tag{9.7}$$

for any $t\in[0,2]$, where $\mathcal{Y}(t,v)$ denotes the inverse of the function $y\mapsto v(t,y)$.

We would like to show now that

$$\sum_{g\in\{F,F^*,F-F^*,\Theta,\Theta^*,B_*',B_*'',V_*',\mathcal{H}\}}\sup_{t\in[0,2]}\|e^{2\delta_0\langle k,\xi\rangle^{1/2}}\tilde{g}(t,k,\xi)\|_{L^2_{k,\xi}}\lesssim\varepsilon, \tag{9.8}$$

for some constant $\delta_0=\delta_0(\beta_0,\vartheta_0)>0$ sufficiently small. Indeed, this follows using again Lemma 3.2 and Lemma 3.1 (i) if $g\in\{F,\Theta,B_*',B_*'',V_*',\mathcal{H}\}$. To bound F^*, $F-F^*$, and Θ^*, we use Green's functions. Indeed, it follows from (2.39) and identity (4.30) that

$$\Psi(v)\phi_k'(t,v)=-\int_{\mathbb{R}}\mathcal{G}_k'(v,v')\frac{F_k(t,v')}{B_0'(v')}e^{ikt(v-v')}\,dv', \tag{9.9}$$

where $\mathcal{G}_k'(v,w)=\Psi(v)\mathcal{G}_k(v,w)\Psi(w)$ as before. Using (4.31)–(4.32), we obtain

$$|(\widetilde{\Psi\phi_k'})(t,\xi)|=C\left|\int_{\mathbb{R}}K(\xi-kt,kt-\zeta)\widetilde{F}_k(t,\zeta)\,d\zeta\right|\lesssim\int_{\mathbb{R}}\frac{e^{-4\delta_0\langle\xi-\zeta\rangle^{1/2}}}{k^2+(\xi-kt)^2}|\widetilde{F}_k(t,\zeta)|\,d\zeta. \tag{9.10}$$

for any $t\in[0,2]$. This gives (9.8) for $g=\Theta^*$, and then for $g=F^*$, using (2.40). This completes the proof of the desired bounds (9.8). In particular, the bounds (2.48) follow from (9.8) if δ_0 is $\varepsilon_1\approx\varepsilon^{2/3}$; see (2.34)–(2.37).

Step 2. Assume now that the solution ω satisfies the bounds in the hypothesis of Proposition 2.2 on a given interval $[0,T]$, $T\geqslant1$. We would like to show that the support of $\omega(t)$ is contained in $\mathbb{T}\times\big[\frac{3}{2}\vartheta_0,1-\frac{3}{2}\vartheta_0\big]$ for any $t\in[0,T]$. Indeed, for this, we notice that only transportation in the y direction, given by the term $u^y\,\partial_y\omega$, could enlarge the support of ω in y outside $[b(2\vartheta_0),b(1-2\vartheta_0)]$. Notice that, on $\mathbb{T}\times[\vartheta_0,1-\vartheta_0]$,

$$u^y(t,x,y)=(\partial_x\psi)(t,x,y)=\partial_z P_{\neq 0}(\Psi\phi)(t,x-tv(t,y),v(t,y)). \tag{9.11}$$

Using the bound on $\mathcal{E}_\Theta$ from (2.49), we can bound, for all $t\in[0,T]$,

$$\sup_{(x,y)\in\mathbb{T}\times[\vartheta_0,1-\vartheta_0]}|u^y(t,x,y)|\lesssim\varepsilon_1\langle t\rangle^{-2}. \tag{9.12}$$

Since the support of $\omega(0)$ is contained in $\mathbb{T}\times[2\vartheta_0,1-2\vartheta_0]$, we can conclude that

$$\operatorname{supp}\omega(t)\subseteq\mathbb{T}\times\left[\tfrac{3}{2}\vartheta_0,1-\tfrac{3}{2}\vartheta_0\right]$$

for any $t\in[0,T]$, as long as ε_1 is sufficiently small.

We can now use Proposition 2.2 and a simple continuity argument to show that, if $\omega_0\in\mathcal{G}^{1,2/3}$ has compact support in $\mathbb{T}\times[2\vartheta_0,1-2\vartheta_0]$ and satisfies the assumptions (1.12), then the solution ω is in $C([0,\infty):\mathcal{G}^{1,3/5})$, has compact support in $[\vartheta_0,1-\vartheta_0]$, and satisfies

$$\|\langle\omega\rangle(t)\|_{H^{10}}\lesssim\varepsilon^{2/3}\quad\text{for all } t\in[0,\infty).$$

Moreover, the variables F, F^*, $F-F^*$, Θ, Θ^*, B'_*, B''_*, V'_*, and $\mathcal{H}$ satisfy the improved bounds (2.50)–(2.51). In particular, since

$$A_k(t,\xi)\geqslant e^{1.1\delta_0\langle k,\xi\rangle^{1/2}}\quad\text{and}\quad A_R(t,\xi)\geqslant A_{NR}(t,\xi)\geqslant e^{1.1\delta_0\langle\xi\rangle^{1/2}},$$

for any $t\in[0,\infty)$ we have

$$\|e^{\delta_0\langle k,\xi\rangle^{1/2}}\widetilde{F}(t,k,\xi)\|_{L^2_{k,\xi}}+\langle t\rangle^2\|ke^{\delta_0\langle k,\xi\rangle^{1/2}}(\widetilde{\Psi\phi})(t,k,\xi)\|_{L^2_{k,\xi}}\lesssim_\delta\varepsilon_1^{3/2}, \tag{9.13}$$

and

$$\|V'_*(t)\|_{\mathcal{G}^{\delta_0,1/2}}+\|B'_*(t)\|_{\mathcal{G}^{\delta_0,1/2}}+\|B''_*(t)\|_{\mathcal{G}^{\delta_0,1/2}}+\langle t\rangle^{3/4}\|\mathcal{H}(t)\|_{\mathcal{G}^{\delta_0,1/2}}\lesssim\varepsilon_1. \tag{9.14}$$

Step 3. We now show that, for any $t\in[0,\infty)$,

$$\langle t\rangle\|\mathcal{H}(t)\|_{\mathcal{G}^{\delta_1,1/2}}+\langle t\rangle^2\|\dot V(t)\|_{\mathcal{G}^{\delta_1,1/2}}\lesssim\varepsilon_1, \tag{9.15}$$

where $\delta_1=\delta_1(\delta_0)>0$. We use equation (2.31), thus

$$\begin{aligned}\partial_t(t\mathcal{H})&=-t\dot V\partial_v\mathcal{H}+tV'\{-\langle\partial_v P_{\neq 0}\phi\partial_z F\rangle+\langle\partial_z\phi\partial_v F\rangle\}\\&=-t\dot V\partial_v\mathcal{H}+tV'\partial_v\langle\partial_z\phi F\rangle.\end{aligned} \tag{9.16}$$

Since $V'=V_*'+B_0'$ and $V''=\frac{1}{2}\partial_v(V')^2$, it follows from (9.14) and Lemma 3.1 that

$$\begin{aligned}\|V'(t)\|_{\widetilde{\mathcal{G}}_K^{1/2}[b(0),b(1)]}+\|V''(t)\|_{\widetilde{\mathcal{G}}_K^{1/2}[b(0),b(1)]}\\ +\|B'(t)\|_{\widetilde{\mathcal{G}}_K^{1/2}[b(0),b(1)]}+\|B''(t)\|_{\widetilde{\mathcal{G}}_K^{1/2}[b(0),b(1)]}\lesssim 1\end{aligned} \tag{9.17}$$

for any $t\in[0,\infty)$, for some $K=K(\delta_0)>0$. Using also (2.32), we have

$$\langle\partial_z\phi F\rangle=(V')^2\langle\partial_z\phi\cdot(\partial_v^2\phi-2t\partial_v\partial_z\phi)\rangle+V''\langle\partial_z\phi\cdot(\partial_v\phi-t\partial_z\phi)\rangle. \tag{9.18}$$

In particular, using the bounds on ϕ in (9.13),

$$\|\langle\partial_z\phi F\rangle(t)\|_{\mathcal{G}^{\delta_0/2,1/2}}\lesssim\varepsilon_1^2\langle t\rangle^{-3}.$$

Using also (9.13)–(9.14) and the identity $\mathcal{H}=tV'\partial_v\dot{V}$, it follows from (9.14) that

$$\|\partial_t(t\mathcal{H})(t)\|_{\mathcal{G}^{\delta_0/2,1/2}}\lesssim\varepsilon_1\langle t\rangle^{-3/2}$$

for any $t\in[0,\infty)$, and the desired bounds (9.15) follow.

As a consequence, we also have the bounds

$$\|v(t)\|_{\mathcal{G}_{K_2}^{1/2}[0,1]}+\langle t\rangle^2\|(\partial_t v)(t)\|_{\mathcal{G}_{K_2}^{1/2}[0,1]}\lesssim 1 \tag{9.19}$$

for any $t\in[0,\infty)$, for some $K_2=K_2(\delta_0)>0$. Indeed, the bounds on v follow from the identity $\partial_y v(t,y)=V'(t,v(t,y))$, the bounds (9.17), and Lemma 3.2. The bounds on $\partial_t v$ then follow using the identity $\partial_t v(t,y)=\dot{V}(t,v(t,y))$, the bounds (9.15), and Lemma 3.2.

Step 4. We now prove the conclusions of the theorem. Notice that

$$\partial_t F-B''\partial_z(\Psi\phi)-V'\partial_v P_{\neq 0}(\Psi\phi)\,\partial_z F+\dot{V}\,\partial_v F+V'\partial_z(\Psi\phi)\,\partial_v F=0, \tag{9.20}$$

using (2.28) and supp $F(t)\subseteq[b(\vartheta_0),b(1-\vartheta_0)]$. Using the bounds (9.13)–(9.15), and (9.17), it follows that

$$\|\partial_t F\|_{\mathcal{G}^{\delta_2,1/2}}\lesssim\varepsilon_1^{3/2}\langle t\rangle^{-2}$$

for some $\delta_2>0$. Moreover, the definitions (2.4)–(2.5) show that

$$\omega(t,x+tb(y)+\Phi(t,y),y)=\omega(t,x+tv(t,y),y)=F(t,x,v(t,y)).$$

Using also (9.19), we have

$$\left\|\frac{d}{dt}[\omega(t,x+tb(y)+\Phi(t,y),y)]\right\|_{\mathcal{G}^{\delta_3,1/2}}\lesssim\varepsilon_1^{3/2}\langle t\rangle^{-2}$$

for some $\delta_3=\delta_3(\delta_0)>0$, and the bounds (1.14) follow.

Moreover, we notice that

$$\psi(t,x,y)=\phi(t,x-tv(t,y),v(t,y)) \tag{9.21}$$

Since $u^y=\partial_x\psi$ and $u^x=-\partial_y\psi$, the bounds (1.18)–(1.19) follow from the bounds on ϕ in (9.13) and the fact that $\psi(t)$ is harmonic in $\mathbb{T}\times\{[0,\vartheta_0]\cup[1-\vartheta_0,1]\}$.

Finally, in order to prove (1.17), we start from the formula $\langle u^x\rangle=-\langle\partial_y\psi\rangle$, thus $\partial_y\langle u^x\rangle=-\langle\omega\rangle$. Therefore, using the evolution equation (1.13),

$$\partial_t\partial_y\langle u^x\rangle=\langle-\partial_t\omega\rangle=\langle u^x\partial_x\omega+u^y\partial_y\omega\rangle=\langle-\partial_y\psi\partial_x\omega+\partial_x\psi\partial_y\omega\rangle=\partial_y\langle\omega\partial_x\psi\rangle.$$

Moreover, since $\psi(t,x,0)=\psi(t,x,1)=0$, for any $t\in[0,\infty)$ we have

$$\int_{[0,1]}\langle u^x\rangle(t,y)\,dy=-\int_{[0,1]}\partial_y\langle\psi\rangle(t,y)\,dy=0. \tag{9.22}$$

Moreover,

$$\langle\omega\partial_x\psi\rangle=\langle\partial_y^2\psi\partial_x\psi\rangle=\partial_y\langle\partial_y\psi\partial_x\psi\rangle.$$

These identities show that

$$\partial_t\langle u^x\rangle=\langle\omega\partial_x\psi\rangle \quad \text{in } [0,\infty)\times[0,1]. \tag{9.23}$$

Using the definitions (2.5), we have

$$\begin{aligned}(\partial_t\langle u^x\rangle)(t,y)&=\frac{1}{2\pi}\int_{\mathbb{T}}\omega(t,x,y)\,\partial_x\psi(t,x,y)\,dx\\&=\frac{1}{2\pi}\int_{\mathbb{T}}F(t,z,v(t,y))\,\partial_z\phi(t,z,v(t,y))\,dz.\end{aligned}$$

Using now (9.18), (9.19), and the bounds on ϕ in (9.13), it follows that

$$\|(\partial_t\langle u^x\rangle)(t)\|_{\mathcal{G}^{\delta_4,1/2}}\lesssim\varepsilon_1^2\langle t\rangle^{-3}$$

for some $\delta_4=\delta_4(\delta_0)>0$. Moreover, using (1.14),

$$\lim_{t\to\infty}\{\partial_y\langle u^x\rangle(t)+\partial_y^2\psi_\infty\}=\lim_{t\to\infty}\{-\langle\omega\rangle(t)+\langle F_\infty\rangle\}=0.$$

The desired conclusion (1.17) follows using also (9.22).

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