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Pixton's formula and Abel–Jacobi theory on the Picard stack

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0. Introduction

0.1. Double ramification cycles

Let $A=(a_1, ..., a_n)$ be a vector of n integers satisfying

$$\sum_{i=1}^{n} a_i = 0.$$

In the moduli space $\mathcal{M}_{g,n}$ of non-singular curves of genus g with n marked points, consider the substack defined by the following classical condition:

$$\left\{ (C, p_1, ..., p_n) \in \mathcal{M}_{g,n} : \mathcal{O}_C\left(\sum_{i=1}^{n} a_i p_i\right) \simeq \mathcal{O}_C \right\}. \tag{0.1}$$

From the point of view of relative Gromov–Witten theory, the most natural compactification of the substack (0.1) is the space $\overline{\mathcal{M}}^{\sim}_{g,A}$ of stable maps to *rubber*: stable maps to $\mathbb{CP}^1$ relative to zero and ∞ modulo the $\mathbb{C}^*$-action on $\mathbb{CP}^1$. The rubber moduli space carries a natural virtual fundamental class $[\overline{\mathcal{M}}^{\sim}_{g,A}]^{\mathrm{vir}}$ of (complex) dimension $2g-3+n$. The pushforward via the canonical morphism

$$\varepsilon: \overline{\mathcal{M}}^{\sim}_{g,A} \longrightarrow \overline{\mathcal{M}}_{g,n}$$

is the *double ramification cycle*

$$\varepsilon_*[\overline{\mathcal{M}}^{\sim}_{g,A}]^{\mathrm{vir}} = \mathsf{DR}_{g,A} \in \mathsf{CH}_{2g-3+n}(\overline{\mathcal{M}}_{g,n}).$$

Double ramification cycles have been studied intensively for the past two decades. Examples of early results can be found in [17], [18], [21], [25], [33], [35], [57]. A complete formula was conjectured by Pixton in 2014 and proven in [42]. For subsequent study and applications, see [4], [16], [20], [26], [28], [37]–[39], [59], [62], [66], [72], [75]. Essential for our work is the formula for double ramification cycles for target varieties in [43].

We refer the reader to [42, §0] and [64, §5] for introductions to the subject. For a classical perspective from the point of view of Abel–Jacobi theory, see [37].

0.2. Twisted double ramification cycles

We develop here a theory which extends the study of double ramification cycles from the moduli space of stable curves $\overline{\mathcal{M}}_{g,n}$ to the Picard stack of curves with line bundles $\mathfrak{Pic}_{g,n}$. An object of $\mathfrak{Pic}_{g,n}$ over $\mathcal{S}$ is a flat family

$$\pi: \mathcal{C} \longrightarrow \mathcal{S}$$

of prestable([1]) n-pointed genus-g curves together with a line bundle

$$\mathcal{L} \longrightarrow \mathcal{C}.$$

The Picard stack $\mathfrak{Pic}_{g,n}$ is an algebraic (Artin) stack which is locally of finite type; see §2.1 for a treatment of foundational issues.

Since the degree of a line bundle is constant in flat families, there is a disjoint union

$$\mathfrak{Pic}_{g,n} = \bigcup_{d\in\mathbb{Z}} \mathfrak{Pic}_{g,n,d},$$

where $\mathfrak{Pic}_{g,n,d}$ is the Picard stack of curves with degree-d line bundles. Let

$$A = (a_1, ..., a_n), \quad \sum_{i=1}^{n} a_i = d,$$

be a vector of integers. The first result of the paper is the construction of a *universal twisted double ramification cycle* in the operational Chow theory([2]) of $\mathfrak{Pic}_{g,n,d}$,

$$\mathsf{DR}^{\mathsf{op}}_{g,A} \in \mathsf{CH}^{g}_{\mathsf{op}}(\mathfrak{Pic}_{g,n,d}).$$

The geometric intuition behind the construction is simple. Let

$$\pi: \mathcal{C} \longrightarrow \mathcal{S}, \quad p_1, ..., p_n: \mathcal{S} \longrightarrow \mathcal{C}, \quad \mathcal{L} \longrightarrow \mathcal{C}$$

be an object of $\mathfrak{Pic}_{g,n,d}$. The class $\mathsf{DR}^{\mathsf{op}}_{g,A}$ should operate as the locus in the base $\mathcal{S}$ heuristically determined by the condition

$$\mathcal{O}_C\left(\sum_{i=1}^{n} a_i p_i\right) \simeq \mathcal{L}|_C.$$

To make the above idea precise, we do *not* use the virtual class of the moduli space of stable maps in Gromov–Witten theory, but rather an alternative approach by partially resolving the classical Abel–Jacobi map. The method follows the path of [37], [39] and may be viewed as a universal Abel–Jacobi construction over the Picard stack. Log geometry based on the stack of tropical divisors constructed in [56] plays a crucial role. Our construction is presented in §3.8.

The basic compatibility of our new operational class

$$\mathsf{DR}^{\mathsf{op}}_{g,A} \in \mathsf{CH}^{g}_{\mathsf{op}}(\mathfrak{Pic}_{g,n,d})$$

([1]) A prestable n-pointed curve is a connected nodal curve with markings at distinct non-singular points. For the entire paper, we avoid the $(g,n)=(1,0)$ case because of non-affine stabilizers.

([2]) All Chow theories in the paper will be taken with $\mathbb{Q}$-coefficients.

with the standard double ramification cycle is as follows. Let

$$A=(a_1,...,a_n),\quad \sum_{i=1}^{n} a_i=0,$$

be given. The universal data

$$\pi\colon \mathcal{C}_{g,n} \longrightarrow \overline{\mathcal{M}}_{g,n},\quad \mathcal{O}\longrightarrow \mathcal{C}_{g,n} \tag{0.2}$$

determine a map $\varphi_{\mathcal{O}}\colon \overline{\mathcal{M}}_{g,n}\to\mathfrak{Pic}_{g,n,0}$. The action of $\mathsf{DR}^{\mathsf{op}}_{g,A}$ on the fundamental class of $\overline{\mathcal{M}}_{g,n}$ corresponding to the family (0.2) then equals the previously defined double ramification cycle

$$\mathsf{DR}^{\mathsf{op}}_{g,A}(\varphi_{\mathcal{O}})([\overline{\mathcal{M}}_{g,n}])=\mathsf{DR}_{g,A}\in\mathsf{CH}_{2g-3+n}(\overline{\mathcal{M}}_{g,n}).$$

More generally, for a vector $A=(a_1,...,a_n)$ of integers satisfying

$$\sum_{i=1}^{n} a_i=k(2g-2),$$

canonically twisted double ramification cycles,

$$\mathsf{DR}_{g,A,\omega^k}\in\mathsf{CH}_{2g-3+n}(\overline{\mathcal{M}}_{g,n}),$$

related to the classical loci

$$\left\{(C,p_1,...,p_n)\in\mathcal{M}_{g,n}:\mathcal{O}_C\left(\sum_{i=1}^{n}a_ip_i\right)\simeq\omega_C^k\right\},$$

have been constructed in [34] for $k=1$ and in [37], [38], [56] for all $k\geqslant 1$. The universal data

$$\pi\colon \mathcal{C}_{g,n} \longrightarrow \overline{\mathcal{M}}_{g,n},\quad \omega_\pi^k\longrightarrow \mathcal{C}_{g,n} \tag{0.3}$$

determine a map $\varphi_{\omega_\pi^k}\colon \overline{\mathcal{M}}_{g,n}\to\mathfrak{Pic}_{g,n,k(2g-2)}$. Here, ω_π is the relative dualizing sheaf of the morphism π.

The action of $\mathsf{DR}^{\mathsf{op}}_{g,A}$ on the fundamental class of $\overline{\mathcal{M}}_{g,n}$ corresponding to the family (0.3) is compatible with the constructions of [34], [37], [38], [56],

$$\mathsf{DR}^{\mathsf{op}}_{g,A}(\varphi_{\omega_\pi^k})([\overline{\mathcal{M}}_{g,n}])=\mathsf{DR}_{g,A,\omega^k}\in\mathsf{CH}_{2g-3+n}(\overline{\mathcal{M}}_{g,n})$$

for all $k\geqslant 1$.

The above compatibilities of $\mathsf{DR}^{\mathsf{op}}_{g,A}$ with the standard and canonically twisted double ramification cycles are proven in §3.7.

THEOREM 0.1. *Let $g\geqslant 0$ and $d\in\mathbb{Z}$. Let $A=(a_1,...,a_n)$ be a vector of integers satisfying*

$$\sum_{i=1}^{n} a_i = d.$$

Logarithmic compactification of the Abel–Jacobi map yields a universal twisted double ramification cycle

$$\mathsf{DR}_{g,A}^{\mathsf{op}} \in \mathsf{CH}^g_{\mathsf{op}}(\mathfrak{Pic}_{g,n,d})$$

which is compatible with the standard double ramification cycle

$$\mathsf{DR}_{g,A,\omega^k} \in \mathsf{CH}_{2g-3+n}(\overline{\mathcal{M}}_{g,n})$$

in case $d=k(2g-2)$ for $k\geqslant 0$.

0.3. Pixton's formula

0.3.1. Prestable graphs

We define the set $\mathsf{G}_{g,n}$ of prestable graphs as follows. A *prestable graph* $\Gamma\in\mathsf{G}_{g,n}$ consists of the data

$$\Gamma = (\mathrm{V},\mathrm{H},\mathrm{L},\mathrm{g}\colon \mathrm{V}\to\mathbb{Z}_{\geqslant 0}, v\colon \mathrm{H}\to\mathrm{V}, \iota\colon \mathrm{H}\to\mathrm{H})$$

satisfying the following properties:

(i) V is a vertex set with a genus function $\mathrm{g}\colon \mathrm{V}\to\mathbb{Z}_{\geqslant 0}$;

(ii) H is a half-edge set equipped with a vertex assignment $v\colon\mathrm{H}\to\mathrm{V}$ and an involution ι;

(iii) E, the edge set, is defined by the 2-cycles of ι in H (self-edges at vertices are permitted);

(iv) L, the set of legs, is defined by the fixed points of ι and is placed in bijective correspondence with a set of n markings;

(v) the pair (V, E) defines a *connected* graph satisfying the genus condition

$$\sum_{v\in\mathrm{V}} \mathrm{g}(v) + h^1(\Gamma) = g.$$

To emphasize Γ, the notation $\mathrm{V}(\Gamma)$, $\mathrm{H}(\Gamma)$, $\mathrm{L}(\Gamma)$, and $\mathrm{E}(\Gamma)$ will also be used for the vertex, half-edges, legs, and edges of Γ, respectively.

An isomorphism between $\Gamma,\Gamma'\in\mathsf{G}_{g,n}$ consists of bijections $\mathrm{V}\to\mathrm{V}'$ and $\mathrm{H}\to\mathrm{H}'$ respecting the structures L, g, v, and ι. Let $\mathrm{Aut}(\Gamma)$ denote the automorphism group of Γ.

While the set of isomorphism classes of prestable graphs is infinite, the set of isomorphism classes of prestable graphs with prescribed bounds on the number of edges is finite.

Let $\mathfrak{M}_{g,n}$ be the algebraic (Artin) stack of prestable curves of genus g with n marked points. A prestable graph Γ determines an algebraic stack $\mathfrak{M}_\Gamma$ of curves with degenerations forced by the graph,

$$\mathfrak{M}_\Gamma = \prod_{v\in \mathrm{V}} \mathfrak{M}_{\mathrm{g}(v),\mathrm{n}(v)},$$

together with a canonical([3]) map

$$j_\Gamma \colon \mathfrak{M}_\Gamma \longrightarrow \mathfrak{M}_{g,n}.$$

Since $\mathfrak{M}_{g,n}$ is smooth and the morphism j_Γ is proper, representable, and lci, we obtain an operational Chow class on the algebraic stack of curves,

$$j_{\Gamma *}[\mathfrak{M}_\Gamma] \in \mathsf{CH}^{|\mathrm{E}(\Gamma)|}_{\mathsf{op}}(\mathfrak{M}_{g,n}).$$

Via the morphism of algebraic stacks,

$$\varepsilon \colon \mathfrak{Pic}_{g,n,d} \longrightarrow \mathfrak{M}_{g,n},$$

$j_{\Gamma *}[\mathfrak{M}_\Gamma]$ also defines an operational Chow class on the Picard stack,

$$\varepsilon^* j_{\Gamma *}[\mathfrak{M}_\Gamma] \in \mathsf{CH}^{|\mathrm{E}(\Gamma)|}_{\mathsf{op}}(\mathfrak{Pic}_{g,n,d}).$$

0.3.2. Prestable graphs with degrees

We will require a refinement of the prestable graphs of §0.3.1 which includes degrees of line bundles.

We define the set $\mathsf{G}_{g,n,d}$ of *prestable graphs of degree d* as follows:

$$\Gamma_\delta = (\Gamma, \delta) \in \mathsf{G}_{g,n,d}$$

consists of the data

- a prestable graph $\Gamma \in \mathsf{G}_{g,n}$;
- a function $\delta \colon \mathrm{V} \to \mathbb{Z}$ satisfying the degree condition

$$\sum_{v\in V} \delta(v) = d.$$

([3]) To define the map, we choose an ordering on the half-edges at each vertex.

The function δ is often called the *multidegree.*

An automorphism of $\Gamma_\delta \in \mathsf{G}_{g,n,d}$ consists of an automorphism of Γ leaving δ invariant. Let $\mathrm{Aut}(\Gamma_\delta)$ denote the automorphism group of Γ_δ.

For $\Gamma_\delta \in \mathsf{G}_{g,n,k}$, let $\mathfrak{M}_\Gamma$ be the algebraic stack of curves defined in §0.3.1 with respect to the underlying prestable graph Γ. Let $\mathfrak{Pic}_{\Gamma_\delta}$ be the Picard stack,

$$\varepsilon \colon \mathfrak{Pic}_{\Gamma_\delta} \longrightarrow \mathfrak{M}_\Gamma,$$

parameterizing curves with degenerations forced by Γ and with line bundles which have degree $\delta(v)$ restriction to the components corresponding to the vertex $v \in \mathrm{V}$. We have a canonical map

$$j_{\Gamma_\delta} \colon \mathfrak{Pic}_{\Gamma_\delta} \longrightarrow \mathfrak{Pic}_{g,n,d}.$$

Since $\mathfrak{Pic}_{g,n,d}$ is smooth and the morphism j_{Γ_δ} is proper, representable, and lci, we obtain an operational Chow class,

$$j_{\Gamma_\delta *}[\mathfrak{Pic}_{\Gamma_\delta}] \in \mathsf{CH}^{|\mathrm{E}(\Gamma)|}_{\mathsf{op}}(\mathfrak{Pic}_{g,n,d}).$$

As operational Chow classes, the following formula holds:

$$\varepsilon^* j_{\Gamma *}[\mathfrak{M}_\Gamma] = \sum_\delta j_{\Gamma_\delta *}[\mathfrak{Pic}_{\Gamma_\delta}] \in \mathsf{CH}^{|\mathrm{E}(\Gamma)|}_{\mathsf{op}}(\mathfrak{Pic}_{g,n,d}), \tag{0.4}$$

where the sum([4]) is over all functions $\delta \colon \mathrm{V} \to \mathbb{Z}$ satisfying the degree condition. Equivalently, we may write (0.4) as

$$\frac{1}{|\mathrm{Aut}(\Gamma)|}\, \varepsilon^* j_{\Gamma *}[\mathfrak{M}_\Gamma] = \sum_{\Gamma_\delta \in \mathsf{G}_{g,n,d}} \frac{1}{|\mathrm{Aut}(\Gamma_\delta)|}\, j_{\Gamma_\delta *}[\mathfrak{Pic}_{\Gamma_\delta}] \in \mathsf{CH}^{|\mathrm{E}(\Gamma)|}_{\mathsf{op}}(\mathfrak{Pic}_{g,n,d}),$$

where the sum on the right-hand side is now over all isomorphism classes of prestable graphs of degree d with underlying prestable graph Γ.

0.3.3. Tautological ψ, ξ, and η classes

The universal curve

$$\pi \colon \mathfrak{C}_{g,n} \longrightarrow \mathfrak{Pic}_{g,n}$$

carries two natural line bundles: the relative dualizing sheaf ω_π and the universal line bundle

$$\mathfrak{L} \longrightarrow \mathfrak{C}_{g,n}.$$

([4]) The sum is infinite, but only finitely many terms are non-zero in any operation.

Let p_i be the ith section of the universal curve, let

$$\mathfrak{S}_i \subset \mathfrak{C}_{g,n}$$

be the corresponding divisor, and let

$$\omega_{\log} = \omega_\pi \left(\sum_{i=1}^{n} \mathfrak{S}_i \right)$$

be the relative log-canonical line bundle with first Chern class $c_1(\omega_{\log})$. Let

$$\xi = c_1(\mathfrak{L})$$

be the first Chern class of $\mathfrak{L}$.

Definition 0.2. The following operational classes on $\mathfrak{Pic}_{g,n}$ are obtained from the following universal structures:

- $\psi_i = c_1(p_i^* \omega_\pi) \in \mathsf{CH}^1_{\mathsf{op}}(\mathfrak{Pic}_{g,n})$;
- $\xi_i = c_1(p_i^* \mathfrak{L}) \in \mathsf{CH}^1_{\mathsf{op}}(\mathfrak{Pic}_{g,n})$;
- $\eta_{a,b} = \pi_*(c_1(\omega_{\log})^a \xi^b) \in \mathsf{CH}^{a+b-1}_{\mathsf{op}}(\mathfrak{Pic}_{g,n})$.

For simplicity in the formulas, we will use the notation

$$\eta = \eta_{0,2} = \pi_*(\xi^2).$$

The standard κ classes are defined by the π pushforwards of powers of $c_1(\omega_{\log})$,

$$\eta_{a,0} = \kappa_{a-1}.$$

Definition 0.3. A *decorated prestable graph* $[\Gamma_\delta, \gamma]$ of degree d is a prestable graph $\Gamma_\delta \in \mathsf{G}_{g,n,d}$ of degree d together with the following decoration data γ:

- each leg $i \in \mathrm{L}$ is decorated with a monomial $\psi_i^a \xi_i^b$;
- each half-edge $h \in \mathrm{H} \setminus \mathrm{L}$ is decorated with a monomial ψ_h^a;
- each edge $e \in \mathrm{E}$ is decorated with a monomial ξ_e^a;
- each vertex in V is decorated with a monomial in the variables $\{\eta_{a,b}\}_{a+b \geqslant 2}$.

In all four cases, the monomial may be be trivial.

Let $\mathsf{DG}_{g,n,d}$ be the set of decorated prestable graphs of degree d. To each decorated graph of degree d,

$$[\Gamma_\delta, \gamma] \in \mathsf{DG}_{g,n,d},$$

we assign the operational class

$$j_{\Gamma_\delta *}[\gamma] \in \mathsf{CH}^*_{\mathsf{op}}(\mathfrak{Pic}_{g,n,d})$$

obtained via the proper representable morphism

$$j_{\Gamma_\delta}\colon \mathfrak{Pic}_{\Gamma_\delta} \longrightarrow \mathfrak{Pic}_{g,n,d}$$

and the action of the decorations.

The action of decorations is described as follows. Given $\Gamma_\delta \in \mathsf{G}_{g,n,k}$, the stack $\mathfrak{Pic}_{\Gamma_\delta}$ admits a morphism(5)

$$\mathfrak{Pic}_{\Gamma_\delta} \longrightarrow \prod_{v\in \mathrm{V}(\Gamma_\delta)} \mathfrak{Pic}_{\mathrm{g}(v),\mathrm{n}(v),\delta(v)}$$

which sends a line bundle $\mathcal{L}$ on a prestable curve $\mathcal{C}$ to its restrictions on the various components,

$$\mathcal{L}|_{\mathcal{C}_v}, \quad v\in \mathrm{V}(\Gamma_\delta).$$

For $v\in\mathrm{V}(\Gamma_\delta)$, we define the operational class $\eta(v)$ on $\mathfrak{Pic}_{\Gamma_\delta}$ as the pullback of the operational class η on the factor $\mathfrak{Pic}_{\mathrm{g}(v),\mathrm{n}(v),\delta(v)}$ above. The operational classes ψ at the markings and ξ at the half-edges are defined similarly.

Definition 0.4. The *tautological classes* in $\mathsf{CH}^*_{\mathsf{op}}(\mathfrak{Pic}_{g,n,d})$ consist of the $\mathbb{Q}$-linear span of the operational classes associated with all $[\Gamma_\delta,\gamma]\in\mathsf{DG}_{g,n,d}$.

By standard analysis [30], the tautological classes have a natural ring structure. Our formula for $\mathsf{DR}^{\mathsf{op}}_{g,A}$ will be a sum of operational classes determined by decorated prestable graphs of degree $d=\sum_{i=1}^n a_i$ (and hence will be tautological).

0.3.4. Weightings mod r

Let $\Gamma_\delta\in\mathsf{G}_{g,n,d}$ be a prestable graph of degree d, and let r be a positive integer.

Definition 0.5. A *weighting mod* r of Γ_δ is a function on the set of half-edges,

$$w\colon \mathrm{H}(\Gamma_\delta) \longrightarrow \{0,1,...,r-1\},$$

which satisfies the following three properties:

(i) for all $i\in\mathrm{L}(\Gamma_\delta)$, corresponding to the marking $i\in\{1,...,n\}$,

$$w(i)=a_i \mod r;$$

(ii) for all $e\in\mathrm{E}(\Gamma_\delta)$, corresponding to two half-edges $h,h'\in\mathrm{H}(\Gamma_\delta)$,

$$w(h)+w(h')=0 \mod r;$$

(5) The fibers of the map are torsors under the group $\mathbb{G}_m^{h^1(\Gamma)}$.

(iii) for all $v \in \mathrm{V}(\Gamma_\delta)$,

$$\sum_{v(h)=v} w(h) = \delta(v) \quad \mathrm{mod}\ r,$$

where the sum is taken over *all* $\mathsf{n}(v)$ half-edges incident to v.

We denote by $\mathsf{W}_{\Gamma_\delta,r}$ the finite set of all possible weightings mod r of Γ_δ. The set $\mathsf{W}_{\Gamma_\delta,r}$ has cardinality $r^{h^1(\Gamma_\delta)}$. We view r as a *regularization parameter.*

0.3.5. Calculation of the twisted double ramification cycle

We denote by $\mathsf{P}^{c,r}_{g,A,d} \in \mathsf{CH}^c_{\mathsf{op}}(\mathfrak{Pic}_{g,n,d})$ the codimension-c([6]) component of the tautological operational class

$$\sum_{\substack{\Gamma_\delta \in \mathsf{G}_{g,n,d}\\ w \in \mathsf{W}_{\Gamma_\delta,r}}} \frac{r^{-h^1(\Gamma_\delta)}}{|\mathrm{Aut}(\Gamma_\delta)|} j_{\Gamma_\delta *}\Bigg[\prod_{i=1}^n \exp\bigg(\frac{1}{2}a_i^2\psi_i + a_i\xi_i\bigg) \prod_{v \in \mathrm{V}(\Gamma_\delta)} \exp\bigg(-\frac{1}{2}\eta(v)\bigg)$$
$$\times \prod_{e=(h,h') \in \mathrm{E}(\Gamma_\delta)} \frac{1}{\psi_h + \psi_{h'}}\bigg(1 - \exp\bigg(-\frac{w(h)w(h')}{2}(\psi_h + \psi_{h'})\bigg)\bigg)\Bigg].$$

Several remarks about the formula are required.

(i) The sum is over all isomorphism classes of prestable graphs of degree d in the set $\mathsf{G}_{g,n,d}$. Only finitely many underlying prestable graphs $\Gamma \in \mathsf{G}_{g,n}$ can contribute in fixed codimension c. However, for each such prestable graph, the above formula has infinitely many summands corresponding to the infinitely many functions

$$\delta \colon \mathrm{V} \longrightarrow \mathbb{Z}$$

which satisfy the degree condition. The operational Chow class $\mathsf{P}^{c,r}_{g,A,d}$ is nevertheless well defined, since only finitely many summands have non-vanishing operation on any given family of curves carrying a degree-d line bundle over a base $\mathcal{S}$ of finite type.

(ii) Once the prestable graph Γ_δ is chosen, we sum over all $r^{h^1(\Gamma_\delta)}$ different weightings $w \in \mathsf{W}_{\Gamma_\delta,r}$.

(iii) Inside the pushforward in the above formula, the first product

$$\prod_{i=1}^n \exp\bigg(\frac{1}{2}a_i^2\psi_{h_i} + a_i\xi_{h_i}\bigg)$$

is over $h \in \mathrm{L}(\Gamma)$ via the correspondence of legs and markings.

([6]) Codimension here is usually called degree. But since we already have line bundle degrees, we use the term codimension for clarity.

(iv) The class $\eta(v)$ is the $\eta_{0,2}$ class of Definition 0.2 associated with the vertex.

(v) The third product is over all $e\in\mathrm{E}(\Gamma_\delta)$. The factor

$$\frac{1}{\psi_h+\psi_{h'}}\left(1-\exp\left(-\frac{w(h)w(h')}{2}(\psi_h+\psi_{h'})\right)\right)$$

is well defined, since

- the denominator formally divides the numerator,
- the factor is symmetric in h and h'.

No edge orientation is necessary.

The following fundamental polynomiality property of $\mathsf{P}^{c,r}_{g,A,d}$ is parallel to Pixton's polynomiality in [42, Appendix] and is a consequence of [42, Proposition 3″].

Proposition 0.6. *For fixed g, A, d, and c and a decorated graph $[\Gamma_\delta,\gamma]$ of degree d, the coefficient of $j_{\Gamma_\delta *}[\gamma]$ in the tautological class*

$$\mathsf{P}^{c,r}_{g,A,d}\in\mathsf{CH}^c_{\mathsf{op}}(\mathfrak{Pic}_{g,n,d})$$

is polynomial in r (for all sufficiently large r).

We denote by $\mathsf{P}^{c}_{g,A,d}$ the value at $r=0$ of the polynomial associated with $\mathsf{P}^{c,r}_{g,A,d}$ by Proposition 0.6. In other words, $\mathsf{P}^{c}_{g,A,d}$ is the *constant* term of the associated polynomial in r.

The main result of the paper is a formula for the universal twisted double ramification cycle in operational Chow.(7)

Theorem 0.7. *Let $g\geqslant 0$ and $d\in\mathbb{Z}$. Let $A=(a_1,...,a_n)$ be a vector of integers with*

$$\sum_{i=1}^n a_i=d.$$

The universal twisted double ramification cycle is calculated by Pixton's formula:

$$\mathsf{DR}^{\mathsf{op}}_{g,A}=\mathsf{P}^{g}_{g,A,d}\in\mathsf{CH}^g_{\mathsf{op}}(\mathfrak{Pic}_{g,n,d}).$$

Theorem 0.7 is the most fundamental formulation of the relationship between Abel–Jacobi theory and Pixton's formula that we know. Certainly, Theorem 0.7 implies the double ramification cycle and X-valued double ramification cycle results of [42], [43]. But since we will use [43] in the proof of Theorem 0.7, we provide no new approach to these older results. However, the additional depth of Theorem 0.7 immediately allows new applications.

(7) Our handling of the prefactor 2^{-g} in [42, Theorem 1] differs here. The factors of 2 are now placed in the definition of $\mathsf{P}^{c,r}_{g,A,d}$ as in [43].

0.4. Vanishing

From his original double ramification cycle formula, Pixton conjectured an associated vanishing property in the tautological ring of the moduli space of curves which was proven by Clader and Janda [21]. The parallel vanishing statement in the tautological ring of the moduli space of stable maps to X was proven in [4]. The most general vanishing statement is the following result.

THEOREM 0.8. *Let $g\geqslant 0$ and $d\in\mathbb{Z}$. Let $A=(a_1,...,a_n)$ be a vector of integers with*

$$\sum_{i=1}^{n} a_i = d.$$

Pixton's vanishing holds in operational Chow:

$$\mathsf{P}^c_{g,A,d} = 0 \in \mathsf{CH}^c_{\mathsf{op}}(\mathfrak{Pic}_{g,n,d}) \quad \textit{for all } c>g.$$

Theorem 0.8 may be viewed as providing relations among tautological classes in the operational Chow ring of the Picard stack — a new direction of study with many open questions.(8) While Theorem 0.8 implies the vanishings of [4], [21], we will use these results in our proof.

0.5. Twisted holomorphic and meromorphic differentials

0.5.1. Fundamental classes

Let $A=(a_1,...,a_n)$ be a vector of zero and pole multiplicities satisfying

$$\sum_{i=1}^{n} a_i = 2g-2.$$

Let $\mathcal{H}_g(A)\subset\mathcal{M}_{g,n}$ be the quasi-projective locus of pointed curves $(C,p_1,...,p_n)$ satisfying the condition

$$\mathcal{O}_C\bigg(\sum_{i=1}^{n} a_i p_i\bigg) \simeq \omega_C.$$

In other words, $\mathcal{H}_g(A)$ is the locus of meromorphic differentials(9) with zero and pole multiplicities prescribed by A. In [28], a compact moduli space of twisted canonical divisors

$$\widetilde{\mathcal{H}}_g(A)\subset\overline{\mathcal{M}}_{g,n}$$

(8) See [67], [77] for tautological relations on the Picard variety over the moduli space of smooth curves.

(9) If all the parts of A are non-negative, then $\mathcal{H}_g(A)$ is the locus of holomorphic differentials.

is constructed which contains $\mathcal{H}_g(A)$ as an open set.

In the strictly meromorphic case, where A contains at least one strictly negative part, $\widetilde{\mathcal{H}}_g(A)$ is of pure codimension g in $\overline{\mathcal{M}}_{g,n}$ by [28, Theorem 3]. A weighted fundamental cycle of $\widetilde{\mathcal{H}}_g(A)$,

$$\mathsf{H}_{g,A} \in \mathsf{CH}_{2g-3+n}(\overline{\mathcal{M}}_{g,n}), \tag{0.5}$$

is constructed in [28, Appendix A] with explicit non-trivial weights on the irreducible components. In the strictly meromorphic case, $\mathcal{H}_g(A) \subset \overline{\mathcal{M}}_{g,n}$ is also of pure codimension g. The closure

$$\overline{\mathcal{H}}_g(A) \subset \overline{\mathcal{M}}_{g,n}$$

contributes to the fundamental class $\mathsf{H}_{g,A}$ with multiplicity 1, but there are additional boundary contributions, see [28, Appendix A].

The universal family over the moduli space of stable curves together with the relative dualizing sheaf,

$$\pi: \mathcal{C}_{g,n} \longrightarrow \overline{\mathcal{M}}_{g,n}, \quad \omega_\pi \longrightarrow \mathcal{C}_{g,n}, \tag{0.6}$$

determine an object of $\mathfrak{Pic}_{g,n,2g-2}$. By [40] and the compatibility of Theorem 0.1, the action of $\mathsf{DR}^{\mathsf{op}}_{g,A}$ on the fundamental class of $\overline{\mathcal{M}}_{g,n}$ equals the weighted fundamental class of $\widetilde{\mathcal{H}}_g(A)$:

$$\mathsf{DR}^{\mathsf{op}}_{g,A}(\varphi_\omega)([\overline{\mathcal{M}}_{g,n}]) = \mathsf{H}_{g,A} \in \mathsf{CH}_{2g-3+n}(\overline{\mathcal{M}}_{g,n}).$$

We can now apply Theorem 0.7 to prove the following result.

THEOREM 0.9. *In the strictly meromorphic case,*

$$\mathsf{H}_{g,A} = \mathsf{P}^g_{g,A,2g-2}[\overline{\mathcal{M}}_{g,n}]$$

for the universal family

$$\pi: \mathcal{C}_{g,n} \longrightarrow \overline{\mathcal{M}}_{g,n}, \quad \omega_\pi \longrightarrow \mathcal{C}_{g,n}.$$

Theorem 0.9 is exactly equivalent to Conjecture A in [28, Appendix A]. Since both the moduli space $\widetilde{\mathcal{H}}_g(A)$ and the weighted fundamental cycle $\mathsf{H}_{g,A}$ have explicit geometric definitions, the result provides a geometric representative of Pixton's cycle class in terms of twisted differentials. Theorem 0.9 is proven in §8.1, where the parallel conjectures [72] for higher differentials are also proven (by parallel arguments).

0.5.2. Closures

Let $A=(a_1, ..., a_n)$ be a vector of integers satisfying

$$\sum_{i=1}^n a_i = 2g-2.$$

A careful investigation of the closure

$$\mathcal{H}_g(A) \subset \overline{\mathcal{H}}_g(A) \subset \overline{\mathcal{M}}_{g,n}$$

is carried out in [8] in both the holomorphic and meromorphic cases. By a simple procedure presented in [28, Appendix], Theorem 0.9 determines the cycle classes of the closures

$$[\overline{\mathcal{H}}_g(A)] \in \mathsf{CH}_*(\overline{\mathcal{M}}_{g,n}).$$

for A in both the holomorphic and meromorphic cases.

A similar procedure determines the corresponding classes for k-differentials, see [72, §3.4] for an explanation. In particular, our results imply that the cycle classes of the closures are tautological(10) for all k (as was previously known only for $k=1$, due to [70]).

In the case of holomorphic differentials, another approach to the class of the closure

$$\overline{\mathcal{H}}_g(A) \subset \overline{\mathcal{M}}_{g,n}$$

is provided by Conjecture A.1 in [65, Appendix] via a limit of Witten's r-spin class. A significant first step in the proof of [65, Conjecture A.1] by Chen, Janda, Ruan, and Sauvaget can be found in [19]. Further progress requires a virtual localization analysis for moduli spaces of stable log maps. An approach to Theorem 0.9 using log stable maps, virtual localization in the log context, and the strategy of [42] also appears possible (once the required moduli spaces and localization formulas are established).

0.6. Invariance properties

The universal twisted double ramification cycle has several basic invariance properties which play an important role in our study.

Recall that an object of $\mathfrak{Pic}_{g,n,d}$ over $\mathcal{S}$ is a flat family of prestable n-pointed genus-g curves together with a line bundle of relative degree d,

$$\pi: \mathcal{C} \longrightarrow \mathcal{S}, \quad p_1, ..., p_n: \mathcal{S} \longrightarrow \mathcal{C}, \quad \mathcal{L} \longrightarrow \mathcal{C}. \tag{0.7}$$

Let $\mathsf{DR}^{\mathsf{op}}_{g,A,\mathcal{L}} \in \mathsf{CH}^g_{\mathsf{op}}(\mathcal{S})$ be the twisted double ramification cycle associated with the above family (0.7) and the vector

$$A = (a_1, ..., a_n), \quad d = \sum_{i=1}^{n} a_i.$$

(10) The precise statement is given in Corollary 8.2 of §8.2.

Invariance I. (Dualizing)

A new object of $\mathfrak{Pic}_{g,n,-d}$ over $\mathcal{S}$ is obtained from (0.7) by dualizing $\mathcal{L}$:

$$\pi: \mathcal{C} \to \mathcal{S}, \quad p_1, ..., p_n : \mathcal{S} \to \mathcal{C}, \quad \mathcal{L}^* \to \mathcal{C}. \tag{0.8}$$

Let $\mathsf{DR}^{\mathsf{op}}_{g,-A,\mathcal{L}^*} \in \mathsf{CH}^g_{\mathsf{op}}(\mathcal{S})$ be the twisted double ramification cycle associated with the new family (0.8) and the vector $-A=(-a_1, ..., -a_n)$. We have the invariance

$$\mathsf{DR}^{\mathsf{op}}_{g,-A,\mathcal{L}^*} = \varepsilon^* \mathsf{DR}^{\mathsf{op}}_{g,A,\mathcal{L}},$$

where $\varepsilon: \mathfrak{Pic}_{g,n,-d} \to \mathfrak{Pic}_{g,n,d}$ is the natural map obtained via dualizing the line bundle.

Invariance II. (Unweighted markings)

Assume that we have an additional section $p_{n+1}: \mathcal{S} \to \mathcal{C}$ of π which yields an object of $\mathfrak{Pic}_{g,n+1,d}$:

$$\pi: \mathcal{C} \to \mathcal{S}, \quad p_1, ..., p_n, p_{n+1}: \mathcal{S} \to \mathcal{C}, \quad \mathcal{L} \to \mathcal{C}. \tag{0.9}$$

Let $A_0 \in \mathbb{Z}^{n+1}$ be the vector obtained by appending zero (as the last coefficient) to A. Let $\mathsf{DR}^{\mathsf{op}}_{g,A_0,\mathcal{L}} \in \mathsf{CH}^g_{\mathsf{op}}(\mathcal{S})$ be the twisted double ramification cycle associated with the new family (0.9) and the vector A_0. We have the invariance

$$\mathsf{DR}^{\mathsf{op}}_{g,A_0,\mathcal{L}} = F^* \mathsf{DR}^{\mathsf{op}}_{g,A,\mathcal{L}},$$

where $F: \mathfrak{Pic}_{g,n+1,d} \to \mathfrak{Pic}_{g,n,d}$ is the map forgetting the last marking.

Invariance III. (Weight translation)

Let $B=(b_1, ..., b_n) \in \mathbb{Z}^n$ satisfy

$$\sum_{i=1}^n b_i = e.$$

Then, the family

$$\pi: \mathcal{C} \longrightarrow \mathcal{S}, \quad p_1, ..., p_n: \mathcal{S} \longrightarrow \mathcal{C}, \quad \mathcal{L}\left(\sum_{i=1}^n b_i p_i\right) \longrightarrow \mathcal{C}. \tag{0.10}$$

defines an object of $\mathfrak{Pic}_{g,n,d+e}$. Let $\mathsf{DR}^{\mathsf{op}}_{g,A+B,\mathcal{L}(\sum_i b_i p_i)} \in \mathsf{CH}^g_{\mathsf{op}}(\mathcal{S})$ be the twisted double ramification cycle associated with the new family (0.10) and the vector $A+B$. We have the invariance

$$\mathsf{DR}^{\mathsf{op}}_{g,A+B,\mathcal{L}(\sum_i b_i p_i)} = \mathsf{DR}^{\mathsf{op}}_{g,A,\mathcal{L}}.$$

Invariance IV. (Twisting by pullback)

Let $\mathcal{B}\to\mathcal{S}$ be any line bundle on the base. By tensoring (0.7) with $\pi^*\mathcal{B}$, we obtain a new object of $\mathfrak{Pic}_{g,n,d}$ over $\mathcal{S}$:

$$\pi:\mathcal{C}\longrightarrow\mathcal{S},\quad p_1,...,p_n:\mathcal{S}\longrightarrow\mathcal{C},\quad \mathcal{L}\otimes\pi^*\mathcal{B}\longrightarrow\mathcal{C}. \tag{0.11}$$

Let $\mathsf{DR}^{\mathsf{op}}_{g,A,\mathcal{L}\otimes\pi^*\mathcal{B}}\in\mathsf{CH}^g_{\mathsf{op}}(\mathcal{S})$ be the twisted double ramification cycle associated with the new family (0.11) and the vector A. We have the invariance

$$\mathsf{DR}^{\mathsf{op}}_{g,A,\mathcal{L}\otimes\pi^*\mathcal{B}}=\mathsf{DR}^{\mathsf{op}}_{g,A,\mathcal{L}}.$$

Invariance V. (Vertical twisting)

Consider a partition of the genus, marking, and degree data:

$$g_1+g_2=g,\quad N_1\sqcup N_2=\{1,...,n\},\quad d_1+d_2=d, \tag{0.12}$$

which is *not* symmetric.(11) Such a partition defines a divisor

$$\Delta_1\in\mathsf{CH}^1(\mathfrak{C}_{g,n,d})$$

in the universal curve over $\mathfrak{Pic}_{g,n,d}$ by twisting by the (g_1,N_1,d_1)-component of a curve with a separating node with separating data (0.12).

By tensoring (0.7) with $\mathcal{O}_{\mathcal{C}}(\Delta_1)$, we obtain a new object of $\mathfrak{Pic}_{g,n,d}$ over $\mathcal{S}$:

$$\pi:\mathcal{C}\longrightarrow\mathcal{S},\quad p_1,...,p_n:\mathcal{S}\longrightarrow\mathcal{C},\quad \mathcal{L}(\Delta_1)\longrightarrow\mathcal{C}. \tag{0.13}$$

Let

$$\mathsf{DR}^{\mathsf{op}}_{g,A,\mathcal{L}(\Delta_1)}\in\mathsf{CH}^g_{\mathsf{op}}(\mathcal{S})$$

be the twisted double ramification cycle associated with the new family (0.13) and the vector A. We have the invariance

$$\mathsf{DR}^{\mathsf{op}}_{g,A,\mathcal{L}(\Delta_1)}=\mathsf{DR}^{\mathsf{op}}_{g,A,\mathcal{L}}. \tag{0.14}$$

For *symmetric* separating data (0.12), equality (0.14) holds with $\Delta_1\subset\mathfrak{C}_{g,n,d}$ defined as the full preimage of the locus $\Delta\subset\mathfrak{Pic}_{g,n,d}$ of curves with a separating node (0.12). Then, equality (0.14) follows from Invariance IV with $\mathcal{B}=\mathcal{O}(\Delta)$.

(11) We require $(g_1,N_1,d_1)\neq(g_2,N_2,d_2)$ so that the two sides of a separating node with separating data (0.12) can be distinguished.

Invariance VI. (Partial stabilization)

Consider a second family of prestable n-pointed genus-g curves over $\mathcal{S}$,

$$\pi'\colon \mathcal{C}' \longrightarrow \mathcal{S}, \quad p'_1, ..., p'_n\colon \mathcal{S} \longrightarrow \mathcal{C}',$$

together with a birational $\mathcal{S}$-morphism

$$f\colon \mathcal{C}' \longrightarrow \mathcal{C}, \quad f \circ p'_i = p_i.$$

A line bundle of relative degree d is defined on $\mathcal{C}'$ by

$$f^*\mathcal{L} \longrightarrow \mathcal{C}'.$$

We require the following property to hold:

If the section p'_i meets the exceptional locus of f, then $a_i = 0$.

Let $\mathsf{DR}^{\mathsf{op}}_{g,A,f^*\mathcal{L}} \in \mathsf{CH}^g_{\mathsf{op}}(\mathcal{S})$ be the twisted double ramification cycle associated with the new family

$$\pi'\colon \mathcal{C}' \longrightarrow \mathcal{S}, \quad p'_1, ..., p'_n\colon \mathcal{S} \longrightarrow \mathcal{C}', \quad f^*\mathcal{L} \longrightarrow \mathcal{C}' \tag{0.15}$$

and the vector A. We have the invariance

$$\mathsf{DR}^{\mathsf{op}}_{g,A,f^*\mathcal{L}} = \mathsf{DR}^{\mathsf{op}}_{g,A,\mathcal{L}}.$$

Theorem 0.7 provides two paths to viewing the above invariance properties. The invariances can be seen either from formal properties of the geometric construction of the universal twisted double ramification cycle or from formal symmetries of Pixton's formula. In fact, all invariances hold not only for the codimension-g part $\mathsf{P}^g_{g,A,d}$ which computes the double ramification cycle, but for the full mixed-degree class $\mathsf{P}^\bullet_{g,A,d}$.

For example, Invariance VI on the formula side says that for the maps

$$\varphi_{\mathcal{L}}, \varphi_{f^*\mathcal{L}}\colon \mathcal{S} \longrightarrow \mathfrak{Pic}_{g,n,d}$$

obtained from the families (0.7) and (0.15), we have

$$\varphi^*_{f^*\mathcal{L}} \mathsf{P}^g_{g,A,d} = \varphi^*_{\mathcal{L}} \mathsf{P}^g_{g,A,d} \in \mathsf{CH}^g_{\mathsf{op}}(\mathcal{S})$$

for $\mathsf{P}^g_{g,A,d} \in \mathsf{CH}^g_{\mathsf{op}}(\mathfrak{Pic}_{g,n,d})$.

Proofs of all of the invariances will be presented in §7. The above invariances (together with geometric definitions when transversality to the Abel–Jacobi map holds) do not characterize([12]) $\mathsf{DR}^{\mathsf{op}}$.

([12]) Further geometric properties are required, see [41, §1.6] for a discussion.

0.7. Universal formula in degree zero

The most efficient statement of the double ramification cycle formula on the Picard stack of curves occurs in the degree $d=0$ case with *no* markings. In order to avoid([13]) the unpointed genus 1 case, let $g\neq 1$.

The specialization of Theorem 0.7 to $d=0$ calculates $\mathsf{DR}^{\mathsf{op}}_{g,\varnothing}$ as the value at $r=0$ of the degree-g part of

$$\exp\left(-\frac{1}{2}\eta\right)\sum_{\substack{\Gamma_\delta\in\mathsf{G}_{g,0,0}\\ w\in\mathsf{W}_{\Gamma_\delta,r}}}\frac{r^{-h^1(\Gamma_\delta)}}{|\mathrm{Aut}(\Gamma_\delta)|}j_{\Gamma_\delta *}\Bigg[\prod_{e=(h,h')\in\mathrm{E}(\Gamma_\delta)}\frac{1}{\psi_h+\psi_{h'}}\\ \times\left(1-\exp\left(-\frac{w(h)w(h')}{2}(\psi_h+\psi_{h'})\right)\right)\Bigg]$$

as an operational Chow class on $\mathfrak{Pic}_g=\mathfrak{Pic}_{g,0,0}$. The full statement of Theorem 0.7 can be recovered from the above $d=0$ specialization via pullback under the map

$$\begin{aligned}\mathfrak{Pic}_{g,n,d}&\longrightarrow\mathfrak{Pic}_g,\\ (C,p_1,...,p_n,\mathcal{L})&\longmapsto\left(C,\mathcal{L}\left(-\sum_{i=1}^n a_ip_i\right)\right).\end{aligned}$$

Indeed, the above map is the composition of the morphism

$$\begin{aligned}\tau_{-A}\colon\mathfrak{Pic}_{g,n,d}&\longrightarrow\mathfrak{Pic}_{g,n,0},\\ (C,p_1,...,p_n,\mathcal{L})&\longmapsto\left(C,p_1,...,p_n,\mathcal{L}\left(-\sum_{i=1}^n a_ip_i\right)\right),\end{aligned}$$

with the morphism

$$F\colon\mathfrak{Pic}_{g,n,0}\longrightarrow\mathfrak{Pic}_g$$

forgetting the markings $p_1,...,p_n$.

- For the $\mathsf{DR}^{\mathsf{op}}_{g,A}$ side of Theorem 0.7, Invariance II implies

$$F^*\mathsf{DR}^{\mathsf{op}}_{g,\varnothing}=\mathsf{DR}^{\mathsf{op}}_{g,\mathbf{0}}$$

for the zero vector $\mathbf{0}\in\mathbb{Z}^n$. Furthermore, Invariance III implies

$$\tau^*_{-A}\mathsf{DR}^{\mathsf{op}}_{g,\mathbf{0}}=\mathsf{DR}^{\mathsf{op}}_{g,A}.$$

- For the $\mathsf{P}^g_{g,A,d}$ side of Theorem 0.7, the corresponding invariance properties of Pixton's formula (discussed in §7) yield the parallel transformation

$$\tau^*_{-A}F^*\mathsf{P}^g_{g,\varnothing,0}=\mathsf{P}^g_{g,A,d}.$$

([13]) For $g=1$, a parallel formula holds for $n=1$ and $A=(0)$.

Therefore, the equality in Theorem 0.7 for general A and d follows from the specialization to $A=\varnothing$ and $d=0$. For certain steps in our proof of Theorem 0.7, the $A=\varnothing$ and $d=0$ geometry is advantageous and is used.

By restricting to suitable open subsets of $\mathfrak{Pic}_g$, we can simplify the $d=0$ formula even further. Let

$$\mathfrak{Pic}_g^{\mathsf{ct}} \subset \mathfrak{Pic}_g$$

be the locus where the curve C is of compact type. We obtain

$$\mathsf{DR}^{\mathsf{op}}_{g,\varnothing}|_{\mathfrak{Pic}_g^{\mathsf{ct}}} = \frac{\theta^g}{g!}, \quad \text{for } \theta = -\frac{1}{2}\left(\eta + \sum_{\Delta} d_\Delta^2 [\Delta]\right), \tag{0.16}$$

where the sum is over the boundary divisors $\Delta \subset \mathfrak{Pic}_g$ on which generically the curve splits into two components carrying line bundles of degrees d_Δ and $-d_\Delta$. The class θ here may be viewed as a universal theta divisor on $\mathfrak{Pic}_g^{\mathsf{ct}}$.

Formula (0.16) was first written on the moduli space of stable curves of compact type in [33], [35]. The operational Chow class $\mathsf{DR}^{\mathsf{op}}_{g,\varnothing}$ on $\mathfrak{Pic}_g$, however, is *not* the power of a divisor.

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1. Notation, conventions, and the plan

1.1. Ground field

In the introduction, the ground field was the complex numbers $\mathbb{C}$. However, for the remainder of the paper, we will work more generally over a field K of characteristic zero. We will make essential use of the results of [43] which are stated over $\mathbb{C}$, but also hold over $\mathbb{Q}$ by the following standard argument:

(i) Both the DR cycle (via the b-Chow approach [37]) and Pixton's class are defined over $\mathbb{Q}$.

(ii) Rational equivalence of cycles uses finitely many subschemes and rational functions and hence descends to a finitely generated $\mathbb{Q}$-subalgebra of $\mathbb{C}$. A non-empty scheme of finite type over $\mathbb{Q}$ has points over some finite extension, and hence the rational equivalence descends to a finite extension of $\mathbb{Q}$.

(iii) The rational equivalence descends (via a Galois argument) further to $\mathbb{Q}$, since we work in Chow with rational coefficients.

By similar arguments, our results are in fact true over $\mathbb{Z}[1/N]$ for a positive integer N depending on the ramification data. Understanding what happens at small primes (or integral Chow groups) is an interesting question.

1.2. Basics

Let K be a ground field of characteristic zero. When we work in the logarithmic category, we assume $\operatorname{Spec} K$ to be equipped with the trivial log structure.

We write $\overline{\mathcal{M}}f$ for the stack of all stable (ordered) marked curves over K and $\mathfrak{M}$ for the stack of all prestable curves with ordered marked points. Both come with natural log structures, and the universal marked curves over these spaces are naturally log curves. For $\overline{\mathcal{M}}$, the log structure is described in [46]. The same construction applies unchanged to $\mathfrak{M}$, see [32, Appendix A]. The natural open immersion

$$\overline{\mathcal{M}} \longrightarrow \mathfrak{M}$$

is strict (though, in contrast to [32], [46], we order the markings of our log curves). We use subscripts g and n to fix the genus and number of markings when necessary.

Let $\mathfrak{C}$ be the universal prestable curve over $\mathfrak{M}$. For efficiency of notation, we will also denote by $\mathfrak{C}$ the universal curve over the various other moduli stacks of curves with additional structure which will appear in the paper. These universal curves are always obtained by pulling-back $\mathfrak{C}$ over $\mathfrak{M}$.

For the convenience of the reader, we provide the key notation in Table 1.

$\mathfrak{M}_{g,n}$	stack of prestable marked curves of genus g with n markings
$\overline{\mathcal{M}}_{g,n}$	stack of stable marked curves of genus g with n markings
$\mathfrak{C}$	universal prestable curve over $\mathfrak{M}$
$A=(a_1,\ldots,a_n)$	$A\in\mathbb{Z}^n$ with $\sum_{i=1}^n a_i$ denoted by d
$\mathfrak{Pic}_{g,n}$	Picard stack (§2.1)
$\mathfrak{Pic}_{g,n}^{\mathrm{rel}}$	relative universal Picard stack (§2.1)
$\mathsf{CH}_{\mathsf{op}}$	operational Chow group (§2.2)
ξ_i	tautological class on $\mathfrak{Pic}_{g,n}$ (§0.3.3)
$\eta_{a,b}$	tautological class on $\mathfrak{Pic}_{g,n}$ (§0.3.3)
ψ_i	tautological class on $\mathfrak{Pic}_{g,n}$ (§0.3.3)
$\mathsf{P}_{g,A,d}^c$	Pixton's cycle (§0.3.5)
$c(\varphi)$	homomorphisms $c(\varphi)\colon \mathsf{CH}_*(B)\to\mathsf{CH}_{*-p}(B)$ given by an operational class $c\in\mathsf{CH}_{\mathsf{op}}^p(\mathfrak{X})$ and $\varphi\colon B\to\mathfrak{X}$ with B finite-type scheme (§2.2)
$\mathsf{DR}_{g,A}^{\mathsf{op}}$	operational DR cycle (§3)

Table 1. Key notation.

1.3. Plan of the paper

In §2 we treat several technical issues related to operational Chow groups of the Picard stack $\mathfrak{Pic}$. In fact, we develop a general theory of operational Chow groups of algebraic stacks which are locally of finite type over K. The theory is certainly known to experts, but for our later results, we will require the precise definitions. In particular, with a proper representable morphism of algebraic stacks, we associate an operational class, which will be the key to constructing the operational double ramification cycle.

The core of the paper starts in §3 where we give three equivalent definitions of the universal double ramification cycle on $\mathfrak{Pic}$. Our first definition is by taking a closure in the spirit of [38] which is simple, but rather difficult to work with. The second is via logarithmic geometry following [56]. The third is a b-Chow definition along the path of [39]. In §3.5, we give an explicit description of the set-theoretic image of the double ramification cycle in $\mathfrak{Pic}$. We prove Theorem 0.1 in §3.7.

In §4, we discuss properties of Pixton's cycle $\mathsf{P}_{g,A,d}^c$ defined in §0.3.5. In particular, formal properties of Pixton's cycle parallel to the invariances of the double ramification cycles are proven. Compatibilities with definitions in previously studied cases are also proven.

In §5, we prove Theorem 0.7, the main result of the paper, by an eventual reduction to the formula of [43] in the case of target $\mathbb{P}^n$ for large n. A crucial step in the proof is the matching of the double ramification cycle defined in [43] via rubber maps with our new universal definition on $\mathfrak{Pic}$ in a suitable sense when the target is $\mathbb{P}^n$. The matching is verified in §6, where we follow the pattern of the proof given in [56] in case the target is a point.

In §7, we prove the invariance properties of §0.6. Theorems 0.8 and 0.9 are proven as a consequence of Theorem 0.7 in §8. The connections between the vanishing result of Theorem 0.8 and past (and future) work is discussed in §8.5.

2. Picard stacks and operational Chow

2.1. The Picard stack and relative Picard space

Our stacks will be with respect to the fppf topology ([63, Definition 9.1.1]). We define the Picard stack $\mathfrak{Pic}_{g,n}$ as the fibered category over $\mathfrak{M}_{g,n}$ whose fiber over a scheme $T \to \mathfrak{M}_{g,n}$ is the groupoid of line bundles on $\mathfrak{C}_{g,n} \times_{\mathfrak{M}_{g,n}} T$ with morphisms given by isomorphisms of line bundles, see [51, Example 14.4.7]. We define the relative Picard space $\mathfrak{Pic}^{\mathrm{rel}}_{g,n}/\mathfrak{M}_{g,n}$ to be the quotient of $\mathfrak{Pic}_{g,n}$ by its relative inertia over $\mathfrak{M}_{g,n}$. Equivalently, $\mathfrak{Pic}^{\mathrm{rel}}_{g,n}$ is the fppf-sheafification of the fibered category of *isomorphism classes* of line bundles on $\mathfrak{C}_{g,n} \times_{\mathfrak{M}_{g,n}} T$, see [14, Chapter 8] and [27, Chapter 9].

Relative representability of $\mathfrak{Pic}^{\mathrm{rel}}_{g,n}/\mathfrak{M}_{g,n}$ by smooth algebraic spaces can be checked locally on $\mathfrak{M}_{g,n}$. It then follows from [3, Appendix], as the curve

$$\mathfrak{C}_{g,n} \longrightarrow \mathfrak{M}_{g,n}$$

is flat, proper, relatively representable by algebraic spaces, and cohomologically flat in dimension zero (reduced and connected geometric fibers). The Picard stack $\mathfrak{Pic}_{g,n}$ is a $\mathbb{G}_m$-gerbe over $\mathfrak{Pic}^{\mathrm{rel}}_{g,n}$, and hence is a (smooth) algebraic stack. In particular, $\mathfrak{Pic}_{g,n}$ is smooth over K of pure dimension $4g-4+n$, and $\mathfrak{Pic}^{\mathrm{rel}}_{g,n}$ is smooth over K of pure dimension $4g-3+n$.

Remark 2.1. We will moreover assume $(g,n)\neq(1,0)$. Then $\mathfrak{M}_{g,n}$, and hence $\mathfrak{Pic}^{\mathrm{rel}}$, $\mathfrak{Pic}$, and anything of Deligne–Mumford type over them, has affine stabilizers, and so is therefore stratified by global quotient stacks in the sense of [49]. The latter property will be important for some intersection-theoretic computations, in particular the proof of Proposition 2.16.

2.2. Operational Chow groups of algebraic stacks

Our goal here is to define the operational Chow group of $\mathfrak{Pic}_{g,n}$, following [29, Chapter 17]. In fact, we construct an operational Chow group for any algebraic stack locally of finite type over a field. The definition is a simple generalization of [29].

Definition 2.2. Let $\mathfrak{Y}$ be a locally finite type algebraic stack over K. Let p be an integer. A bivariant class c in the pth operational Chow group $\mathsf{CH}^p_{\mathsf{op}}(\mathfrak{Y})$ is a collection of homomorphisms

$$c(\varphi)^m\colon \mathsf{CH}_m(B) \longrightarrow \mathsf{CH}_{m-p}(B)$$

for all maps $\varphi\colon B\to\mathfrak{Y}$ where B is a scheme of finite type over K, and for all integers m, compatible with proper pushforward, flat pullback, and Gysin homomorphisms for regular embeddings (conditions (C1)–(C3) in [29, §17.1]).

For a class $\alpha\in\mathsf{CH}_m(B)$, we will sometimes write $c(\alpha)$ in place of $c(\varphi)^m(\alpha)$, if the morphism φ is clear.

Such a definition for the operational Chow group of a Deligne–Mumford stack is given in [24]. To be able to use Chow groups on algebraic stacks as defined in [49] for algebraic stacks of finite type over a field, we will use the following result.

LEMMA 2.3. *Let $f\colon\mathfrak{A}\to\mathfrak{B}$ be a morphism over a field K, where $\mathfrak{B}$ is an algebraic stack locally of finite type over K, and $\mathfrak{A}$ is an algebraic stack of finite type over K. Then, there exists a factorization of f via a commutative diagram*

$$\begin{tikzcd} \mathfrak{A} \arrow[rr, "f"] \arrow[dr] & & \mathfrak{B} \\ & \mathfrak{B}', \arrow[ur] & \end{tikzcd}$$

where $\mathfrak{B}'\to\mathfrak{B}$ is an open immersion and $\mathfrak{B}'$ is quasi-compact (and hence of finite type).

Proof. We cover $\mathfrak{B}$ by affine flat finite presentation morphisms $\{V_i\to\mathfrak{B}\}_{i\in I}$. Let U_i be the image of V_i in $\mathfrak{B}$, for all i. The U_i are open, and the $f^{-1}(U_i)$ cover $\mathfrak{A}$. As $\mathfrak{A}$ is quasi-compact, there is a finite subset $J\subset I$ such that $\{f^{-1}(U_i)\}_{i\in J}$ covers $\mathfrak{A}$. Then,

$$\mathfrak{B}' = \bigcup_{i\in J} U_i$$

defines the required factorization. □

For representable morphisms (representable by algebraic spaces) $f\colon\mathfrak{X}\to\mathfrak{Y}$, we can also define the operational Chow group $\mathsf{CH}^p_{\mathsf{op}}(\mathfrak{X}\to\mathfrak{Y})$ as a collection of morphisms

$$c(\varphi)^m\colon \mathsf{CH}_m(B) \longrightarrow \mathsf{CH}_{m-p}(B\times_{\mathfrak{Y}}\mathfrak{X})$$

for all maps $\varphi\colon B\to\mathfrak{Y}$, where B is a scheme of finite type over K, and for all integers m, compatible with proper pushforward, flat pullback, and Gysin homomorphisms. We have

$$\mathsf{CH}^p_{\mathsf{op}}(\mathfrak{X}) = \mathsf{CH}^p_{\mathsf{op}}(\mathrm{id}\colon \mathfrak{X}\to\mathfrak{X}).$$

We have products, pullbacks, and proper representable pushforwards on these operational Chow groups of algebraic stacks as described in [29, §17.2] satisfying the properties described there.

Remark 2.4. Even for B a scheme and $\pi\colon\mathfrak{X}\to\mathfrak{Y}$ representable, the fiber product $B\times_{\mathfrak{Y}}\mathfrak{X}$ can be an algebraic space. Therefore, some care is needed when generalizing classical constructions such as the product

$$\mathsf{CH}^a_{\mathsf{op}}(\mathfrak{X})\times\mathsf{CH}^b_{\mathsf{op}}(\pi\colon\mathfrak{X}\to\mathfrak{Y})\longrightarrow\mathsf{CH}^{a+b}_{\mathsf{op}}(\pi\colon\mathfrak{X}\to\mathfrak{Y}).$$

Indeed, for $c\in\mathsf{CH}^a_{\mathsf{op}}(\mathfrak{X})$ and $d\in\mathsf{CH}^b_{\mathsf{op}}(\pi\colon\mathfrak{X}\to\mathfrak{Y})$, we want to define

$$c\cdot d\in\mathsf{CH}^{a+b}_{\mathsf{op}}(\pi\colon\mathfrak{X}\to\mathfrak{Y}).$$

For a map $\varphi\colon B\to\mathfrak{Y}$ with B a finite-type scheme fitting in a pullback diagram

$$\begin{tikzcd} B\times_{\mathfrak{Y}}\mathfrak{X} \arrow[r, "\varphi'"] \arrow[d] & \mathfrak{X} \arrow[d, "\pi"] \\ B \arrow[r, "\varphi"] & \mathfrak{Y}, \end{tikzcd}$$

we want to define the induced map

$$(c\cdot d)(\varphi)^m\colon\mathsf{CH}_m(B)\longrightarrow\mathsf{CH}_{m-a-b}(B\times_{\mathfrak{Y}}\mathfrak{X})$$

as the composition

$$\mathsf{CH}_m(B)\xrightarrow{d(\varphi)^m}\mathsf{CH}_{m-b}(B\times_{\mathfrak{Y}}\mathfrak{X})\xrightarrow{c(\varphi')^{m-b}}\mathsf{CH}_{m-a-b}(B\times_{\mathfrak{Y}}\mathfrak{X}).$$

But, a priori, the map $c(\varphi')^{m-b}$ does not make sense, since the domain $B\times_{\mathfrak{Y}}\mathfrak{X}$ of φ' is an algebraic space. However, given a collection

$$c=c(\varphi')^n\colon\mathsf{CH}_n(B')\longrightarrow\mathsf{CH}_{n-a}(B')$$

for all finite-type schemes B' with $\varphi'\colon B'\to\mathfrak{X}$, we can construct a collection of maps

$$c(\varphi')^n\colon\mathsf{CH}_n(B')\longrightarrow\mathsf{CH}_{n-a}(B')$$

for $\varphi' \colon B' \to \mathfrak{X}$, with B' a finite-type algebraic space via [76, §5.1]. Indeed, for each integral closed substack $Z \subset B'$, [76, §5.1] defines an action of c on $[Z]$ which is independent of the chosen cover of the algebraic space by a scheme. This action induces a map

$$\mathsf{CH}_n(B') \longrightarrow \mathsf{CH}_{n-a}(B'),$$

which commutes with proper pushforward and flat morphisms via [76, Lemma 5.3] and is compatible with Gysin homomorphisms. Applying this to

$$\varphi' \colon B' = B \times_{\mathfrak{Y}} \mathfrak{X} \longrightarrow \mathfrak{X}$$

with $n = m - b$ gives the desired map $c(\varphi')^{m-b}$.

For a proper representable morphism $\pi \colon \mathfrak{X} \to \mathfrak{Y}$ which is flat of relative dimension q, the pushforward

$$\mathsf{CH}^*_{\mathsf{op}}(\pi \colon \mathfrak{X} \to \mathfrak{Y}) \longrightarrow \mathsf{CH}^*_{\mathsf{op}}(\mathfrak{Y})$$

can be extended to a pushforward

$$\mathsf{CH}^p_{\mathsf{op}}(\mathfrak{X}) \longrightarrow \mathsf{CH}^{p-q}_{\mathsf{op}}(\mathfrak{Y})$$

as follows. Because π is flat, the pullback π^* gives a natural element in

$$\mathsf{CH}^{-q}_{\mathsf{op}}(\pi \colon \mathfrak{X} \to \mathfrak{Y}),$$

and then we can compose

$$\mathsf{CH}^p_{\mathsf{op}}(\mathfrak{X}) \longrightarrow \mathsf{CH}^p_{\mathsf{op}}(\mathfrak{X}) \times \mathsf{CH}^{-q}_{\mathsf{op}}(\pi \colon \mathfrak{X} \to \mathfrak{Y})$$

given by $c \mapsto (c, \pi^*)$ with the product and the pushforward maps

$$\mathsf{CH}^p_{\mathsf{op}}(\mathfrak{X}) \times \mathsf{CH}^{-q}_{\mathsf{op}}(\pi \colon \mathfrak{X} \to \mathfrak{Y}) \longrightarrow \mathsf{CH}^{p-q}_{\mathsf{op}}(\pi \colon \mathfrak{X} \to \mathfrak{Y}) \longrightarrow \mathsf{CH}^{p-q}_{\mathsf{op}}(\mathfrak{Y}),$$

yielding the desired pushforward map

$$\pi_* \colon \mathsf{CH}^p_{\mathsf{op}}(\mathfrak{X}) \longrightarrow \mathsf{CH}^{p-q}_{\mathsf{op}}(\mathfrak{Y}).$$

This may for example be applied to the universal curve $\pi \colon \mathfrak{C}_{g,n} \to \mathfrak{Pic}_{g,n}$. A similar construction also works for π proper representable and lci. This pushforward map commutes with pullback of operational classes.

2.3. Relationship to usual Chow groups

Let $\mathfrak{Y}$ be a locally finite type algebraic stack over K, and $(\mathfrak{U}_i)_{i\in\mathbb{N}}$ an increasing sequence of finite-type open substacks of $\mathfrak{Y}$ with

$$\bigcup_i \mathfrak{U}_i = \mathfrak{Y}.$$

In particular, for a finite-type scheme B/K, every map $B\to\mathfrak{Y}$ factors via some $\mathfrak{U}_i$. We have pullback maps

$$\mathsf{CH}^*_{\mathsf{op}}(\mathfrak{Y}) \longrightarrow \mathsf{CH}^*_{\mathsf{op}}(\mathfrak{U}_i) \tag{2.1}$$

which induce a map

$$\Phi\colon \mathsf{CH}^*_{\mathsf{op}}(\mathfrak{Y}) \longrightarrow \lim_i \mathsf{CH}^*_{\mathsf{op}}(\mathfrak{U}_i) \tag{2.2}$$

to the inverse limit of the $\mathsf{CH}^*_{\mathsf{op}}(\mathfrak{U}_i)$, with transition maps given by pullback of operational classes along open immersions.

LEMMA 2.5. *The map*

$$\Phi\colon \mathsf{CH}^*_{\mathsf{op}}(\mathfrak{Y}) \longrightarrow \lim_i \mathsf{CH}^*_{\mathsf{op}}(\mathfrak{U}_i)$$

is an isomorphism of abelian groups.

Proof. We first show injectivity. Let c in $\mathsf{CH}^*_{\mathsf{op}}(\mathfrak{Y})$ with $\Phi(c)=0$. For every $B\to\mathfrak{Y}$ with B/K of finite type, we get a map

$$c(B/\mathfrak{Y})\colon \mathsf{CH}_*(B) \longrightarrow \mathsf{CH}_*(B).$$

There exists i such that the map $B\to\mathfrak{Y}$ factors via $\mathfrak{U}_i$. Then, $\Phi(c)=0$ implies that

$$c(B/\mathfrak{U}_i)\colon \mathsf{CH}_*(B) \longrightarrow \mathsf{CH}_*(B)$$

is the zero map. By definition of the pullback,

$$c(B/\mathfrak{Y}) = c(B/\mathfrak{U}_i).$$

Next, we show surjectivity. Suppose that we have a compatible collection

$$c_i \in \mathsf{CH}^*_{\mathsf{op}}(\mathfrak{U}_i).$$

We will build $c\in\mathsf{CH}^*_{\mathsf{op}}(\mathfrak{Y})$ as follows. Let $B\to\mathfrak{Y}$ with B/K of finite type. There exists N such that, for all $i\geqslant N$, the map $B\to\mathfrak{Y}$ factors via $\mathfrak{U}_i$. Then, for all $i\geqslant N$, we have maps

$$c(B/\mathfrak{U}_i)\colon \mathsf{CH}_*(B) \longrightarrow \mathsf{CH}_*(B),$$

and the compatibly means $c_i(B/\mathfrak{U}_i)=c_j(B/\mathfrak{U}_i)$ for all $i,j\geqslant N$. We define $c=c_N$, which clearly is sent by Φ to the c_i.

To conclude, we must check that c satisfies the axioms of an operational class. This follows easily from the fact that each $B\to\mathfrak{Y}$ factors via some $\mathfrak{U}_i$. $\square$

LEMMA 2.6. ([76, Proposition 5.6]) *Let $\mathfrak{Y}$ be a smooth finite-type Deligne–Mumford stack over K of pure dimension d, and let $\iota: \mathfrak{Y} \to \mathfrak{Y}$ be the identity. Then, for $m \geqslant 0$, the map*

$$\begin{aligned} \mathsf{CH}^m_{\mathsf{op}}(\mathfrak{Y}) &\longrightarrow \mathsf{CH}_{d-m}(\mathfrak{Y}), \\ \alpha &\longmapsto \alpha(\iota)([\mathfrak{Y}]), \end{aligned} \tag{2.3}$$

is an isomorphism.

Combining Lemmas 2.5 and 2.6, we immediately obtain the following result.

COROLLARY 2.7. *Let $\mathfrak{Y}$ be a smooth Deligne–Mumford stack over K of pure dimension d, and let $\mathfrak{U}_i$ be a sequence of finite-type open substacks with $\bigcup_{i \in I} \mathfrak{U}_i = \mathfrak{Y}$. Then, the natural map*

$$\Phi: \mathsf{CH}^m_{\mathsf{op}}(\mathfrak{Y}) \longrightarrow \lim_{i \in I} \mathsf{CH}_{d-m}(\mathfrak{U}_i) \tag{2.4}$$

obtained by combining (2.2) and (2.3) is an isomorphism.

As a final remark,([14]) we note that there exists a map of Chow groups in the opposite direction of (2.3) in greater generality. Let $\mathfrak{Y}$ be a smooth algebraic stack of finite type over K and of pure dimension d which has a stratification([15]) by quotient stacks. Then, there exists a map

$$\Psi: \mathsf{CH}_{d-m}(\mathfrak{Y}) \longrightarrow \mathsf{CH}^m_{\mathsf{op}}(\mathfrak{Y})$$

from the Chow group $\mathsf{CH}_*(\mathfrak{Y})$ constructed in [49] to the operational Chow group of $\mathfrak{Y}$ defined as follows. Given $\varphi: B \to \mathfrak{Y}$ with B a finite-type scheme, let

$$\varphi_B: B \longrightarrow B \times \mathfrak{Y}$$

be the diagonal morphism. Since $\mathfrak{Y}$ is smooth, φ_B is representable and is a local complete intersection of codimension d. For $\beta \in \mathsf{CH}_{d-m}(\mathfrak{Y})$, we define

$$\begin{aligned} \Psi(\beta)(\varphi): \mathsf{CH}_*(B) &\longrightarrow \mathsf{CH}_{*-m}(B), \\ \alpha &\longmapsto \varphi_B^!(\alpha \times \beta), \end{aligned}$$

where $\alpha \times \beta \in \mathsf{CH}_{*+d-m}(B \times \mathfrak{Y})$ is the exterior product of α and β as defined in [49, §3.2]. The collection of maps $\Psi(\beta)(\varphi)$ defines an element

$$\Psi(\beta) \in \mathsf{CH}^m_{\mathsf{op}}(\mathfrak{Y}).$$

For $\mathfrak{Y}$ a Deligne–Mumford stack, the map Ψ is the inverse of the map (2.3). However, for an arbitrary smooth algebraic stack $\mathfrak{Y}$, we do *not* know whether Ψ is injective or surjective.

([14]) We thank A. Kresch for related discussion.
([15]) See [49] for the precise definition. The property is always satisfied for Deligne–Mumford stacks.

2.4. Constructing an operational Chow class

2.4.1. Overview

Given a vector $A\in\mathbb{Z}^n$ of ramification data satisfying

$$\sum_{i=1}^{n} a_i = d,$$

we will construct in §3.3 a stack $\mathbf{Div}_{g,A}$ together with a proper representable Abel–Jacobi map

$$\mathbf{Div}_{g,A} \longrightarrow \mathfrak{Pic}_{g,n,d}.$$

We wish to define the twisted universal double ramification cycle $\mathsf{DR}^{\mathsf{op}}_{g,A}$ as the pushforward of the fundamental class of $\mathbf{Div}_{g,A}$ to $\mathfrak{Pic}_{g,n,d}$. However, two basic issues must be settled to carry out the construction:

- The stack $\mathbf{Div}_{g,A}$ is not Deligne–Mumford and is not quasi-compact, so the existence of a well-behaved fundamental class in the operational Chow group is not clear.
- The proper representable pushforward of [6, Appendix B] is only defined between finite-type-stacks, and so cannot be applied directly.(16)

To solve these problems, we provide here a very general construction which associates with a suitable proper morphism

$$a\colon \mathfrak{X} \longrightarrow \mathfrak{Y}$$

an operational Chow class on $\mathfrak{Y}$ which plays the role of the pushforward of the fundamental class of $\mathfrak{X}$. In §3.3, we will apply the result to construct the universal twisted double ramification cycle $\mathsf{DR}^{\mathsf{op}}_{g,A}$. We also verify certain basic properties such as invariance of the class under proper birational maps which will be important in §6.

2.4.2. Construction

Let $\mathfrak{X}$ and $\mathfrak{Y}$ be algebraic stacks locally of finite type over a field K and suppose we have a proper morphism

$$a\colon \mathfrak{X} \longrightarrow \mathfrak{Y}$$

of Deligne–Mumford type. Suppose further that $\mathfrak{Y}$ is smooth of pure dimension $\dim\mathfrak{Y}$ over the field, and $\mathfrak{X}$ is of pure dimension $\dim\mathfrak{X}$. Let

$$e = \dim\mathfrak{Y} - \dim\mathfrak{X}.$$

We will construct an operational Chow class associated with $\mathfrak{X}$ in $\mathsf{CH}^e_{\mathsf{op}}(\mathfrak{Y})$.

(16) Chow theory of non-finite-type algebraic stacks will be developed in [6, Appendix A].

For all finite-type schemes B with a morphism $\varphi\colon B\to\mathfrak{Y}$, and for all integers m, we must define maps

$$c(\varphi)^m\colon \mathsf{CH}_m(B)\longrightarrow \mathsf{CH}_{m-e}(B),$$

which are compatible under proper pushforward and flat pullback and satisfy commutativity (properties (C1)–(C3) of [29, §17.1]).

Let $[V]\in\mathsf{CH}_m(B)$ be an irreducible cycle, and let $i_V\colon V\to B$ be the inclusion. Let $V\to B\to\mathfrak{Y}$ be factored as in Lemma 2.3 into $V\to\mathfrak{Y}'\to\mathfrak{Y}$, where $\mathfrak{Y}'$ is of finite type and $\mathfrak{Y}'\to\mathfrak{Y}$ is an open immersion.

We form the diagram

$$\begin{tikzcd} \mathfrak{X}'\times_{\mathfrak{Y}'}V \arrow[r] \arrow[d, "\psi_V"] & \mathfrak{X}'\times V \arrow[d, "a\times\mathrm{id}"] \\ V \arrow[r, "\varphi_V"] & \mathfrak{Y}'\times V, \end{tikzcd} \tag{2.5}$$

where $\mathfrak{X}'$ is the inverse image of $\mathfrak{Y}'$ under a. Each stack in this diagram is of finite type, and therefore has a Chow group in the sense of [49]. Since $\mathfrak{Y}$ is smooth over K, the map φ_V is a regular embedding of codimension $\dim\mathfrak{Y}$. Also φ_V is unramified, and hence a regular local embedding, so Kresch's contruction yields a map

$$\varphi_V^!\colon \mathsf{CH}_m(\mathfrak{X}'\times V)\longrightarrow \mathsf{CH}_{m-\dim\mathfrak{Y}}(\mathfrak{X}'\times_{\mathfrak{Y}'}V).$$

In particular, $[\mathfrak{X}'\times V]$ is a class in dimension $\dim\mathfrak{X}+m$, so the class $\varphi_V^!([\mathfrak{X}\times V])$ lies in $\mathsf{CH}_{m-e}(\mathfrak{X}\times_{\mathfrak{Y}}V)$. The morphism ψ_V is proper and of Deligne–Mumford type, and so by [6, Appendix B] we have a pushforward ψ_{V*}.

Definition 2.8. We define a class $a_{\mathrm{op}}[\mathfrak{X}]\in\mathsf{CH}_{\mathsf{op}}(\mathfrak{Y})$ via the formula

$$\begin{aligned} c(\varphi)^m\colon Z_m(B)&\longrightarrow \mathsf{CH}_{m-e}(B),\\ [V]&\longmapsto i_{V*}\psi_{V*}\varphi_V^!([\mathfrak{X}'\times V]). \end{aligned}$$

We must verify that this construction passes to rational equivalence, is independent of the choices made, and satisfies the properties (C1)–(C3). After verifying in Lemma 2.9 independence on the choice of factorization, we follow the logic in [29]: we verify in Lemmas 2.11–2.13 of §2.4.3 that the properties (C1)–(C3) hold on the level of cycles, and finally in Lemma 2.15 we use these to show that the construction passes to rational equivalence.

LEMMA 2.9. *The class $c(\varphi)^m([V])$ defined above is independent of the chosen factorization $V\to\mathfrak{Y}'\to\mathfrak{Y}$.*

Proof. Let $V \to \mathfrak{Y}' \to \mathfrak{Y}$ and $V \to \mathfrak{Y}'' \to \mathfrak{Y}$ be two such factorizations. By considering $\mathfrak{Y}' \cap \mathfrak{Y}''$ inside $\mathfrak{Y}$, we may restrict to the case where one is contained in the other. So we suppose there is an open immersion

$$\iota \colon \mathfrak{Y}'' \longrightarrow \mathfrak{Y}'.$$

Let $j \colon \mathfrak{X}'' \to \mathfrak{X}'$ be the induced map. Consider the diagram

$$\begin{tikzcd}
 & \mathfrak{X}'' \times_{\mathfrak{Y}''} V \arrow[rr] \arrow[dl, "\psi''_V"'] \arrow[dd, dashed, "j \times \mathrm{id}"] & & \mathfrak{X}'' \times V \arrow[dl, "a \times \mathrm{id}"'] \arrow[dd, "j \times \mathrm{id}"] \\
V \arrow[rr, "\varphi''_V"'] \arrow[dd, "\mathrm{id}"'] & & \mathfrak{Y}'' \times V \arrow[dd, "\iota \times \mathrm{id}"] & \\
 & \mathfrak{X}' \times_{\mathfrak{Y}'} V \arrow[rr] \arrow[dl, "\psi'_V"] & & \mathfrak{X}' \times V \arrow[dl, "a \times \mathrm{id}"] \\
V \arrow[rr, "\varphi'_V"'] & & \mathfrak{Y}' \times V, &
\end{tikzcd} \tag{2.6}$$

where $\mathfrak{X}'' \times_{\mathfrak{Y}''} V \to \mathfrak{X}' \times_{\mathfrak{Y}'} V$ is an isomorphism. We must show that

$$\psi'_{V*}(\varphi'_V)^!([\mathfrak{X}' \times V]) = \psi''_{V*}(\varphi''_V)^!([\mathfrak{X}'' \times V]).$$

Because φ''_V and φ'_V are both regular embeddings of the same codimension and the front square commutes, by the same proof as for [29, Theorem 6.2 (c)], we obtain

$$(\varphi''_V)^!([\mathfrak{X}'' \times V]) = (\varphi'_V)^!([\mathfrak{X}'' \times V]).$$

Therefore,

$$\psi''_{V*}(\varphi''_V)^!([\mathfrak{X}'' \times V]) = \psi''_{V*}(\varphi'_V)^!([\mathfrak{X}'' \times V]) = \psi''_{V*}(\varphi'_V)^!((j \times \mathrm{id})^*[\mathfrak{X}' \times V]), \tag{2.7}$$

since $j \times \mathrm{id}$ is an open immersion, so in particular flat, and the flat pullback of the fundamental class is the fundamental class itself. By the compatibility of the flat pullback and Gysin maps [49, §3.1], we obtain that (2.7) is equal to

$$\psi''_{V*}(j \times \mathrm{id})^*(\varphi'_V)^!([\mathfrak{X}' \times V]).$$

By commutativity of the left side of the cube and because of the pullback pushforward formula, we obtain

$$\psi''_{V*}(\varphi''_V)^!([\mathfrak{X}'' \times V]) = \psi''_{V*}(j \times \mathrm{id})^*(\varphi'_V)^!([\mathfrak{X}' \times V]) = (\psi'_V)_*(\varphi'_V)^!([\mathfrak{X}' \times V])$$

as required. □

2.4.3. Compatibility

We will now check that the maps $c(\varphi)^m$ defined in §2.4.2 are compatible under proper pushforward and flat pullback, and satisfy commutativity (properties (C1)–(C3) of [29, §17.1]).

The proper pushforward along DM-type maps between finite-type algebraic stacks over a field which are stratified by global quotient stacks is defined in [6, Appendix B]. We cannot use the results of [49] for compatibility with the proper pushforward, since Kresch discusses only projective pushforward. Nevertheless, we now show that the proper pushforward for DM-type maps of finite-type algebraic stacks over a field as defined in [6, Appendix B] is compatible with the Gysin maps of [49].

Proposition 2.10. *For a pullback diagram of algebraic stacks of finite type over K,*

$$\begin{tikzcd} \mathfrak{X}'' \arrow[r, "i''"] \arrow[d, "q"] & \mathfrak{Y}'' \arrow[d, "p"] \\ \mathfrak{X}' \arrow[r, "i'"] \arrow[d, "g"] & \mathfrak{Y}' \arrow[d, "f"] \\ \mathfrak{X} \arrow[r, "i"] & \mathfrak{Y}, \end{tikzcd}$$

where i is a regular local embedding of codimension d, p is a proper DM*-type morphism, and $\mathfrak{Y}'$ is stratified by global quotient stacks, we have*

$$i^! p_*(\alpha) = q_*(i^! \alpha)$$

for all $\alpha \in \mathsf{CH}_*(\mathfrak{Y}'')$.

Proof. A class α on $\mathfrak{Y}''$ is represented by a projective map $z'' \colon Z'' \to \mathfrak{Y}''$, a vector bundle $E'' \to Z''$ and a class $[V]$ in the naive Chow group of E'' represented by $V \subset E$.

We want to push α forward via the construction of [6, Appendix B]: by [6, Theorem B.17], it suffices to treat the case where $E'' \to Z'' \to \mathfrak{Y}''$ fits in a pullback diagram of the form

$$\begin{tikzcd} E'' \arrow[r] \arrow[d, "p''"] & Z'' \arrow[r, "z''"] \arrow[d, "p'"] & \mathfrak{Y}'' \arrow[d, "p"] \\ E' \arrow[r] & Z' \arrow[r, "z'"] & \mathfrak{Y}', \end{tikzcd} \tag{2.8}$$

where z' is projective. Then, the pushforward is defined by simply pushing forward on the level of bundles, so

$$p_*(z'', [V]) = (z', p''_*([V])).$$

If $W = p''(V)$, then $p''_*([V]) = \deg(V/W)[W]$.

Let N be the normal bundle $N_{\mathfrak{X}}\mathfrak{Y}$. The Gysin maps constructed in [49, §3.1] are described explicitly on level of representatives as follows: $i^!(z', [W])$ is represented by $[C_{W\times_{\mathfrak{Y}}\mathfrak{X}}W]$ as a class on the bundle

$$N|_{Z'\times_{\mathfrak{Y}}\mathfrak{X}} \oplus E'|_{Z'\times_{\mathfrak{Y}}\mathfrak{X}}$$

with the projective map $\tilde{z}' \colon Z' \times_{\mathfrak{Y}} \mathfrak{X} \to \mathfrak{X}'$ induced by z'. Hence,

$$i^! p_*(z'', [V]) = (\tilde{z}', \deg(V/W)[C_{W\times_{\mathfrak{Y}}\mathfrak{X}}W]).$$

Next, we study $q_* i^!(z'', [V])$. To start, $i^!(z'', [V])$ is represented by $[C_{V\times_{\mathfrak{Y}}\mathfrak{X}}V]$ as class on the bundle

$$N|_{Z''\times_{\mathfrak{Y}}\mathfrak{X}} \oplus E''|_{Z''\times_{\mathfrak{Y}}\mathfrak{X}}$$

with the projective map $Z'' \times_{\mathfrak{Y}} \mathfrak{X} \to \mathfrak{X}''$ induced by z''. We push $i^!(z'', [V])$ forward via the construction of [6, Appendix B]. We complete the diagram

$$\begin{tikzcd}
N|_{Z''\times_{\mathfrak{Y}}\mathfrak{X}} \oplus E''|_{Z''\times_{\mathfrak{Y}}\mathfrak{X}} \arrow[r] & Z'' \times_{\mathfrak{Y}} \mathfrak{X} \arrow[r, "\tilde{z}''"] & \mathfrak{X}'' \arrow[d, "q"] \\
 & & \mathfrak{X}'
\end{tikzcd}$$

to a pullback diagram so that we can pushforward on the levels of bundles:

$$\begin{tikzcd}
 & E'' \arrow[rr] \arrow[dd, "p''"] & & Z'' \arrow[rr, "z''"] \arrow[dd, "p'"] & & \mathfrak{Y}'' \arrow[dd, "p"] \\
N|_{Z''\times_{\mathfrak{Y}}\mathfrak{X}} \oplus E''|_{Z''\times_{\mathfrak{Y}}\mathfrak{X}} \arrow[rr] & & Z'' \times_{\mathfrak{Y}} \mathfrak{X} \arrow[ur] \arrow[rr, "\tilde{z}''"'] \arrow[dd, "\tilde{p}'"] & & \mathfrak{X}'' \arrow[ur, "i''"] \arrow[dd, "q"] & \\
 & E' \arrow[rr] & & Z' \arrow[rr, "z'"] & & \mathfrak{Y}' \arrow[dd, "f"] \\
N|_{Z'\times_{\mathfrak{Y}}\mathfrak{X}} \oplus E'|_{Z'\times_{\mathfrak{Y}}\mathfrak{X}} \arrow[rr] & & Z' \times_{\mathfrak{Y}} \mathfrak{X} \arrow[ur] \arrow[rr, "\tilde{z}'"] & & \mathfrak{X}' \arrow[ur, "i'"] \arrow[dd, "g"] & \\
 & & & & & \mathfrak{Y}. \\
 & & & & \mathfrak{X} \arrow[ur, "i"] &
\end{tikzcd}$$

The map $p' \colon Z'' \to Z'$ induces a map $\tilde{p}' \colon Z'' \times_{\mathfrak{Y}} \mathfrak{X} \to Z' \times_{\mathfrak{Y}} \mathfrak{X}$, and the square with q, $\tilde{p}'$, $\tilde{z}'$, and $\tilde{z}''$ is a pullback (as the pullback of a pullback square).

There is a map

$$q'' \colon N|_{Z''\times_{\mathfrak{Y}}\mathfrak{X}} \oplus E''|_{Z''\times_{\mathfrak{Y}}\mathfrak{X}} \longrightarrow N|_{Z'\times_{\mathfrak{Y}}\mathfrak{X}} \oplus E'|_{Z'\times_{\mathfrak{Y}}\mathfrak{X}}$$

induced by p'' such that it forms a pullback square, and so the pushforward along q is simply

$$q_*(i^!(z'',[V])) = q_*(\tilde{z}'',[C_{V\times_{\mathfrak{Y}}\mathfrak{X}}V]) = (\tilde{z}', q''_*([C_{V\times_{\mathfrak{Y}}\mathfrak{X}}V])).$$

The proof then reduces to comparing $q''_*([C_{V\times_{\mathfrak{Y}}\mathfrak{X}}V])$ and $\deg(V/W)([C_{W\times_{\mathfrak{Y}}\mathfrak{X}}W])$, which follows from [76, Lemma 3.15]. The result is a completely local statement and therefore extends from the setting of schemes to the setting of Deligne–Mumford stacks which we need here. $\square$

Let $\varphi\colon B\to\mathfrak{Y}$ and $\varphi'\colon B'\to\mathfrak{Y}$ be morphisms from finite-type schemes, and let

$$h\colon B' \longrightarrow B$$

be a $\mathfrak{Y}$-morphism. By Definition 2.8, we have morphisms

$$c(\varphi)^m\colon Z_m(B)\longrightarrow \mathsf{CH}_{m-e}(B) \quad\text{and}\quad c(\varphi')^m\colon Z_m(B')\longrightarrow \mathsf{CH}_{m-e}(B').$$

If h is proper, we have a pushforward map([17])

$$h_*\colon Z_m(B')\longrightarrow Z_m(B)$$

which descends to Chow.

Lemma 2.11. *If h is proper, then*

$$c(\varphi)^m\circ h_* = h_*\circ c(\varphi h)^m.$$

Proof. Let $[V']\in Z_m(B')$ for an irreducible cycle V'. Let $V=h(V')$. By definition,

$$h_*([V']) = \deg(V'/V)[V].$$

Let $\mathfrak{Y}'$ be a factorization of $V'\to V\to\mathfrak{Y}$. We have

$$(\mathrm{id}\times h)_*([\mathfrak{X}'\times V']) = \deg(V'/V)[\mathfrak{X}'\times V].$$

Via the commutative diagram

$$\begin{tikzcd}
 & & \mathfrak{X}'\times_{\mathfrak{Y}'}V' \arrow[rr] \arrow[dl, "\psi_{V'}"'] \arrow[dd, dashed, "\mathrm{id}\times h"] & & \mathfrak{X}'\times V' \arrow[dl] \arrow[dd, dashed, "\mathrm{id}\times h"] \\
 & V' \arrow[rr, "\varphi_{V'}"] \arrow[dl, "i_{V'}"'] \arrow[dd, "h"] & & \mathfrak{Y}'\times V' \arrow[dd, dashed, "\mathrm{id}\times h"] & \\
B' \arrow[dd, "h"] & & \mathfrak{X}'\times_{\mathfrak{Y}'}V \arrow[rr] \arrow[dl, "\psi_V"] & & \mathfrak{X}'\times V \arrow[dl] \\
 & V \arrow[rr, "\varphi_V"'] \arrow[dl, "i_V"] & & \mathfrak{Y}'\times V & \\
B \arrow[d, "\varphi"] & & & & \\
\mathfrak{Y}, & & & &
\end{tikzcd} \tag{2.9}$$

([17]) We use the same notation for the proper pushforward on the Chow groups.

we compute

$$\begin{aligned} c(\varphi)^m(h_*(\alpha)) &= c(\varphi)^m(\deg(V'/V)[V]) \\ &= i_{V*}\psi_{V*}\varphi^!_V(\deg(V'/V)[\mathfrak{X}'\times V]) \\ &= i_{V*}\psi_{V*}\varphi^!_V((\mathrm{id}\times h)_*[\mathfrak{X}'\times V']) \\ &= h_* i_{V'*}\psi_{V'*}\varphi^!_{V'}([\mathfrak{X}'\times V']) \\ &= h_* c(\varphi h)^m(\alpha), \end{aligned}$$

where the final line follows from compatibility of the Gysin map with proper representable pushforward (Proposition 2.10). □

If $h\colon B'\to B$ is flat of relative dimension n, we have a pullback map

$$h^*\colon Z_m(B)\longrightarrow Z_{m+n}(B')$$

which descends to Chow.

LEMMA 2.12. *If h is flat of relative dimension n,*

$$c(\varphi h)^{m+n}\circ h^* = h^*\circ c(\varphi)^m.$$

Proof. Let $[V]\in Z_m(B)$ for an irreducible cycle V. Let

$$V' = h^{-1}(V),$$

so $[V']=h^*([V])$. Let $\mathfrak{Y}'$ be a factorization of $V'\to V\to\mathfrak{Y}$. We have

$$(\mathrm{id}\times h)^*([\mathfrak{X}'\times V]) = [\mathfrak{X}'\times V'].$$

Via the commutative diagram (2.9), we compute

$$\begin{aligned} c(\varphi h)^{m+n}([V']) &= i_{V'*}\psi_{V'*}\varphi^!_{V'}([\mathfrak{X}'\times V']) \\ &= i_{V'*}\psi_{V'*}\varphi^!_{V'}((\mathrm{id}\times h)^*[\mathfrak{X}'\times V]) \\ &= h^* i_{V*}\psi_{V*}\varphi^!_V([\mathfrak{X}'\times V]) \\ &= h^* c(\varphi)^m([V]), \end{aligned}$$

where the final equality follows from the compatibility of the Gysin map with flat pullback for a morphism of finite-type algebraic stacks ([49, §3.1]) and the pullback and pushforward formulas. □

Let $\varphi: B \to \mathfrak{Y}$ be a morphism from a finite-type scheme as above. Let

$$g: B \longrightarrow Z$$

be a morphism of finite-type schemes, and let $i: Z' \to Z$ be a regular embedding of codimension f. Form the fiber square

$$\begin{tikzcd} B' \arrow[r] \arrow[d, "i'"] & Z' \arrow[d, "i"] \\ B \arrow[r, "g"] \arrow[d, "\varphi"] & Z \\ \mathfrak{Y}. & \end{tikzcd}$$

Let V be an irreducible cycle in B with inverse image $V' = (i')^{-1}(V)$. We choose a representative $i^![V] = \sum_j n_j [V'_j]$ in $Z_{m-f}(V')$.

Lemma 2.13. *We have*

$$i^! c(\varphi)^m([V]) = c(\varphi i')^{m-f}\left(\sum_j n_j [V'_j]\right).$$

Remark 2.14. In particular, once we have shown that the maps $c(\varphi)^m$ pass to rational equivalence, Lemma 2.13 will imply

$$i^! c(\varphi)^m(\alpha) = c(\varphi i')^{m-f}(i^! \alpha)$$

for $\alpha \in \mathsf{CH}_m(B)$.

Proof. From the equality $i^![V] = \sum n_j [V'_j]$, we deduce that

$$i^![\mathfrak{X}' \times V] = \sum n_j [\mathfrak{X}' \times V'_j].$$

Via the commutative diagram[18]

$$\begin{tikzcd}[column sep=small] & & & \mathfrak{X}' \times_{\mathfrak{Y}'} V' \arrow[rr] \arrow[dl, "\psi_{V'}"'] \arrow[dd, dashed, "\mathrm{id} \times i'"] & & \mathfrak{X}' \times V' \arrow[dl] \arrow[dd, dashed, "\mathrm{id} \times i'"] \\ & & V' \arrow[rr, "\varphi_{V'}"'] \arrow[dll] \arrow[dl, "i_{V'}"] \arrow[dd, "i'"] & & \mathfrak{Y}' \times V' \arrow[dd, dashed, "\mathrm{id} \times i'"] & \\ Z' \arrow[dd, "i"] & B' \arrow[l] \arrow[dd, "i'"] & & \mathfrak{X}' \times_{\mathfrak{Y}'} V \arrow[rr] \arrow[dl, "\psi_V"] & & \mathfrak{X}' \times V \arrow[dl] \\ & & V \arrow[rr, "\varphi_V"'] \arrow[dll] \arrow[dl, "i_V"] & & \mathfrak{Y}' \times V & \\ Z & B \arrow[l, "g"] \arrow[d, "\varphi"] & & & & \\ & \mathfrak{Y}, & & & & \end{tikzcd} \tag{2.10}$$

([18]) We should also add another layer of the diagram for the V'_j.

we compute

$$i^! c(\varphi)^m([V]) = i^! i_{V*}\psi_{V*}\varphi^!_V([\mathfrak{X}'\times V])$$

and

$$c(\varphi i')^{m-f}\left(\sum_j n_j[V'_j]\right) = \sum n_j i_{V'_j*}\psi_{V'_j*}\varphi^!_{V'_j}([\mathfrak{X}'\times V'_j]) = i_{V'*}\psi_{V'*}\varphi^!_{V'}(i^![\mathfrak{X}'\times V]).$$

We deduce equality of these expressions by using the compatibility of Gysin maps with proper pushforward (Proposition 2.10) and then the commutativity of Gysin maps from [49] to obtain $\varphi^!_{V'} i^! = i^! \varphi^!_V$. □

LEMMA 2.15. *The morphisms $c(\varphi)^m$ from Definition 2.8 pass to rational equivalence,*

$$c(\varphi)^m\colon \mathsf{CH}_m(B) \longrightarrow \mathsf{CH}_{m-e}(B).$$

Proof. The proof is now completely analogous to [29, Theorem 17.1]. □

2.4.4. Properties

The class constructed in Definition 2.8 is invariant under proper birational maps in the following sense.

PROPOSITION 2.16. *Let $f\colon \mathfrak{W}\to\mathfrak{X}$ be a proper DM-type birational morphism of locally-finite-type algebraic stacks over K of pure dimension, with $\mathfrak{X}$ stratified by global quotient stacks. Then,*

$$(a\circ f)_{\mathrm{op}}[\mathfrak{W}] = a_{\mathrm{op}}[\mathfrak{X}] \in \mathsf{CH}^e_{\mathsf{op}}(\mathfrak{Y}),$$

where $(a\circ f)_{\mathrm{op}}[\mathfrak{W}]$ and $a_{\mathrm{op}}[\mathfrak{X}]$ are the operational classes constructed in Definition 2.8 with respect to $a\circ f\colon \mathfrak{W}\to\mathfrak{Y}$ and $a\colon \mathfrak{X}\to\mathfrak{Y}$.

Proof. The proper pushforward of the fundamental class along f is the fundamental class, as the map f is birational and hence of degree 1.

Choose a factorization $V\to\mathfrak{Y}'\to\mathfrak{Y}$. Denote by $\mathfrak{X}'$ the pullback of $\mathfrak{X}$ along a and by $\mathfrak{W}'$ the the pullback of $\mathfrak{X}'$ along f. As in (2.5), we form a pullback diagram

$$\begin{tikzcd}
\mathfrak{W}'\times_{\mathfrak{Y}'} V \arrow[r] \arrow[d, "\tilde f"] & \mathfrak{W}'\times V \arrow[d, "f\times \mathrm{id}"] \\
\mathfrak{X}'\times_{\mathfrak{Y}'} V \arrow[r] \arrow[d, "\psi_V"] & \mathfrak{X}'\times V \arrow[d, "a\times \mathrm{id}"] \\
V \arrow[r, "\varphi_V"] & \mathfrak{Y}'\times V.
\end{tikzcd} \tag{2.11}$$

Proposition 2.10 then yields

$$\begin{aligned}\psi_{V*}\varphi_V^!([\mathfrak{X}'\times V]) &= \psi_{V*}\varphi_V^!((f\times \mathrm{id})_*[\mathfrak{W}'\times V])\\ &= \psi_{V*}\tilde f_*\varphi_V^!([\mathfrak{W}'\times V])\\ &= (\psi_V\tilde f)_*\varphi_V^!([\mathfrak{W}'\times V]),\end{aligned}$$

which is the required equality. □

We will also require a flat pullback property. Let $\mathfrak{X}$, $\mathfrak{Y}$, and $\mathfrak{Z}$ be pure-dimensional algebraic stacks of locally finite type over K, with $\mathfrak{Y}$ and $\mathfrak{Z}$ smooth. Suppose that we have a fiber diagram

$$\begin{tikzcd} \mathfrak{X}\times_{\mathfrak{Y}}\mathfrak{Z} \arrow[r, "\tilde a"] \arrow[d, "\tilde f"] & \mathfrak{Z} \arrow[d, "f"] \\ \mathfrak{X} \arrow[r, "a"] & \mathfrak{Y}, \end{tikzcd}$$

where $a\colon\mathfrak{X}\to\mathfrak{Y}$ is a proper DM-type morphism, and $f\colon\mathfrak{Z}\to\mathfrak{Y}$ is flat *and* lci,([19]) with $\mathfrak{Z}$ stratified by global quotient stacks.

LEMMA 2.17. *In* $\mathsf{CH}_{\mathsf{op}}(\mathfrak{Z})$, *we have*

$$\tilde a_{\mathrm{op}}[\mathfrak{X}\times_{\mathfrak{Y}}\mathfrak{Z}] = f^*a_{\mathrm{op}}[\mathfrak{X}].$$

Proof. Let $\mathfrak{X}'$, $\mathfrak{Y}'$, and $\mathfrak{Z}'$ denote appropriate finite-type factorizations as in Definition 2.8. Then, we compare the two operational classes via the following diagram:

$$\begin{tikzcd} & (\mathfrak{X}'\times_{\mathfrak{Y}'}\mathfrak{Z}')\times_{\mathfrak{Z}'}V \arrow[r] \arrow[d, "\sim"] & \mathfrak{X}'\times_{\mathfrak{Y}'}\mathfrak{Z}'\times V \arrow[dd, "\tilde a\times\mathrm{id}"'] \arrow[dr, "\tilde f\times\mathrm{id}"] & \\ & \mathfrak{X}'\times_{\mathfrak{Y}'}V \arrow[rr] \arrow[d, "\psi_V"] & & \mathfrak{X}'\times V \arrow[d, "a\times\mathrm{id}"] \\ & V \arrow[r, "\varphi_V"] \arrow[dl, "i_V"] \arrow[rr, bend right, "(f\circ\varphi)_V"'] & \mathfrak{Z}'\times V \arrow[r, "f\times\mathrm{id}"] & \mathfrak{Y}'\times V \\ B. & & & \end{tikzcd}$$

By definition,

$$\tilde a_{\mathrm{op}}[\mathfrak{X}\times_{\mathfrak{Y}}\mathfrak{Z}](\varphi)([V]) = i_{V*}\psi_{V*}\varphi_V^!([\mathfrak{X}'\times_{\mathfrak{Y}'}\mathfrak{Z}'\times V])$$

and

$$f^*a_{\mathrm{op}}[\mathfrak{X}](\varphi)([V]) = a_{\mathrm{op}}[\mathfrak{X}](f\circ\varphi)([V]) = i_{V*}\psi_{V*}(f\circ\varphi)_V^!([\mathfrak{X}'\times V]).$$

([19]) A flat and lci map is called *syntomic*.

Since f is lci, the above expression is equal to

$$i_{V*}\psi_{V*}\varphi_V^!(f\times\mathrm{id})^!([\mathfrak{X}'\times V]).$$

Because f is also flat, we see, as in [29, Proposition 6.6 (b)] that we obtain

$$i_{V*}\psi_{V*}\varphi_V^!(\tilde{f}\times\mathrm{id})^*([\mathfrak{X}'\times V])=i_{V*}\psi_{V*}\varphi_V^!([\mathfrak{X}'\times_{\mathfrak{Y}'}\mathfrak{Z}'\times V]),$$

which yields the required equality. □

3. The universal double ramification cycle

3.1. Overview

We fix a genus g, a number of markings n, and a vector $A=(a_1,...,a_n)\in\mathbb{Z}^n$ of ramification data satisfying

$$\sum_{i=1}^{n} a_i = d.$$

We define here the associated universal twisted double ramification cycle class in the operational Chow group of the universal Picard stack $\mathfrak{Pic}_{g,n,d}$. The operational class is the class associated with a certain proper representable morphism

$$\mathbf{Div}_{g,A}\longrightarrow\mathfrak{Pic}_{g,n,d},$$

using the theory of §2.4. Our goal here is to define the stack $\mathbf{Div}_{g,A}$ over $\mathfrak{Pic}_{g,n,d}$.

We will present in §§3.2–3.4 three essentially equivalent definitions of the universal twisted double ramification cycle, which yield the same operational class:

- a definition in §3.2 by closing the Abel–Jacobi section, which is simple to state but difficult to handle;
- an intrinsic logarithmic definition in §3.3 following Marcus–Wise [56];
- a slight variation of the log definition in §3.4, which facilitates comparison to the spaces of rubber maps.

After analyzing the set-theoretic closure of the Abel–Jacobi section in §3.5, the equality of the three resulting classes will be shown in §3.6. In §3.8, we briefly discuss the lift of universal twisted double ramification cycle to operational b-Chow.

3.2. $\mathsf{DR}^{\mathsf{op}}_{g,A}$ by closure

We define the Abel–Jacobi section σ of $\mathfrak{Pic}_{g,n,d} \to \mathfrak{M}_{g,n}$ by

$$\begin{aligned} \sigma: \mathfrak{M}_{g,n} &\longrightarrow \mathfrak{Pic}_{g,n,d}, \\ (C, p_1, ..., &\longmapsto \mathcal{O}_C\left(\sum_{i=1}^{n} a_i p_i\right). \end{aligned} \tag{3.1}$$

The section σ is not a closed immersion (both because of the $\mathbb{G}_m$-automorphism groups of line bundles and because the image is not closed). However, σ is quasi-compact and relatively representable by schemes, and hence admits a well-defined *schematic image* (we use that the formation of the schematic image is compatible with flat base change, see [73, Lemma 081I]). The schematic image is the smallest closed reduced substack through which σ factors.

Since the schematic image $\bar{\sigma}$ is a closed substack of pure dimension,

$$\iota: \bar{\sigma} \longrightarrow \mathfrak{Pic}_{g,n,d},$$

we obtain an operational class $\iota_{\mathsf{op}}[\bar{\sigma}]$ by Definition 2.8. Our first definition of the universal twisted double ramification cycles is via the schematic image of σ:

$$\mathsf{DR}^{\mathsf{op}}_{g,A} = \iota_{\mathsf{op}}[\bar{\sigma}] \in \mathsf{CH}^{g}_{\mathsf{op}}(\mathfrak{Pic}_{g,n,d}). \tag{3.2}$$

Let

$$\mathfrak{Pic}_{\underline{0}} \hookrightarrow \mathfrak{Pic}$$

be the open substack consisting of line bundles having degree zero on every irreducible component of every geometric fiber (*multidegree* $\underline{0}$), and let

$$\mathfrak{Pic}^{\mathrm{rel}}_{\underline{0}} \hookrightarrow \mathfrak{Pic}^{\mathrm{rel}}$$

be defined analogously. We have a commutative diagram in which all squares are pullbacks:

$$\begin{tikzcd}
(B\mathbb{G}_m)_{\mathfrak{M}_{g,n}} \arrow[r] \arrow[d, equal] & \mathfrak{M}_{g,n} \arrow[d, equal] \arrow[rrdd, "e=\mathcal{O}_C"] & & \\
\bar{\sigma}^{\underline{0}} \arrow[r] \arrow[rrd] \arrow[dd] & \bar{\sigma}^{\underline{0}}_{\mathsf{rel}} \arrow[rrd] \arrow[dd] & & \\
& & \mathfrak{Pic}_{g,n,\underline{0}} \arrow[r] \arrow[dd] & \mathfrak{Pic}^{\mathrm{rel}}_{g,n,\underline{0}} \arrow[dd] \\
\bar{\sigma} \arrow[r] \arrow[rrd] & \bar{\sigma}_{\mathsf{rel}} \arrow[rrd] & & \\
& & \mathfrak{Pic}_{g,n,0} \arrow[r] & \mathfrak{Pic}^{\mathrm{rel}}_{g,n,0}.
\end{tikzcd} \tag{3.3}$$

Let $(C/B, p_1, ..., p_n)$ be a prestable curve over a scheme B of finite type over K. Let L be a line bundle on C such that

$$L\Big(-\sum_{i=1}^{n} a_i p_i\Big)$$

is of multidegree $\underline{0}$ for $A=(a_1, ..., a_n)\in\mathbb{Z}^n$. The data

$$C \longrightarrow B, \quad L\Big(-\sum_{i=1}^{n} a_i p_i\Big) \longrightarrow C$$

determine a map

$$\varphi: B \longrightarrow \mathfrak{Pic}_{g,n,\underline{0}},$$

and we form a pullback diagram

$$\begin{tikzcd} B' \arrow[r] \arrow[d, "\psi"] & \mathfrak{M}_{g,n} \arrow[d, "e"] \\ B \arrow[r, "\varphi^{\mathsf{rel}}"] & \mathfrak{Pic}^{\mathrm{rel}}_{g,n,\underline{0}}. \end{tikzcd} \tag{3.4}$$

Since $\mathfrak{M}_{g,n}$ is smooth and $\mathfrak{Pic}^{\mathrm{rel}}_{g,n,\underline{0}}$ is separated, the map e is a regular embedding.

LEMMA 3.1. *In the multidegree $\underline{0}$ case, we have*

$$\mathsf{DR}^{\mathsf{op}}_{g,A}(\varphi)([B]) = \psi_* e^![B]. \tag{3.5}$$

Proof. We begin by expanding the diagram (3.4) to

$$\begin{tikzcd} B' \arrow[r] \arrow[d, "\psi"] & \mathfrak{M}_{g,n}\times B \arrow[r, "f'"] \arrow[d] & \mathfrak{M}_{g,n} \arrow[d, "e"] \\ B \arrow[r, "\varphi'"] & \mathfrak{Pic}^{\mathrm{rel}}_{g,n,\underline{0}}\times B \arrow[r, "f"] & \mathfrak{Pic}^{\mathrm{rel}}_{g,n,\underline{0}}. \end{tikzcd} \tag{3.6}$$

Since $\mathfrak{Pic}_{g,n}\to\mathfrak{Pic}^{\mathrm{rel}}_{g,n}$ is smooth, we deduce from Lemma 2.17 and diagram (3.3) that

$$\mathsf{DR}^{\mathsf{op}}_{g,A}(\varphi)([B]) = \psi_*\varphi'^![\mathfrak{M}_{g,n}\times B]. \tag{3.7}$$

We then compute

$$\begin{aligned} \mathsf{DR}^{\mathsf{op}}_{g,A}(\varphi)([B]) &= \psi_*\varphi'^![\mathfrak{M}_{g,n}\times B] \\ &= \psi_*\varphi'^! f'^*[\mathfrak{M}_{g,n}] \\ &= \psi_*\varphi'^! f'^* e^![\mathfrak{Pic}^{\mathrm{rel}}_{g,n,\underline{0}}] \\ &= \psi_*\varphi'^! e^! f^*[\mathfrak{Pic}^{\mathrm{rel}}_{g,n,\underline{0}}] \\ &= \psi_*\varphi'^! e^![\mathfrak{Pic}^{\mathrm{rel}}_{g,n,\underline{0}}\times B] \\ &= \psi_* e^!\varphi'^![\mathfrak{Pic}^{\mathrm{rel}}_{g,n,\underline{0}}\times B] \\ &= \psi_* e^![B]. \end{aligned}$$

□

In particular, if the intersection of B with the unit section in $\mathfrak{Pic}^{\mathrm{rel}}_{g,n,\underline{0}}$ is transversal, then we simply take the naive intersection in $\mathfrak{Pic}^{\mathrm{rel}}_{g,n,\underline{0}}$ and push it down to B.

3.3. Logarithmic definition of $\mathsf{DR}^{\mathsf{op}}$

3.3.1. Overview of log divisors

We begin by recalling various results and definitions from log geometry. We refer the reader to [47] for basics on log geometry and [56] for the details of what we do here. While log geometry will not play a substantial role elsewhere in the paper, it will reappear in §6.

Given a log scheme $S=(S, M_S)$, we write

$$\mathbb{G}_m^{\mathsf{log}}(S)=\Gamma(S, M_S^{\mathsf{gp}}) \quad \text{and} \quad \mathbb{G}_m^{\mathsf{trop}}(S)=\Gamma(S, \overline{M}_S^{\mathsf{gp}}),$$

which we call the logarithmic and tropical multiplicative groups. Both can naturally be extended to presheaves on the category $\mathbf{LSch}_S$ of log schemes over S, and both admit log smooth covers by log schemes (with subdivisions $\mathbb{P}^1$ and $[\mathbb{P}^1/\mathbb{G}_m]$, respectively). A *log* (*tropical*) line on S is a $\mathbb{G}_m^{\mathsf{log}}$ ($\mathbb{G}_m^{\mathsf{trop}}$) torsor on S, for the strict étale topology.

Definition 3.2. (See [56, Definition 4.6]) Let C be a logarithmic curve over a logarithmic scheme S. A *logarithmic divisor* on C is a tropical line P over S and an S-morphism $C\to P$.

Let $\mathbf{Div}_g^{\mathsf{rel}}$ be the stack(20) in the strict étale topology on logarithmic schemes whose S-points are triples (C, P, α), where C is a logarithmic curve of genus g over S, P is a tropical line over S, and

$$\alpha\colon C\longrightarrow P$$

is an S-morphism.

If S is a geometric log point and C/S is a log curve, then the set of isomorphism classes of $\mathbf{Div}_g^{\mathsf{rel}}(S)$ is given by $\pi_*\overline{M}_C^{\mathsf{gp}}/\overline{M}_S^{\mathsf{gp}}$. At the markings, an element of $\pi_*\overline{M}_C^{\mathsf{gp}}/\overline{M}_S^{\mathsf{gp}}$ determines an element of the groupified relative characteristic monoid $\mathbb{Z}$ (for those who prefer a tropical perspective, this can be viewed as the outgoing slope at the marking).

Definition 3.3. Let $\mathbf{Div}_{g,A}^{\mathsf{rel}}$ be the (open and closed) substack of $\mathbf{Div}_g^{\mathsf{rel}}$ consisting of those triples where the curve carries exactly n markings and where on each geometric fiber the outgoing slopes at the markings correspond to A (our log curves come with an ordering of their markings as explained in §1.2).

(20) The stack $\mathbf{Div}_g^{\mathsf{rel}}$ was denoted $\mathbf{Div}_g$ in [56], but we wish to reserve the latter notation for a certain $\mathbb{G}_m$-gerbe over $\mathbf{Div}_g^{\mathsf{rel}}$ which will play a much more prominent role in our paper.

Remark 3.4. It is natural to ask for a description of the functor of points of the underlying (non-logarithmic) stack of $\mathbf{Div}^{\mathsf{rel}}_{g,A}$ as a fibered category over $\mathfrak{M}_{g,n}$. However, we expect that such a description will not be simple. A closely related problem is solved in [12], and the intricacy of the resulting definition suggests that the path will not be easy.

3.3.2. Abel–Jacobi map

Given a log curve $\pi\colon C\to S$ of genus g, the right-derived pushforward to S of the standard exact sequence

$$1 \longrightarrow \mathcal{O}_C^\times \longrightarrow M_C^{\mathsf{gp}} \longrightarrow \overline{M}_C^{\mathsf{gp}} \longrightarrow 1 \tag{3.8}$$

yields a natural map

$$\pi_*\overline{M}_C^{\mathsf{gp}} \longrightarrow R^1\pi_*\mathcal{O}_C^\times,$$

which factors via the quotient

$$\pi_*\overline{M}_C^{\mathsf{gp}}/\overline{M}_S^{\mathsf{gp}} = \mathbf{Div}^{\mathsf{rel}}_g(S).$$

We therefore obtain a *relative Abel–Jacobi* map

$$\mathsf{AJ}^{\mathsf{rel}}\colon \mathbf{Div}^{\mathsf{rel}}_g \longrightarrow \mathfrak{Pic}^{\mathrm{rel}}_g,$$

which restricts to maps

$$\mathsf{AJ}^{\mathsf{rel}}\colon \mathbf{Div}^{\mathsf{rel}}_{g,A} \longrightarrow \mathfrak{Pic}^{\mathrm{rel}}_{g,n,d}.$$

For a first example, suppose S is a geometric log point with $\overline{M}_S=\mathbb{N}$. The data then determines to first order a deformation of the curve over a DVR (which we take generically smooth), and the section of $\pi_*\overline{M}_C^{\mathsf{gp}}/\overline{M}_S^{\mathsf{gp}}$ gives the multiplicities of components in the special fiber and the twists by the markings.

For another example, consider what happens over the locus of (strict) smooth curves. Writing $N/\mathcal{M}^{\mathsf{log}}_{g,n}$ for the stack of markings (finite étale), we see $\mathbf{Div}^{\mathsf{rel}}_g$ is just the category of locally constant functions from N to $\mathbb{Z}$ — in other words, choices of outgoing slope/weight on each leg. The Abel–Jacobi map yields

$$\mathcal{O}_C\left(\sum_{i=1}^n a_i p_i\right)$$

where the p_i are the markings and a_i are the weights. In particular, we see that

$$\mathbf{Div}^{\mathsf{rel}}_{g,A} \to \mathcal{M}^{\mathsf{log}}_{g,n}$$

is birational (as we fixed an ordering of the markings) and log étale.

Definition 3.5. Let $\mathbf{Div}_g$ be the fiber product

$$\mathbf{Div}_g^{\mathsf{rel}} \times_{\mathfrak{Pic}_g^{\mathrm{rel}}} \mathfrak{Pic}_g. \tag{3.9}$$

More concretely, an S-point of $\mathbf{Div}_g$ is a quadruple $(C, P, \alpha, \mathcal{L})$, where (C, P, α) is an S-point of $\mathbf{Div}_g^{\mathsf{rel}}$, and $\mathcal{L}$ is a line bundle on C satisfying([21])

$$[\mathcal{L}] = \mathsf{AJ}^{\mathsf{rel}}(C, P, \alpha) \in \mathfrak{Pic}_g^{\mathrm{rel}}(S).$$

We will denote by AJ the resulting Abel–Jacobi map

$$\mathbf{Div}_g \to \mathfrak{Pic}_g.$$

Observe that $\mathbf{Div}_g$ is a $\mathbb{G}_m$-gerbe over $\mathbf{Div}_g^{\mathsf{rel}}$, just as $\mathfrak{Pic}_{g,n,d}$ is a $\mathbb{G}_m$-gerbe over $\mathfrak{Pic}_{g,n,d}^{\mathrm{rel}}$. Analogously, we define

$$\mathbf{Div}_{g,A} = \mathbf{Div}_{g,A}^{\mathsf{rel}} \times_{\mathfrak{Pic}_{g,n,d}^{\mathrm{rel}}} \mathfrak{Pic}_{g,n,d} \quad \text{and} \quad \mathsf{AJ}\colon \mathbf{Div}_{g,A} \longrightarrow \mathfrak{Pic}_{g,n,d}. \tag{3.10}$$

We summarize the key properties of the Abel–Jacobi map. These are proven in [56] for $\mathsf{AJ}^{\mathsf{rel}}$, and are stable under base change.

PROPOSITION 3.6. *The Abel–Jacobi map*

$$\mathsf{AJ}\colon \mathbf{Div}_{g,A} \longrightarrow \mathfrak{Pic}_{g,n,d}$$

is proper, relatively representable by algebraic spaces, and is a monomorphism of log stacks.

We obtain an operational class $\mathsf{AJ}_{\mathsf{op}}[\mathbf{Div}_{g,A}]$ associated by Definition 2.8 with the Abel–Jacobi map AJ. Our second definition of the universal twisted double ramification cycles is via AJ:

$$\mathsf{DR}_{g,A}^{\mathsf{op}} = \mathsf{AJ}_{\mathsf{op}}[\mathbf{Div}_{g,A}] \in \mathsf{CH}_{\mathsf{op}}^{g}(\mathfrak{Pic}_{g,n,d}). \tag{3.11}$$

The equivalence of definitions (3.2) and (3.11) will be proven in §3.6.

3.3.3. Description of $\mathbf{Div}_g$ with log line bundles

Our approach to $\mathbf{Div}_g$ in Definition 3.5 via a fiber product is indirect. While it will not be used in the paper, a more conceptual path is to consider the stack $\mathbf{Div}_g'$ whose objects are tuples

$$(C/S, \mathcal{P}, \alpha),$$

([21]) Here, $[\cdot]$ denotes the equivalence class under the relations of isomorphism and tensoring with the pullback of a line bundle from the base.

where C/S is a log curve, $\mathcal{P}$ is a *logarithmic* line on S (a $\mathbb{G}_m^{\mathsf{log}}$ torsor), and α is a map from C to the *tropical* line P on S induced from $\mathcal{P}$ by the exact sequence (3.8). There is a natural map

$$\mathbf{Div}'_g \longrightarrow \mathbf{Div}_g^{\mathsf{rel}}.$$

We can see α as a section of the tropicalization of the pullback of $\mathcal{P}$ to C. As such, by the sequence (3.8), α induces a $\mathbb{G}_m$-torsor on C, giving us an Abel–Jacobi map $\mathbf{Div}'_g \to \mathfrak{Pic}_g$. Together these maps induce a map

$$\mathbf{Div}'_g \longrightarrow \mathbf{Div}_g$$

to the fiber product, and a local computation verifies that this is an isomorphism.

The above discussion points([22]) towards a definition of the double ramification cycle via the *logarithmic Picard functor* of [60], which we hope will be pursued in future.

3.4. Logarithmic rubber definition of $\mathsf{DR}^{\mathsf{op}}$

Marcus and Wise introduce a slight variant $\mathbf{Rub}_g^{\mathsf{rel}}$ of the stack $\mathbf{Div}_g^{\mathsf{rel}}$ which parameterizes pairs (C, P, α), where P is a tropical line on S and $\alpha: C \to P$ is an S-morphism *such that, on each geometric fiber over* S, *the values taken by* α *on the irreducible components of* C *are totally ordered in* $(\overline{M}_S^{\mathsf{gp}})_s$, with the ordering given by declaring the elements of $(\overline{M}_S)_s$ to be the non-negative elements.

The space $\mathbf{Rub}_{g,A}$, defined via pullback

$$\mathbf{Rub}_{g,A} \cong \mathbf{Div}_{g,A} \times_{\mathbf{Div}_{g,A}^{\mathsf{rel}}} \mathbf{Rub}_{g,A}^{\mathsf{rel}},$$

is pure dimensional and comes with a proper birational map

$$\mathbf{Rub}_{g,A} \longrightarrow \mathbf{Div}_{g,A}. \tag{3.12}$$

The stack $\mathbf{Rub}_{g,A}$ will play an important role in the comparison to classes coming from stable map spaces in §6.

We obtain an operational class $\mathsf{AJ}_{\mathsf{op}}^{\mathsf{rub}}[\mathbf{Rub}_{g,A}]$ associated by Definition 2.8 with

$$\mathsf{AJ}^{\mathsf{rub}}: \mathbf{Rub}_{g,A} \longrightarrow \mathfrak{Pic}_{g,n,d}$$

([22]) The *unit section* of $\mathfrak{Pic}$ is given by the stack of $\mathbb{G}_m$ torsors on the base. Similarly, the *unit section* of the *logarithmic* Picard stack $\mathfrak{LogPic}_g$ is given by the stack of $\mathbb{G}_m^{\mathsf{log}}$ torsors on the base. The natural map $\mathfrak{Pic}_g \to \mathfrak{LogPic}_g$ is neither injective nor surjective: a logarithmic line bundle comes from a line bundle if and only if the associated tropical line bundle is trivial, and a choice of trivialization of that tropical line bundle then determines a lift to a line bundle. Hence, we see that $\mathbf{Div}'_g$ is precisely the pullback of the unit section of $\mathfrak{LogPic}_g$ to $\mathfrak{Pic}_g$.

obtained by composing (3.12) with AJ. Our third definition of the universal twisted double ramification cycles is via $\mathsf{AJ}^{\mathsf{rub}}$:

$$\mathsf{DR}^{\mathsf{op}}_{g,A} = \mathsf{AJ}^{\mathsf{rub}}_{\mathsf{op}}[\mathbf{Rub}_{g,A}] \in \mathsf{CH}^{g}_{\mathsf{op}}(\mathfrak{Pic}_{g,n,d}). \tag{3.13}$$

The equivalence with the first two definitions will be proven in §3.6.

3.5. The image of the Abel–Jacobi map

The set theoretic image of the Abel–Jacobi map

$$\mathsf{AJ}\colon \mathbf{Div}_{g,A} \longrightarrow \mathfrak{Pic}_{g,n,d}$$

can be characterized in terms of a condition on twisted divisors similar to the conditions of [28] for the moduli spaces $\widetilde{\mathcal{H}}_g(A)$ twisted canonical divisors.

Given a prestable graph Γ_δ of degree d, a *twist* on Γ_δ is a function $I\colon H(\Gamma)\to\mathbb{Z}$ which satisfies the following properties:

(i) for all $j\in\mathrm{L}(\Gamma_\delta)$, corresponding to the marking $j\in\{1,...,n\}$, one has

$$I(j)=a_j;$$

(ii) for all $e\in\mathrm{E}(\Gamma_\delta)$, corresponding to two half-edges $h,h'\in\mathrm{H}(\Gamma_\delta)$, one has

$$I(h)+I(h')=0;$$

(iii) for all $v\in\mathrm{V}(\Gamma_\delta)$, one has

$$\sum_{v(h)=v} I(h)=\delta(v),$$

where the sum is taken over *all* $\mathsf{n}(v)$ half-edges incident to v;

(iv) there is no strict cycle([23]) in Γ.

Let $(C,p_1,...,p_n)$ together with a line bundle $\mathcal{L}\to C$ of degree d be a geometric point of $\mathfrak{Pic}_{g,n,d}$. Let Γ_δ be the prestable graph of C decorated with the degrees $\delta(v)$ of the line bundle $\mathcal{L}$ restricted to the components C_v of C. Given a twist I on Γ_δ, let

$$\eta_I\colon C_I \longrightarrow C$$

([23]) A *strict cycle* is a sequence $\vec{e}_i=(h_i,h'_i)$, $i=1,...,\ell$, of directed edges in Γ forming a closed path in Γ such that $I(h_i)\geqslant 0$ for all i, and there exists at least one i with $I(h_i)>0$. Condition (iv) corresponds to the combination of the vanishing, sign, and transitivity conditions for twists in [28, §0.3].

be the partial normalization of C at all nodes $q \in C$ corresponding to edges $e=(h,h')$ of Γ with

$$I(h) = -I(h') \neq 0.$$

Denote by $q_h, q_{h'} \in C_I$ the preimages of q corresponding to the half-edges h and h', respectively. Denote by $\hat{p}_i \in C_I$ the unique preimage of the ith marking $p_i \in C$.

We say the point $(C, p_1, ..., p_n, \mathcal{L})$ of $\mathfrak{Pic}_{g,n,d}$ satisfies the *twisted divisor condition* for the integer vector A if and only if there exists a twist I on Γ_δ such that, on the partial normalization C_I of C, there exists an isomorphism of line bundles

$$\eta_I^* \mathcal{L} \cong \mathcal{O}_C\left(\sum_{i=1}^{n} a_i \hat{p}_i + \sum_{h \in H(\Gamma)} I(h) q_h\right). \tag{3.14}$$

For $\mathcal{L}=\omega_C$, this exactly corresponds to the notion [28, Definition 1] of a twisted canonical divisor.

PROPOSITION 3.7. *A geometric point $(C, p_1, ..., p_n, \mathcal{L})$ of $\mathfrak{Pic}_{g,n,d}$ lies in the image of the Abel–Jacobi map* $\mathsf{AJ}\colon \mathbf{Div}_{g,A} \to \mathfrak{Pic}_{g,n,d}$ *if and only if the twisted divisor condition for the vector A is satisfied.*

Proof. We may suppose that K is a separably closed field and $(C/S, p_1, ..., p_n)$ is a prestable curve over K. We must show that the twisted divisor condition is equivalent to the existence of a log structure on C/S satisfying the following property: *C/S is a log curve with markings given by the p_i which admits a global section α of $\overline{M}_C^{\mathsf{gp}}$ with outgoing slope at p_i given by a_i.*

Suppose that such a log structure exists. From the log structure, we can determine a twist. With each leg we associate the outgoing slope of α on the corresponding leg. For an edge $\{h, h'\}$, we define $\ell(\{h,h'\})$ to be the element of $\overline{M}_S$ associated via the data of the log morphism $C \to S$ with the node of C corresponding to $\{h,h'\}$. If $\{h,h'\}$ is an edge with half-edge h attached to a vertex u and the opposite half-edge h' attached to v, we set the integer $I(h)$ to be the unique integer such that

$$\alpha(u) + I(h) \cdot \ell(\{h,h'\}) = \alpha(v) \in \overline{M}_S^{\mathsf{gp}}. \tag{3.15}$$

That such an $I(h)$ exists follows from the structure of $\overline{M}_C^{\mathsf{gp}}$.

Next, we verify that I is a twist. Conditions (i) and (ii) are immediate from the construction. We deduce condition (iv) because by following a strict cycle starting at some vertex u and applying (3.15) along each edge would yield $\alpha(u) < \alpha(u)$, which is impossible. Condition (iii) is immediate from the twisted divisor condition (3.14) and the fact that isomorphic line bundles have the same degree, so this will be proven once we have checked the latter condition.

For the latter condition, we must work a little harder. To start, we claim that there exists a morphism $\overline{M}_S \to \mathbb{N}$ which does not send the label of any edge to zero. Indeed, $\overline{M}_s^{\mathsf{gp}}$ injects into its groupification which is a finitely generated torsion-free abelian group, and hence isomorphic to $\mathbb{Z}^m$. Since $\overline{M}_S$ is sharp([24]) and finitely generated, the non-zero elements of its image in $\mathbb{Z}^m$ land in some strict half-space of $\mathbb{Z}^m$ cut out by a linear equation with integral coefficients. Such a half-space admits a map to $\mathbb{N}$ such that the only preimage of zero is zero.

After base changing over S along such a map, we may assume that $\overline{M}_S^{\mathsf{gp}} = \mathbb{N}$, and that all edges have non-zero label. We obtain a first-order map passing through our given point,

$$S = \operatorname{Spec} K[[t]] \longrightarrow \mathbf{Div}_{g,A}$$

for which the induced prestable curve C/S is generically smooth. On the curve C, we define a Weil divisor D by assigning to an irreducible component v the integer $\alpha(v)$. The divisor D is then Cartier by (3.15), which still applies after base change, and hence determines a line bundle $\mathcal{O}_C(-D)$, which is exactly the image of the Abel–Jacobi map. In particular, the bundle $\mathcal{L}$ is (up to isomorphism) given by restricting $\mathcal{O}_C(-D)$ to the central fiber, so it suffices to verify (3.14) for the latter bundle, which is a standard local calculation on a prestable curve over a discrete valuation ring.

Conversely, suppose the twisted divisor condition is satisfied. We must build a log structure and a suitable section $\alpha \in \overline{M}_C^{\mathsf{gp}}(C)$. We could try equipping $(C/S, p_1, ..., p_n)$ with its minimal log structure (see §1.2), but then the section α is unlikely to exist — if there are no separating edges then there are no non-constant sections of $\overline{M}_C^{\mathsf{gp}}$. Instead, we will construct a log structure by deforming the curve.

First, we claim that there exists an assignment of a positive integer $\ell(e) \in \mathbb{Z}_{>0}$ to each edge and of an integer $d(v) \in \mathbb{Z}$ to each vertex such that the following condition is satisfied:

$$\begin{gathered} \textit{if } e = \{h, h'\} \textit{ is an edge with } h \textit{ attached to } u \textit{ and } h' \textit{ to } v, \\ \textit{then } d(u) + I(h) \cdot \ell(e) = d(v). \end{gathered} \tag{*}$$

A twist I on Γ induces a binary relation $\preceq$ on $V(\Gamma)$ by

$$u \preceq v \iff \text{there is an edge } e = \{h, h'\} \text{ with } h \text{ at } u,\ h' \text{ at } v, \text{ and } I(h) \geqslant 0.$$

The fact that Γ contains no strict cycles is equivalent by [74] to the existence of an extension of $\preceq$ to a total preorder on $V(\Gamma)$ (a reflexive, total, and transitive binary relation). Hence, there exists a level function $d_0: \mathrm{V}(\Gamma) \to \mathbb{Z}$ such that

$$u \preceq v \iff u \text{ and } v \text{ are connected by an edge and } d_0(u) \leqslant d_0(v). \tag{3.16}$$

([24]) A monoid is *sharp* if zero is indecomposable: $a+b=0$ implies $a=b=0$.

We define

$$L = lcm\{I(h) : h \in H(\Gamma), I(h) > 0\}.$$

Then, $d(v)=Ld_0(v)$ still has property (3.16) and, for any edge $e=\{h,h'\}$ with h attached to u and h' to v, we have two cases:

- $I(h)=0$, in which case all edges $\{\tilde{h},\tilde{h}'\}$ connecting u and v must satisfy $I(\tilde{h})=0$ (due to the strict cycle condition), so we can set $\ell(e)=1$;
- $I(h)\neq 0$, in which case the number

$$\ell(e) = \frac{d(v)-d(u)}{I(h)}$$

is indeed a positive integer (since d has values in $L\cdot\mathbb{Z}$).

Clearly, the functions d and ℓ thus constructed satisfy the condition above.

Such d and ℓ are far from unique, but we pick them. Consider then the space of all smoothings $\mathcal{C}$ of C over $K[[t]]$ such that the thickness([25]) of $\mathcal{C}$ at the node corresponding to edge e is $\ell(e)$. Given such a smoothing $\mathcal{C}$, we construct a vertical Weil divisor D by assigning to the irreducible component corresponding to vertex v the weight $d(v)$. The divisor D is then Cartier by the condition (*). Set

$$\mathcal{L}_{\mathcal{C}} = \mathcal{O}_{\mathcal{C}}(D)|_C \otimes \mathcal{O}_C\Bigg(\sum_i a_i p_i\Bigg).$$

The smoothing $\mathcal{C}$ also induces a log structure on C by taking the divisorial log structure of the special fiber. The twist I then determines an element α of $\pi_*\overline{M}_C^{\mathsf{gp}}/\overline{M}_S^{\mathsf{gp}}$. Applying the Abel–Jacobi map to α recovers $\mathcal{L}_{\mathcal{C}}$.

One can readily verify that $\mathcal{L}_{\mathcal{C}}$ satisfies (3.14) by a local computation, but we need to show more: the smoothing $\mathcal{C}$ can be chosen so that $\mathcal{L}_{\mathcal{C}}$ is isomorphic to the line bundle $\mathcal{L}$ that we started with. The space of such smoothings $\mathcal{C}$ naturally surjects onto

$$\bigoplus_{\substack{e=\{h,h'\}\\ I(h)\neq 0}} (\mathcal{O}_{C_I}(h)\otimes\mathcal{O}_{C_I}(h'))^{\otimes \ell(e)}, \tag{3.17}$$

where the 1-dimensional K-vector space

$$(\mathcal{O}_{C_I}(h)\otimes\mathcal{O}_{C_I}(h'))^{\otimes \ell(e)}$$

corresponds exactly to the ways to glueing the two branches of $\eta_I^*\mathcal{L}$ together at the points h and h'. In other words, by moving over the space (3.17), we can recover *all* ways of glueing $\eta_I^*\mathcal{L}$ to a line bundle on C. In particular, we can recover $\mathcal{L}$, and hence we can realize $\mathcal{L}$ as $\mathcal{L}_{\mathcal{C}}$ for some smoothing $\mathcal{C}$, as required. □

([25]) The local equation of the node is $xy=t^r$ for some positive integer r which we call the *thickness* of the node.

3.6. Proof of the equivalence of the definitions

The equivalence of the classes coming from $\mathbf{Div}_{g,A}$ and from $\mathbf{Rub}_{g,A}$ is immediate by applying Proposition 2.16. We must compare the latter two with the class defined by (3.2) via the schematic image. We will require the following two easy results.

LEMMA 3.8. *Let $U \hookrightarrow \mathbf{Div}_{g,A}$ denote the open locus where the log curve is classically smooth. Then, U is schematically dense in $\mathbf{Div}_{g,A}$.*

Proof. Since $\mathbf{Div}_{g,A} \to \mathcal{M}_{g,n}^{\mathsf{log}}$ is log étale, we deduce that $\mathbf{Div}_{g,A}$ is log regular. In particular, $\mathbf{Div}_{g,A}$ is reduced, and the locus where the log structure is trivial is dense. □

LEMMA 3.9. *The Abel–Jacobi map $\mathsf{AJ}\colon \mathbf{Div}_{g,A} \to \mathfrak{Pic}_{g,n,d}$ factors through the inclusion $\bar{\sigma} \to \mathfrak{Pic}_{g,n,d}$, and the induced map*

$$\mathbf{Div}_{g,A} \longrightarrow \bar{\sigma}$$

is proper and birational.

Proof. That $\mathbf{Div}_{g,a} \to \mathfrak{Pic}_{g,n,d}$ factors through $\bar{\sigma} \to \mathfrak{Pic}$ is immediate from Lemma 3.8 and the definition of the schematic image. The induced map $\mathbf{Div}_{g,A} \to \bar{\sigma}$ is proper since $\mathbf{Div}_{g,A}$ is proper over $\mathfrak{Pic}_{g,n,d}$, and is birational since it is an isomorphism over the locus of smooth curves. □

By another application of Proposition 2.16, the definitions of $\mathsf{DR}^{\mathsf{op}}$ via $\mathbf{Div}_{g,A}$ and the schematic image are equivalent.

3.7. Proof of Theorem 0.1

Let $k \geqslant 0$, and let $A = (a_1, ..., a_n)$ be a vector of integers satisfying

$$\sum_{i=1}^{n} a_i = k(2g-2).$$

There are three definitions in the literature for the classical twisted double ramification cycle

$$\mathsf{DR}_{g,A,\omega^k} \in \mathsf{CH}_{2g-3+n}(\overline{\mathcal{M}}_{g,n})$$

on the moduli space of stable curves:

- via birational modifications of $\overline{\mathcal{M}}_{g,n}$ [37];
- via the closure of the image of the Abel–Jacobi section [38];
- via logarithmic geometry and the stack $\mathbf{Div}_g^{\mathsf{rel}}$ [56].

All three are shown to be equivalent in [37], [38]. For the proof of Theorem 0.1, we choose the definition of [56], as this will give the shortest path.

For $d=k(2g-2)$, let $\varphi:\overline{\mathcal{M}}_{g,n}\to\mathfrak{Pic}_{g,n,d}$ be the morphism associated with the data

$$\pi:\mathcal{C}_{g,n}\longrightarrow\overline{\mathcal{M}}_{g,n},\quad \omega_\pi^k\longrightarrow\mathcal{C}_{g,n}. \tag{3.18}$$

To prove Theorem 0.1, we must show that

$$\mathsf{DR}^{\mathsf{op}}_{g,A}(\varphi)([\overline{\mathcal{M}}_{g,n}])=\mathsf{DR}_{g,A,\omega^k},$$

where $[\overline{\mathcal{M}}_{g,n}]$ is the fundamental class.

We form the pullback diagram

$$\begin{tikzcd} \mathbf{Div}_{g,A}\times_{\mathfrak{Pic}_{g,n,d}}\overline{\mathcal{M}}_{g,n} \arrow[r] \arrow[d, "\psi"] & \mathbf{Div}_{g,A}\times\overline{\mathcal{M}}_{g,n} \arrow[d, "a\times\mathrm{id}"] \\ \overline{\mathcal{M}}_{g,n} \arrow[r, "\varphi'=\varphi\times\mathrm{id}"] & \mathfrak{Pic}_{g,n,d}\times\overline{\mathcal{M}}_{g,n}. \end{tikzcd} \tag{3.19}$$

Following Definition 2.8, we have

$$\mathsf{DR}^{\mathsf{op}}_{g,A}(\varphi)([\overline{\mathcal{M}}_{g,n}])=\psi_*(\varphi')^!([\mathbf{Div}_{g,A}\times\overline{\mathcal{M}}_{g,n}]).$$

The construction is equivalent to the definition of the class $\mathsf{DR}_{g,A,\omega^k}$ in [56] after making the standard translation between the Gysin pullback and the virtual fundamental class as in [11, Example 7.6].

3.8. The double ramification cycle in b-Chow

The construction of the double ramification cycle in [37] naturally yielded a more refined object: a b-cycle([26]) on $\overline{\mathcal{M}}_{g,n}$ which pushes down to the usual double ramification cycle on $\overline{\mathcal{M}}_{g,n}$. The refined cycle was shown in [39] to have better properties with respect to intersection products than the usual double ramification cycle. By considering rational sections of the multidegree-zero relative Picard space over $\mathfrak{Pic}_{g,n,d}$, we can in an analogous way define a b-cycle on $\mathfrak{Pic}_{g,n,d}$ refining the universal twisted double ramification cycle introduced here. In future work, we will show that this refined universal cycle is compatible with intersection products in the sense of [39] and that the *toric contact cycles* of [68] can be obtained by pulling back these products.

([26]) An element of the colimit of the Chow groups of smooth blowups of $\overline{\mathcal{M}}_{g,n}$ with transition maps given by pullback.

4. Pixton's formula

4.1. Reformulation

Recall the cycle $\mathsf{P}^c_{g,A,d} \in \mathsf{CH}^c_{\mathsf{op}}(\mathfrak{Pic}_{g,n,d})$ defined in §0.3.5. We write

$$\mathsf{P}^\bullet_{g,A,d} = \sum_{c=0}^{\infty} \mathsf{P}^c_{g,A,d} \in \prod_{c=0}^{\infty} \mathsf{CH}^c_{\mathsf{op}}(\mathfrak{Pic}_{g,n,d})$$

for the associated mixed dimensional class. We will rewrite the formula for $\mathsf{P}^\bullet_{g,A,d}$ in a more convenient form for computation.

Several factors in the formula of §0.3.5 can be pulled out of the sum over graphs and weightings. We require the following four definitions:

- Let $\mathsf{G}^{\mathsf{se}}_{g,n,d}$ be the set of graphs in $\mathsf{G}_{g,n,d}$ having exactly two vertices connected by a single edge. Such graphs are thus described by a partition

$$(g_1, I_1, \delta_1 \,|\, g_2, I_2, \delta_2) \tag{4.1}$$

of the genus, the marking set, and the degree of the universal line bundle.

- Given a vector $A = (a_1, ..., a_n) \in \mathbb{Z}^n$ satisfying

$$\sum_{i=1}^{n} a_i = d$$

and $\Gamma_\delta \in \mathsf{G}^{\mathsf{se}}_{g,n,d}$ corresponding to the partition (4.1), we define

$$c_A(\Gamma_\delta) = -\left(\delta_1 - \sum_{i\in I_1} a_i\right)^2 = -\left(\delta_2 - \sum_{i\in I_2} a_i\right)^2.$$

- For $\Gamma_\delta \in \mathsf{G}^{\mathsf{se}}_{g,n,d}$, we write

$$[\Gamma_\delta] = \frac{1}{|\mathrm{Aut}(\Gamma_\delta)|} j_{\Gamma_\delta *}[\mathfrak{Pic}_{\Gamma_\delta}]$$

for the class of the boundary divisor of $\mathfrak{Pic}_{g,n,d}$ associated with Γ_δ.

- Let $\mathsf{G}^{\mathsf{nse}}_{g,n,d}$ be the set of graphs in $\mathsf{G}_{g,n,d}$ such that every edge is non-separating.

PROPOSITION 4.1. *The class $\mathsf{P}^\bullet_{g,A,d}$ is the constant term in r of*

$$\begin{aligned}&\exp\left(\frac{1}{2}\left(-\eta + \sum_{i=1}^{n} 2a_i\xi_i + a_i^2\psi_i + \sum_{\Gamma_\delta \in \mathsf{G}^{\mathsf{se}}_{g,n,d}} c_A(\Gamma_\delta)[\Gamma_\delta]\right)\right)\\ &\times \sum_{\substack{\Gamma_\delta \in \mathsf{G}^{\mathsf{nse}}_{g,n,d}\\ w\in \mathsf{W}_{\Gamma_\delta,r}}} \frac{r^{-h^1(\Gamma_\delta)}}{|\mathrm{Aut}(\Gamma_\delta)|} j_{\Gamma_\delta *}\left[\prod_{e=(h,h')\in \mathrm{E}(\Gamma_\delta)} \frac{1}{\psi_h+\psi_{h'}}\left(1-\exp\left(-\frac{w(h)w(h')}{2}(\psi_h+\psi_{h'})\right)\right)\right],\end{aligned} \tag{4.2}$$

for $r \gg 0$.

In the proof of Proposition 4.1, we will use the following computation which provides an interpretation for parts of the formula (4.2) and which will be used again in §8.

LEMMA 4.2. *Let* $A=(a_1,...,a_n)\in\mathbb{Z}^n$ *with* $\sum_{i=1}^n a_i=d$. *For the line bundle* $\mathcal{L}$ *on the universal curve*

$$\pi\colon \mathfrak{C}_{g,n,d}\longrightarrow \mathfrak{Pic}_{g,n,d},$$

with universal sections $p_1,...,p_n$, *we define*

$$\mathcal{L}_A=\mathcal{L}\bigg(-\sum_{i=1}^n a_i[p_i]\bigg).$$

Then, we have

$$-\pi_*c_1(\mathcal{L}_A)^2=-\eta+\sum_{i=1}^n 2a_i\xi_i+a_i^2\psi_i.$$

Proof. The result follows from the definitions of the classes η and ξ_i:

$$-\pi_*c_1(\mathcal{L}_A)^2=-\pi_*\bigg(c_1(\mathcal{L})^2+\sum_{i=1}^n -2a_ic_1(\mathcal{L})|_{[p_i]}+a_i^2[p_i]^2\bigg)=-\eta+\sum_{i=1}^n 2a_i\xi_i+a_i^2\psi_i,$$

where, for the self-intersection $[p_i]^2$, we have used that the first Chern class of the normal bundle of p_i is given by $-\psi_i$. □

Proof of Proposition 4.1. We denote by

$$\Phi_a(x)=\frac{1}{x}\bigg(1-\exp\bigg(-\frac{a}{2}x\bigg)\bigg)=\sum_{m=0}^{\infty}(-1)^m\bigg(\frac{a}{2}\bigg)^{m+1}\frac{1}{(m+1)!}x^m=\frac{a}{2}-\frac{a^2}{8}x+...$$

the power series appearing in the edge-terms of Pixton's formula.

As a first step, we show that the constant term in r of

$$\begin{aligned}&\exp\bigg(\frac{1}{2}\sum_{\Gamma_\delta\in\mathsf{G}^{\mathsf{se}}_{g,n,d}}c_A(\Gamma_\delta)[\Gamma_\delta]\bigg)\\ &\quad\times\sum_{\substack{\Gamma_\delta\in\mathsf{G}^{\mathsf{nse}}_{g,n,d}\\ w\in\mathsf{W}_{\Gamma_\delta,r}}}\frac{r^{-h^1(\Gamma_\delta)}}{|\mathrm{Aut}(\Gamma_\delta)|}j_{\Gamma_\delta*}\prod_{e=(h,h')\in\mathrm{E}(\Gamma_\delta)}\Phi_{w(h)w(h')}(\psi_h+\psi_{h'})\end{aligned}\tag{4.3}$$

and the constant term in r of

$$\sum_{\substack{\Gamma_\delta\in\mathsf{G}_{g,n,d}\\ w\in\mathsf{W}_{\Gamma_\delta,r}}}\frac{r^{-h^1(\Gamma_\delta)}}{|\mathrm{Aut}(\Gamma_\delta)|}j_{\Gamma_\delta*}\prod_{e=(h,h')\in\mathrm{E}(\Gamma_\delta)}\Phi_{w(h)w(h')}(\psi_h+\psi_{h'})\tag{4.4}$$

are equal. The formula (4.4) is a linear combination of boundary strata decorated by edge-terms $(\psi_h+\psi_{h'})^{m(e)}$ for non-negative integers $m(e)$, $e\in\mathrm{E}(\Gamma_\delta)$, that is, terms of the form

$$j_{\Gamma_\delta *}\prod_{e=(h,h')\in\mathrm{E}(\Gamma_\delta)}(\psi_h+\psi_{h'})^{m(e)}. \tag{4.5}$$

A first consequence of the combinatorial rules for computing intersections in the tautological ring([27]) of $\mathfrak{Pic}_{g,n,d}$ is that (4.3) is also a linear combination of such terms. The decorations $(\psi_h+\psi_{h'})^{m(e)}$ on separating edges $e=(h,h')$ appear naturally in the self-intersection formula for the boundary divisors $[\Gamma_\delta]$ since, for $\Gamma_\delta\in\mathsf{G}^{\mathsf{se}}_{g,n,d}$, the Chern class of the normal bundle of j_{Γ_δ} is given by $-(\psi_h+\psi_{h'})$.

We show that the coefficients of the term (4.5) in (4.3) and (4.4) have the same constant term in r. In (4.4), the coefficient is given by

$$\sum_{w\in\mathsf{W}_{\Gamma_\delta,r}}\frac{r^{-h^1(\Gamma_\delta)}}{|\mathrm{Aut}(\Gamma_\delta)|}\prod_{e=(h,h')\in\mathrm{E}(\Gamma_\delta)}(-1)^{m(e)}\left(\frac{w(h)w(h')}{2}\right)^{m(e)+1}\frac{1}{(m(e)+1)!}. \tag{4.6}$$

On the other hand, let $e_1,...,e_\ell\in E(\Gamma_\delta)$ be the separating edges of Γ_δ, and let $\bar\Gamma_\delta\in\mathsf{G}^{\mathsf{nse}}_{g,n,d}$ be the graph obtained from Γ_δ by contracting these separating edges. Each separating edge e_j corresponds to a unique graph $(\Gamma_j)_{\delta_j}\in\mathsf{G}^{\mathsf{se}}_{g,n,d}$ obtained by contracting all edges of Γ_δ except for e_j.

In the product (4.3), the intersection rules of the tautological ring of $\mathfrak{Pic}_{g,n,d}$ imply that we obtain multiples of the term (4.5) by combining

- for $j=1,...,\ell$, a total of $m(e_j)+1$ terms $[(\Gamma_j)_{\delta_j}]$ from expanding the power series

$$\exp\left(\frac{1}{2}\sum_{\Gamma_\delta\in\mathsf{G}^{\mathsf{se}}_{g,n,d}}c_A(\Gamma_\delta)[\Gamma_\delta]\right),$$

- the terms associated with $\bar\Gamma_\delta\in\mathsf{G}^{\mathsf{nse}}_{g,n,d}$ in the second factor.

Let $M=\sum_{j=1}^{\ell}(m(j)+1)$. Then, (4.5) appears in (4.3) with coefficient

$$\begin{aligned}\frac{1}{M!}\binom{M}{m(e_1)+1,...,m(e_\ell)+1}\left(\prod_{j=1}^{\ell}\left(\frac{c_A((\Gamma_j)_{\delta_j})}{2}\right)^{m(e_j)+1}(-1)^{m(e_j)}\right)\frac{|\mathrm{Aut}(\bar\Gamma_\delta)|}{|\mathrm{Aut}(\Gamma_\delta)|}\\ \times\sum_{w\in\mathsf{W}_{\bar\Gamma_\delta,r}}\frac{r^{-h^1(\bar\Gamma_\delta)}}{|\mathrm{Aut}(\bar\Gamma_\delta)|}\prod_{e=(h,h')\in\mathrm{E}(\bar\Gamma_\delta)}(-1)^{m(e)}\left(\frac{w(h)w(h')}{2}\right)^{m(e)+1}\frac{1}{(m(e)+1)!}.\end{aligned} \tag{4.7}$$

([27]) See [30] for the original treatment of the tautological ring of $\overline{\mathcal{M}}_{g,n}$. A corresponding treatment for $\mathfrak{M}_{g,n}$ will be given in [6], [7]. See also [43, §1.1 and §1.7].

To show the equality of (4.6) and (4.7), we combine a number of observations. First, for the multinomial coefficients, we have

$$\frac{1}{M!}\binom{M}{m(e_1)+1,\ldots,m(e_\ell)+1}=\prod_{j=1}^{\ell}\frac{1}{(m(e_j)+1)!}.$$

Second, for the graph morphism $\Gamma_\delta\to\bar{\Gamma}_\delta$ contracting the separating edges, the following statements hold.

- We have an equality of Betti numbers $h^1(\Gamma_\delta)=h^1(\bar{\Gamma}_\delta)$.
- For the separating edges $e_j=(h_j,h_j')$ of Γ_δ, splitting the graph according to the partition $(g_1,I_1,\delta_1\,|\,g_2,I_2,\delta_2)$, the value of every weighting $w\in\mathsf{W}_{\Gamma_\delta,r}$ is uniquely determined on h_j and h_j', since

$$w(h_j)=\delta_1-\sum_{i\in I_1}a_i \pmod r \quad\text{and}\quad w(h_j')=\delta_2-\sum_{i\in I_2}a_i \pmod r.$$

Hence, the constant term in r of $w(h_j)w(h_j')$ is precisely given by $c_A((\Gamma_j)_{\delta_j})$.

- Concerning the non-separating edges for fixed Γ_δ with contraction $\Gamma_\delta\to\bar{\Gamma}_\delta$, the map $W_{\Gamma_\delta,r}\to W_{\bar{\Gamma}_\delta,r}$ given by restricting weightings $w\in W_{\Gamma_\delta,r}$ to the remaining half-edges $H(\bar{\Gamma}_\delta)\subset H(\Gamma_\delta)$ is a bijection.

The combination of these facts proves equality of (4.6) and (4.7), and hence the equality of (4.3) and (4.4).

To conclude the proof, we must show that the remaining part of the exponential term of (4.2) can be drawn into the graph sum. Using the projection formula, this identity is equivalent to showing that

$$(j_{\Gamma_\delta})^*\exp\left(\frac{1}{2}\left(-\eta+\sum_{i=1}^{n}2a_i\xi_i+a_i^2\psi_i\right)\right)=\prod_{v\in \mathrm{V}(\Gamma_\delta)}\exp\left(-\frac{1}{2}\eta(v)\right)\prod_{i=1}^{n}\exp\left(\frac{1}{2}a_i^2\psi_i+a_i\xi_i\right),$$

which immediately reduces to showing

$$(j_{\Gamma_\delta})^*\left(-\eta+\sum_{i=1}^{n}(2a_i\xi_i+a_i^2\psi_i)\right)=-\sum_{v\in \mathrm{V}(\Gamma_\delta)}\eta(v)+\sum_{i=1}^{n}(2a_i\xi_i+a_i^2\psi_i).$$

By Lemma 4.2,

$$-\eta+\sum_{i=1}^{n}(2a_i\xi_i+a_i^2\psi_i)=-\pi_*c_1(\mathcal{L}_A)^2.$$

Now, consider the diagram of universal curves

$$\begin{tikzcd}
\prod_{v\in \mathrm{V}(\Gamma)}\mathfrak{C}_{\mathrm{g}(v),\mathrm{n}(v),\delta(v)} \arrow[d] & \mathfrak{C}'_{\Gamma_\delta} \arrow[l] \arrow[r, "G"] \arrow[d, "\pi'_{\Gamma_\delta}"] & \mathfrak{C}_{\Gamma_\delta} \arrow[r, "J_{\Gamma_\delta}"] \arrow[d, "\pi_{\Gamma_\delta}"] & \mathfrak{C}_{g,n,d} \arrow[d, "\pi"] \\
\prod_{v\in \mathrm{V}(\Gamma)}\mathfrak{Pic}_{\mathrm{g}(v),\mathrm{n}(v),\delta(v)} & \mathfrak{Pic}_{\Gamma_\delta} \arrow[l] \arrow[r, equal] & \mathfrak{Pic}_{\Gamma_\delta} \arrow[r, "j_{\Gamma_\delta}"] & \mathfrak{Pic}_{g,n,d}.
\end{tikzcd}$$

where the left and right squares are cartesian, and the map G is the gluing map identifying sections of $\mathfrak{C}'_{\Gamma_\delta}\to\mathfrak{Pic}_{\Gamma_\delta}$ corresponding to pairs of half-edges forming an edge. This map G is proper, representable, and birational.

The space $\mathfrak{C}'_{\Gamma_\delta}$ is a disjoint union of universal curves

$$\pi'_{\Gamma_\delta,v}\colon \mathfrak{C}'_{\Gamma_\delta,v}\longrightarrow \mathfrak{Pic}_{\Gamma_\delta}$$

for $v\in\mathrm{V}(\Gamma)$, and the bundle $G^*J^*_{\Gamma_\delta}\mathcal{L}_A$ restricted to the component $\mathfrak{C}'_{\Gamma_\delta,v}$ is equal to the pullback of the line bundle $\mathcal{L}_{v,A_v}$ from the factor $C_{\mathrm{g}(v),\mathrm{n}(v),\delta(v)}$ (where A_v is the vector formed by numbers a_i for i a marking at v, extended by zero on the half-edges at v). Then, using the projection formula together with Proposition 2.16, we have

$$\begin{aligned}(j_{\Gamma_\delta})^*\pi_*c_1(\mathcal{L}_A)^2 &= (\pi_{\Gamma_\delta})_*J^*_{\Gamma_\delta}c_1(\mathcal{L}_A)^2\\ &= (G\circ\pi_{\Gamma_\delta})_*(G\circ\pi_{\Gamma_\delta})^*c_1(\mathcal{L}_A)^2\\ &= \sum_{v\in\mathrm{V}(\Gamma_\delta)}(\pi'_{\Gamma_\delta,v})_*c_1(\mathcal{L}_{v,A_v})^2\\ &= \sum_{v\in\mathrm{V}(\Gamma_\delta)}\eta(v)+\sum_{i=1}^n a_i^2\psi_i+2a_i\xi_i,\end{aligned}$$

where, for the last equality, we again use Lemma 4.2. □

In the case $n=0$ and $d=0$, the formula $\mathsf{P}^\bullet_{g,\varnothing,0}$ takes a slightly simpler shape: it is the $r=0$ term of the formula

$$\exp\Big(-\frac{1}{2}\eta\Big)\\ \times\sum_{\substack{\Gamma_\delta\in\mathsf{G}_{g,0,0}\\ w\in\mathsf{W}_{\Gamma_\delta,r}}}\frac{r^{-h^1(\Gamma_\delta)}}{|\mathrm{Aut}(\Gamma_\delta)|}j_{\Gamma_\delta *}\Bigg[\prod_{e=(h,h')\in\mathrm{E}(\Gamma_\delta)}\frac{1}{\psi_h+\psi_{h'}}\Bigg(1-\exp\Bigg(-\frac{w(h)w(h')}{2}(\psi_h+\psi_{h'})\Bigg)\Bigg)\Bigg]. \tag{4.8}$$

As explained in §0.7, the full formula $\mathsf{P}^\bullet_{g,A,d}$ can be reconstructed from $\mathsf{P}^\bullet_{g,\varnothing,0}$.

4.2. Comparison to Pixton's k-twisted formula

Given $k\geqslant 0$ and a vector $A=(a_1,\dots,a_n)\in\mathbb{Z}^n$ satisfying

$$\sum_i a_i=k(2g-2),$$

let $\tilde{A}=(\tilde{a}_1,\dots,\tilde{a}_n)$ be the vector with entries $\tilde{a}_i=a_i+k$. Denote by

$$P_g^{c,k}(\tilde{A})\in\mathsf{CH}^c(\overline{\mathcal{M}}_{g,n})$$

Pixton's original formula defined in [42, §1.1].

In the $k{=}0$ case, $A{=}\tilde{A}$, and $2^{-g}P_g^{g,0}(\tilde{A})$ is the class originally conjectured by Pixton to equal the double ramification cycle associated with the vector $\tilde{A}$. Compatibility with the formula for the universal twisted double ramification cycle is given by the following result.

PROPOSITION 4.3. *Via the map* $\varphi_{\omega_\pi^k}\colon \overline{\mathcal{M}}_{g,n}\to\mathfrak{Pic}_{g,n,k(2g-2)}$ *associated with the universal data*

$$\pi\colon \mathcal{C}_{g,n}\longrightarrow\overline{\mathcal{M}}_{g,n},\quad \omega_\pi^k\longrightarrow\mathcal{C}_{g,n},$$

the class $\mathsf{P}^c_{g,A,k(2g-2)}$ *acts as*

$$\mathsf{P}^c_{g,A,k(2g-2)}(\varphi_{\omega_\pi^k})([\overline{\mathcal{M}}_{g,n}])=2^{-c}P_g^{c,k}(\tilde{A}) \tag{4.9}$$

for every $c\geqslant 0$.

Proof. The left-hand side of (4.9) is obtained from $\mathsf{P}^c_{g,A,k(2g-2)}$ by substituting

$$\mathcal{L}=\omega_\pi^{\otimes k} \tag{4.10}$$

in the formula and taking the action. A factor 2^{-c} arises on the left-hand side, since all terms in $\mathsf{P}^c_{g,A,k(2g-2)}$ increasing the codimension of the cycle naturally come with corresponding negative powers of 2 (which is placed as a prefactor on the right-hand side in [42, §1.1]).

Under the substitution (4.10), the edge terms and weightings modulo r in the two formulas naturally correspond to each other. Using Proposition 4.1 and Lemma 4.2, we must show that

$$\exp\biggl(-\frac{1}{2}\pi_*c_1\biggl(\omega_\pi^{\otimes k}\biggl(-\sum_{i=1}^n a_i[p_i]\biggr)\biggr)^2\biggr)=\exp\biggl(-\frac{1}{2}\biggl(k^2\kappa_1-\sum_{i=1}^n\tilde{a}_i^2\psi_i\biggr)\biggr),$$

where again $[p_i]$ denotes the class of the image of the section $p_i\colon\overline{\mathcal{M}}_{g,n}\to\mathcal{C}_{g,n}$. Defining

$$\omega_\pi^{\mathsf{log}}=\omega_\pi\biggl(\sum_i p_i\biggr),$$

we see that

$$\begin{aligned}c_1\biggl(\omega_\pi^{\otimes k}\biggl(-\sum_{i=1}^n a_i[p_i]\biggr)\biggr)^2&=\biggl(kc_1(\omega_{\log})-\sum_{i=1}^n\tilde{a}_i[p_i]\biggr)^2\\&=k^2c_1(\omega_{\log})^2-2k\sum_{i=1}^n\tilde{a}_ic_1(\omega_{\log})|_{[p_i]}+\sum_{i=1}^n\tilde{a}_i^2[p_i]^2.\end{aligned}$$

After pushing forward, the first term gives $k^2\kappa_1$, the second vanishes (since $\omega_{\log}$ restricts to zero on the section p_i), and the third gives $-\sum_i\tilde{a}_i^2\psi_i$, as desired. $\square$

4.3. Comparison to Pixton's formula with targets

Let X be a non-singular projective variety over K. The moduli space $\overline{\mathcal{M}}_{g,n}(X,\beta)$ parameterizes stable maps

$$f\colon (C,p_1,...,p_n) \longrightarrow X$$

from genus-g, n-pointed curves C to X of degree $\beta \in H_2(X,\mathbb{Z})$. The moduli space carries a virtual fundamental class

$$[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{\mathsf{vir}} \in \mathsf{CH}_{\mathrm{vdim}(g,n,\beta)}(\overline{\mathcal{M}}_{g,n}(X,\beta)),$$

where

$$\mathrm{vdim}(g,n,\beta) = (\dim X - 3)(1-g) + \int_\beta c_1(X) + n.$$

See [11] for the construction of virtual fundamental classes.

Given the data of a line bundle $\mathcal{L}$ on X and a vector $A=(a_1,...,a_n)\in\mathbb{Z}^n$ satisfying

$$\int_\beta c_1(\mathcal{L}) = \sum_{i=1}^n a_i,$$

a double ramification cycle

$$\mathsf{DR}_{g,A}(X,\mathcal{L}) \in \mathsf{CH}_{\mathrm{vdim}(g,n,\beta)-g}(\overline{\mathcal{M}}_{g,n}(X,\beta))$$

virtually compactifying the locus of maps $f\colon (C,p_1,...,p_n)\to X$ with

$$f^*\mathcal{L} \cong \mathcal{O}_C\left(\sum_{i=1}^n a_i p_i\right)$$

is defined in [43]. Furthermore, the authors define the notion of tautological classes inside the operational Chow ring $\mathsf{CH}^*_{\mathsf{op}}(\overline{\mathcal{M}}_{g,n}(X,\beta))$ of $\overline{\mathcal{M}}_{g,n}(X,\beta)$. The main result of [43] is a Pixton formula for a codimension-g tautological class whose action on $[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{\mathsf{vir}}$ yields $\mathsf{DR}_{g,A}(X,\mathcal{L})$.

We define a morphism

$$\begin{aligned}\varphi_{\mathcal{L}}\colon \overline{\mathcal{M}}_{g,n,\beta}(X) &\longrightarrow \mathfrak{Pic}_{g,A,d},\\ f &\longmapsto (C,p_1,...,p_n,f^*\mathcal{L}).\end{aligned}$$

The compatibility result here is

$$\mathsf{DR}_{g,A}(X,\mathcal{L}) = \varphi_{\mathcal{L}}^*\mathsf{P}^g_{g,A,d}([\overline{\mathcal{M}}_{g,n}(X,\beta)]^{\mathsf{vir}}). \tag{4.11}$$

The equality follows by an exact matching of the definition of $\mathsf{P}^g_{g,\mathbf{A},d}$ in §0.3.5 (after pullback by $\varphi_{\mathcal{L}}^*$) with the Pixton formula in the main result of [43].

In fact, the compatibility (4.11) represented the starting point for our investigation of the universal twisted double ramification cycle here.

5. Proof of Theorem 0.7

5.1. Overview

We prove here the main result of the paper: for $A=(a_1, ..., a_n)\in\mathbb{Z}^n$ satisfying

$$\sum_{i=1}^n a_i = d,$$

the universal twisted double ramification cycle is calculated by Pixton's formula

$$\mathsf{DR}^{\mathsf{op}}_{g,A} = \mathsf{P}^g_{g,A,d} \in \mathsf{CH}^g_{\mathsf{op}}(\mathfrak{Pic}_{g,n,d}).$$

The result is an equality in the operational Chow group, and therefore an equality on every finite-type family of prestable curves. Given a prestable curve $C\to B$ and a line bundle $\mathcal{L}$ on C of relative degree d, we obtain a map

$$\varphi_{\mathcal{L}}: B \longrightarrow \mathfrak{Pic}_{g,n,d}.$$

We must prove that

$$\mathsf{DR}^{\mathsf{op}}_{g,A}(\varphi_{\mathcal{L}}) = \mathsf{P}^g_{g,A,d}(\varphi_{\mathcal{L}}): \mathsf{CH}_*(B) \longrightarrow \mathsf{CH}_{*-g}(B). \tag{5.1}$$

As explained in §0.7, the result for general $A\in\mathbb{Z}^n$ can be reduced to the case $n=0$, $d=0$, though the case of arbitrary A will be important in the proof, as we proceed through a sequence of special cases. We recall that this reduction used the invariances II and III for the double ramification cycle and Pixton's formula. Note that these will be proved separately and independent of Theorem 0.7 in §7, so no circular reasoning occurs.

5.2. On an open subset of $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^l, \beta)$

As before, let $A=(a_1, ..., a_n)\in\mathbb{Z}^n$ with

$$\sum_{i=1}^n a_i = d.$$

We consider here the target $X=\mathbb{P}^l$. Let β be the class of d times a line in $\mathbb{P}^l$. Let

$$\mathcal{C} \longrightarrow \overline{\mathcal{M}}_{g,n}(\mathbb{P}^l, \beta)$$

be the universal curve over the moduli of stable maps to $\mathbb{P}^l$, let

$$f: \mathcal{C} \longrightarrow \mathbb{P}^l$$

be the universal map, and let $\mathcal{L}=f^*\mathcal{O}_X(1)$.

We have a tautological map

$$\varphi_{\mathcal{L}}\colon \overline{\mathcal{M}}_{g,n}(\mathbb{P}^l,\beta) \longrightarrow \mathfrak{Pic}_{g,n,d}. \tag{5.2}$$

We would like to prove an equality of operational classes

$$\varphi_{\mathcal{L}}^*\mathsf{DR}_{g,A}^{\mathsf{op}} = \varphi_{\mathcal{L}}^*\mathsf{P}_{g,A,d}^g \in \mathsf{CH}_{\mathsf{op}}^g(\overline{\mathcal{M}}_{g,n}(\mathbb{P}^l,\beta)).$$

We will apply the main result of [43] which relates the double ramification cycle there to Pixton's formula. However, only the action of $\varphi_{\mathcal{L}}^*\mathsf{P}_{g,A,d}^g$ on the virtual fundamental class $[\overline{\mathcal{M}}_{g,n}(\mathbb{P}^l,\beta)]^{\mathsf{vir}}$ is computed in [43]. Since we are interested here in the full operational class $\varphi_{\mathcal{L}}^*\mathsf{P}_{g,A,d}^g$, our first idea is to restrict to the open locus

$$\overline{\mathcal{M}}_{g,n}(\mathbb{P}^l,\beta)' \hookrightarrow \overline{\mathcal{M}}_{g,n}(\mathbb{P}^l,\beta)$$

where (on each geometric fiber) we have $H^1(C,\mathcal{L})=0$.

LEMMA 5.1. *On the smooth Deligne–Mumford stack $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^l,\beta)'$, the fundamental and virtual fundamental classes coincide.*

Proof. It suffices to show that $H^1(C,f^*T_{\mathbb{P}^l})=0$ on $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^l,\beta)'$. Pulling back the Euler exact sequence on $\mathbb{P}^l$ via f yields

$$0 \longrightarrow \mathcal{O}_C \longrightarrow \bigoplus_1^{l+1} f^*\mathcal{O}_{\mathbb{P}^l}(1) \longrightarrow f^*T_{\mathbb{P}^l} \longrightarrow 0. \tag{5.3}$$

Taking cohomology yields the exact sequence

$$\bigoplus_1^{l+1} H^1(C,f^*\mathcal{O}_{\mathbb{P}^l}(1)) \longrightarrow H^1(C,f^*T_{\mathbb{P}^l}) \longrightarrow H^2(C,\mathcal{O}_C). \tag{5.4}$$

But $H^1(C,f^*\mathcal{O}_{\mathbb{P}^l}(1))=0$ by assumption, and $H^2(C,\mathcal{O}_C)=0$ for dimension reasons. □

The next lemma depends on a careful comparison of the logarithmic and rubber approaches to double ramification cycles, which will be postponed to §6.

LEMMA 5.2. *Let ${\varphi'}_{\mathcal{L}}$ be the restriction of $\varphi_{\mathcal{L}}$ to $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^l,\beta)'$. We have an equality of operational classes*

$${\varphi'}_{\mathcal{L}}^*\mathsf{DR}_{g,A}^{\mathsf{op}} = {\varphi'}_{\mathcal{L}}^*\mathsf{P}_{g,A,d}^g \in \mathsf{CH}_{\mathsf{op}}^g(\overline{\mathcal{M}}_{g,n}(\mathbb{P}^l,\beta)'). \tag{5.5}$$

Proof. By Lemma 2.6, the two sides of (5.5) are equal if and only if their actions on the fundamental class $[\overline{\mathcal{M}}_{g,n}(\mathbb{P}^l,\beta)']$ are equal in $\mathsf{CH}_*(\overline{\mathcal{M}}_{g,n}(\mathbb{P}^l,\beta)')$. By (4.11), the action of the right-hand side of (5.5) on

$$[\overline{\mathcal{M}}_{g,n}(\mathbb{P}^l,\beta)'] = [\overline{\mathcal{M}}_{g,n}(\mathbb{P}^l,\beta)']^{\mathsf{vir}}$$

equals the restriction of $\mathsf{DR}_{g,A}(\mathbb{P}^l,\mathcal{L})$ to $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^l,\beta)'$.

The cycle $\mathsf{DR}_{g,A}(\mathbb{P}^l,\mathcal{L})$ is defined in [43] as the pushforward of the virtual fundamental class of the space of rubber maps.(28) By Proposition 6.12 of §6.5, the restriction of $\mathsf{DR}_{g,A}(X,\mathcal{L})$ to $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^l,\beta)'$ is equal to $\varphi'^*_{\mathcal{L}}\mathsf{DR}^{\mathsf{op}}_{g,A}([\overline{\mathcal{M}}_{g,n}(\mathbb{P}^l,\beta)'])$. □

5.3. For sufficiently positive line bundles

Let $\pi: C\to B$ be an n-pointed prestable curve over a scheme of finite type over K. Let $\mathcal{L}$ on C be a line bundle of relative degree d. Let

$$A=(a_1,...,a_n)\in\mathbb{Z}^n$$

with $\sum_{i=1}^n a_i=d$. The line bundle $\mathcal{L}$ induces a map

$$\varphi_{\mathcal{L}}: B\longrightarrow \mathfrak{Pic}_{g,n,d}.$$

We say that $\mathcal{L}$ is *relatively sufficiently positive* if $\mathcal{L}$ is relatively base-point free and satisfies $R^1\pi_*\mathcal{L}=0$.

LEMMA 5.3. *Let $\mathcal{L}$ be a line bundle which is relatively sufficiently positive. Then, we have an equality*

$$\mathsf{DR}^{\mathsf{op}}_{g,A}(\varphi_{\mathcal{L}}) = \mathsf{P}^g_{g,A,d}(\varphi_{\mathcal{L}}): \mathsf{CH}_*(B)\longrightarrow \mathsf{CH}_{*-g}(B). \tag{5.6}$$

Proof. For any finite-type scheme B, the union of irreducible components of B maps properly and surjectively to B. Thus, the pushforward from the Chow groups of the irreducible components to that of B is surjective, and hence it suffices to show the equality (5.6) of maps of Chow groups for B irreducible.

By relative sufficient positivity,

$$R\pi_*\mathcal{L}=\pi_*\mathcal{L}$$

(28) Rubber maps will be discussed in §6.3.

is a vector bundle on B of rank N. For a positive integer l, we define

$$E_l = \bigoplus_1^{l+1} R\pi_* \mathcal{L}, \tag{5.7}$$

a vector bundle on B of rank $r{=}N(l{+}1)$. Let $U_l \subseteq E_l$ denote the open locus of linear systems which are base-point free. Via pullback along $\psi\colon U_l \to B$, we obtain a map

$$\psi^*\colon \mathsf{CH}_*(B) \longrightarrow \mathsf{CH}_{*+r}(U_l).$$

We claim that, for $l{>}\dim B$, the pullback (5.7) is injective. To prove the injectivity, we show that the boundary $E_l \backslash U_l$ has codimension in E_l greater than $\dim B$. Since $E_l \to B$ is flat with irreducible target, it suffices to bound the codimension on each geometric fiber over B: for a prestable curve C/K and a sufficiently positive line bundle $\mathcal{L}$ on C, we must show that the locus in $\bigoplus_1^{l+1} H^0(C, \mathcal{L})$ consisting of base-point-free linear systems has a complement of codimension greater than $\dim B$.

Since $\mathcal{L}$ is base-point free on C, the dimension of the locus in $\bigoplus_1^{l+1} H^0(C, \mathcal{L})$, where the linear system has a base point at some given $p \in C$ is $(N-1)(l+1)$. Hence, as p varies, the complement of the base-point-free locus in $\bigoplus_1^{l+1} H^0(C, \mathcal{L})$ has dimension at most $1+(N-1)(l+1)$. So the codimension is at least

$$N(l+1)-1-(N-1)(l+1) = l.$$

We have a canonical map $g\colon C \times_B U_l \to \mathbb{P}^l$ with $g^*\mathcal{O}_{\mathbb{P}^l}(1) = \mathcal{L}$, which induces a map

$$U_l \longrightarrow \overline{\mathcal{M}}_{g,n}(\mathbb{P}^l, \beta),$$

which factors via the locus

$$\overline{\mathcal{M}}_{g,n}(\mathbb{P}^l, \beta)' \subset \overline{\mathcal{M}}_{g,n}(\mathbb{P}^l, d),$$

where $H^1(C, f^*\mathcal{O}_{\mathbb{P}^l}(1)) = 0$. By construction, the composition

$$U_l \xrightarrow{\psi} B \xrightarrow{\varphi_{\mathcal{L}}} \mathfrak{Pic}_{g,n,d}$$

then factors through the map

$$\overline{\mathcal{M}}_{g,n}(\mathbb{P}^l, \beta)' \longrightarrow \mathfrak{Pic}_{g,n,d}$$

induced by the line bundle $f^*\mathcal{O}_{\mathbb{P}^l}(1)$ as before. In other words, we have a commutative diagram

$$\begin{tikzcd}
U_l \arrow[r] \arrow[d, "\psi"] & \overline{\mathcal{M}}_{g,n}(\mathbb{P}^l, \beta)' \arrow[d, "\varphi_{f^*\mathcal{O}_{\mathbb{P}^l}(1)}"] \\
B \arrow[r, "\varphi_{\mathcal{L}}"] & \mathfrak{Pic}_{g,n,d}.
\end{tikzcd}$$

Lemma 5.2 then implies that

$$(\mathsf{DR}^{\mathsf{op}}_{g,A} - \mathsf{P}^g_{g,A,d})(\varphi_{\mathcal{L}} \circ \psi) \colon \mathsf{CH}_*(U_l) \longrightarrow \mathsf{CH}_{*-g}(U_l) \tag{5.8}$$

is the zero map, and we conclude the proof of the lemma from the commutative diagram

$$\begin{tikzcd} \mathsf{CH}_{*+r}(U) \arrow[r, "(\mathsf{DR}^{\mathsf{op}}_{g,A} - \mathsf{P}^g_{g,A,d})(\varphi_{\mathcal{L}} \circ \psi)"] & \mathsf{CH}_{*+r-g}(U) \\ \mathsf{CH}_*(B) \arrow[u, hook, "\psi^*"] \arrow[r, "(\mathsf{DR}^{\mathsf{op}}_{g,A} - \mathsf{P}^g_{g,A,d})(\varphi_{\mathcal{L}})"] & \mathsf{CH}_{*-g}(B). \arrow[u, hook, "\psi^*"'] \end{tikzcd}$$

□

5.4. With sufficiently many sections

Let $\pi\colon C \to B$ be an n-pointed prestable curve with markings $p_1, ..., p_n$ over a scheme of finite type over K. Let $\mathcal{L}$ on C be a line bundle of relative degree d. Let

$$A = (a_1, ..., a_n) \in \mathbb{Z}^n$$

with $\sum_{i=1}^n a_i = d$. The line bundle $\mathcal{L}$ induces a map

$$\varphi_{\mathcal{L}} \colon B \longrightarrow \mathfrak{Pic}_{g,n,d}.$$

LEMMA 5.4. *For every geometric fiber of C/B, suppose that the complement of the union of irreducible components which carry markings is a disjoint union of trees of non-singular rational curves on which $\mathcal{L}$ is trivial. Then, we have an equality*

$$\mathsf{DR}^{\mathsf{op}}_{g,A}(\varphi_{\mathcal{L}}) = \mathsf{P}^g_{g,A,d}(\varphi_{\mathcal{L}}) \colon \mathsf{CH}_*(B) \longrightarrow \mathsf{CH}_{*-g}(B). \tag{5.9}$$

Proof. We can choose $A' = (a'_1, ..., a'_n)$ with entries

$$a'_i \gg 0, \quad \sum_{i=1}^n a'_i = d',$$

large enough so that

$$\mathcal{L}' = \mathcal{L}\left(\sum_{i=1}^n a'_i p_i\right)$$

is relatively sufficiently positive (by Riemann–Roch for singular curves).

We obtain an associated map

$$\varphi_{\mathcal{L}'}\colon B \longrightarrow \mathfrak{Pic}_{g,n,d+d'}.$$

By §5.3,

$$\mathsf{DR}^{\mathsf{op}}_{g,A+A'}(\varphi_{\mathcal{L}'}) = \mathsf{P}^g_{g,A+A',d+d'}(\varphi_{\mathcal{L}'})\colon \mathsf{CH}_*(B) \longrightarrow \mathsf{CH}_{*-g}(B). \tag{5.10}$$

Invariance III of §0.6 (proven in §7) implies that

$$\mathsf{DR}^{\mathsf{op}}_{g,A+A'}(\varphi_{\mathcal{L}'}) = \mathsf{DR}^{\mathsf{op}}_{g,A}(\varphi_{\mathcal{L}}),$$
$$\mathsf{P}^g_{g,A+A',d+d'}(\varphi_{\mathcal{L}'}) = \mathsf{P}^g_{g,A,d}(\varphi_{\mathcal{L}}),$$

which together with (5.10) finishes the proof. □

5.5. Proof in the general case

To conclude the proof of Theorem 0.7, will use the invariances of §0.6 (proven in §7). As discussed in §5.1, we can reduce to showing the result in the case $n=0$, $d=0$.([29])

Let B be an irreducible scheme of finite type over K. Let $\pi\colon C\to B$ a prestable curve, and let $\mathcal{L}$ on C be a line bundle of relative degree zero. The line bundle $\mathcal{L}$ induces a map

$$\varphi_{\mathcal{L}}\colon B \longrightarrow \mathfrak{Pic}_{g,0,0}.$$

LEMMA 5.5. *There exist an alteration*([30]) $B'\to B$ *and a destabilization*

$$C' \longrightarrow C\times_B B' \tag{5.11}$$

such that C' *admits sections* $p_1, ..., p_m$ *and satisfies the following property*:

$$(C'/B', p_1, ..., p_m)$$

is a family of m*-pointed prestable curves and, for every geometric fiber of* C'/B'*, the complement of the union of irreducible components which carry markings is a disjoint union of trees of non-singular rational curves which are contracted by the morphism* (5.11).

([29]) In genus $g=1$, we follow a slightly modified strategy, since there we must avoid the case $n=0$ for technical reasons (see Remark 2.1). Instead, we can use the invariances to reduce to the case $g=1$, $n=1$, and $A=(0)$. All the proofs below generalize in a straightforward way, since the vector $A=(0)$ does not affect the line bundles involved.

([30]) An alteration here is a proper, surjective, generically finite morphism between irreducible schemes.

Proof. We first claim, after an alteration $\widehat{B}\to B$, there exists a multisection([31])

$$Z\subset C_{\widehat{B}}=C\times_B\widehat{B}\longrightarrow\widehat{B}$$

satisfying the following two conditions:

(i) Over the generic point of $\widehat{B}$, Z is contained in the smooth locus of $C_{\widehat{B}}\to\widehat{B}$.

(ii) Every component of every geometric fiber of $C_{\widehat{B}}\to\widehat{B}$ carries at least two distinct étale multisection points in the smooth locus. In other words, the étale locus of $Z\to\widehat{B}$ meets the smooth locus of every component of every geometric fiber of $C_{\widehat{B}}\to\widehat{B}$ in at least two points.

To prove the above claim, we observe that, for every geometric point b of B, there exists an étale map $U_p\to B$ and a factorization $U_p\to C$ whose image meets the smooth locus of every irreducible component of every geometric fiber in some Zariski neighbourhood $V_b\subseteq B$ of b at least twice. Choose a finite set of b such that the V_b cover B, define U to be the union of the U_b, and define Z' to be the closure of the image of U in C. Then $Z'\to B$ is proper and generically finite. Let

$$\widehat{B}\longrightarrow B$$

be a modification which flattens Z' (see [69]), and let Z be the strict transform of Z' over $\widehat{B}$. Then

$$Z\longrightarrow\widehat{B}$$

is proper, flat, and generically finite, and hence finite — so condition (i) is satisfied. Moreover, U already satisfies condition (ii), and the strict transform of a flat map is just the fiber product, and hence Z also satisfies condition (ii).

Let $\widetilde{B}\to\widehat{B}$ be an alteration such that over $\widetilde{B}$ the multisection Z becomes a disjoint union of sections. In other words, the pullback

$$C_{\widetilde{B}}=C\times_B\widetilde{B}\longrightarrow\widetilde{B}$$

has sections $\tilde{p}_1,...,\tilde{p}_m$ such that, as a set, the preimage of Z is given by the union of the images of sections $\tilde{p}_1,...,\tilde{p}_m$. Such a $\widetilde{B}$ exists([32]) by [44, Lemma 5.6]. We may assume that the sections $\tilde{p}_i$ are pairwise disjoint over the generic point of $\widetilde{B}$.

By assumption (i) above, the family $C_{\widetilde{B}}\to\widetilde{B}$ with sections $\tilde{p}_1,...,\tilde{p}_m$ is generically a stable m-pointed curve (since every component has at least *two* of the sections). We therefore obtain a rational map

$$\widetilde{B}\dashrightarrow\overline{\mathcal{M}}_{g,m}.$$

([31]) By a multisection of $C_{\widehat{B}}\to\widehat{B}$, we mean a closed substack $Z\subset C_{\widehat{B}}$ such that $Z\to\widehat{B}$ is finite and flat.

([32]) The base $\widehat{B}$ is excellent, since it is of finite type over a field.

Let $B'\to\widetilde{B}$ be a blow-up resolving the indeterminacy of this map([33])

$$\begin{tikzcd} B' \arrow[r] \arrow[d] & \overline{\mathcal{M}}_{g,m} \\ \widetilde{B} \arrow[ur, dashed] & \end{tikzcd} \tag{5.12}$$

and let $C'\to B'$ with sections $p_1,...,p_m: B'\to C'$ be the pullback of the universal curve over $\overline{\mathcal{M}}_{g,n}$ to B'. Let

$$C_{B'} = C\times_B B'$$

be the pullback of C/B under

$$B' \longrightarrow \widetilde{B} \longrightarrow \widehat{B} \longrightarrow B.$$

Then, we have a map $f: C'\to C_{B'}$ fitting in a commutative diagram

$$\begin{tikzcd} C' \arrow[rr, "f"] \arrow[dr] & & C_{B'} \arrow[dl] \\ & B' \arrow[ul, bend left, "p_i"] \arrow[ur, bend right, "\tilde{p}_i"'] & \end{tikzcd} \tag{5.13}$$

such that f is a partial destabilization. On geometric fibers of $C'\to B'$, f collapses trees of rational curves to either nodes or coincident sections $\tilde{p}_i$ on the geometric fibers of $C_{B'}$.

To conclude, we must show that, for every geometric point $b\in B'$ and every irreducible component $D\subset C'_b$ which is not contracted by f, we can find a marking

$$p_i(b)\in D.$$

The image of D under f is a component of $(C_{B'})_b$. By condition (ii) above, $f(D)$ has at least one $\tilde{p}_i(b)$ in the smooth locus of $f(D)$ pairwise distinct from all other $\tilde{p}_j(b)$. Since there are no components of C'_b which collapse to $\tilde{p}_i(b)$, we must have $p_i(b)\in D$. □

LEMMA 5.6. *We have*

$$\mathsf{DR}^{\mathsf{op}}_{g,\varnothing}(\varphi_{\mathcal{L}}) = \mathsf{P}^g_{g,\varnothing,0}(\varphi_{\mathcal{L}}): \mathsf{CH}_*(B) \longrightarrow \mathsf{CH}_{*-g}(B).$$

Proof. We apply Lemma 5.5 to the family C/B to obtain

$$h: B' \longrightarrow B, \quad C' \longrightarrow C_{B'}.$$

([33]) As usual, this blowup is constructed by taking the closure of the graph and flattening. Then we check that this ensures the existence of the map to $C_{B'}$ as written below.

Let $\mathcal{L}'$ be the pullback of $\mathcal{L}$ to C'. After applying Lemma 5.4 with $A=\mathbf{0}\in\mathbb{Z}^m$, we obtain

$$\mathsf{DR}^{\mathsf{op}}_{g,\mathbf{0}}(\varphi_{\mathcal{L}'})=\mathsf{P}^g_{g,\mathbf{0},d}(\varphi_{\mathcal{L}'})\colon \mathsf{CH}_*(B')\longrightarrow \mathsf{CH}_{*-g}(B'). \tag{5.14}$$

Since h is proper and surjective, for any $\alpha\in\mathsf{CH}_*(B)$ there exists $\alpha'\in\mathsf{CH}_*(B')$ satisfying $h_*\alpha'=\alpha$. If any operational class maps α' to zero, then it maps α to zero because the operation commutes with h_*.

It therefore suffices to prove that

$$(\mathsf{DR}^{\mathsf{op}}_{g,\varnothing}-\mathsf{P}^g_{g,\varnothing,0})(\varphi_{\mathcal{L}})\circ h_* \tag{5.15}$$

is the zero map on $\mathsf{CH}_*(B')$. By the compatibilities of operational classes, we have

$$(\mathsf{DR}^{\mathsf{op}}_{g,\varnothing}-\mathsf{P}^g_{g,\varnothing,0})(\varphi_{\mathcal{L}})\circ h_*=h_*(\mathsf{DR}^{\mathsf{op}}_{g,\varnothing}-\mathsf{P}^g_{g,\varnothing,0})(\varphi_{\mathcal{L}}\circ h),$$

and the proof below will in fact show that

$$(\mathsf{DR}^{\mathsf{op}}_{g,\varnothing}-\mathsf{P}^g_{g,\varnothing,0})(\varphi_{\mathcal{L}}\circ h)=0.$$

By (5.14), we need only show

$$\mathsf{DR}^{\mathsf{op}}_{g,\varnothing}(\varphi_{\mathcal{L}}\circ h)=\mathsf{DR}^{\mathsf{op}}_{g,\mathbf{0}}(\varphi_{\mathcal{L}'})\colon \mathsf{CH}_*(B')\longrightarrow \mathsf{CH}_{*-g}(B'), \tag{5.16}$$

$$\mathsf{P}^g_{g,\varnothing,0}(\varphi_{\mathcal{L}}\circ h)=\mathsf{P}^g_{g,\mathbf{0},0}(\varphi_{\mathcal{L}'})\colon \mathsf{CH}_*(B')\longrightarrow \mathsf{CH}_{*-g}(B'). \tag{5.17}$$

For the map $F\colon\mathfrak{Pic}_{g,n,0}\to\mathfrak{Pic}_{g,0,0}$ forgetting the markings, Invariance II from §0.6 for the double ramification cycle and the Pixton formula shows that we have

$$\mathsf{DR}^{\mathsf{op}}_{g,\mathbf{0}}(\varphi_{\mathcal{L}'})=\mathsf{DR}^{\mathsf{op}}_{g,\varnothing}(F\circ\varphi_{\mathcal{L}'})\quad\text{and}\quad \mathsf{P}^g_{g,\mathbf{0},0}(\varphi_{\mathcal{L}'})=\mathsf{P}^g_{g,\varnothing,0}(F\circ\varphi_{\mathcal{L}'}).$$

So, we are reduced to showing that

$$\mathsf{DR}^{\mathsf{op}}_{g,\varnothing}(\varphi_{\mathcal{L}}\circ h)=\mathsf{DR}^{\mathsf{op}}_{g,\varnothing}(F\circ\varphi_{\mathcal{L}'})\quad\text{and}\quad \mathsf{P}^g_{g,\varnothing,0}(\varphi_{\mathcal{L}}\circ h)=\mathsf{P}^g_{g,\varnothing,0}(F\circ\varphi_{\mathcal{L}'}). \tag{5.18}$$

The claims (5.18) follow from Invariance VI of §0.6. As before, let $C_{B'}$ be the pullback of C under h, and let $\mathcal{L}_{B'}$ be the pullback of $\mathcal{L}$ to $C_{B'}$. The map

$$\varphi_{\mathcal{L}}\circ h\colon B'\longrightarrow\mathfrak{Pic}_{g,0,0}$$

is induced by the data

$$C_{B'}\longrightarrow B',\quad \mathcal{L}_{B'}\longrightarrow C_{B'},$$

whereas $F\circ\varphi_{\mathcal{L}'}\colon B'\to\mathfrak{Pic}_{g,0,0}$ is induced by

$$C'\longrightarrow B',\quad \mathcal{L}'\longrightarrow C'.$$

By construction, we have a partial destabilization $C'\to C_{B'}$ over B', and the line bundle $\mathcal{L}'$ is the pullback of $\mathcal{L}_{B'}$ under this map. Hence, the equalities (5.18) follow from Invariance VI of §0.6. $\square$

6. Comparing rubber and log spaces

6.1. Overview

Our goal here is to compare the stack of stable rubber maps associated with a line bundle $\mathcal{L}$ on a target X (introduced by Li [54] and studied by Graber–Vakil [31]) to the stack $\mathbf{Rub}_{g,A}$ of Marcus–Wise (see §3.3) and our operational class $\mathsf{DR}^{\mathsf{op}}_{g,A}$. Rubber maps are reviewed in §6.3 and connected to the logarithmic space in §6.4. The relationship between the construction of Marcus–Wise and $\mathsf{DR}^{\mathsf{op}}_{g,A}$ is Lemma 6.11 of §6.5.2. The comparison to the class of Graber–Vakil is carried out in [56], in the case where the target X is a point. We require the case where

$$X = \mathbb{P}^l \quad \text{and} \quad \mathcal{L} = \mathcal{O}(1),$$

but only over the unobstructed locus

$$\overline{\mathcal{M}}_{g,n}(\mathbb{P}^l, d)' \subset \overline{\mathcal{M}}_{g,n}(\mathbb{P}^l, d),$$

see §5.2. We will treat the case of a general non-singular projective target X, since restricting to $\mathbb{P}^l$ provides no simplification (though the unobstructed locus may be rather small for general X). The final comparison result is Proposition 6.12 in §6.5.3.

6.2. Refined definition of the logarithmic rubber space

As described in §3.4, Marcus and Wise define $\mathbf{Rub}^{\mathsf{rel}}_g$ to be the moduli space of pairs (C, P, α), where P is a tropical line on S and

$$\alpha \colon C \longrightarrow P$$

is an S-morphism such that on each geometric fiber over S the values taken by α on the irreducible components of C are totally ordered in $(\overline{M}^{\mathsf{gp}}_S)_s$. However, with the above definition, certain key results of their paper (in particular concerning the comparison to spaces of rubber maps) are not correct as stated.

To explain the problem, we restrict to the case where the base C is a geometric log point. Subdividing P at the images of the vertices of C under the map α yields a *divided tropical line* Q (in the language of [56]). It is asserted in the discussion above [56, Proposition 5.5.2] that the fiber product $C \times_P Q$ is again a log curve over S, which, in general, is not true. For example, take $\overline{M}_S$ to be the sub-monoid of $\mathbb{Z}^2$ generated by $(1,1)$, $(1,0)$, and $(1,-1)$, and C and α to be as illustrated in Figure 1. In the fiber product, the edge with length $(1,0)$ must be subdivided into two shorter edges, but $(1,0)$ is an irreducible element of $\overline{M}_S$. In fact, *failure of divisibility* is the only thing that can go wrong.

LEMMA 6.1. *Let* (C, P, α) *be a point of* $\mathbf{Rub}_g^{\mathsf{rel}}$ *over a geometric log point* B, *and let* Q *be obtained from* P *by subdividing at the image of* α. *Then, the following conditions are equivalent*:

(i) *The fiber product* $C \times_P Q$ *is a log curve over* B.

(ii) *Let* e *be an edge of* Γ_C *between vertices* u *and* v (*satisfying* $\alpha(v) \geqslant \alpha(u)$) *with length* $\ell_e \in \overline{M}_S$ *and slope*

$$\kappa_e = \frac{\alpha(v) - \alpha(u)}{\ell_e}.$$

Then, for every $y \in \mathrm{Image}(\alpha)$ *with* $\alpha(u) < y < \alpha(v)$, *the monoid* $\overline{M}_B$ *contains the element*

$$\frac{y - \alpha(u)}{\kappa_e}.$$

Proof. The characteristic monoid at a singular point with length ℓ_e is given by the monoid

$$\{(a, b) \in \overline{M}_b^2 : \ell_e \mid a - b\}. \tag{6.1}$$

Taking the fiber product over P with Q subdivides the characteristic monoid at the element

$$\frac{y - \alpha(u)}{\kappa_e} \in \overline{M}_B^{\mathsf{gp}} \otimes_{\mathbb{Z}} \mathbb{Q}.$$

If $(y - \alpha(u))/\kappa_e$ lies in $\overline{M}_B$, then the fiber product is easily seen to be a log curve. If not, then the subdivision is not even reduced. $\square$

Definition 6.2. We define $\widetilde{\mathbf{Rub}}^{\mathsf{rel}}$ to be the full subcategory of $\mathbf{Rub}^{\mathsf{rel}}$ consisting of objects (C, P, α) which, on each geometric fiber over B, satisfy the equivalent conditions of Lemma 6.1. We define $\widetilde{\mathbf{Rub}}$ to be the fiber product of $\widetilde{\mathbf{Rub}}^{\mathsf{rel}}$ over $\mathfrak{Pic}^{\mathrm{rel}}$ with $\mathfrak{Pic}$.

Remark 6.3. The double ramification cycle $\mathsf{DR}^{\mathsf{op}}$ can be defined as the operational class induced by the map $\mathbf{Rub} \to \mathfrak{Pic}$ following Definition 2.8. Applying the same definition to the composite map $\widetilde{\mathbf{Rub}} \to \mathfrak{Pic}$ yields the same operational class, by Proposition 2.16.

6.3. The stack of prestable rubber maps

Let $\mathfrak{M}(X)$ be the stack of maps from marked prestable curves to X. An S-point of $\mathfrak{M}(X)$ is a pair

$$(C/S, f : C \to X),$$

where C/S is prestable with markings. To simplify notation, we will often suppress the markings.

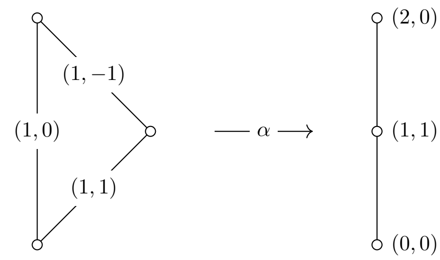


Figure 1. A point of **Rub**.

The space of rubber maps associated with a line bundle $\mathcal{L}$ on X is summarized in [43]: *a map to rubber with target X is a map to a rubber chain of $\mathbb{CP}^1$-bundles $\mathbb{P}(\mathcal{O}_X \oplus \mathcal{L})$ over X attached along their zero and ∞ divisors.*

To facilitate our comparison, we begin by writing the definition explicitly. Let $\mathbf{P}$ denote the projective bundle $\mathbb{P}(\mathcal{O}_X \oplus \mathcal{L})$. The map collapsing the fibers,

$$\rho \colon \mathbf{P} \longrightarrow X \tag{6.2}$$

admits two sections $r_0, r_\infty \colon X \to \mathbf{P}$ corresponding to $\mathcal{O}_X$ and $\mathcal{L}$, respectively.

Definition 6.4. An $(X, \mathcal{L})$-rubber target $(R/S, \rho, r_0, r_\infty)$ is flat, proper, and finitely presented

$$R \longrightarrow S$$

and a collapsing map $\rho \colon R \to X_S$ with two sections

$$r_0, r_\infty : X_S \to R$$

satisfying the following properties:

(i) Every geometric fiber R_s is isomorphic over X_s to a finite chain

$$\mathbf{P} \cup \mathbf{P} \cup \ldots \cup \mathbf{P} \tag{6.3}$$

with the components attached successively along the respective zero and ∞ divisors. The collapsing maps (6.2) on the components together define

$$\rho_s \colon R_s \longrightarrow X_s.$$

The zero and ∞ sections of ρ_s are determined, respectively, by the zero section of first component and the ∞ section of last components of the chain (6.3).

(ii) Étale locally near every point $s \in S$, the data of $(R/S, \rho, r_0, r_\infty)$ is pulled back from a versal deformation space described by Li [53] with one dimension for every component of the singular locus of (6.3).

Definition 6.5. The stack $\mathbf{Rub}^{\mathsf{pre}}(X,\mathcal{L})$ of prestable rubber maps to $\mathcal{L}$ is a fibered category over $\mathfrak{M}(X)$ whose fiber over a map $S\to\mathfrak{M}(X)$ consists of three pieces of data:

(i) a prestable curve $\widetilde{C}/S$ and a partial stabilization([34]) map $\tau\colon\widetilde{C}\to C_S$ which is allowed to contract genus-zero components with two special points;

(ii) an $(X,\mathcal{L})$-rubber target $(R/S,\rho,r_0,r_\infty)$;

(iii) a map $\tilde{f}\colon\widetilde{C}\to R$ for which the following diagram commutes:

$$\begin{tikzcd} \widetilde{C} \arrow[r, "\tilde{f}"] \arrow[d, "\tau"] & R \arrow[d, "\rho"] \\ C_S \arrow[r, "f"] & X_S. \end{tikzcd} \tag{6.4}$$

The map $\tilde{f}$ in (iii) is finite over the singularities of R/X_S and predeformable.([35]) Moreover, over each geometric point $s\in S$, the image $\tilde{f}(\widetilde{C}_s)$ meets every component of R_s.

An isomorphism between two objects

$$(\widetilde{C}\to C_S, R, r_0, r_\infty, \widetilde{C}\to R) \quad\text{and}\quad (\widetilde{C}'\to C_S, R', r_0', r_\infty', \widetilde{C}'\to R')$$

over $S\to\mathfrak{M}(X)$ is given by the data of isomorphisms

$$\widetilde{C}'\xrightarrow{\sim}\widetilde{C}$$

over C_S and

$$R'\xrightarrow{\sim}R$$

over X_S, compatible with the markings and such that the diagram

$$\begin{tikzcd} \widetilde{C}' \arrow[r, "\sim"] \arrow[d] & \widetilde{C} \arrow[d] \\ R' \arrow[r, "\sim"] & R \end{tikzcd}$$

commutes. We leave the definition of the cartesian morphisms to the careful reader.

Suppose now that we fix a genus g and a vector of integers A of length n. We define the stack $\mathbf{Rub}^{\mathsf{pre}}_{g,A}(X,\mathcal{L})$ with objects being tuples

$$(\tau\colon(\widetilde{C},p_1,...,p_n)\to C_S, R/X_S, \tilde{f}\colon\widetilde{C}\to R), \tag{6.5}$$

where $(\widetilde{C},p_1,...,p_n)$ is a prestable curve of genus g with n markings. The data (6.5) are as for $\mathbf{Rub}^{\mathsf{pre}}(X,\mathcal{L})$. Moreover, the following statements hold:

- if $a_i>0$, $p_i\in\widetilde{C}$ is mapped to the zero-divisor with ramification degree a_i;
- if $a_i<0$, $p_i\in\widetilde{C}$ is mapped to the ∞-divisor with ramification degree $-a_i$;
- if $a_i=0$, $p_i\in\widetilde{C}$ is mapped to the smooth locus of R away from the zero-divisor and from the ∞-divisor.

([34]) C_S is not necessarily a stable curve.
([35]) See [53].

6.4. Comparison to the logarithmic space

The pullback of $\mathcal{L}$ from X to the universal curve over $\mathfrak{M}(X)$ induces a map $\mathfrak{M}(X)\to\mathfrak{Pic}$. The key comparison result is the following.

Proposition 6.6. *The stack* $\mathbf{Rub}^{\mathsf{pre}}_{g,A}(X,\mathcal{L})$ *is naturally isomorphic to the fiber product of* $\widetilde{\mathbf{Rub}}_{g,A}$ *over* $\mathfrak{Pic}$ *with* $\mathfrak{M}(X)$ *along the map induced by* $\mathcal{L}$:

$$\mathbf{Rub}^{\mathsf{pre}}_{g,A}(X,\mathcal{L})\cong\widetilde{\mathbf{Rub}}_{g,A}\times_{\mathfrak{Pic}}\mathfrak{M}(X).$$

Proof. The right-hand side comes with a built-in log structure, but the left-hand side does not. Our isomorphism will be between the underlying stacks. Our proof is based on the discussion above [56, Proposition 5.5.2], and we will use the language of *divided tropical lines* of [56].

We begin by building a map from the right to the left. We are given a log curve C/S, a tropical line $\mathcal{P}$ on S, a map $\alpha: C\to\mathcal{P}$ whose image is totally ordered, and a map $f: C\to X$, such that $f^*\mathcal{L}$ lies in the isomorphism class $\mathcal{O}_C(\alpha)$.

sThe $\mathbb{G}_m^{\mathsf{trop}}$-torsor $\mathcal{P}$ is rigidified by the least element among the images of the irreducible components of C (here we use the total ordering condition), and hence comes with a canonical $\mathbb{G}_m$-torsor $P\to\mathcal{P}$ (if we use the rigidification to identify $\mathcal{P}=\mathbb{G}_m^{\mathsf{trop}}$, then $P=\mathbb{G}_m^{\mathsf{log}}$). The pullback α^*P gives a *canonical* $\mathbb{G}_m$-torsor on C, which is isomorphic to $f^*\mathcal{L}^*$ up to pullback from S. In other words, the bundle $\alpha^*P\otimes f^*\mathcal{L}^\vee$ descends to a line bundle on S which we denote $\mathcal{M}$.

The images of the irreducible components of C yield a subdivision $\mathcal{Q}$ of $\mathcal{P}$, and we define a destabilization $\widetilde{C}=C\times_{\mathcal{P}}\mathcal{Q}$ of C, which is a log curve over S, by Lemma 6.1. This $\mathcal{Q}$ comes with a canonical $\mathbb{G}_m$-torsor Q by pulling back P from $\mathcal{P}$; this Q is then a 2-marked semistable genus-zero curve by [56, Proposition 5.2.4]. We define an $(X,\mathcal{L})$-rubber target R over S by the formula

$$R=\mathrm{Hom}((\mathcal{L}\otimes\mathcal{M})^*,Q).$$

Here, we pull back and take $\mathbb{G}_m$-equivariant homomorphisms over X_S.

Write $\tilde{f}:\widetilde{C}\to X$. We need a predeformable map $\widetilde{C}\to R$, equivalently an equivariant logarithmic map $\tilde{f}^*\mathcal{L}^*\to Q$ over X_S. It is enough to give a map $f^*\mathcal{L}\to P$ (since then we can tensor over $\mathcal{P}$ with $\mathcal{Q}$), which reduces to writing down an element of

$$\begin{aligned}\mathrm{Hom}_C(f^*(\mathcal{L}\otimes\mathcal{M}),\alpha^*P)&=\mathrm{Hom}_C(f^*\mathcal{L}\otimes f^*\mathcal{L}^\vee\otimes\alpha^*P,\alpha^*P)\\&=\mathrm{Hom}_C(\alpha^*P,\alpha^*P),\end{aligned}\tag{6.6}$$

which contains the identity. The scheme-theoretic map is predeformable as it comes from a logarithmic map, see [48].

Finally we check that no component of $\widetilde{C}$ is mapped to a non-smooth point of R and that every component is hit. The target R is constructed by subdividing $f^*\mathcal{L}$ at images of components of C, and then $\widetilde{C}$ is constructed by subdividing C at points lying over these divisions, so both assertions are clear.

Now, we construct a map from left to right. Given a prestable rubber map to $\mathcal{L}$ over a base S, we first need to equip S with a suitable log structure.

The curve R/X_S is a map $X_S \to \mathfrak{M}^{\mathsf{ss}}_{0,2}$, giving a (minimal) log structure on X_S by pullback. Lemma 6.7 below shows that this log structure descends to S. The curve R/X_S now carries the structure of a log curve, and similarly the quotient $[R/\mathbb{G}_m]$ descends to S (again by the Lemma 6.7), determining our tropical line $\mathcal{P}$—which evidently satisfies the divisibility condition in Lemma 6.1.

It remains to verify that the map $\widetilde{C} \to R$ descends to a map $C \to \mathcal{P}$, and that the total ordering condition is satisfied. Write

$$\tau \colon \widetilde{C} \longrightarrow C.$$

By the proof of [56, Proposition 5.5.2], we see that $R \to \tau_* \tau^* R$ is an isomorphism, and hence the map descends as required. The condition that no components are mapped to the nodes implies that the values of α on the irreducible of C are a subset of the irreducible components of R, in particular are totally ordered. □

LEMMA 6.7. *Let $(R/S, \rho, r_0, r_\infty)$ be an $(X, \mathcal{L})$-rubber target. Then, there exists a (minimal) log structure on S such that R/X_S can be equipped with the structure of a log curve making X_S strict over S. The quotient log stack $[R/\mathbb{G}_m]$ descends to a divided tropical line on S.*

Proof. The curve R/X_S with markings r_i is prestable, and hence admits a minimal log structure. We must verify that the resulting log structure on X_S descends to S. After a finite extension of K, we may assume that X has a K point, so that

$$\pi \colon X_S \longrightarrow S$$

admits a section $x \colon S \to X_S$, and we can equip S with the pullback log structure. It remains to construct an isomorphism $\pi^* x^* M_{X_S} \to M_{X_S}$. We start by building a map from left to right.

We first build a map on the level of characteristic monoids. The characteristic monoid at a geometric point $t \in X_S$ is given by $\mathbb{N}^\ell$, where ℓ is the length of the chain of projective lines of R over t. Crucially, the irreducible elements of $\mathbb{N}^\ell$ come with a total order, given by proximity of the corresponding singularity to the r_0 marking.

This rigidifies the characteristic monoid, so as we move along the fiber over $\pi(t)$ the characteristic monoids are *canonically* identified. We obtain canonical identifications

$$(\overline{x^* M_{X_S}})_{\pi(t)} \xrightarrow{\sim} (\overline{M}_{X_S})_t,$$

which give an isomorphism

$$\pi^* x^* \overline{M}_{X_S} \xrightarrow{\sim} \overline{M}_{X_S}.$$

To construct an isomorphism of log structures, we will use the perspective of [13, §3.1] that a log structure is a monoidal functor from the groupified characteristic monoid to the stack of line bundles. The rubber target is by definition pulled back from Li's versal deformation spaces, so it suffices to construct our map in that setting. We may therefore assume that S is regular and the locus of non-smooth curves is a reduced divisor in X_S. Since our map will be canonical, we may further shrink S to be atomic.(36) Then, $\overline{M}_{X/S}$ is generated by its global sections, and there is a natural isomorphism of sheaves on X_S:

$$\varphi\colon \mathbb{N}^\ell \xrightarrow{\sim} \overline{M}_{X/S},$$

where ℓ is the number of singular points in the fiber of C over any point of X_S lying over the closed stratum of S. Given $1 \leqslant i \leqslant \ell$, write D_i for the Cartier divisor in X_S, where the singularity at distance i from the first marking persists. Then φ sends the ith generator of $\mathbb{N}^\ell$ to the section corresponding to the line bundle $\mathcal{O}_{X_S}(D_i)$. To build the required map of log structures

$$\pi^* x^* M_{X_S} \xrightarrow{\sim} M_{X_S},$$

we must construct an isomorphism

$$\pi^* x^* \mathcal{O}_{X_S}(D_i) \xrightarrow{\sim} \mathcal{O}_{X_S}(D_i).$$

Condition (i) of Definition 6.4 implies that the underlying point set of D_i is a union of fibers of X_S/S. Since S is regular and X_S is smooth over S, it follows that

$$D_i = \pi^* x^* D_i$$

giving the required isomorphism.

The quotient log stack

$$[R/\mathbb{G}_m]$$

is a divided tropical line on X_S with divisions coming from the divisors D_i. We may identify the underlying tropical line with $\mathbb{G}_m^{\mathsf{trop}}$ by specifying that the smallest element in the sequence of divisions is mapped to zero. We have already established that these divisions D_i descend to S, and hence so does the divided tropical line. $\square$

(36) [2, Definition 2.2.4].

After restriction to the locus where the infinitesimal automorphisms are trivial, we obtain a stable version of Lemma 6.6. Let $\overline{\mathcal{M}}_{g,n}(X,\beta)$ denote the stack of stable maps from n-pointed curves to X representing the class β. The line bundle $\mathcal{L}$ determines a map

$$\overline{\mathcal{M}}_{g,n}(X,\beta) \longrightarrow \mathfrak{Pic},$$

and we can pullback $\widetilde{\mathbf{Rub}}_{g,A}$ as before. Let

$$\mathbf{Rub}_{g,A}(X,\mathcal{L}) \subset \mathbf{Rub}^{\mathsf{pre}}_{g,A}(X,\mathcal{L})$$

be the locus where the infinitesimal automorphisms are trivial.

LEMMA 6.8. *The stack $\mathbf{Rub}_{g,A}(X,\mathcal{L})$ is the fiber product of $\widetilde{\mathbf{Rub}}_{g,A}$ over $\mathfrak{Pic}$ with $\overline{\mathcal{M}}_{g,n}(X,\beta)$ along the map given by $\mathcal{L}$:*

$$\mathbf{Rub}_{g,A}(X,\mathcal{L}) \cong \widetilde{\mathbf{Rub}}_{g,A} \times_{\mathfrak{Pic}} \overline{\mathcal{M}}_{g,n}(X,\beta).$$

Next, we will compare the virtual fundamental classes on these spaces. We will carry out the comparison on a smaller open locus.

(i) We let $\overline{\mathcal{M}}_{g,n}(X,\beta)'$ be the open locus of maps $(C, f\colon C \to X)$ in $\overline{\mathcal{M}}_{g,n}(X,\beta)$ where

$$H^1(C, f^*\mathcal{L}) = 0.$$

(ii) We define

$$\mathbf{Rub}_{g,A}(X,\mathcal{L})' = \mathbf{Rub}_{g,A}(X,\mathcal{L}) \times_{\overline{\mathcal{M}}_{g,n}(X,\beta)} \overline{\mathcal{M}}_{g,n}(X,\beta)'.$$

In §5.2, we considered the case $X = \mathbb{P}^l$ and showed that this unobstructed locus is large enough to control the cycles relevant to Theorem 0.7. For general X, the unobstructed locus might be very small (and possibly empty).

6.5. Comparing the virtual classes

6.5.1. Overview

We begin by briefly discussing of several spaces which will be relevant in setting up the obstruction theories. Let

$$\mathfrak{M}^{\mathsf{ss}}_{g,n} \subset \mathfrak{M}_{g,n}$$

be the semistable locus (where every rational curve has at least two distinguished points). We write $\mathscr{T}$ for the algebraic stack with log structure which parameterizes tropical lines with at least one division. There are natural maps

$$\mathfrak{M}^{\mathsf{ss}}_{0,2} \longrightarrow \mathscr{T} \quad \text{and} \quad \mathfrak{M}^{\mathsf{ss}}_{0,2} \longrightarrow B\mathbb{G}_m,$$

the former defined by dividing $\mathbb{G}_m^{\mathsf{trop}}$ at 1 and at the smoothing parameters of the nodes, and the latter defined by the normal bundle at the first marking. The induced map

$$\mathfrak{M}_{0,2}^{\mathsf{ss}} \longrightarrow \mathscr{T} \times B\mathbb{G}_m \tag{6.7}$$

is an isomorphism by [1, Proposition 3.3.3].

As $\mathbf{Rub}^{\mathsf{rel}}$ is the moduli stack of tuples $(C, \alpha: C \to \mathcal{P})$, where $\mathcal{P}$ is a tropical line and the images of the irreducible components of C are totally ordered, there is a natural map

$$\mathbf{Rub}^{\mathsf{rel}} \longrightarrow \mathscr{T} \tag{6.8}$$

sending $(C, \alpha: C \to \mathcal{P})$ to the tropical line $\mathcal{P}$ with the division given by the images of the irreducible components of C.

We will construct a map

$$\mathbf{Rub}_{g,A}(X, \mathcal{L})' \longrightarrow \mathfrak{M}_{0,2}^{\mathsf{ss}} \tag{6.9}$$

lifting the morphism (6.8) by the following argument. A point of $\mathbf{Rub}_{g,A}(X, \mathcal{L})'$ is a tuple

$$(C, \alpha: C \to \mathcal{P}, f: C \to X),$$

where $f^*\mathcal{L}$ lies in the class([37]) $[\mathcal{O}_C(\alpha)]$. However, as $\mathcal{P}$ is divided, there is a unique isomorphism $\mathcal{P} \xrightarrow{\sim} \mathbb{G}_m^{\mathsf{trop}}$, where the smallest division maps to zero. The universal $\mathbb{G}_m$ torsor $\mathbb{G}_m^{\mathsf{log}} \to \mathbb{G}_m^{\mathsf{trop}}$ pulls back to a well-defined $\mathbb{G}_m$-torsor $\mathcal{O}_C^*(\alpha)$ on C, and the difference $f^*\mathcal{L}^* \otimes_{\mathcal{O}_C^*} \mathcal{O}_C^*(-\alpha)$ descends to a $\mathbb{G}_m$-torsor on S by the construction of $\mathbf{Rub}_{g,A}(X, \mathcal{L})'$ as a fiber product. The $\mathbb{G}_m$-torsor on S induces a map $\mathbf{Rub}_{g,A}(X, \mathcal{L})' \to B\mathbb{G}_m$. Combined with the map $\mathbf{Rub}_{g,A}(X, \mathcal{L})' \to \mathscr{T}$ via (6.8), we obtain the map (6.9).

The space $\mathbf{Rub}_{g,A}(X, \mathcal{L})'$ carries three virtual fundamental classes by the following three constructions:

(i) The class $\mathsf{DR}_{g,A}^{\mathsf{op}}(\varphi_{\mathcal{L}})([\overline{\mathcal{M}}_{g,n}(X, \beta)'])$ obtained by applying $\mathsf{DR}_{g,A}^{\mathsf{op}}$ to the (virtual) fundamental class of $\overline{\mathcal{M}}_{g,n}(X, \beta)'$ via the map $\varphi_{\mathcal{L}}$.

(ii) The class obtained from a 2-step obstruction theory described by Marcus and Wise [56] for the map $\mathbf{Rub}_{g,A}(X, \mathcal{L})' \to \mathfrak{M}_{g,n} \times \mathscr{T}$.

(iii) A class coming from a 2-step obstruction theory studied by Graber and Vakil [31] for the map

$$\mathbf{Rub}_{g,A}(X, \mathcal{L})' \to \overline{\mathcal{M}}_{g,n}(X, \beta)' \times \mathfrak{M}_{0,2}^{\mathsf{ss}}.$$

We will prove that (i)–(iii) are all equal.

([37]) Here, $[\mathcal{O}_C(\alpha)]$ is an equivalence class under isomorphisms and tensoring with pullbacks from S.

6.5.2. $\mathsf{DR}^{\mathsf{op}}$ and the obstruction theory of Marcus–Wise

The obstruction theory of Marcus–Wise is a 2-*step* obstruction theory, a notion which we now recall. Unless otherwise stated, by *perfect obstruction theory* we mean an obstruction theory which is perfect in amplitude $[-1,0]$.

Definition 6.9. A 2-*step* obstruction theory for a map $f\colon X\to S$ consists of a factorization

$$X \longrightarrow Y \longrightarrow S$$

together with perfect relative obstruction theories for X/Y and for Y/S.

A 2-step obstruction theory induces a virtual pullback by composition.([38]) If S has a fundamental class $[S]$, the virtual pullback of $[S]$ is the *virtual fundamental class* of X associated with the 2-step obstruction theory.

We first recall the 2-step obstruction theory of [56] in the case when X is a point. We have a diagram

$$\begin{tikzcd} \mathbf{Rub}_{g,A}(\mathsf{pt},\mathcal{O})' \arrow[r] \arrow[dr] & \widetilde{\mathbf{Rub}}_{g,A} \arrow[d] \\ & \mathfrak{M}_{g,n}\times\mathscr{T}. \end{tikzcd} \tag{6.10}$$

A perfect relative obstruction theory for the horizontal map is given in [56, §5.6.3], and for the vertical map in [56, Proposition 5.6.5.3]; while the reader might expect that these arguments apply to $\mathbf{Rub}_{g,A}$, rather than the root stack $\widetilde{\mathbf{Rub}}_{g,A}$, Marcus and Wise in fact assume in both constructions the divisibility conditions of Lemma 6.1, and hence their constructions in fact apply to $\widetilde{\mathbf{Rub}}_{g,A}$ (and not to $\mathbf{Rub}_{g,A}$). This 2-step obstruction theory coincides with the rubber theory of Graber–Vakil, as shown in [56, §5.6.6]. Moreover, the virtual fundamental class obtained equals the operational class of $\mathbf{Div}_{g,A}$, see [56, Theorem 5.6.1]; again, these results all assume the divisibility condition of Lemma 6.1, and hence apply to $\widetilde{\mathbf{Rub}}_{g,A}$ in place of $\mathbf{Rub}_{g,A}$.

Returning to the case of arbitrary $(X,\mathcal{L})$, we can construct a similar commutative diagram

$$\begin{tikzcd} \mathbf{Rub}_{g,A}(X,\mathcal{L})' \arrow[r] \arrow[dr] & \widetilde{\mathbf{Rub}}_{g,A} \arrow[d] \\ & \mathfrak{M}_{g,n}\times\mathscr{T}. \end{tikzcd} \tag{6.11}$$

([38]) A 2-step obstruction theory also induces a perfect obstruction theory for X/S in amplitude $[-2,0]$, but we will not use the latter construction.

The vertical map is unchanged, and so again has a perfect relative obstruction theory by [56, Proposition 5.6.5.3]; in fact the morphism is a local complete intersection, and the obstruction theory of [56, Proposition 5.6.5.3] is just the relative tangent complex.

We need to supply a perfect obstruction theory for the horizontal map

$$\widetilde{\mathbf{Rub}}_{g,A}(X,\mathcal{L})' \longrightarrow \mathbf{Rub}_{g,A},$$

which we can factor as

$$\mathbf{Rub}_{g,A}(X,\mathcal{L})' \longrightarrow \overline{\mathcal{M}}_{g,n}(X,\beta)' \times_{\mathfrak{M}_{g,n}} \widetilde{\mathbf{Rub}}_{g,A} \longrightarrow \widetilde{\mathbf{Rub}}_{g,A}. \tag{6.12}$$

The second map is a base change of the unobstructed map $\overline{\mathcal{M}}_{g,n}(X,\beta)' \to \mathfrak{M}_{g,n}$, and hence is unobstructed. For the first map, consider the pullback square

$$\begin{tikzcd} \mathbf{Rub}_{g,A}(X,\mathcal{L})' \arrow[r] \arrow[d] & \overline{\mathcal{M}}_{g,n}(X,\beta)' \arrow[d] \\ \overline{\mathcal{M}}_{g,n}(X,\beta)' \times_{\mathfrak{M}_{g,n}} \widetilde{\mathbf{Rub}}_{g,A} \arrow[r] & \overline{\mathcal{M}}_{g,n}(X,\beta)' \times_{\mathfrak{M}_{g,n}} \mathfrak{Pic}_{g,n}. \end{tikzcd} \tag{6.13}$$

The right vertical arrow is a section of a base change of the smooth morphism

$$\mathfrak{Pic}_{g,n} \longrightarrow \mathfrak{M}_{g,n},$$

and as such is lci and has a perfect relative obstruction theory given by the relative tangent complex $R^1\pi_*\mathcal{O}_C$. Pullback yields a corresponding perfect obstruction theory for the left vertical arrow. This gives a 2-step obstruction theory for the composite map

$$\mathbf{Rub}_{g,A}(X,\mathcal{L})' \longrightarrow \widetilde{\mathbf{Rub}}_{g,A},$$

from which we obtain a virtual fundamental class following [55].

The discussion here is a very slight generalization of the obstruction theory constructed in [56, Proposition 5.6.3.1].

Definition 6.10. The 2-step obstruction theory for the diagonal map of (6.11),

$$\mathbf{Rub}_{g,A}(X,\mathcal{L})' \longrightarrow \mathfrak{M}_{g,n} \times \mathscr{T}$$

is the *Marcus–Wise* obstruction theory.

LEMMA 6.11. *The pushforward along*

$$\psi\colon \mathbf{Rub}_{g,A}(X,\mathcal{L})' \longrightarrow \overline{\mathcal{M}}_{g,n}(X,\beta)'$$

of the virtual fundamental class of the Marcus–Wise theory on $\mathbf{Rub}_{g,A}(X,\mathcal{L})'$ *equals the class* $\mathsf{DR}^{\mathsf{op}}_{g,A}(\varphi_{\mathcal{L}})([\overline{\mathcal{M}}_{g,n}(X,\beta)'])$ *obtained via the map*

$$\varphi_{\mathcal{L}}\colon \overline{\mathcal{M}}_{g,n}(X,\beta)' \longrightarrow \mathfrak{Pic}_{g,n}.$$

Proof. From $\varphi_{\mathcal{L}}$, we obtain maps

$$\begin{aligned}\varphi'_{\mathcal{L}}&\colon \overline{\mathcal{M}}_{g,n}(X,\beta)' \longrightarrow \overline{\mathcal{M}}_{g,n}(X,\beta)'\times_{\mathfrak{M}_{g,n}}\mathfrak{Pic}_{g,n},\\ \varphi''_{\mathcal{L}}&\colon \overline{\mathcal{M}}_{g,n}(X,\beta)' \longrightarrow \overline{\mathcal{M}}_{g,n}(X,\beta)'\times\mathfrak{Pic}_{g,n}.\end{aligned}$$

Both are lci morphisms because $\mathfrak{Pic}_{g,n}/\mathfrak{M}_{g,n}$ is smooth. By Definition 2.8 and §3.6, we have

$$\begin{aligned}\mathsf{DR}^{\mathsf{op}}_{g,A}(\varphi_{\mathcal{L}})([\overline{\mathcal{M}}_{g,n}(X,\beta)']) &= \psi_*(\varphi''_{\mathcal{L}})^![\overline{\mathcal{M}}_{g,n}(X,\beta)'\times\widetilde{\mathbf{Rub}}_{g,A}]\\ &= \psi_*(\varphi'_{\mathcal{L}})^![\overline{\mathcal{M}}_{g,n}(X,\beta)'\times_{\mathfrak{M}_{g,n}}\widetilde{\mathbf{Rub}}_{g,A}].\end{aligned}$$

The virtual fundamental class of the Marcus–Wise theory is the virtual pullback of the fundamental class of $\mathfrak{M}_{g,n}\times\mathscr{T}$ along the composition

$$\mathbf{Rub}_{g,A}(X,\mathcal{L})' \xrightarrow{(1)} \overline{\mathcal{M}}_{g,n}(X,\beta)'\times_{\mathfrak{M}_{g,n}}\widetilde{\mathbf{Rub}}_{g,A} \xrightarrow{(2)} \widetilde{\mathbf{Rub}}_{g,A} \xrightarrow{(3)} \mathfrak{M}_{g,n}\times\mathscr{T}. \quad (6.14)$$

The map (3) is lci and the obstruction theory is the relative tangent complex, so the pullback of the fundamental class is the fundamental class of $\widetilde{\mathbf{Rub}}_{g,A}$. The map (2) is unobstructed, so the (virtual) pullback of the fundamental class is again the fundamental class. The obstruction theory of the map (1) is defined by pulling back the relative tangent complex of the lci morphism

$$\overline{\mathcal{M}}_{g,n}(X,\beta)' \longrightarrow \overline{\mathcal{M}}_{g,n}(X,\beta)'\times_{\mathfrak{M}_{g,n}}\mathfrak{Pic}_{g,n}$$

via the pullback square

$$\begin{tikzcd} \mathbf{Rub}_{g,A}(X,\mathcal{L})' \arrow[r] \arrow[d] & \overline{\mathcal{M}}_{g,n}(X,\beta)'\times_{\mathfrak{M}_{g,n}}\widetilde{\mathbf{Rub}}_{g,A} \arrow[d] \\ \overline{\mathcal{M}}_{g,n}(X,\beta)' \arrow[r, "\varphi'_{\mathcal{L}}"] & \overline{\mathcal{M}}_{g,n}(X,\beta)'\times_{\mathfrak{M}_{g,n}}\mathfrak{Pic}_{g,n}, \end{tikzcd} \quad (6.15)$$

so the virtual pullback of the fundamental class of $\overline{\mathcal{M}}_{g,n}(X,\beta)'\times_{\mathfrak{M}_{g,n}}\widetilde{\mathbf{Rub}}_{g,A}$ is equal to the Gysin pullback

$$(\varphi'_{\mathcal{L}})^![\overline{\mathcal{M}}_{g,n}(X,\beta)'\times_{\mathfrak{M}_{g,n}}\widetilde{\mathbf{Rub}}_{g,A}]. \qquad \square$$

6.5.3. Marcus–Wise and Graber–Vakil

As recalled above, Marcus–Wise define a 2-step obstruction theory for the map

$$\mathbf{Rub}_{g,A}(X,\mathcal{L})' \longrightarrow \mathfrak{M}_{g,n} \times \mathscr{T}.$$

Graber and Vakil consider an obstruction theory for the map

$$\mathbf{Rub}_{g,A}(X,\mathcal{L})' \longrightarrow \overline{\mathcal{M}}_{g,n}(X,\beta)' \times \mathfrak{M}^{\mathsf{ss}}_{0,2}. \tag{6.16}$$

We wish to show an equality of the corresponding virtual fundamental classes on $\mathbf{Rub}_{g,A}(X,\mathcal{L})'$. Since

$$\mathfrak{M}^{\mathsf{ss}}_{0,2} = \mathscr{T} \times B\mathbb{G}_m,$$

and that the maps $\overline{\mathcal{M}}_{g,n}(X,\beta)' \to \mathfrak{M}_{g,n}$ and $B\mathbb{G}_m \to \operatorname{Spec} K$ are unobstructed, we have an unobstructed map

$$\overline{\mathcal{M}}_{g,n}(X,\beta)' \times \mathfrak{M}^{\mathsf{ss}}_{0,2} \longrightarrow \mathfrak{M}_{g,n} \times \mathfrak{M}^{\mathsf{ss}}_{0,2} \longrightarrow \mathfrak{M}_{g,n} \times \mathscr{T}.$$

Our final step is therefore to compare the obstruction theories (and thereby the corresponding virtual pullbacks) between Marcus–Wise and Graber–Vakil [31], [54]. We will match the obstruction spaces when the base S is a point. The full matching of deformation theories is similar and will be treated in [36]. The claims are also required for [56].

Suppose we are given the data of a point in $\mathbf{Rub}_{g,A}(X,\mathcal{L})'(S)$:

$$(\tau\colon \widetilde{C} \to C_S, R/X_S, \varphi\colon \widetilde{C} \to R, f\colon C_S \to X_S, p_1,\ldots,p_n, f^*\mathcal{L} \xrightarrow{\sim} \mathcal{O}_C(\alpha)). \tag{6.17}$$

Proposition 6.12. *The restriction to $\overline{\mathcal{M}}_{g,n}(X,\beta)'$ of the class $\mathsf{DR}_{g,A}(X,\mathcal{L})$ of [43] is equal to the class obtained by letting $\mathsf{DR}^{\mathsf{op}}_{g,A}$ act on the fundamental class of $\overline{\mathcal{M}}_{g,n}(X,\beta)'$ via the map induced by $\mathcal{L}$.*

Proof. The primary obstruction of Graber–Vakil lies in

$$H^0(\widetilde{C}, \varphi^{-1}\,\mathcal{E}\mathrm{xt}^1(\Omega_{R/X_S}(\log D), \mathcal{O}_R)). \tag{6.18}$$

Here, D is the divisor on R given by the sum of the two markings r_0 and r_∞, and $\Omega_{R/X_S}(\log D)$ is the sheaf of relative 1-forms on R/X_S allowed logarithmic poles along D (a coherent sheaf on R). The obstruction space (6.18) is isomorphic to the product of the deformation spaces of the nodes of R and coincides with the obstruction space for the map

$$\widetilde{\mathbf{Rub}}_{g,A} \longrightarrow \mathfrak{M}_{g,n} \times \mathscr{T}$$

(the vertical arrow in (6.11)), coming from [56, Proposition 5.6.5.3] (where they assume the and divisibility conditions of Lemma 6.1, hence the results apply to $\widetilde{\mathbf{Rub}}_{g,A}$ and not to $\mathbf{Rub}_{g,A}$).

Suppose that the primary obstruction vanishes. Denote by

$$T_{R/X_S} = \mathcal{H}\mathrm{om}_{\mathcal{O}_R}(\Omega_{R/X_S}, \mathcal{O}_R)$$

the relative tangent sheaf. There is a secondary obstruction in

$$H^1(\widetilde{C}, \varphi^\dagger(T_{R/X_S})), \tag{6.19}$$

where the $\varphi^\dagger$ is the torsion-free part of φ^*, see [31] and [56]. The obstruction space (6.19) is the image of the obstruction space

$$H^1(C_S, f^*T_R) = H^1(C_S, \mathcal{O}_{C_S})$$

for the map

$$\mathbf{Rub}_{g,A}(X, \mathcal{L})' \longrightarrow \widetilde{\mathbf{Rub}}_{g,A}$$

(the horizontal arrow in (6.11)), coming from [56, Proposition 5.6.3.1].

When both of these obstructions vanish, the deformations are a torsor under

$$H^0(\widetilde{C}, \varphi^\dagger T_{R/X_S}),$$

an extension of the first term of the obstruction complex in [56, Proposition 5.6.5.3] by the first term of the obstruction complex of [56, Proposition 5.6.4.1]. The comparison of the obstruction theories is complete. □

7. Invariance properties

7.1. Overview

We prove here the invariance properties of the universal twisted double ramification cycle as presented in §0.6.

We start with an object of $\varphi_{\mathcal{L}}: \mathcal{S} \to \mathfrak{Pic}_{g,n,d}$ given by a flat family of prestable n-pointed genus-g curves together with a line bundle of relative degree d,

$$\pi: \mathcal{C} \longrightarrow \mathcal{S}, \quad p_1, ..., p_n: \mathcal{S} \longrightarrow \mathcal{C}, \quad \mathcal{L} \longrightarrow \mathcal{C}. \tag{7.1}$$

Let $\mathsf{DR}^{\mathsf{op}}_{g,A,\mathcal{L}} = \varphi^*_{\mathcal{L}} \mathsf{DR}^{\mathsf{op}}_{g,A} \in \mathsf{CH}^g_{\mathsf{op}}(\mathcal{S})$ be the twisted double ramification cycle associated with the above family (7.1) and the vector

$$A = (a_1, ..., a_n), \quad d = \sum_{i=1}^{n} a_i.$$

Theorem 0.7 asserts that $\mathsf{DR}^{\mathsf{op}}_{g,A}$ is equal to the tautological class

$$\mathsf{P}^g_{g,A,d} \in \mathsf{CH}^g_{\mathsf{op}}(\mathfrak{Pic}_{g,n,d}).$$

If we assume Theorem 0.7, we have a choice of proving the invariance properties either for $\mathsf{DR}^{\mathsf{op}}_{g,A}$ or for the formula in tautological classes. In fact, since both sides of Invariances II, III, and VI are used *in the proof* of Theorem 0.7, we will have to prove these two sides separately in each of these cases. In fact, we will do this for all the invariances, as each side yields interesting perspectives. Also, we will show the invariances of Pixton's formula hold not just for the codimension-g part $\mathsf{P}^g_{g,A,d}$, but for the full mixed degree class

$$\mathsf{P}^\bullet_{g,A,d} \in \prod_{c=0}^{\infty} \mathsf{CH}^c_{\mathsf{op}}(\mathfrak{Pic}_{g,n,d}).$$

7.2. Proof of Invariance I (Dualizing)

We want to show the invariance

$$\mathsf{DR}^{\mathsf{op}}_{g,-A,\mathcal{L}^*} = \varepsilon^* \mathsf{DR}^{\mathsf{op}}_{g,A,\mathcal{L}},$$

where $\varepsilon : \mathfrak{Pic}_{g,n,-d} \to \mathfrak{Pic}_{g,n,d}$ is the natural map obtained via dualizing the line bundle. It is enough to show the invariance

$$\mathsf{DR}^{\mathsf{op}}_{g,-A} = \varepsilon^* \mathsf{DR}^{\mathsf{op}}_{g,A} \tag{7.2}$$

of the universal twisted double ramification cycles. The invariance (7.2) can be deduced by applying Lemma 2.17 to the following commutative diagram of morphisms, where the horizontal morphisms are the corresponding Abel–Jacobi maps and the vertical morphisms are isomorphisms:

$$\begin{tikzcd} \mathbf{Div}_{g,A} \arrow[r] \arrow[d, "\hat{\varepsilon}"] & \mathfrak{Pic}_{g,n,d} \arrow[d, "\varepsilon"] \\ \mathbf{Div}_{g,-A} \arrow[r] & \mathfrak{Pic}_{g,n,-d}. \end{tikzcd}$$

Here, in the language of §3.3, the morphism $\hat{\varepsilon}$ is induced by the natural map

$$\pi_* \overline{M}^{\mathsf{gp}}_C / \overline{M}^{\mathsf{gp}}_S \longrightarrow \pi_* \overline{M}^{\mathsf{gp}}_C / \overline{M}^{\mathsf{gp}}_S$$

given by inversion in $\overline{M}^{\mathsf{gp}}_C$.

We now prove the invariance

$$\mathsf{P}^\bullet_{g,-A,-d} = \varepsilon^* \mathsf{P}^\bullet_{g,A,d} \tag{7.3}$$

using the formulas for these cycles from Proposition 4.1. The equality is then implied by the following observations:

- We write

$$\mathcal{L}_A = \mathcal{L}\left(-\sum_{i=1}^n a_i p_i\right)$$

for the twisted universal line bundle on the universal curve $\pi\colon \mathfrak{C}\to\mathfrak{Pic}_{g,n,d}$, and we use Lemma 4.2 to obtain

$$\begin{aligned}\varepsilon^*\left(-\eta+\sum_{i=1}^n 2a_i\xi_i+a_i^2\psi_i\right) &= -\varepsilon^*\pi_*c_1(\mathcal{L}_A)^2 = -\pi_*c_1((\mathcal{L}_A)^*)^2\\ &= -\pi_*(-c_1(\mathcal{L}_A))^2 = -\pi_*c_1(\mathcal{L}_A)^2\\ &= -\eta+\sum_{i=1}^n 2a_i\xi_i+a_i^2\psi_i.\end{aligned}$$

- Given a prestable graph Γ_δ describing a stratum in $\mathfrak{Pic}_{g,n,-d}$, the map ε sends this stratum isomorphically to the stratum of $\Gamma_{-\delta}$ (with an associated commutative diagram of gluing morphisms over $\mathfrak{M}_{g,n}$). Combined with the equality $c_A(\Gamma_\delta)=c_{-A}(\Gamma_{-\delta})$ for $\Gamma_\delta\in\mathsf{G}^{\mathsf{se}}_{g,n,d}$, we see that the first line of formula (4.2) for $\mathsf{P}^\bullet_{g,A,d}$ has the desired invariance.
- For the sum over graphs and weightings, we clearly have $h^1(\Gamma_\delta)=h^1(\Gamma_{-\delta})$ and $\mathrm{Aut}(\Gamma_\delta)=\mathrm{Aut}(\Gamma_{-\delta})$. Moreover, we have a natural bijection of the admissible weightings modulo r,

$$\begin{aligned}\mathsf{W}_{\Gamma_\delta,r} &\longrightarrow \mathsf{W}_{\Gamma_{-\delta},r},\\ w &\longmapsto (h\mapsto r-w(h) \bmod r).\end{aligned}$$

The map of weightings leaves the edge terms of the formula (4.2) invariant since they only depend on products $w(h)w(h')$ for and edge (h,h')—which are of the form $a(r-a)$ and thus sent to $(r-a)a$.

Therefore, the formula of Proposition 4.1 applied to the two sides of (7.3) yields the same result.

7.3. Proof of Invariance II (Unweighted markings)

Assume we have an additional section $p_{n+1}\colon\mathcal{S}\to\mathcal{C}$ of π which yields an object of $\mathfrak{Pic}_{g,n+1,d}$,

$$\pi\colon \mathcal{C}\longrightarrow\mathcal{S},\quad p_1,\dots,p_n,p_{n+1}\colon\mathcal{S}\longrightarrow\mathcal{C},\quad \mathcal{L}\longrightarrow\mathcal{C}. \tag{7.4}$$

Then, for the vector $A_0\in\mathbb{Z}^{n+1}$ obtained by appending zero (as the last coefficient) to A, we want to show the invariance

$$\mathsf{DR}^{\mathsf{op}}_{g,A_0,\mathcal{L}} = \mathsf{DR}^{\mathsf{op}}_{g,A,\mathcal{L}}. \tag{7.5}$$

For the map $\mathfrak{M}_{g,n+1} \to \mathfrak{M}_{g,n}$ induced by forgetting the last marking, we have a diagram of cartesian squares

$$\begin{tikzcd}
\mathbf{Div}_{g,A_0} \arrow[r] \arrow[d] & \mathbf{Div}_{g,A} \arrow[d] \\
\mathfrak{Pic}_{g,n+1,d} \arrow[r, "F"] \arrow[d] & \mathfrak{Pic}_{g,n,d} \arrow[d] \\
\mathfrak{M}_{g,n+1} \arrow[r] & \mathfrak{M}_{g,n},
\end{tikzcd} \tag{7.6}$$

where the morphism F is syntomic. In particular, for $\mathbf{0} \in \mathbb{Z}^n$ the zero vector, the stack $\mathbf{Div}_{g,\mathbf{0}}$ can be obtained by pulling back $\mathbf{Div}_{g,\varnothing}$ from $\mathfrak{Pic}_{g,0,0}$. Then, as a consequence of the above cartesian square and the definition of the double ramification cycle, Lemma 2.17 yields

$$F^* \mathsf{DR}^{\mathsf{op}}_{g,A} = \mathsf{DR}^{\mathsf{op}}_{g,A_0}. \tag{7.7}$$

Since the morphisms

$$\mathcal{S} \longrightarrow \mathfrak{Pic}_{g,n,d} \quad \text{and} \quad \mathcal{S} \longrightarrow \mathfrak{Pic}_{g,n+1,d}$$

used to define $\mathsf{DR}^{\mathsf{op}}_{g,A,\mathcal{L}}$ and $\mathsf{DR}^{\mathsf{op}}_{g,A_0,\mathcal{L}}$ fit in a diagram

$$\begin{tikzcd}
& \mathcal{S} \arrow[dl] \arrow[dr] & \\
\mathfrak{Pic}_{g,n+1,d} \arrow[rr, "F"] & & \mathfrak{Pic}_{g,n,d},
\end{tikzcd}$$

the equation (7.7) immediately proves the invariance (7.5).

We now prove the corresponding invariance

$$F^* \mathsf{P}^{\bullet}_{g,A,d} = \mathsf{P}^{\bullet}_{g,A_0,d} \tag{7.8}$$

of Pixton's formula. First, since the map F does not change the curve or the line bundle, we have a cartesian diagram

$$\begin{tikzcd}
\mathfrak{C}_{g,n+1,d} \arrow[r, "\widehat{F}"] \arrow[d, "\pi'"] & \mathfrak{C}_{g,n,d} \arrow[d, "\pi"] \\
\mathfrak{Pic}_{g,n+1,d} \arrow[r, "F"] & \mathfrak{Pic}_{g,n,d}.
\end{tikzcd}$$

The universal line bundles $\mathcal{L}_{g,n,d}$ and $\mathcal{L}_{g,n+1,d}$ on $\mathfrak{C}_{g,n,d}$ and $\mathfrak{C}_{g,n+1,d}$ satisfy

$$\widehat{F}^*\mathcal{L}_{g,n,d} = \mathcal{L}_{g,n+1,d}.$$

Similarly, for the canonical line bundles of π and π' we have $\widehat{F}^*\omega_\pi = \omega_{\pi'}$. Combining these facts, we see that F pulls back the operational classes η, ξ_i, and ψ_i on $\mathfrak{Pic}_{g,n,d}$ to the corresponding classes on $\mathfrak{Pic}_{g,n+1,d}$.

Next, given a graph $\Gamma_\delta \in \mathsf{G}_{g,n,d}$, we have a fiber diagram

$$\begin{tikzcd}[column sep=large, row sep=large]
\coprod_{v\in \mathrm{V}(\Gamma)} \mathfrak{Pic}_{\Gamma_{v,\delta}} \arrow[r, "\coprod_v j_{\Gamma_{v,\delta}}"] \arrow[d] & \mathfrak{Pic}_{g,n+1,d} \arrow[d, "\pi"] \\
\mathfrak{Pic}_{\Gamma_\delta} \arrow[r, "j_{\Gamma_\delta}"] & \mathfrak{Pic}_{g,n,d},
\end{tikzcd} \tag{7.9}$$

where, for $v\in \mathrm{V}(\Gamma)$, we denote by $\Gamma_{v,\delta} \in \mathsf{G}_{g,n+1,d}$ the graph obtained from Γ_δ by adding marking $n+1$ at v and leaving the remaining data fixed.

Using the expression (4.2) given in Proposition 4.1, we conclude the proof of (7.8) by the following observations:

- The invariance of η, ξ_i, and ψ_i under F implies that

$$F^*\left(-\eta + \sum_{i=1}^{n} 2a_i\xi_i + a_i^2\psi_i\right) = -\eta + \sum_{i=1}^{n+1} 2a_i\xi_i + a_i^2\psi_i,$$

where we use $a_{n+1}=0$.

- From the fiber diagram (7.9) and the equality $c_A(\Gamma_\delta) = c_{A_0}(\Gamma_{v,\delta})$ for $\Gamma_\delta \in \mathsf{G}^{\mathsf{se}}_{g,n,d}$ and any $v\in \mathrm{V}(\Gamma)$, the sum over the terms $c_A(\Gamma_\delta)[\Gamma_\delta]$ pulls back correctly.
- For all $\Gamma_\delta \in \mathsf{G}_{g,n,d}$ and $v\in\mathrm{V}(\Gamma)$, we have $h^1(\Gamma_\delta) = h^1(\Gamma_{v,\delta})$ since the Betti number is independent of the position of the markings. Moreover, the automorphism group $\mathrm{Aut}(\Gamma_\delta)$ acts on $\mathrm{V}(\Gamma_\delta)$ and, by the orbit-stabilizer formula, the size of the orbit $\mathrm{Aut}(\Gamma_\delta)\cdot v$ of v and the size of its stabilizer $\mathrm{Aut}(\Gamma_\delta)_v$ satisfy

$$|\mathrm{Aut}(\Gamma_\delta)\cdot v| = \frac{|\mathrm{Aut}(\Gamma_\delta)|}{|\mathrm{Aut}(\Gamma_\delta)_v|}. \tag{7.10}$$

The stabilizer $\mathrm{Aut}(\Gamma_\delta)_v$ is exactly equal to the automorphism group $\mathrm{Aut}(\Gamma_{v,\delta})$ of the graph $\Gamma_{v,\delta}$, since the marking $n+1$ at v forces this vertex to be fixed.

- As Γ_δ runs through $\mathsf{G}^{\mathsf{nse}}_{g,n,d}$, the graphs $\Gamma_{v,\delta}$ run through $\mathsf{G}^{\mathsf{nse}}_{g,n+1,d}$. The equality (7.10) precisely implies that the corresponding graph sums (weighted by the inverse size of automorphism groups) correspond to each other under pullback by F.
- Finally, the weightings $\mathsf{W}_{\Gamma_\delta,r}$ and $\mathsf{W}_{\Gamma_{v,\delta},r}$ are naturally bijective.

Combining the above observations, we see that also the second line of (4.2) transforms under pullback of F as expected.

7.4. Proof of Invariance III (Weight translation)

Let $B=(b_1,...,b_n)\in\mathbb{Z}^n$ satisfy $\sum_{i=1}^n b_i=e$. Then, for the family

$$\pi:\mathcal{C}\longrightarrow\mathcal{S},\quad p_1,...,p_n:\mathcal{S}\longrightarrow\mathcal{C},\quad \mathcal{L}\bigg(\sum_{i=1}^n b_ip_i\bigg)\longrightarrow\mathcal{C}, \tag{7.11}$$

defining an object of $\mathfrak{Pic}_{g,n,d+e}$, we want to show the invariance

$$\mathsf{DR}^{\mathsf{op}}_{g,A+B,\mathcal{L}(\sum_i b_ip_i)}=\mathsf{DR}^{\mathsf{op}}_{g,A,\mathcal{L}}. \tag{7.12}$$

To show this, consider the smooth map

$$\begin{aligned}\tau_B:\mathfrak{Pic}_{g,n,d}&\longrightarrow\mathfrak{Pic}_{g,n,d+e},\\ \mathcal{L}&\longmapsto\mathcal{L}\bigg(\sum_{i=1}^n b_ip_i\bigg).\end{aligned}$$

over $\mathfrak{M}_{g,n}$. We have a natural cartesian diagram

$$\begin{tikzcd}\mathbf{Div}_{g,A} \arrow[r] \arrow[d] & \mathbf{Div}_{g,A+B} \arrow[d] \\ \mathfrak{Pic}_{g,n,d} \arrow[r, "\tau_B"] & \mathfrak{Pic}_{g,n,d+e}.\end{tikzcd} \tag{7.13}$$

In particular, for any ramification data A, we can obtain $\mathbf{Div}_{g,A}$ from $\mathbf{Div}_{g,\mathbf{0}}$ by such translations. The diagram above together with Lemma 2.17 implies

$$\tau_B^*\mathsf{DR}^{\mathsf{op}}_{g,A+B}=\mathsf{DR}^{\mathsf{op}}_{g,A}. \tag{7.14}$$

Since the morphisms

$$\mathcal{S}\longrightarrow\mathfrak{Pic}_{g,n,d}\quad\text{and}\quad\mathcal{S}\longrightarrow\mathfrak{Pic}_{g,n,d+e}$$

used to define $\mathsf{DR}^{\mathsf{op}}_{g,A,\mathcal{L}}$ and $\mathsf{DR}^{\mathsf{op}}_{g,A+B,\mathcal{L}(\sum_i b_ip_i)}$ fit in a diagram

$$\begin{tikzcd} & \mathcal{S} \arrow[dl] \arrow[dr] & \\ \mathfrak{Pic}_{g,n,d} \arrow[rr, "\tau_B"] & & \mathfrak{Pic}_{g,n,d+e},\end{tikzcd}$$

the equation (7.14) immediately proves the invariance (7.12).

Now, we prove the invariance

$$\tau_B^* \mathsf{P}^\bullet_{g,A+B,d+e} = \mathsf{P}^\bullet_{g,A,d} \tag{7.15}$$

for Pixton's formula. Recall the notation

$$\mathcal{L}_A = \mathcal{L}\Big(-\sum_i a_i[p_i]\Big)$$

for the twisted universal line bundles from Lemma 4.2, observe that in the cartesian diagram

$$\begin{tikzcd} \mathfrak{C}_{g,n,d} \arrow[r, "\hat{\tau}_B"] \arrow[d, "\pi'"] & \mathfrak{C}_{g,n,d+e} \arrow[d, "\pi"] \\ \mathfrak{Pic}_{g,n,d} \arrow[r, "\tau_B"] & \mathfrak{Pic}_{g,n,d+e} \end{tikzcd}$$

we have

$$\hat{\tau}_B^* \mathcal{L}_{A+B} = \mathcal{L}_A.$$

By Lemma 4.2, we have

$$\begin{aligned} \tau_B^*\Big(-\eta + \sum_{i=1}^n 2(a_i+b_i)\xi_i + (a_i+b_i)^2\psi_i\Big) &= -\tau_B^* \pi_* c_1(\mathcal{L}_{A+B})^2 \\ &= -\pi'_* c_1(\mathcal{L}_A)^2 = -\eta + \sum_{i=1}^n 2a_i\xi_i + a_i^2\psi_i, \end{aligned}$$

which shows the compatibility of the first part of formula (4.2) for $\mathsf{P}^\bullet_{g,A+B,d+e}$.

For the second part, we can combine the exponential of the graph sum over $\mathsf{G}^{\mathsf{se}}_{g,n,d+e}$ with the graph sum over $\mathsf{G}^{\mathsf{nse}}_{g,n,d+e}$ in (4.2) to recover the sum

$$\sum_{\substack{\Gamma_\delta \in \mathsf{G}_{g,n,d+e} \\ w \in \mathsf{W}_{\Gamma_\delta,r}}} \frac{r^{-h^1(\Gamma_\delta)}}{|\mathrm{Aut}(\Gamma_\delta)|} j_{\Gamma_\delta *}\Big[\prod_{e=(h,h')\in \mathrm{E}(\Gamma_\delta)} \frac{1}{\psi_h+\psi_{h'}}\Big(1-\exp\Big(-\frac{w(h)w(h')}{2}(\psi_h+\psi_{h'})\Big)\Big)\Big] \tag{7.16}$$

over all graphs in $\mathsf{G}_{g,n,d+e}$, as in the proof of Proposition 4.1. It will be more convenient to simply show the compatibility of the full graph sum (7.16) under pullback by τ_B.

Given a graph $\Gamma_\delta \in \mathsf{G}_{g,n,d+e}$, denote by $\delta^B\colon \mathrm{V}(\Gamma) \to \mathbb{Z}$ the map defined by

$$\delta^B(v) = \delta(v) - \sum_{\substack{i \text{ marking} \\ \text{at } v}} b_i.$$

We have a fiber diagram

$$\begin{tikzcd} \mathfrak{Pic}_{\Gamma_{\delta^B}} \arrow[r, "j_{\Gamma_{\delta^B}}"] \arrow[d] & \mathfrak{Pic}_{g,n,d} \arrow[d, "\tau_B"] \\ \mathfrak{Pic}_{\Gamma_\delta} \arrow[r, "j_{\Gamma_\delta}"] & \mathfrak{Pic}_{g,n,d+e}. \end{tikzcd} \tag{7.17}$$

The proof that (7.16) pulls back under τ_B as desired follows from the following observations:

- As Γ_δ runs through $\mathsf{G}_{g,n,d+e}$, the graphs Γ_{δ^B} run through $\mathsf{G}_{g,n,d}$. From the definitions, we verify that the conditions defining admissible weightings w mod r for Γ_δ and Γ_{δ^B} are identical (the shift from δ to δ^B cancels the shift from $A+B$ to A).
- Since the underlying graphs of Γ_δ and Γ_{δ^B} agree, we have $h^1(\Gamma_\delta)=h^1(\Gamma_{\delta^B})$. Concerning the automorphisms, they appear to take into account the degree functions δ and δ^B on the graphs. But any vertex v such that $\delta(v)\neq\delta^B(v)$ must carry a marking and thus must anyway be fixed under an automorphism. Hence, $\mathrm{Aut}(\Gamma_\delta)=\mathrm{Aut}(\Gamma_{\delta^B})$.
- To conclude using the diagram (7.17), we observe that the map

$$\mathfrak{Pic}_{\Gamma_{\delta^B}} \longrightarrow \mathfrak{Pic}_{\Gamma_\delta}$$

appearing there is a map over $\mathfrak{M}_\Gamma$ (since only the line bundle is changed). Hence, the classes ψ_h and $\psi_{h'}$ appearing in the edge terms of (7.16) are invariant.

7.5. Proof of Invariance IV (Twisting by pullback)

Let $\mathcal{B}\to\mathcal{S}$ be a line bundle on the base. We obtain a new object of $\mathfrak{Pic}_{g,n,d}$ over $\mathcal{S}$ by tensoring (7.1) with $\pi^*\mathcal{B}$:

$$\pi\colon \mathcal{C} \longrightarrow \mathcal{S}, \quad p_1,\dots,p_n\colon \mathcal{S} \longrightarrow \mathcal{C}, \quad \mathcal{L}\otimes\pi^*\mathcal{B} \longrightarrow \mathcal{C}. \tag{7.18}$$

We want to show the invariance

$$\mathsf{DR}^{\mathsf{op}}_{g,A,\mathcal{L}\otimes\pi^*\mathcal{B}} = \mathsf{DR}^{\mathsf{op}}_{g,A,\mathcal{L}}.$$

The universal twisted double ramification cycle $\mathsf{DR}^{\mathsf{op}}_{g,A}$ is the class associated with the Abel–Jacobi map

$$\mathsf{AJ}\colon \mathbf{Div}_{g,A} \longrightarrow \mathfrak{Pic}_{g,n},$$

and the latter is constructed (Definition 3.5) by pulling back the morphism

$$\mathsf{AJ}^{\mathsf{rel}}\colon \mathbf{Div}^{\mathsf{rel}}_{g,A} \longrightarrow \mathfrak{Pic}^{\mathrm{rel}}_{g,n}.$$

Thus, by Lemma 2.17, the corresponding cycle $\mathsf{DR}^{\mathsf{op}}_{g,A}$ is a pullback of $\mathsf{AJ}^{\mathsf{rel}}_{\mathsf{op}}[\mathbf{Div}^{\mathsf{rel}}_{g,A}]$ from $\mathfrak{Pic}^{\mathrm{rel}}_{g,n,d}$. But twisting the family (7.1) by a line bundle pulled back from the base does not change the map to $\mathfrak{Pic}^{\mathrm{rel}}_{g,n,d}$, and so does not change the resulting operational class, proving the invariance.

We now prove the invariance

$$\varphi^*_{\mathcal{L}\otimes\pi^*\mathcal{B}}\mathsf{P}^\bullet_{g,A,d} = \varphi^*_{\mathcal{L}}\mathsf{P}^\bullet_{g,A,d}.$$

Using that $\mathsf{P}^\bullet_{g,A,d}$ is a pullback of $\mathsf{P}^\bullet_{g,\varnothing,0}$ as described in §0.7, it suffices to show that the cycle $\mathsf{P}^\bullet_g$ on $\mathfrak{Pic}_{g,0,0}$ pulls back to the same expression under the two maps

$$\varphi_{\mathcal{L}_A}, \varphi_{\mathcal{L}_A\otimes\pi^*\mathcal{B}} \colon \mathcal{S} \longrightarrow \mathfrak{Pic}_{g,0,0}$$

induced by

$$\mathcal{L}_A = \mathcal{L}\left(-\sum_{i=1}^n a_i p_i\right) \quad \text{and} \quad \mathcal{L}_A \otimes \pi^*\mathcal{B},$$

respectively. We will use formula (4.8) for $\mathsf{P}^\bullet_g$ and show that both parts of the formula are invariant separately.

- For the term $\exp\left(-\frac{1}{2}\eta\right)$, we use Lemma 4.2 to obtain

$$\begin{aligned}
\varphi^*_{\mathcal{L}_A\otimes\pi^*\mathcal{B}}\eta &= \pi_* c_1(\mathcal{L}_A\otimes\pi^*\mathcal{B})^2 \\
&= \pi_*(c_1(\mathcal{L}_A)^2 + 2c_1(\mathcal{L}_A)\pi^*c_1(\mathcal{B}) + \pi^*c_1(\mathcal{B})^2) \\
&= \pi_*(c_1(\mathcal{L}_A)^2) + 2c_1(\mathcal{B})\underbrace{\pi_*c_1(\mathcal{L}_A)}_{=0} + c_1(\mathcal{B})^2\underbrace{\pi_*1}_{=0} \\
&= \varphi^*_{\mathcal{L}_A}\eta,
\end{aligned}$$

where $\pi_*c_1(\mathcal{L}_A)$ vanishes since $\mathcal{L}_A$ has degree zero, and π_*1 vanishes for dimension reasons. The vertex term is therefore invariant under twisting by $\pi^*\mathcal{B}$.

- For the graph sum

$$\sum_{\substack{\Gamma_\delta\in\mathsf{G}_{g,0,0}\\ w\in\mathsf{W}_{\Gamma_\delta,r}}} \frac{r^{-h^1(\Gamma_\delta)}}{|\mathrm{Aut}(\Gamma_\delta)|} j_{\Gamma_\delta *}\left[\prod_{e=(h,h')\in\mathrm{E}(\Gamma_\delta)} \frac{1}{\psi_h+\psi_{h'}}\left(1-\exp\left(-\frac{w(h)w(h')}{2}(\psi_h+\psi_{h'})\right)\right)\right] \tag{7.19}$$

in (4.8), we show that it is a pullback under the morphism

$$\mathfrak{Pic}_{g,0,0} \longrightarrow \mathfrak{Pic}^{\mathrm{rel}}_{g,0,0}, \tag{7.20}$$

which finishes the proof since the compositions of $\varphi_{\mathcal{L}_A}$ and $\varphi_{\mathcal{L}_A\otimes\pi^*\mathcal{B}}$ with the morphism (7.20) agree. For a prestable graph Γ, we have cartesian diagrams

$$\begin{tikzcd} \coprod_\delta \mathfrak{Pic}_{\Gamma_\delta} \arrow[r, "\amalg_\delta j_{\Gamma_\delta}"] \arrow[d] & \mathfrak{Pic}_{g,0,0} \arrow[d] \\ \coprod_\delta \mathfrak{Pic}^{\mathrm{rel}}_{\Gamma_\delta} \arrow[r, "\amalg_\delta j^{\mathsf{rel}}_{\Gamma_\delta}"] \arrow[d] & \mathfrak{Pic}^{\mathrm{rel}}_{g,0,0} \arrow[d] \\ \mathfrak{M}_\Gamma \arrow[r, "j_\Gamma"] & \mathfrak{M}_g. \end{tikzcd} \tag{7.21}$$

In formula (7.19), the edge terms use only the classes ψ_h and $\psi_{h'}$, which are pullbacks from $\mathfrak{M}_\Gamma$. Therefore, (7.19) is the pullback under (7.20) of the identical formula with j_{Γ_δ} replaced with $j^{\mathsf{rel}}_{\Gamma_\delta}$.

7.6. Proof of Invariance V (Vertical twisting)

Consider the boundary divisor Δ of $\mathfrak{Pic}_{g,n,d}$ given by the partition

$$g_1+g_2=g, \quad N_1\sqcup N_2=\{1,...,n\}, \quad d_1+d_2=d,$$

which is not symmetric. In

$$\mathfrak{C}_{g,n,d}\longrightarrow \mathfrak{Pic}_{g,n,d},$$

let Δ_1 and Δ_2 be, respectively, the (g_1,N_1,d_1) and (g_2,N_2,d_2) components of the universal curve over Δ. Then, we have a morphism

$$\Phi_{\Delta_1}\colon \mathfrak{Pic}_{g,n,d}\longrightarrow \mathfrak{Pic}_{g,n,d}$$

associated with the twisted line bundle $\mathcal{L}(\Delta_1)$ on the universal curve

$$\mathfrak{C}_{g,n,d}\longrightarrow \mathfrak{Pic}_{g,n,d}.$$

Remark 7.1. The map Φ_{Δ_1} is *not* an isomorphism. Indeed, the map is equal to the identity away from $\Delta\subset\mathfrak{Pic}_{g,n,d}$, but it sends the generic point of Δ to the generic point of the boundary divisor

$$\tilde{\Delta}=\Delta(g_1,N_1,d_1-1|g_2,N_2,d_2+1),$$

which itself is fixed under Φ_{Δ_1}. Hence, Φ_{Δ_1} is not injective, though it is easily seen to be étale.

We want to show the invariance

$$\Phi_{\Delta_1}^{*}\mathsf{DR}_{g,A}^{\mathsf{op}}=\mathsf{DR}_{g,A}^{\mathsf{op}}. \tag{7.22}$$

Using the data

$$g_1+g_2=g,\quad N_1\sqcup N_2=\{1,...,n\},\quad d_1+d_2=d,$$

we will define a map

$$\Phi'\colon \mathbf{Div}_{g,A}\longrightarrow \mathbf{Div}_{g,A}.$$

In fact, we will define a map $\mathbf{Div}_{g,A}^{\mathsf{rel}}\to\mathbf{Div}_{g,A}^{\mathsf{rel}}$, and then lift it to $\mathbf{Div}_{g,A}$ by fiber product with $\mathfrak{Pic}$. The invariance (7.22) will be deduced from Φ'.

Suppose we are given a map $S\to\mathbf{Div}_{g,A}^{\mathsf{rel}}$ compatible with C/S defined by a $\overline{M}_S^{\mathsf{gp}}$ torsor $\mathcal{P}$ on S and a map $\alpha\colon C\to\mathcal{P}$. The divisor Δ determines an element of the characteristic sheaf of the log structure on $\mathfrak{Pic}_{g,n,d}$, which pulls back under the composition

$$S\longrightarrow \mathbf{Div}_{g,A}^{\mathsf{rel}}\xrightarrow{\mathsf{AJ}}\mathfrak{Pic}_{g,n,d}$$

to an element $\delta\in\overline{M}_S(S)$. All lifts of δ to $M_S(S)$ generate the same ideal sheaf on S, whose closed subscheme is exactly the pullback Δ_S of Δ. We write

$$\Delta_1',\Delta_2'\subset C$$

for the two components of the universal curve over $\Delta_S\subset S$.

We define a new map $(\mathcal{P}',\alpha')\colon S\to\mathbf{Div}_{g,A}^{\mathsf{rel}}$ as follows. We take the same torsor $\mathcal{P}'=\mathcal{P}$. On the locus $C_1\hookrightarrow C$ which is the complement of Δ_1', we define $\alpha'=\alpha$. On the locus $C_2\hookrightarrow C$ which is the complement of Δ_2', we define $\alpha'=\alpha-\delta$. Since δ vanishes on the overlap $C_1\cap C_2$, we just have to check that the defined section extends from $C_1\cup C_2$ to the whole of C (across the separating node $\Delta_1'\cap\Delta_2'$). The extension can be checked étale locally, and then the claim follows from the local description of the log structure.

We have defined a map $\Phi'\colon\mathbf{Div}_{g,A}\to\mathbf{Div}_{g,A}$, and we verify easily that the diagram

$$\begin{tikzcd} \mathbf{Div}_{g,A} \arrow[r, "\mathsf{AJ}"] \arrow[d, "\Phi'"] & \mathfrak{Pic}_{g,n,d} \arrow[d, "\Phi_{\Delta_1}"] \\ \mathbf{Div}_{g,A} \arrow[r, "\mathsf{AJ}"] & \mathfrak{Pic}_{g,n,d} \end{tikzcd} \tag{7.23}$$

commutes. We will prove (7.23) is a pullback square which by Lemma 2.17 yields the invariance (7.22).

To prove that (7.23) is a pullback square, since the horizontal arrows are monomorphisms, we need to show the following: given $(\mathcal{P},\alpha)\in\mathbf{Div}^{\mathsf{rel}}_{g,A}(S)$, a line bundle $\mathcal{L}$ on C and an isomorphism $\mathsf{AJ}(\mathcal{P},\alpha)\xrightarrow{\sim}\Phi_{\Delta_1}(\mathcal{L})$, there exists $(\mathcal{P}_0,\alpha_0)\in\mathbf{Div}^{\mathsf{rel}}_{g,A}(S)$ such that

$$\mathsf{AJ}(\mathcal{P}_0,\alpha_0)\cong\mathcal{L}.$$

If the element $(\mathcal{P},\alpha)$ has only one preimage under Φ', we are done by commutativity. If there are two preimages (the only other case), the Abel–Jacobi images differ by a twist by Δ_1', and the bundle $\mathcal{L}$ will determine which one we choose. More formally, by uniqueness, we may assume S to be strictly henselian local, then Φ' has exactly one preimage whenever $\Delta(g_1,N_1,d_1-1\,|\,g_2,N_2,d_2+1)$ does not meet S, and the result is clear as the diagram commutes. If, on the other hand, $\Delta(g_1,N_1,d_1-1\,|\,g_2,N_2,d_2+1)$ does meet S, then the two preimages under Φ' will have multidegrees

$$(d_1,d_2)\quad\text{and}\quad(d_1-1,d_2+1),$$

and only one of these can be sent by the Abel–Jacobi map to $\mathcal{L}$.

We now prove the invariance

$$\Phi_{\Delta_1}^*\mathsf{P}^\bullet_{g,A,d}=\mathsf{P}^\bullet_{g,A,d}.$$

We will use Proposition 4.1 and prove that the two lines of formula (4.2) are separately invariant.

- For the exponential term, we must show that the divisor

$$-\eta+\sum_{i=1}^n(2a_i\xi_i+a_i^2\psi_i)+\sum_{\Gamma_\delta\in\mathsf{G}^{\mathsf{se}}_{g,n,d}}c_A(\Gamma_\delta)[\Gamma_\delta]$$

is invariant. By Lemma 4.2, we see

$$-\pi_*c_1(\mathcal{L}_A)^2=-\eta+\sum_{i=1}^n(2a_i\xi_i+a_i^2\psi_i).$$

After pulling back via Φ_{Δ_1}, we obtain

$$\begin{aligned}-\pi_*c_1(\mathcal{L}_A(\Delta_1))^2&=-\pi_*(c_1(\mathcal{L}_A)^2+2c_1(\mathcal{L}_A)\Delta_1+\Delta_1^2)\\&=-\pi_*(c_1(\mathcal{L}_A)^2)-2\deg(\mathcal{L}_A|_{\Delta_1})\pi_*\Delta_1-\pi_*(\Delta_1\cdot(\pi^*\Delta-\Delta_2))\\&=-\pi_*(c_1(\mathcal{L}_A)^2)-2\left(d_1-\sum_{i\in N_1}a_i\right)\Delta+\Delta.\end{aligned}$$

We have used

$$\pi^*\Delta = \Delta_1 + \Delta_2$$

and the fact that the intersection of Δ_1 and Δ_2 has degree 1 over Δ.

For the pullback of the linear combination

$$\sum_{\Gamma_\delta \in \mathsf{G}^{\mathsf{se}}_{g,n,d}} c_A(\Gamma_\delta)[\Gamma_\delta]$$

of boundary divisors, recall the divisor $\tilde{\Delta} = \Delta(g_1, N_1, d_1-1 \,|\, g_2, N_2, d_2+1)$, which satisfies

$$\tilde{\Delta} = \Phi_{\Gamma_1}(\tilde{\Delta}) = \Phi_{\Gamma_1}(\Delta).$$

We see $\Phi^*_{\Delta_1}[\Gamma_\delta] = [\Gamma_\delta]$ for all boundary divisors $[\Gamma_\delta]$ different from $[\Delta]$ and $[\tilde{\Delta}]$. Moreover,

$$\Phi^*_{\Delta_1}[\tilde{\Delta}] = [\tilde{\Delta}] + [\Delta] \quad \text{and} \quad \Phi^*_{\Delta_1}[\Delta] = 0.$$

Writing Γ_Δ and $\Gamma_{\tilde{\Delta}}$ for the graphs associated with Δ and $\tilde{\Delta}$, respectively, we see that

$$\Phi^*_{\Delta_1} \sum_{\Gamma_\delta \in \mathsf{G}^{\mathsf{se}}_{g,n,d}} c_A(\Gamma_\delta)[\Gamma_\delta] = \sum_{\Gamma_\delta \in \mathsf{G}^{\mathsf{se}}_{g,n,d}} c_A(\Gamma_\delta)[\Gamma_\delta] + (c_A(\Gamma_{\tilde{\Delta}}) - c_A(\Gamma_\Delta))[\Delta].$$

After expanding the last term further, we obtain the coefficient

$$c_A(\Gamma_{\tilde{\Delta}}) - c_A(\Gamma_\Delta) = -\left(d_1 - 1 - \sum_{i \in I_1} a_i\right)^2 + \left(d_1 - \sum_{i \in I_1} a_i\right)^2 = 2\left(d_1 - \sum_{i \in I_1} a_i\right) - 1,$$

which exactly balances out the error term we obtained in the pullback of

$$-\pi_* c_1(\mathcal{L}_A(\Delta_1))^2.$$

We have finished the proof of the invariance of the exponential term in (4.8).

• For the invariance of the sum over $\Gamma_\delta \in \mathsf{G}^{\mathsf{nse}}_{g,n,d}$, we claim that given any graph Γ_δ, we have the diagram

$$\begin{tikzcd}
\coprod_{\sum_v \delta(v) = d} \mathfrak{Pic}_{\Gamma_\delta} \arrow[r, "\amalg_\delta j_{\Gamma_\delta}"] \arrow[d, "\Phi_{\Delta_1,\Gamma}"] & \mathfrak{Pic}_{g,n,d} \arrow[d, "\Phi_{\Delta_1}"] \\
\coprod_{\sum_v \delta(v) = d} \mathfrak{Pic}_{\Gamma_\delta} \arrow[r, "\amalg_\delta j_{\Gamma_\delta}"] \arrow[d] & \mathfrak{Pic}_{g,n,d} \arrow[d] \\
\mathfrak{M}_\Gamma \arrow[r, "j_\Gamma"] & \mathfrak{M}_{g,n}.
\end{tikzcd} \tag{7.24}$$

The lower and outer diagrams are cartesian as we have seen in §0.3.2, thus the upper diagram is also cartesian. While for a general graph Γ_δ the map $\Phi_{\Delta_1,\Gamma}$ induces a non-trivial map on the set of components of

$$\coprod_{\delta} \mathfrak{Pic}_{\Gamma_\delta},$$

for Γ having only non-separating edges, we obtain

$$\Phi_{\Delta_1,\Gamma}\colon \mathfrak{Pic}_{\Gamma_\delta} \longrightarrow \mathfrak{Pic}_{\Gamma_\delta}$$

over $\mathfrak{M}_\Gamma$. The classes ψ_h (for $h\in H(\Gamma)$) are pullbacks from $\mathfrak{M}_\Gamma$, in particular $\Phi^*_{\Delta_1,\Gamma}\psi_h=\psi_h$ for all such h. As a result, each term

$$j_{\Gamma_\delta *}\left[\prod_{e=(h,h')\in \mathrm{E}(\Gamma_\delta)} \frac{1}{\psi_h+\psi_{h'}}\left(1-\exp\left(-\frac{w(h)w(h')}{2}(\psi_h+\psi_{h'})\right)\right)\right]$$

in the sum over $\Gamma_\delta\in\mathsf{G}^{\mathsf{nse}}_{g,n,d}$ and $w\in\mathsf{W}_{\Gamma_\delta,r}$ in formula (4.2) is indeed invariant.

7.7. Proof of Invariance VI (Partial stabilization)

7.7.1. The stack $\mathfrak{N}$

We begin by introducing a stack $\mathfrak{N}$ that allows us to reformulate Invariance VI as an equality of operational classes on $\mathfrak{N}$. The stack $\mathfrak{N}$ parameterizes data

$$f\colon C'\longrightarrow C, \quad \mathcal{L}/C,$$

where f is a map of prestable genus-g curves which is a partial stabilization (a surjection which contracts some unstable rational components of C') and $\mathcal{L}$ is a line bundle on C of degree zero. By a small extension of the arguments of [50], $\mathfrak{N}$ is an algebraic stack and comes with two maps to $\mathfrak{Pic}_{g,0,0}$ given by

$$\begin{aligned} \ell\colon \mathfrak{N} &\longrightarrow \mathfrak{Pic}_{g,0,0},\\ (f\colon C'\to C,\mathcal{L}) &\longmapsto (C,\mathcal{L}),\end{aligned}$$

and

$$\begin{aligned} \ell'\colon \mathfrak{N} &\longrightarrow \mathfrak{Pic}_{g,0,0},\\ (f\colon C'\to C,\mathcal{L}) &\longmapsto (C',f^*\mathcal{L}).\end{aligned}$$

Lemma 7.2. *We have*

$$\ell^*\mathsf{DR}^{\mathsf{op}}_{g,\mathbf{0}}=(\ell')^*\mathsf{DR}^{\mathsf{op}}_{g,\mathbf{0}},\quad \ell^*\mathsf{P}^\bullet_g=(\ell')^*\mathsf{P}^\bullet_g. \tag{7.25}$$

This lemma will be proven in the next subsection, but first we show that Invariance VI is implied by the lemma.

For Invariance VI, we are given the data

$$C'\xrightarrow{f}C\longrightarrow\mathcal{S},\quad \mathcal{L}\longrightarrow C,\quad p_1,...,p_n:\mathcal{S}\longrightarrow C,\quad p'_1,...,p'_n:\mathcal{S}\longrightarrow C',$$

with f a partial stabilization satisfying $f\circ p'_i=p_i$ and a vector $A\in\mathbb{Z}^n$ satisfying $a_i=0$ if p'_i meets the exceptional locus of f. Invariance VI then says that for the maps

$$\varphi_{\mathcal{L}},\varphi_{f^*\mathcal{L}}:\mathcal{S}\longrightarrow\mathfrak{Pic}_{g,n,d}$$

induced by

$$C\longrightarrow\mathcal{S},\quad p_1,...,p_n:\mathcal{S}\longrightarrow C,\quad \mathcal{L}\longrightarrow C,$$

and

$$C'\longrightarrow\mathcal{S},\quad p'_1,...,p'_n:\mathcal{S}\longrightarrow C',\quad f^*\mathcal{L}\longrightarrow C,$$

respectively, we have

$$\varphi^*_{\mathcal{L}}\mathsf{DR}^{\mathsf{op}}_{g,A}=\varphi^*_{f^*\mathcal{L}}\mathsf{DR}^{\mathsf{op}}_{g,A}\quad\text{and}\quad\varphi^*_{\mathcal{L}}\mathsf{P}^\bullet_{g,A,d}=\varphi^*_{f^*\mathcal{L}}\mathsf{P}^\bullet_{g,A,d}.$$

The condition $a_i=0$ if p'_i meets the exceptional locus of f implies the equality

$$f^*\biggl(\mathcal{L}\biggl(-\sum_{i=1}^n a_ip_i\biggr)\biggr)=(f^*\mathcal{L})\biggl(-\sum_{i=1}^n a_ip'_i\biggr)$$

of line bundles on C'. Denote by $f:\mathcal{S}\to\mathfrak{N}$ the map associated with the data

$$C'\xrightarrow{f}C\longrightarrow\mathcal{S},\quad \mathcal{L}\biggl(-\sum_{i=1}^n a_ip_i\biggr)\longrightarrow C.$$

Writing

$$\mathcal{L}_A=\mathcal{L}\biggl(-\sum_{i=1}^n a_ip_i\biggr)\quad\text{and}\quad f^*\mathcal{L}_A=(f^*\mathcal{L})\biggl(-\sum_{i=1}^n a_ip'_i\biggr),$$

we obtain a commutative diagram

$$\begin{tikzcd}
& \mathcal{S} \arrow[ddl, "\varphi_{\mathcal{L}_A}"'] \arrow[d, "g"] \arrow[ddr, "\varphi_{f^*\mathcal{L}_A}"] & \\
& \mathfrak{N} \arrow[dl, "\ell"] \arrow[dr, "\ell'"'] & \\
\mathfrak{Pic}_{g,0,0} & & \mathfrak{Pic}_{g,0,0}.
\end{tikzcd} \tag{7.26}$$

From the arguments presented in §0.7, it follows that

$$\varphi_{\mathcal{L}}^{*}\mathsf{DR}^{\mathsf{op}}_{g,A} = \varphi_{\mathcal{L}_A}^{*}\mathsf{DR}^{\mathsf{op}}_{g,\varnothing} \quad \text{and} \quad \varphi_{f^*\mathcal{L}}^{*}\mathsf{DR}^{\mathsf{op}}_{g,A} = \varphi_{f^*\mathcal{L}_A}^{*}\mathsf{DR}^{\mathsf{op}}_{g,\varnothing},$$

with parallel equations for $\mathsf{P}^{\bullet}_{g,A,d}$ and $\mathsf{P}^{\bullet}_{g,\varnothing,0}$. Assuming (7.25), we have

$$\varphi_{\mathcal{L}}^{*}\mathsf{DR}^{\mathsf{op}}_{g,A} = \varphi_{\mathcal{L}_A}^{*}\mathsf{DR}^{\mathsf{op}}_{g,\varnothing} = g^*\ell^*\mathsf{DR}^{\mathsf{op}}_{g,\varnothing} = g^*\ell'^*\mathsf{DR}^{\mathsf{op}}_{g,\varnothing} = \varphi_{f^*\mathcal{L}_A}^{*}\mathsf{DR}^{\mathsf{op}}_{g,\varnothing} = \varphi_{f^*\mathcal{L}}^{*}\mathsf{DR}^{\mathsf{op}}_{g,A},$$

and similarly for $\mathsf{P}^{\bullet}_{g,A,d}$. Thus (7.25) implies Invariance VI.

7.7.2. Invariance for $\mathfrak{N}$

Here we prove Lemma 7.2; it follows immediately from (7.25) in Lemmas 7.4 and 7.6 below. We start with a preliminary result.

LEMMA 7.3. *The map*

$$\begin{aligned} \ell\colon \mathfrak{N} &\longrightarrow \mathfrak{Pic}_{g,0,0}, \\ (f\colon C' \to C, \mathcal{L}) &\longmapsto (C, \mathcal{L}), \end{aligned}$$

is syntomic,[39] *and the map*

$$\begin{aligned} \ell'\colon \mathfrak{N} &\longrightarrow \mathfrak{Pic}_{g,0,0}, \\ (f\colon C' \to C, \mathcal{L}) &\longmapsto (C', f^*\mathcal{L}). \end{aligned}$$

is smooth.

Proof. The stack of partial stabilizations $(f\colon C' \to C)$ has smooth charts given by $\overline{\mathcal{M}}_{g,n+m}$, where $(C, p_1, \dots, p_n, q_1, \dots, q_m)$ maps to the contraction map from C to the stabilization of $(C, p_1, \dots, p_n)$ by forgetting the q markings. Charts for $\mathfrak{N}$ are then given by

$$\overline{\mathcal{M}}_{g,n+m} \times_{\overline{\mathcal{M}}_{g,n}} \mathfrak{Pic}_{\overline{\mathcal{M}}_{g,n+1}/\overline{\mathcal{M}}_{g,n}} \xrightarrow{G} \mathfrak{N}.$$

Charts for the map ℓ are given by the composition of the top horizontal arrows in the commutative diagram

$$\begin{tikzcd} \square \arrow[r] \arrow[d] \arrow[rr, bend left, "\ell"] & \mathfrak{Pic}_{\overline{\mathcal{M}}_{g,n+1}/\overline{\mathcal{M}}_{g,n}} \arrow[r] \arrow[d] & \mathfrak{Pic}_g \arrow[d] \\ \overline{\mathcal{M}}_{g,n+m} \arrow[r] & \overline{\mathcal{M}}_{g,n} \arrow[r] & \mathfrak{M}_g. \end{tikzcd} \tag{7.27}$$

(39) Flat and lci.

Both squares here are pullbacks, the bottom right horizontal map is smooth, and the bottom left horizontal map is syntomic. Hence ℓ is syntomic, using that syntomicity is a flat-local property on the target.

Charts for the map ℓ' are given by commutative diagrams

$$\begin{tikzcd} \overline{\mathcal{M}}_{g,n+m} \times_{\overline{\mathcal{M}}_{g,n}} \mathfrak{Pic}_{\overline{\mathcal{M}}_{g,n+1}/\overline{\mathcal{M}}_{g,n}} \arrow[r, "F"] \arrow[d, "G"] & \mathfrak{Pic}_{\overline{\mathcal{M}}_{g,n+m+1}/\overline{\mathcal{M}}_{g,n+m}} \arrow[d] \\ \mathfrak{N} \arrow[r, "\ell'"] & \mathfrak{Pic}_g. \end{tikzcd} \tag{7.28}$$

The right vertical arrow is a base change of the smooth map $\overline{\mathcal{M}}_{g,n+m} \to \mathfrak{M}_g$, so once we have shown F to be smooth, we can conclude using that smoothness is a flat-local property on the target.

In fact, F is an open immersion: F is isomorphic to the inclusion of the locus

$$U \hookrightarrow \mathfrak{Pic}_{\overline{\mathcal{M}}_{g,n+m+1}/\overline{\mathcal{M}}_{g,n+m}}$$

of line bundles which are trivial on the contracted rational components. We must verify the induced map

$$F' \colon \overline{\mathcal{M}}_{g,n+m} \times_{\overline{\mathcal{M}}_{g,n}} \mathfrak{Pic}_{\overline{\mathcal{M}}_{g,n+1}/\overline{\mathcal{M}}_{g,n}} \longrightarrow U$$

is an isomorphism. The source and target are smooth over $\overline{\mathcal{M}}_{g,n+m}$. On each geometric fiber over $\overline{\mathcal{M}}_{g,n+m}$, the map F' is an isomorphism via the explicit description of the Jacobian of a prestable curve. But then F' is flat (by the fiberwise criterion), is unramified (by a pointwise check), and is universally injective (again by a pointwise check), and hence is an isomorphism. □

LEMMA 7.4. *We have*

$$\ell^* \mathsf{DR}^{\mathsf{op}}_{g,\mathbf{0}} = (\ell')^* \mathsf{DR}^{\mathsf{op}}_{g,\mathbf{0}} \in \mathsf{CH}^g_{\mathsf{op}}(\mathfrak{N}).$$

Proof. The maps ℓ and ℓ' are syntomic by Lemma 7.3. It therefore suffices by Lemma 2.17 to construct an isomorphism

$$\varphi \colon \ell^* \mathbf{Div}_{g,\mathbf{0}} \xrightarrow{\sim} (\ell')^* \mathbf{Div}_{g,\mathbf{0}}$$

of stacks over $\mathfrak{N}$ and even to construct the isomorphism on the level of $\mathbf{Div}^{\mathsf{rel}}$.

An object of $\ell^* \mathbf{Div}^{\mathsf{rel}}_{g,\mathbf{0}}$ consists of a stabilization map $f \colon C' \to C$, a line bundle $\mathcal{L}$ on C, a $\mathbb{G}_m^{\mathsf{trop}}$ torsor $\mathcal{P}$ on S, and a map $\alpha \colon C \to \mathcal{P}$ such that $\mathcal{O}(\alpha)$ and $\mathcal{L}$ are isomorphic up to pullback from the base. An object of $(\ell')^*\mathbf{Div}^{\mathsf{rel}}_{g,\mathbf{0}}$ consists of almost the same data, but α is replaced by any $\alpha'\colon C'\to\mathcal{P}$, with $\mathcal{O}(\alpha')$ and $f^*\mathcal{L}$ isomorphic up to pullback. We then define the map φ simply by composing, setting $\alpha'=f\circ\alpha$.

We must show that φ is an isomorphism. Suppose we are given the data $f\colon C'\to C$, $\mathcal{L}$ on C, $\mathcal{P}$, $\alpha'\colon C'\to\mathcal{P}$, with $\mathcal{O}(\alpha')\cong f^*\mathcal{L}$ up to pullback. Since the degree of $f^*\mathcal{L}$ vanishes on components contracted by f, we see that the same is true of the degree of $\mathcal{O}(\alpha')$ — the slopes of the restriction of α' to the contracted graph of the curve are linear on edges. In other words, the restriction is still piecewise linear with integer slopes, hence we can set α to be the restriction. $\square$

More work will be required to prove the second equality

$$\ell^*\mathsf{P}^\bullet_g=(\ell')^*\mathsf{P}^\bullet_g.$$

We start by considering the morphism ℓ. For a prestable graph Γ_δ of degree zero, consider the diagram

$$\begin{tikzcd}
\prod_{v\in\mathrm{V}(\Gamma)}\mathfrak{N}_{\mathrm{g}(v),\mathrm{n}(v),\delta(v)} \arrow[d, "\prod_v \ell_v"] & \mathfrak{N}'_{\Gamma_\delta} \arrow[l] \arrow[r, "G"] \arrow[d] & \mathfrak{N}_{\Gamma_\delta} \arrow[r, "J_{\Gamma_\delta}"] \arrow[d, "\ell_{\Gamma_\delta}"] & \mathfrak{N} \arrow[d, "\ell"] \\
\prod_{v\in\mathrm{V}(\Gamma)}\mathfrak{Pic}_{\mathrm{g}(v),\mathrm{n}(v),\delta(v)} & \mathfrak{Pic}_{\Gamma_\delta} \arrow[l] \arrow[r, equal] & \mathfrak{Pic}_{\Gamma_\delta} \arrow[r, "j_{\Gamma_\delta}"] & \mathfrak{Pic}_{g,0,0},
\end{tikzcd} \tag{7.29}$$

where the left and the right squares are pullbacks and the middle square is commutative.

- The stacks $\mathfrak{N}_{\mathrm{g}(v),\mathrm{n}(v),\delta(v)}$ are the natural generalizations of $\mathfrak{N}$ to the case of marked curves and line bundles of arbitrary total degrees.
- The stack $\mathfrak{N}_{\Gamma_\delta}$ parameterizes data

$$(C_v)_{v\in\mathrm{V}(\Gamma)},\quad \mathcal{L}/C,\quad f\colon C'\longrightarrow C,$$

where f is a partial stabilization and $\mathcal{L}$ is a multidegree δ line bundle on

$$C=\bigsqcup_{v\in\mathrm{V}(\Gamma)}C_v.$$

- The stack $\mathfrak{N}'_{\Gamma_\delta}$ parameterizes data

$$(C_v)_{v\in\mathrm{V}(\Gamma)},\quad \mathcal{L}/C,\quad (f_v\colon C'_v\to C_v)_{v\in\mathrm{V}(\Gamma)}.$$

- The gluing map $G\colon\mathfrak{N}'_{\Gamma_\delta}\to\mathfrak{N}_{\Gamma_\delta}$ sending $(f_v\colon C'_v\to C_v)_{v\in\mathrm{V}(\Gamma)}$ to

$$f\colon\bigsqcup_v C'_v=C'\longrightarrow C=\bigsqcup_v C_v$$

is proper, representable and birational.

Properness of G can be checked using the valuative criterion. The difference between $\mathfrak{N}_{\Gamma_\delta}$ and $\mathfrak{N}'_{\Gamma_\delta}$ is that, in the first space, we have sections of the non-smooth locus of $\mathfrak{C}$ for each half-edge (telling us where to cut apart the curve), whereas, for the second, the sections go to the non-smooth locus of $\mathfrak{C}'$. Fibres of G correspond to choices of lifts of these sections along

$$f\colon \mathfrak{C}' \longrightarrow \mathfrak{C}.$$

Existence and uniqueness of such lifts follows from properness of f and of the inclusion of the non-smooth locus. Representability of G is a short argument showing G is injective on stabilizer groups. Birationality follows since G is an isomorphism over the dense open locus where f is an isomorphism. Let

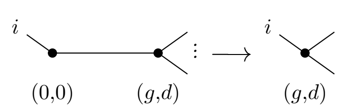
$$\hat{J}_{\Gamma_\delta}\colon \mathfrak{N}'_{\Gamma_\delta} \longrightarrow \mathfrak{N}$$

be the composition of G and J_{Γ_δ}.

In the following lemma, we compare pullback formulas under ℓ and ℓ' for the stacks $\mathfrak{N}_{g,n,d}$ above. Denote by $\psi_i=\ell^*\psi_i$ and $\psi_i'=(\ell')^*\psi_i\in\mathsf{CH}^1_{\mathsf{op}}(\mathfrak{N}_{g,n,d})$ the pullbacks of ψ-classes under ℓ and ℓ', respectively.

LEMMA 7.5. (i) *$\ell^*\eta=(\ell')^*\eta$.*

(ii) *$\psi_i=\psi_i'-D_i$, where D_i is the class in $\mathsf{CH}^1_{\mathsf{op}}(\mathfrak{N}_{g,n,d})$ associated with the boundary divisor of $\mathfrak{N}_{g,n,d}$ generically parameterizing a partial stabilization*



.

Proof. (i) There are two pairs of universal curves with line bundle $(\mathfrak{C},\mathfrak{L})$ and $(\mathfrak{C}',\mathfrak{L}')$ over the stack $\mathfrak{N}_{g,n,d}$:

$$\begin{tikzcd} \mathfrak{C}' \arrow[rr, "f"] \arrow[dr, "\pi'"'] & & \mathfrak{C} \arrow[dl, "\pi"] \\ & \mathfrak{N}_{g,n,d}, & \end{tikzcd}$$

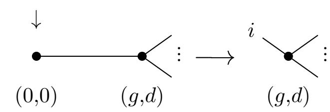
with sections

$$\sigma_i\colon \mathfrak{N}_{g,n,d} \longrightarrow \mathfrak{C} \quad\text{and}\quad \sigma_i'\colon \mathfrak{N}_{g,n,d} \to \mathfrak{C}'.$$

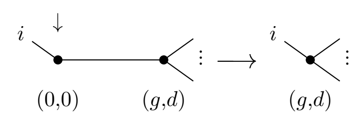
Because $f_*[\mathfrak{C}']=[\mathfrak{C}]$, we have

$$\ell^*\eta=\pi_*(c_1(\mathfrak{L})^2)=\pi_*(c_1(\mathfrak{L})^2 f_*[\mathfrak{C}'])=\pi'_*(c_1(f^*\mathfrak{L})^2)=(\ell')^*\eta.$$

(ii) Let D_0' be the divisor



in $\mathfrak{C}'$, and let D_i' be the divisor



in $\mathfrak{C}'$. Here, the arrows pointing to the vertices with genus and degree zero indicate which component of the universal curve over the corresponding boundary divisor in $\mathfrak{N}_{g,n,d}$ we take. The divisors $D_0', D_1', \ldots, D_n'$ are precisely the divisorial loci in $\mathfrak{C}'$ which are contracted by the map $f\colon \mathfrak{C}' \to \mathfrak{C}$. Then,

$$\ell^*\psi_i = c_1(\sigma_i^*\omega_\pi) = (\sigma_i')^* c_1(f^*\omega_\pi) = (\sigma_i')^* c_1\left(\omega_{\pi'}\left(-D_0' - \sum_{i=1}^n D_i'\right)\right) = (\ell')^*\psi_i - D_i,$$

where the sections are denoted by σ_i. $\square$

For the morphism ℓ', form the fiber diagram

$$\begin{tikzcd}
\mathfrak{Pic}'_{\Gamma_\delta} \arrow[r, "J'_{\Gamma_\delta}"] \arrow[d] & \mathfrak{N} \arrow[d, "\ell'"] \\
\mathfrak{Pic}_{\Gamma_\delta} \arrow[r, "j_{\Gamma_\delta}"] & \mathfrak{Pic}_{g,0,0}.
\end{tikzcd} \tag{7.30}$$

By definition, the fiber product $\mathfrak{Pic}'_{\Gamma_\delta}$ parameterizes data

$$(C_v')_{v\in \mathrm{V}(\Gamma)}, \quad \mathcal{L}'/C' = \bigsqcup_v C_v', \quad f\colon C' \longrightarrow C, \quad \mathcal{L}/C, \quad f^*\mathcal{L} \xrightarrow{\sim} \mathcal{L}',$$

which simplifies to

$$(C_v')_{v\in \mathrm{V}(\Gamma)}, \quad f\colon \bigsqcup_v C_v' = C' \longrightarrow C, \quad \mathcal{L}/C.$$

On the other hand, the stack $\mathfrak{N}'_{\Gamma_\delta}$ parameterizes

$$(C_v)_{v\in \mathrm{V}(\Gamma)}, \quad \mathcal{L}/C, \quad (f_v\colon C_v' \to C_v)_{v\in \mathrm{V}(\Gamma)}.$$

There is a subtle difference here. For $\mathfrak{Pic}'_{\Gamma_\delta}$, the map f is allowed to contract entire components C_v' to points, whereas in the second case the target C_v is always 1-dimensional.

Our next step is to show that

$$\mathfrak{Pic}'_{\Gamma_\delta} \cong \coprod_{\Gamma_\delta \to \widetilde{\Gamma}_\delta} \mathfrak{M}_{\Gamma_\delta \to \widetilde{\Gamma}_\delta}. \tag{7.31}$$

More precisely, the connected components of $\mathfrak{Pic}'_{\Gamma_\delta}$ are in bijective correspondence to partial stabilizations

$$\Gamma_\delta \to \widetilde{\Gamma}_\delta.$$

We will prove, given a vertex $v \in \mathrm{V}(\Gamma)$ which can be contracted (with $g(v)=0$, $n(v) \leqslant 2$, and $\delta(v)=0$), that the locus of points in $\mathfrak{Pic}'_{\Gamma_\delta}$ where $f\colon C' \to C$ contracts C'_v is open and closed.

For a vertex v with $n(v)=2$, the universal curve $\mathfrak{C}'_v \to \mathfrak{Pic}'_{\Gamma_\delta}$ has two sections (corresponding to the half-edges at v), and the locus where C'_v is contracted equals the locus where the sections coincide, which is closed since $\mathfrak{C}'_v \to \mathfrak{Pic}'_{\Gamma_\delta}$ is separated.

To show closedness of the locus where C'_v is *not* contracted, assume that we have a family

$$(C'_{v,S})_{v \in \mathrm{V}(\Gamma)}, \quad f\colon \coprod_v C'_{v,S} = C'_S \longrightarrow C_S, \quad \mathcal{L}/C_S,$$

of $\mathfrak{Pic}'_{\Gamma_\delta}$ over the spectrum S of a strictly henselian DVR such that the fiber $C'_{v,\eta}$ of $C'_{v,S}$ over the generic point η of S is not contracted by f. We want to show that then also the fiber $C'_{v,L}$ over the closed point L of S is not contracted. By assumption, $C'_{v,\eta}$ maps to a union $C_{v,\eta}$ of components of the fiber C_η of C_S over η. Then $C_{v,\eta}$ specializes to a union $C_{v,L}$ of components of C_L. Since f is proper, f maps the closure $C'_{v,S}$ of $C'_{v,\eta}$ to the closure of its image $C_{v,\eta}$. Since $C_{v,L}$ is still positive-dimensional, the curve $C'_{v,L}$ is indeed not contracted. For related arguments, see the proof of [50, Proposition 2.2].

For a vertex v with $n(v)=1$, the universal curve $\mathfrak{C}'_v \to \mathfrak{Pic}'_{\Gamma_\delta}$ has a single section σ_h. On the one hand, the locus where C'_v is contracted is exactly the locus where $f\colon \mathfrak{C}' \to \mathfrak{C}$ maps σ_h to the smooth locus of $\mathfrak{C} \to \mathfrak{Pic}'_{\Gamma_\delta}$, thus it is open. On the other hand, it is also the preimage under σ_h of the exceptional locus of $\mathfrak{C}' \to \mathfrak{C}$, and thus closed.

We have proven that the connected components of $\mathfrak{Pic}'_{\Gamma_\delta}$ are in bijective correspondence to partial stabilizations $\Gamma_\delta \to \widetilde{\Gamma}_\delta$. But a point

$$(C'_v)_{v \in \mathrm{V}(\Gamma)}, \quad f\colon \coprod_v C'_v = C' \to C, \quad \mathcal{L}/C,$$

on the corresponding component is equivalent to the data of any collection of curves C'_v for $v' \in \mathrm{V}(\Gamma) \setminus \mathrm{V}(\widetilde{\Gamma})$, which are contracted by f, together with a point

$$(C'_v)_{v \in \mathrm{V}(\widetilde{\Gamma})}, \quad f\colon (C'_v \to C_v), \quad \mathcal{L}/C = \coprod_{v \in V(\widetilde{\Gamma})} C_v,$$

of $\mathfrak{N}'_{\widetilde{\Gamma}_\delta}$. Hence, we have a isomorphism

$$\mathfrak{N}_{\Gamma_\delta\to\widetilde{\Gamma}_\delta}\cong\mathfrak{N}'_{\widetilde{\Gamma}_\delta}\times\prod_{v\in \mathrm{V}(\Gamma)\setminus\mathrm{V}(\widetilde{\Gamma})}\mathfrak{M}_{0,\mathrm{n}(v)},$$

where, in the last expression, $n(v)$ is necessarily 1 or 2.

For each partial stabilization $\Gamma_\delta\to\widetilde{\Gamma}_\delta$, we denote by

$$J_{\Gamma_\delta\to\widetilde{\Gamma}_\delta}:\mathfrak{N}_{\Gamma_\delta\to\widetilde{\Gamma}_\delta}\longrightarrow\mathfrak{N}$$

the restriction of J'_{Γ_δ} to $\mathfrak{N}_{\Gamma_\delta\to\widetilde{\Gamma}_\delta}$.

LEMMA 7.6. *We have*

$$\ell^*\mathsf{P}_g^\bullet=(\ell')^*\mathsf{P}_g^\bullet\in\prod_{c=0}^{\infty}\mathsf{CH}^c_{\mathsf{op}}(\mathfrak{N})\quad\textit{for all } c\geqslant 0.$$

Proof. We will use formula (4.8) for $\mathsf{P}_g^\bullet$. By Lemma 7.5, the terms $\exp\left(-\frac{1}{2}\eta\right)$ have identical pullback under ℓ and ℓ'. We can therefore focus on the sum over graphs and weighting mod r.

We start with a few remarks about the combinatorial factors in $\mathsf{P}_g^\bullet$ which will arise in the proof. Let $\Gamma_\delta\to\widetilde{\Gamma}_\delta$ be a partial stabilization, then the Betti numbers agree:

$$h^1(\Gamma_\delta)=h^1(\widetilde{\Gamma}_\delta).$$

Given r, the map $W_{\Gamma_\delta,r}\to W_{\widetilde{\Gamma}_\delta,r}$ of admissible weightings mod r (induced by the inclusion $H(\widetilde{\Gamma})\to H(\Gamma)$ of half-edge sets) is a bijection.

Moreover, if the map $\Gamma_\delta\to\widetilde{\Gamma}_\delta$ only contracts components with

$$(\mathrm{g}(v),\mathrm{n}(v),\delta(v))=(0,2,0),$$

there is a canonical isomorphism $\mathrm{Aut}(\Gamma_\delta)\cong\mathrm{Aut}(\widetilde{\Gamma}_\delta)$. On the other hand, in the formula for $\mathsf{P}_g^\bullet$, every term such that Γ_δ has a vertex with

$$(\mathrm{g}(v),\mathrm{n}(v),\delta(v))=(0,1,0)$$

necessarily vanishes. Indeed, the half-edge h at this vertex must have $w(h)=0$ such that the corresponding edge term vanishes.

To keep the notation concise, we write Φ_a for the power-series

$$\Phi_a(x)=\frac{1}{x}\left(1-\exp\left(-\frac{a}{2}x\right)\right)=\sum_{m=0}^{\infty}(-1)^m\left(\frac{a}{2}\right)^{m+1}\frac{1}{(m+1)!}x^m=\frac{a}{2}-\frac{a^2}{8}x+\ldots$$

appearing in the edge terms of $\mathsf{P}_g^\bullet$. Moreover, given a graph $\widetilde{\Gamma}_\delta$ with a half-edge h incident to a vertex v, denote by ψ_h and ψ'_h the classes on $\mathfrak{M}'_{\widetilde{\Gamma}_\delta}$ pulled back from $\mathfrak{M}_{\mathrm{g}(v),\mathrm{n}(v),\delta(v)}$ in the diagram (7.29). Similarly, given a partial stabilization $\Gamma_\delta \to \widetilde{\Gamma}_\delta$, the space $\mathfrak{M}_{\Gamma_\delta \to \widetilde{\Gamma}_\delta}$ contains $\mathfrak{M}'_{\widetilde{\Gamma}_\delta}$ as a factor, and hence the notation also makes sense on $\mathfrak{M}_{\Gamma_\delta \to \widetilde{\Gamma}_\delta}$ (provided h is a half-edge of $\widetilde{\Gamma}_\delta$).

Let us first compute the pullback of the graph sum in $\mathsf{P}_g^{\bullet,r}$ via $\ell' \colon \mathfrak{M} \to \mathfrak{Pic}_{g,0,0}$. Using the diagram (7.30) and the decomposition (7.31), we see that

$$
\begin{aligned}
&(\ell')^* \sum_{\Gamma_\delta, w} \frac{r^{-h^1(\Gamma_\delta)}}{|\mathrm{Aut}(\Gamma_\delta)|} j_{\Gamma_\delta *}\Bigg[\prod_{e=(h,h')\in \mathrm{E}(\Gamma_\delta)} \Phi_{w(h)w(h')}(\psi_h+\psi_{h'})\Bigg] \\
&\quad = \sum_{\Gamma_\delta \to \widetilde{\Gamma}_\delta, w} \frac{r^{-h^1(\Gamma_\delta)}}{|\mathrm{Aut}(\Gamma_\delta)|} J_{\Gamma_\delta \to \widetilde{\Gamma}_\delta *}\Bigg[\prod_{e=(h,h')\in \mathrm{E}(\Gamma_\delta)} \Phi_{w(h)w(h')}(\psi'_h+\psi'_{h'})\Bigg]. \qquad (7.32)
\end{aligned}
$$

In the second line, the sum is over all partial stabilizations $\Gamma_\delta \to \widetilde{\Gamma}_\delta$.

Second, we compute the pullback of the graph sum in $\mathsf{P}_g^{\bullet,r}$ via $\ell \colon \mathfrak{M} \to \mathfrak{Pic}_{g,0,0}$:

$$
\begin{aligned}
&\ell^* \sum_{\widetilde{\Gamma}_\delta, w} \frac{r^{-h^1(\widetilde{\Gamma}_\delta)}}{|\mathrm{Aut}(\widetilde{\Gamma}_\delta)|} j_{\widetilde{\Gamma}_\delta *}\Bigg[\prod_{e=(h,h')\in \mathrm{E}(\widetilde{\Gamma}_\delta)} \Phi_{w(h)w(h')}(\psi_h+\psi_{h'})\Bigg] \\
&\quad = \sum_{\widetilde{\Gamma}_\delta, w} \frac{r^{-h^1(\widetilde{\Gamma}_\delta)}}{|\mathrm{Aut}(\widetilde{\Gamma}_\delta)|} \hat{J}_{\widetilde{\Gamma}_\delta *}\Bigg[\prod_{e=(h,h')\in \mathrm{E}(\widetilde{\Gamma}_\delta)} \Phi_{w(h)w(h')}(\psi_h+\psi_{h'})\Bigg]. \qquad (7.33)
\end{aligned}
$$

We use here the right fiber diagram in (7.29) together with the fact that G is proper, representable, and birational. So, by Proposition 2.16, the pushforward of fundamental classes under $J_{\widetilde{\Gamma}_\delta}$ and $\hat{J}_{\widetilde{\Gamma}_\delta}$ agree.

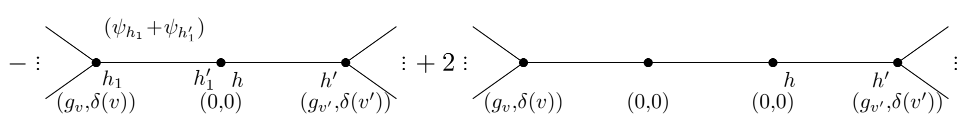
To compare to the formula for the pullback under ℓ', we use

$$\psi_h + \psi_{h'} = \psi'_h - D_h + \psi'_{h'} - D_{h'}$$

on $\mathfrak{M}'_{\widetilde{\Gamma}_\delta}$, by Lemma 7.5. The next step of the proof is to use the self-intersection formula for D_h and $D_{h'}$ (similar to the formula described in [30, Appendix A]) to expand the edge term

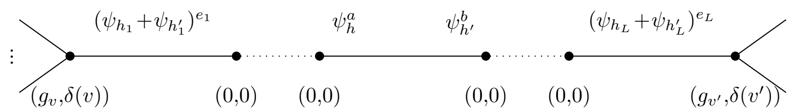
$$\Phi_{w\,w'}(\psi'_h - D_h + \psi'_{h'} - D_{h'}) = \sum_{m=0}^{\infty} (-1)^m \left(\frac{w\,w'}{2}\right)^{m+1} \frac{1}{(m+1)!} (\psi'_h - D_h + \psi'_{h'} - D_{h'})^m.$$

For example, $(D_h)^2$ is equal to



and similarly for $(D_{h'})^2$.

The result will be a linear combination of terms



, (7.34)

where the edge (h,h') is at position ℓ in the above chain $(1\leqslant\ell\leqslant L)$. The total degree of this term (before the pushforward by $\hat{J}_{\tilde{\Gamma}_\delta}$) is

$$m=\sum_{j\neq\ell}(e_j+1)+a+b.$$

The total coefficient of this particular term in

$$\Phi_{w\,w'}(\psi'_h-D_h+\psi'_{h'}-D_{h'})$$

is then

$$\underbrace{(-1)^m\left(\frac{w\,w'}{2}\right)^{m+1}\frac{1}{(m+1)!}}_{\text{coefficient in }\Phi_{w\,w'}}\cdot\binom{m}{e_1+1,\dots,a,b,\dots,e_L+1}\underbrace{(-1)^{L-1}}_{\substack{\text{excess intersection}\\ \text{of }-D_h\text{ and }-D_{h'}}},$$

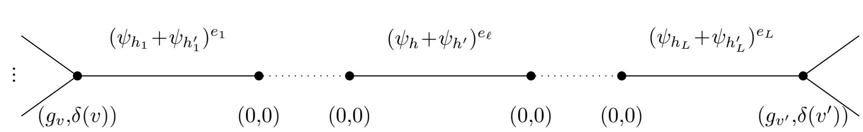
where the multinomial coefficient comes from the expansion of

$$(\psi'_h-D_h+\psi'_{h'}-D_{h'})^m.$$

Writing $e_\ell=a+b$, we can simplify to obtain

$$\frac{1}{m+1}\left(\prod_{j=1}^{L}(-1)^{e_j}\left(\frac{w\,w'}{2}\right)^{e_j+1}\frac{1}{(e_j+1)!}\right)\binom{e_\ell}{a}(e_\ell+1).$$

Summing over all choices $a+b=e_\ell$ for fixed e_ℓ, the coefficient of the term



(7.35)

is exactly equal to

$$\frac{e_\ell+1}{m+1}\left(\prod_{j=1}^{L}(-1)^{e_j}\left(\frac{w\,w'}{2}\right)^{e_j+1}\frac{1}{(e_j+1)!}\right).$$

Pushing forward by $\hat{J}_{\widetilde{\Gamma}_\delta}$, we forget where in the chain above the edge (h,h') has been. Summing over the L possible positions and using

$$m+1=\sum_{\ell=1}^{L}(e_\ell+1),$$

we obtain the coefficient

$$\prod_{j=1}^{L}\underbrace{(-1)^{e_j}\left(\frac{w\,w'}{2}\right)^{e_j+1}\frac{1}{(e_j+1)!}}_{\text{coefficient of } x^{e_j} \text{ in } \Phi_{w\,w'}(x)}.$$

From the above discussion, we see that (7.33) equals

$$\sum_{\Gamma_\delta\to\widetilde{\Gamma}_\delta,w}\frac{r^{-h^1(\widetilde{\Gamma}_\delta)}}{|\mathrm{Aut}(\widetilde{\Gamma}_\delta)|}J_{\Gamma_\delta\to\widetilde{\Gamma}_\delta *}\Bigg[\prod_{e=(h,h')\in \mathrm{E}(\Gamma_\delta)}\Phi_{w(h)w'(h)}(\psi'_h+\psi'_{h'})\Bigg].$$

The sum goes over stabilizations $\Gamma_\delta\to\widetilde{\Gamma}_\delta$ contracting chains of curves with

$$(g,n,d)=(0,2,0).$$

By the previous remarks concerning the combinatorial factors, we have

$$h^1(\widetilde{\Gamma}_\delta)=h^1(\Gamma_\delta) \quad \text{and} \quad |\mathrm{Aut}(\widetilde{\Gamma}_\delta)|=|\mathrm{Aut}(\Gamma_\delta)|.$$

The sum does not change if we allow arbitrary stabilizations $\Gamma_\delta\to\widetilde{\Gamma}_\delta$, since for Γ_δ having a vertex with $(g,n,d)=(0,1,0)$, the summand automatically vanishes. Thus the sum above equals the term computed in (7.32). □

8. Applications

8.1. Proofs of Theorem 0.9 and Conjecture A

We start by recalling notions presented in §0.5, but now in the more general setting of k-differentials. Let $A=(a_1,...,a_n)$ be a vector of zero and pole multiplicities satisfying

$$\sum_{i=1}^{n}a_i=k(2g-2).$$

Let $\mathcal{H}_g^k(A)\subset\mathcal{M}_{g,n}$ be the closed (generally non-proper) locus of pointed curves

$$(C,p_1,...,p_n)$$

satisfying the condition

$$\mathcal{O}_C\Bigg(\sum_{i=1}^n a_i p_i\Bigg) \simeq \omega_C^{\otimes k}.$$

In other words, $\mathcal{H}_g^k(A)$ is the locus of (possibly) meromorphic k-differentials with zero and pole multiplicities prescribed by A. In paper [28], a compact moduli space of twisted k-canonical divisors

$$\widetilde{\mathcal{H}}_g^k(A) \subset \overline{\mathcal{M}}_{g,n}$$

is constructed extending $\mathcal{H}_g^k(A)=\widetilde{\mathcal{H}}_g^k(A)\cap\mathcal{M}_{g,n}$ to the boundary of $\overline{\mathcal{M}}_{g,n}$.

For $k\geqslant 1$ and A not of the form $A=k\cdot A'$ with a vector A' of non-negative integers, the locus $\widetilde{\mathcal{H}}_g^k(A)$ is of pure codimension g in $\overline{\mathcal{M}}_{g,n}$ by [28, Theorem 3] (for $k=1$) and [72, Theorem 1.1] (for $k>1$). A weighted fundamental cycle of $\widetilde{\mathcal{H}}_g^k(A)$,

$$\mathsf{H}_{g,A}^k \in \mathsf{CH}_{2g-3+n}(\overline{\mathcal{M}}_{g,n}), \tag{8.1}$$

is constructed in [28, Appendix A] and [72, §3.1] with explicit non-trivial weights on the irreducible components. The closure

$$\overline{\mathcal{H}}_{g,A}^k \subset \overline{\mathcal{M}}_{g,n}$$

contributes to the weighted fundamental class $\mathsf{H}_{g,A}^k$ with multiplicity 1, but there are additional boundary contributions, as described in the references above.

The weighted fundamental class $\mathsf{H}_{g,A}^k$ was conjectured in [28], [72] to equal the class given by Pixton's formula for the double ramification cycle. To state the conjecture, consider the shifted([40]) vector

$$\tilde{A} = (a_1+k, ..., a_n+k).$$

Conjecture A. For $k\geqslant 1$ and A not of the form $A=k\cdot A'$ with a vector A' of non-negative integers, we have an equality

$$\mathsf{H}_{g,A}^k = 2^{-g} P_g^{g,k}(\tilde{A}),$$

where $P_g^{g,k}(\tilde{A})$ is Pixton's cycle class defined in [42, §1.1].

By combining Theorem 0.7 with previous results of [40], we can now prove the conjecture.

([40]) The shift is needed since Pixton's original formula worked with powers of the log-canonical line bundle $\omega_C^{\mathsf{log}}=\omega_C(\sum_{i=1}^n p_i)$ instead of ω_C.

Theorem 8.1. *Conjecture* A *is true.*

Proof. By [40, Theorem 1.1], the weighted fundamental class $\mathsf{H}^k_{g,A}$ is equal to the double ramification cycle $\mathsf{DR}_{g,A,\omega^k}$ constructed in [37]. By Theorem 0.1, $\mathsf{DR}_{g,A,\omega^k}$ is in turn given by the action of $\mathsf{DR}^{\mathsf{op}}_{g,A}$ on the fundamental class of $\overline{\mathcal{M}}_{g,n}$ via the morphism $\varphi_{\omega^k_\pi}\colon \overline{\mathcal{M}}_{g,n} \to \mathfrak{Pic}_{g,n,k(2g-2)}$ associated with the family

$$\pi\colon \mathcal{C}_{g,n} \longrightarrow \overline{\mathcal{M}}_{g,n}, \quad \omega^k_\pi \longrightarrow \mathcal{C}_{g,n}.$$

By Theorem 0.7, the class $\mathsf{DR}^{\mathsf{op}}_{g,A}$ is computed by the tautological class

$$\mathsf{P}^g_{g,A,d} \in \mathsf{CH}^g_{\mathsf{op}}(\mathfrak{Pic}_{g,n,k(2g-2)}).$$

By Proposition 4.3, the action of $\mathsf{P}^g_{g,A,d}$ on $[\overline{\mathcal{M}}_{g,n}]$ is indeed given by Pixton's original formula $2^{-g}P^{g,k}_g(\tilde{A})$, finishing the proof. □

The steps of the proof of Theorem 8.1 are summarized as follows:

$$\begin{aligned}
\mathsf{H}^k_{g,A} &= \mathsf{DR}_{g,A,\omega^k} && \text{([40, Theorem 1.1])}\\
&= \mathsf{DR}^{\mathsf{op}}_{g,A}(\varphi_{\omega^k_\pi})([\overline{\mathcal{M}}_{g,n}]) && \text{(Theorem 0.1)}\\
&= \mathsf{P}^g_{g,A,d}(\varphi_{\omega^k_\pi})([\overline{\mathcal{M}}_{g,n}]) && \text{(Theorem 0.7)}\\
&= 2^{-g}P^{g,k}_g(\tilde{A}). && \text{(Proposition 4.3).}
\end{aligned}$$

The result provides a completely geometric representative of Pixton's cycle in terms of twisted k-differentials. Theorem 0.9 of §0.5 is the $k=1$ case of Theorem 8.1.

8.2. Closures

Let $A=(a_1, \ldots, a_n)$ be a vector of integers satisfying

$$\sum_{i=1}^n a_i = k(2g-2).$$

A careful investigation of the closure

$$\mathcal{H}^k_g(A) \subset \overline{\mathcal{H}}^k_g(A) \subset \overline{\mathcal{M}}_{g,n}$$

is carried out in [8] and [9]. By a simple method presented in [28, Appendix A] and [72, §3.4], Theorem 8.1 easily determines the cycle classes of the closures

$$[\overline{\mathcal{H}}^k_g(A)] \in \mathsf{CH}_*(\overline{\mathcal{M}}_{g,n})$$

for the following two cases:

- $k=1$ and all a_i are non-negative, when $\overline{\mathcal{H}}^k_g(A)$ has pure codimension $g-1$;
- $k \geqslant 1$ and A is not of the form $A=k\cdot A'$ with a vector A' of non-negative integers, when $\overline{\mathcal{H}}^k_g(A)$ has pure codimension g.

In particular, from the recursive formula for $[\overline{\mathcal{H}}_g^k(A)]$ and the fact that Pixton's cycle on $\overline{\mathcal{M}}_{g,n}$ is tautological, the following is immediate.

COROLLARY 8.2. *The cycles* $[\overline{\mathcal{H}}_g^k(A)]$ *are tautological classes in* $\mathsf{CH}_*(\overline{\mathcal{M}}_{g,n})$.

In the case $k=1$, Corollary 8.2 was known by work of Sauvaget [70], who gave a different approach to $[\overline{\mathcal{H}}_g^1(A)]$ in terms of tautological classes. The recursive formulas for $[\overline{\mathcal{H}}_g^k(A)]$ from Corollary 8.2 have been implemented([41]) in the software [23] for computations in the tautological ring of $\overline{\mathcal{M}}_{g,n}$.

Another application of Conjecture A is presented in the recent paper [71] by Sauvaget. The paper studies moduli spaces of flat surfaces of genus g with conical singularities at marked points $p_1, ..., p_n$. The singularities have fixed cone angles $2\pi\alpha_i$, for $\alpha_1, ..., \alpha_n \in \mathbb{R}$, summing to $2g-2+n$. If all α_i are rational, the spaces of flat surfaces naturally contain $\mathcal{H}_g^k(kA)$, for

$$A = (\alpha_i - 1)_{i=1}^n,$$

as closed subsets (for k sufficiently divisibly). These subsets equidistribute (with respect to natural measures) as $k \to \infty$. Using the equidistribution, Sauvaget is able to apply the recursive expression for $\overline{\mathcal{H}}_g^k(kA)$ from Conjecture A to derive an explicit recursion for the volumes of the moduli spaces of flat surfaces.

8.3. *k*-twisted DR cycles with targets

We define here *k-twisted double ramification cycles with targets* via the class $\mathsf{DR}_{g,A}^{\mathsf{op}}$.

Let X be a non-singular projective variety with line bundle $\mathcal{L}$ and an effective curve class $\beta \in H_2(X, \mathbb{Z})$. Let

$$d_\beta = \int_\beta c_1(\mathcal{L}).$$

Let $k \in \mathbb{Z}$ and $A = (a_1, ..., a_n) \in \mathbb{Z}^n$ satisfy

$$d_\beta + k(2g-2+n) = \sum_{i=1}^n a_i.$$

Consider the morphism

$$\varphi: \overline{\mathcal{M}}_{g,n}(X, \beta) \longrightarrow \mathfrak{Pic}_{g,n,d_\beta + k(2g-2+n)}$$

([41]) In the ongoing project [22], the authors study formulas for Euler characteristics of strata of differentials in terms of intersection numbers on the compactification of these strata constructed in [10]. The implementation of $[\overline{\mathcal{H}}_g^1(A)]$ has played a role in corroborating their formulas.

defined by the universal data

$$\pi : \mathcal{C}_{g,n,\beta} \longrightarrow \overline{\mathcal{M}}_{g,n}(X,\beta), \quad f^*\mathcal{L}\otimes\omega_{log}^{\otimes k} \longrightarrow \mathcal{C}_{g,n,\beta}, \tag{8.2}$$

where $f : \mathcal{C}_{g,n,\beta} \to X$ is the universal map.

Definition 8.3. The *k-twisted X-valued double ramification cycle* is defined by

$$\mathsf{DR}^k_{g,n,\beta}(X,\mathcal{L}) = \mathsf{DR}^{\mathsf{op}}_{g,A}(\varphi)([\overline{\mathcal{M}}_{g,n}(X,\beta)]^{\mathsf{vir}}) \in \mathsf{CH}_{\mathrm{vdim}(g,n,\beta)-g}(\overline{\mathcal{M}}_{g,n}(X,\beta)).$$

In the notation of [43, §0.4], let $\mathsf{P}^{c,k,r}_{g,A,\beta}(X,\mathcal{L})$ be the codimension c part of the following expression

$$\sum_{\substack{\Gamma\in\mathsf{G}_{g,n,\beta}(X)\\ w\in\mathsf{W}_{\Gamma,r,k}(X)}} \frac{r^{-h^1(\Gamma)}}{|\mathrm{Aut}(\Gamma)|} j_{\Gamma_\delta *}\Bigg[\prod_{i=1}^n \exp\left(\frac{1}{2}a_i^2\psi_i + a_i\xi_i\right)$$
$$\times \prod_{v\in\mathrm{V}(\Gamma)} \exp\left(-\frac{1}{2}\eta(v) - k\eta_{1,1}(v) - \frac{k^2}{2}\eta_{0,2}(v)\right)$$
$$\times \prod_{e=(h,h')\in\mathrm{E}(\Gamma)} \frac{1}{\psi_h+\psi_{h'}}\left(1-\exp\left(-\frac{w(h)w(h')}{2}(\psi_h+\psi_{h'})\right)\right)\Bigg].$$

The definition of the admissible k-weightings $w\in\mathsf{W}_{\Gamma,r,k}(X)$ is similar to that in §0.3.4, but with the condition (iii) replaced by

$$k(2g(v)-2+n(v)) + \int_{\beta(v)} c_1(\mathcal{L}) = \sum_{v(h)=v} w(h) \quad \text{for } v\in\mathrm{V}(\Gamma).$$

As in the case $k=0$ discussed in [43, Proposition 1], the class $\mathsf{P}^{c,k,r}_{g,A,\beta}(X,\mathcal{L})$ is polynomial in r for all sufficiently large r. Denote by $\mathsf{P}^{c,k}_{g,A,\beta}(X,\mathcal{L})$ the value at $r=0$ of this polynomial.

By Theorem 0.7 and a slight generalization of the procedure for pulling back Chow cohomology classes from $\mathfrak{Pic}_{g,n,d_\beta+k(2g-2+n)}$ to $\overline{\mathcal{M}}_{g,n}(X,\beta)$ described in [43, §1.5], we have

$$\begin{aligned}\mathsf{DR}^k_{g,n,\beta}(X,\mathcal{L}) &= \mathsf{DR}^{\mathsf{op}}_{g,A}(\varphi)([\overline{\mathcal{M}}_{g,n}(X,\beta)]^{\mathsf{vir}})\\ &= \mathsf{P}^g_{g,A,d_\beta+k(2g-2+n)}(\varphi)([\overline{\mathcal{M}}_{g,n}(X,\beta)]^{\mathsf{vir}})\\ &= \mathsf{P}^{g,k}_{g,A,\beta}(X,\mathcal{L}).\end{aligned}$$

8.4. Proof of Theorem 0.8

For all $c>g$, we will prove

$$\mathsf{P}^c_{g,A,d} = 0 \in \mathsf{CH}^c_{\mathsf{op}}(\mathfrak{Pic}_{g,n,d}). \tag{8.3}$$

The path is parallel to the proof of Theorem 0.7.

By definition, the claim is equivalent to showing that the map

$$\mathsf{P}^c_{g,A,d}(\varphi)\colon \mathsf{CH}_*(B) \longrightarrow \mathsf{CH}_{*-c}(B) \tag{8.4}$$

is zero for every morphism $\varphi\colon B \to \mathfrak{Pic}_{g,n,d}$ from an (irreducible) finite-type scheme B corresponding to the data

$$C \longrightarrow B, \quad \mathcal{L} \longrightarrow C.$$

Retracing the steps of §5 (and using the invariance Lemma 7.6 for the codimension-c part $\mathsf{P}^c_{g,A,d}$ of Pixton's formula), we can reduce to the situation where $\mathcal{L}$ on C is relatively sufficiently positive with respect to $C \to B$. As in §5.3, we can then find

$$\psi\colon U_l \longrightarrow B$$

such that ψ^* is injective on Chow groups and such that the composition

$$U_l \longrightarrow B \longrightarrow \mathfrak{Pic}_{g,n,d}$$

factors through $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^l, d)'$. By [4, Theorem 3.2], we have the vanishing

$$\mathsf{P}^c_{g,A,d}(\mathbb{P}^l, \mathcal{O}(1)) = 0 \in \mathsf{CH}_{\mathrm{vdim}(g,n,d)-c}(\overline{\mathcal{M}}_{g,n}(\mathbb{P}^l, d)).$$

The same combination of Lemma 2.6 and the injectivity of ψ^* then shows the desired vanishing of the map (8.4).

8.5. Connections to past and future results

The relations of Theorem 0.8 generalize several previous results. For $g=0$ and $c=1$, the vanishing (8.3) was observed in [45, Proposition 1.2]. In fact, in genus zero, there are many connections to past equations, see [4, §4] for a full discussion with many examples including classical equations and the relations of [52].

Randal-Williams [67] proves a vanishing result in cohomology with integral coefficients on the locus $\mathfrak{Pic}^{\mathsf{sm}}_{g,0,d}$ of smooth curves for every $d \in \mathbb{Z}$. We can recover a version of Randal-William's vanishing in operational Chow with $\mathbb{Q}$-coefficients which extends to all of $\mathfrak{Pic}_{g,0,d}$. By Proposition 4.1 and Lemma 4.2, Pixton formula's on the locus $\mathfrak{Pic}^{\mathsf{sm}}_{g,0,0}$ takes the simple form

$$\mathsf{P}^c_{g,\varnothing,0} = \frac{1}{c!}(\mathsf{P}^1_{g,\varnothing,0})^c, \quad \mathsf{P}^1_{g,\varnothing,0} = -\frac{1}{2}\pi_*(c_1(\mathcal{L})^2)$$

for the universal curve and the universal line bundle

$$\pi\colon \mathfrak{C} \longrightarrow \mathfrak{Pic}_{g,0,0}, \quad \mathcal{L} \longrightarrow \mathfrak{C}.$$

We claim that, up to scaling, the relation

$$\Omega^{g+1} = 0$$

of [67, Theorem A] is exactly the restriction of the pullback of the relation

$$(\mathsf{P}^1_{g,\varnothing,0})^{g+1} = (g+1)!\,\mathsf{P}^{g+1}_{g,\varnothing,0} = 0 \in \mathsf{CH}^{g+1}_{\mathsf{op}}(\mathfrak{Pic}_{g,0,0})$$

under the morphism

$$\begin{aligned} \mathfrak{Pic}_{g,0,d} &\longrightarrow \mathfrak{Pic}_{g,0,0}, \\ (C,\mathcal{L}) &\longmapsto (C, \mathcal{L}^{\otimes 2g-2} \otimes \omega_C^{\otimes(-d)}). \end{aligned}$$

Indeed, over the locus of smooth curves, the pullback of $\mathsf{P}^1_{g,\varnothing,0}$ is given by

$$\begin{aligned} &-\tfrac{1}{2}\pi_*(c_1(\mathcal{L}^{\otimes 2g-2} \otimes \omega_C^{\otimes(-d)})^2) \\ &\qquad = -\tfrac{1}{2}((2g-2)^2\pi_*(c_1(\mathcal{L})^2) - 2d(2g-2)\pi_*(c_1(\mathcal{L})c_1(\omega_\pi)) + d^2\pi_*(c_1(\omega_\pi)^2)), \end{aligned}$$

which matches the definition of Ω given in [67, Theorem A] up to scalars.

In Gromov–Witten theory, pulling back (8.3) under the morphisms

$$\overline{\mathcal{M}}_{g,n}(X,\beta) \longrightarrow \mathfrak{Pic}_{g,n,d}$$

described in §8.3 and capping with the virtual class $[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{\mathsf{vir}}$ simply recovers the known vanishing

$$\mathsf{P}^{c,k}_{g,A,\beta}(X,\mathcal{L}) = 0 \in \mathsf{CH}_{\mathrm{vdim}(g,n,\beta)-c}(\overline{\mathcal{M}}_{g,n}(X,\beta)) \tag{8.5}$$

for $c > g$ proven in [4]. However, there are new applications for *reduced* Gromov–Witten theory. Indeed, for a target X having a non-degenerate holomorphic 2-form, the virtual class of $\overline{\mathcal{M}}_{g,n}(X,\beta)$ vanishes when $\beta \neq 0$. To define invariants for such targets, the reduced class

$$[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{\mathsf{red}} \in \mathsf{CH}_*(\overline{\mathcal{M}}_{g,n}(X,\beta))$$

is used instead; see [15], [58]. By pulling back (8.3) and capping with $[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{\mathsf{red}}$, we obtain new relations among reduced Gromov–Witten invariants. An application to the Gromov–Witten theory of K3 surfaces will appear in [5] related to conjectures of [61].

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