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Soliton resolution for the radial critical wave equation in all odd space dimensions

by

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1. Introduction

Consider the wave equation on $\mathbb{R}^N$, $N\geqslant 3$, with an energy-critical focusing non-linearity:

$$\partial_t^2 u - \Delta u = |u|^{4/(N-2)} u, \tag{1.1}$$

and initial data

$$\vec{u}\restriction_{t=0} = (u_0, u_1) \in \mathcal{H}, \tag{1.2}$$

where $\vec{u}:=(u,\partial_t u)$ and $\mathcal{H}:=\dot{H}^1(\mathbb{R}^N)\times L^2(\mathbb{R}^N)$. The equation is locally well posed in $\mathcal{H}$ (see e.g. [48], [43], [4]): for any initial data $(u_0,u_1)\in\mathcal{H}$, there exists a unique maximal solution $\vec{u}\in C^0((T_-,T_+),\mathcal{H})$. The energy

$$E(\vec{u}(t)) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla_{t,x} u(t,x)|^2\,dx - \frac{N-2}{2N}\int_{\mathbb{R}^N} |u(t,x)|^{2N/(N-2)}\,dx$$

of a solution is conserved, where

$$\nabla u = (\partial_{x_j} u)_{1\leqslant j\leqslant N} \quad \text{and} \quad \nabla_{t,x} u = (\partial_t u, \nabla u).$$

The equation (1.1) has the following scaling invariance. For $f\in\dot{H}^1(\mathbb{R}^N)$ and $\lambda>0$, we denote

$$f_{(\lambda)}(x) = \frac{1}{\lambda^{N/2-1}} f\left(\frac{x}{\lambda}\right).$$

If u is a solution of (1.1), then

$$\frac{1}{\lambda^{N/2-1}} u\left(\frac{t}{\lambda},\frac{x}{\lambda}\right) = u_{(\lambda)}\left(\frac{t}{\lambda}, x\right)$$

is also a solution.

As many other non-linear dispersive equations, equation (1.1) admits solitary waves (or solitons) that are well-localized solutions traveling at a fixed speed. The soliton resolution conjecture predicts that any global solution of this type of equations decouples asymptotically as a sum of decoupled solitons, a radiative term (typically, a solution to a linear equation) and a term going to zero in the energy space. For finite-time blow-up solutions, a similar decomposition should hold depending on the nature of the blow-up.

Our main result (Theorem 1 below) is the soliton resolution for equation (1.1), when N is odd and (u_0, u_1) is radial. To put this result into perspective, we start with a discussion on the soliton resolution conjecture for general non-linear dispersive equations.

This conjecture arose from numerical simulations and the theory of integrable systems. It was observed in 1955 by Fermi, Pasta and Ulam [28] in one of the first numerical experiments that a discretization of a wave equation with a quadratic non-linearity leads to localized, soliton-like solutions. In 1965, Zabusky and Kruskal [67] highlighted numerically the emergence of solitons and multisolitons solutions of the completely integrable KdV. This explained the result in [28], as Kruskal found that, as the spacial mesh in the discretization tends to zero, the solutions of the Fermi–Pasta–Ulam problem converge to solutions of the KdV equation; see [47]. We refer to [35] for a survey on numerical work.

The first theoretical results in the direction of the soliton resolution were obtained for the completely integrable KdV, mKdV and 1-dimensional cubic NLS, using the method of inverse scattering. Namely, for KdV, a solution with smooth initial data decaying sufficiently fast at infinity decomposes, for positive x, as a finite sum of solitons and a term going to zero at infinity (see [26], [25]). Note that this is only a partial result, due to the restriction on the initial data and also to the fact that the dispersive component, localized in $\{x<0\}$ is not completely described. We refer to [58] for mKdV, and [68], [60], [59], [52], [2] for cubic NLS in one space dimension. A characteristic feature of these integrable systems, already observed in [67], is that the collision between solitons is elastic: a solution behaving as a sum of solitons as $t\to+\infty$ also behaves as a sum of solitons, with the same parameters, as $t\to-\infty$.

Very few complete results are known for non-integrable models. A typical dispersive partial differential equation for which the soliton resolution is believed to hold unconditionally is the energy critical wave maps. For this equation, the first known related results were "bubble" theorems, stating that any solution developing a singularity in finite or infinite time converges locally in space, along a sequence of times, to a soliton (see [5], [63] for the equivariant case, [61] for the general case). Using the "channels of energy" method coming from our previous works [18], [21], that is closely related to the techniques that we will develop in this article, it was proved that the soliton resolution holds for wave maps, in an equivariant setting, with an additional assumption ruling out a multi-soliton configuration [7], [8], and that it holds for a sequence of times without this condition (see [6] and [39]). The limiting case of a pure 2-soliton is treated in [38], where it is shown in particular that the collision between the two solitons is inelastic.

For wave maps without symmetry assumption, but with the same S^2 target, a weak form of the soliton resolution was proved along a sequence of times (see [32]), and the complete resolution is only known close to the ground-state [17].

The proof of the soliton resolution conjecture seems out of reach for other non-integrable non-linear dispersive equations, such as non-linear Schrödinger and Klein–Gordon equations. Known results include scattering below a threshold given by the ground state of the equation (see e.g. [42], [15], [34], [12]), local study close to the ground state solution (see [51]), and in some particular cases the existence of a global compact attractor (see [65]). We refer to the introduction of [16] for a more complete discussion and more references on the subject.

Going back to equation (1.1), it is known that, if $\|(u_0,u_1)\|_{\mathcal{H}}$ is sufficiently small, then $T_+=\infty$ and the solution scatters to a linear solution. It is also well known that in general finite-energy solutions to equation (1.1) may blow up in finite time. Indeed, using the finite speed of propagation for equation (1.1) to localize ODE-type blow-up solutions, one can easily construct solutions $\vec{u}$ with $T_+<\infty$ and $\|\vec{u}(t)\|_{\mathcal{H}}\to\infty$ as $t\to T_+$. These solutions are called *type*-I blow-up solutions. It is expected that these solutions, after a self-similar change of variable, satisfy a decomposition similar to the soliton resolution. This type of result is only known in the 1-dimensional setting (see [50] and references therein) and very little is known in the energy-critical case (see [13] for a local study).

To rule out the ODE-type behavior, we will focus on solutions that are bounded in the energy space, i.e. such that

$$\sup_{t\in[0,T_+)} \|\vec{u}(t)\|_{\mathcal{H}}<\infty. \tag{1.3}$$

The dynamics of these solutions is very rich. Apart from the scattering solutions mentioned above, equation (1.1) admits also various types of finite-energy steady states $Q\in\dot{H}^1$, i.e.

$$-\Delta Q=|Q|^{4/(N-2)}Q \quad \text{in } \mathbb{R}^N \tag{1.4}$$

(see [11], [53], [54]). Among them, a distinguished role is played by the *ground state*

$$W:=\left(1+\frac{|x|^2}{N(N-2)}\right)^{1-N/2}, \tag{1.5}$$

which is, as a consequence of [55], [29], the unique $\dot{H}^1$ radial solution of (1.4) on $\mathbb{R}^N$, up to scaling and sign change, and the non-zero solution of (1.4) with least energy (see [64]).

Stationary solutions are not the only global, non-scattering solutions. It is indeed possible to construct solutions of the form

$$u(t,x) = W_{(\lambda(t)}(x) + v_L(t,x),$$

where v_L is a small solution of the free wave equation: see [44] (for $\lambda(t)=1$) and [14] ($\lambda(t)=t^\eta$, $|\eta|$ small). There also exist, at least in high space dimensions, global solutions that are asymptotically of the form $W + W_{(\lambda(t))}$, where $\lambda(t)$ goes to zero as t goes to infinity (see [37]).

There are also solutions blowing up in finite time that are bounded in the energy space. These solutions are called *type*-II blow-up solutions. In [46], [33], [45] and [36], type-II blow-up solutions of the form of a rescaled ground state plus a small dispersive term were constructed. More precisely, the solution is given by

$$u(t,x) = W_{(\lambda(t)}(x) + \varepsilon(t,x),$$

where

$$\frac{\lambda(t)}{T_+ - t} \to 0^+ \quad \text{as } t \to T_+,$$

and $\vec{\varepsilon}(t) = (\varepsilon, \partial_t \varepsilon)$ is small in the energy space. It is expected that multi-soliton concentration is also possible for type-II blow-up solutions, and it is an open problem to construct such a solution.

In the radial setting, W is the unique steady state, and thus the only soliton up to sign change and scaling. The soliton resolution conjecture predicts that any radial solution that does not blow up with a type-I blow-up decomposes asymptotically as a sum of $\pm W$, decoupled by time-dependent scalings, a radiation term and a term going to zero in the energy space. The radiation term should be a solution to the linear wave equation in the global case, and a fixed element of $\mathcal{H}$ in the finite-time blow-up case. We note that all the solutions mentioned above are in accordance with this conjecture. The resolution was proved in [21] by the authors, for $N=3$. For other dimensions (still in the radial case), soliton resolution is only known along a sequence of times; see [9], [57] and [39]. For the non-radial setting, for a sequence of times, see [16].

With the method of proof used in [9], [57], [39], [16], relying on monotonicity laws giving convergence only after averaging in time, we cannot hope for more than a decomposition for a particular sequence of times. The difficulty in obtaining the resolution for all times is illustrated by the harmonic map heat flow equation, for which the decomposition for a sequence of times is known, but the soliton resolution for all times does not hold in full generality because of an example of Topping [66].

The soliton resolution for radial solutions of (1.1) holds in full generality [21] when $N=3$. The key fact in the proof is the following dispersive estimate for radial non-zero solutions u of (1.1), with $(u_0,u_1)\neq(\pm W_{(\lambda)},0)$, $N=3$. Assume (for simplicity) that u exists globally in time. Then,

$$\sum_{\pm}\lim_{t\to\pm\infty}\int_{r\geqslant|t|}|\nabla_{t,x}u(t,x)|^2\,dx>0. \tag{1.6}$$

The proof of (1.6) relies fundamentally, among other things, on the following "energy channel" property of radial solution v of the linear wave equation in space dimension 3 (see [18], [21]). Let $R>0$,

$$P(R)=\left\{\left(\frac{a}{r},0\right):a\in\mathbb{R}\right\}\subset\mathcal{H}(R):=(\dot{H}^1\times L^2)(\{x\in\mathbb{R}^3:|x|>R\}).$$

Then,

$$\sum_{\pm}\lim_{t\to\pm\infty}\int_{|x|>R+|t|}|\partial_{t,r}v(t)|^2\,dx\geqslant\frac{1}{2}\|\Pi^{\perp}_{P(R)}(v_0,v_1)\|^2_{\mathcal{H}(R)}, \tag{1.7}$$

where $\Pi^{\perp}_{P(R)}$ denotes the orthogonal projection onto the orthogonal complement of $P(R)$ in $\mathcal{H}(R)$. The analogue of (1.7) for higher odd dimension was obtained in [41], [40], but the exceptional subspace $P(R)$ is replaced by a finite-dimensional subspace of $\mathcal{H}(R)$ with dimension increasing to infinity with N. The fact that the dimension of $P(R)$ is strictly greater than 1 for $N\geqslant5$ is responsible for the failure of this method, since we only have here the 1-parameter scaling invariance to deal with this failure, to start the proof. Let us mention however that it is possible, using (1.7), to prove that in odd space dimensions $N\geqslant5$, any radial solution of (1.1) that does not satisfy (1.6) is asymptotically close, for large r, to one of the elements of $P(R)$ (see [24]).

The radial solution u of (1.1) is said to be a *pure multi-soliton* (asymptotically as $t\to\pm\infty$) when there exist $J\geqslant2$ scaling parameters

$$0<\lambda_J(t)\ll\ldots\ll\lambda_2(t)\ll\lambda_1(t)$$

and signs $(\iota_j)_j\in\{\pm1\}^J$ such that

$$\vec{u}(t,x)=\sum_{j=1}^{J}(\iota_jW_{(\lambda_j(t))},0)+o(1),\quad t\to\pm\infty,$$

where $o(1)$ goes to zero in $\mathcal{H}$ (see [37] for an example of pure multi-soliton). If u is both a pure multi-soliton as $t\to+\infty$ and $t\to-\infty$, we say that the collision between the solitons is *elastic*. In space dimension 3, the fact that (1.6) is valid for any non-stationary solution u rules out elastic collisions, in stark contrast to the integrable case [35]. One of the main results in this work is a slightly weaker form of (1.6), namely that a radial solution u of (1.1) that stays close to a sum of decoupled solitons for a sufficiently long time must satisfy

$$\sum_{\pm} \lim_{t\to\pm\infty} \int_{r\geqslant |t|-M} |\nabla_{t,x} u(t,x)|^2 \, dx > 0, \tag{1.8}$$

for some large $M>0$ (see Propositions 5.1 and 6.1). As a consequence, there does not exist a radial solution of (1.1) that is a pure multi-soliton both as $t\to\infty$ and $t\to-\infty$, i.e. the collision of solitons is inelastic for (1.1), when N is odd. The proof of this property depends heavily on the "energy channels" property for the linearized wave equation

$$\left(\partial_t^2 - \Delta - \frac{N+2}{N-2} W^{4/(N-2)}\right) u = 0,$$

established in [23]. The M in (1.8) is needed to eliminate the extra dimensions arising from P in (1.7), when $N\geqslant 5$, odd.

The main result in this work, namely full resolution for (1.1), N odd, in the radial case (assuming bounded energy norm) combines the sequence of times result in [57] with a strengthened version of (1.8). The result of [57] allows us to reduce ourselves to studying the dynamics close to a sum of solitons plus a dispersive term, and the strengthened version of (1.8) allows us to take advantage of the fact that the collision of two or more solitons produce dispersion, which then gives the full decomposition. We view this as a "road map" to attack soliton resolution in non-integrable settings.

We now turn to the main results of this paper. If a and b are integers with $a<b$, we denote $[\![a,b]\!]=[a,b]\cap\mathbb{N}$.

THEOREM 1. *Assume that $N\geqslant 5$ is odd. Let u be a radial solution of (1.1), with maximal time of existence T_+, such that*

$$\sup_{0\leqslant t<T_+} \|\vec{u}(t)\|_{\mathcal{H}} < \infty. \tag{1.9}$$

Then, there exist $J\geqslant 0$, signs $(\iota_j)_j\in\{\pm1\}^J$ and scaling parameters $(\lambda_j)_j\in(0,\infty)^J$ such that

$$\lim_{t\to T_+} \frac{\lambda_j(t)}{\lambda_{j+1}(t)} = +\infty \quad \textit{for all } j\in[\![1,J-1]\!]$$

and the following statements hold.

- *(Type-II blow-up case) If $T_+<\infty$, then $J\geqslant 1$ and there exists $(v_0,v_1)\in\mathcal{H}$ such that*

$$\lim_{t\to T_+} \left\| \vec{u}(t) - (v_0,v_1) - \sum_{j=1}^{J} (\iota_j W_{(\lambda_j)}, 0) \right\|_{\mathcal{H}} = 0.$$

Furthermore,

$$\lim_{t\to T_+}\frac{\lambda_1(t)}{T_+-t}=0.$$

• (*Global in time case*) *If* $T_+=+\infty$, *then there exists a solution* v_L *of the linear wave equation such that*

$$\lim_{t\to+\infty}\left\|\vec{u}(t)-\vec{v}_L(t)-\sum_{j=1}^{J}(\iota_j W_{(\lambda_j)},0)\right\|_{\mathcal{H}}=0.$$

Furthermore, if $J\geqslant 1$,

$$\lim_{t\to+\infty}\frac{\lambda_1(t)}{t}=0.$$

As mentioned above, the proof of Theorem 1 yields the fact that the collision between radial solitons in odd space dimension $N\geqslant 5$ is inelastic.

THEOREM 2. *Assume that* $N\geqslant 5$ *is odd. Let* u *be a radial, global solution of* (1.1) *such that*

$$\sup_{t\in\mathbb{R}}\|\vec{u}(t)\|_{\mathcal{H}}<\infty$$

and

$$\sum_{\pm\infty}\lim_{t\to\pm\infty}\int_{|x|>|t|-A}|\nabla_{t,x}u(t,x)|^2\,dx=0 \quad \text{for all } A>0. \tag{1.10}$$

Then, $(u_0,u_1)=(0,0)$ *or there exist* $\lambda>0$ *and* $\iota\in\{\pm1\}$ *such that* $(u_0,u_1)=(\iota W_{(\lambda)},0)$.

Note that (1.10) exactly means that the linear component v_L is identically zero both as $t\to+\infty$ and $t\to-\infty$, so that Theorem 2 rules out being asymptotically a multi-soliton at both $t=+\infty$ and $t=-\infty$, in stark contrast to the completely integrable case [35]. For results on inelastic soliton collisions for equation (1.1) without a radiality assumption, see [49].

The outline of the paper is as follows. The preliminary §2 is mainly devoted to the Cauchy theory for equation (1.1). We recall well-posedness results from [30], [31], [43], and more particularly from [4], where the high-dimensional case is treated. Using finite speed of propagation, we also recall a local and global Cauchy theory for the equation (1.1) in the exterior of a wave cone $\{|x|>R+|t|\}$, $R\geqslant 0$, as developed in [24]. §3 concerns the bound from below of the exterior energy for linear equations with a potential. After recalling the main result of [23], we state and prove an exterior energy bound for the linearized operator close to a multi-soliton. In §4 we consider solutions of the equation (1.1) such that

$$\lim_{t\to+\infty}\int_{|x|>|t|+R}|\nabla_{t,x}u|^2\,dx=0,$$

for some fixed $R>0$. We recall from [24] that the initial data of these solutions (that we call non-radiative solutions) have a prescribed asymptotic behaviour. We also consider the case of non-radiative solutions that are close to a multi-soliton, proving a bound from below of the exterior scaling parameter λ_1. In §5, we reduce the proof of the soliton resolution to the study of a finite-dimensional dynamical system on the scaling parameters λ_j and some of the coefficients arising in the expansion of the solution. Finally, in §6, we prove a blow-up/ejection result for this dynamical system and conclude the proof. §7 is dedicated to a short sketch of the proof of Theorem 2, which is a byproduct of part of the proof of Theorem 1. A few computations are gathered in the appendix.

2. Preliminaries

2.1. Notation

We denote $\dot{H}^1=\dot{H}^1(\mathbb{R}^N)$, $L^2=L^2(\mathbb{R}^N)$, $\mathcal{H}=\dot{H}^1\times L^2$. If $\lambda>0$, $f\in\dot{H}^1$ and $g\in L^2$, we let

$$f_{(\lambda)}(x)=\frac{1}{\lambda^{N/2-1}}f\left(\frac{x}{\lambda}\right) \quad \text{and} \quad g_{[\lambda]}(x)=\frac{1}{\lambda^{N/2}}g\left(\frac{x}{\lambda}\right),$$

so that

$$\|f_{(\lambda)}\|_{\dot{H}^1}=\|f\|_{\dot{H}^1} \quad \text{and} \quad \|g_{[\lambda]}\|_{L^2}=\|g\|_{L^2}.$$

If A is a space of distributions on $\mathbb{R}^N$, we will denote by A_{rad} the subspace of A consisting of the elements of A that are radial. We will, without making a distinction, consider a radial function as depending on the variable $x\in\mathbb{R}^N$ or the variable $r=|x|$.

If Ω is an open subset of $\mathbb{R}^n$ ($n=N$ or $n=N+1$), and $A=A(\mathbb{R}^n)$ is a Banach space of distributions on $\mathbb{R}^n$, we recall that $A(\Omega)$ is the set of restrictions of elements of A to Ω, with the norm

$$\|u\|_{A(\Omega)}:=\inf_{\tilde{u}}\|\tilde{u}\|_{A(\mathbb{R}^n)},$$

where the infimum is taken over all $\tilde{u}\in A(\mathbb{R}^n)$ such that $\tilde{u}\restriction_{\Omega}=u$. To lighten notation, if $R>0$ and $n=N$, we will set

$$A(R):=A_{\mathrm{rad}}(\{x\in\mathbb{R}^N:|x|>R\}).$$

We will mainly use this notation with $\mathcal{H}$, so that $\mathcal{H}(R)$ is the space of radial distributions (u_0,u_1) defined for $r>R$ such that

$$u_0\in L^{2N/(N-2)}((R,+\infty),r^{N-1}dr), \qquad \int_R^{\infty}(\partial_r u_0)^2 r^{N-1}\,dr<\infty$$

and

$$u_1\in L^2((R,+\infty),r^{N-1}\,dr).$$

We will often consider solutions of the wave equation in the exterior of wave cones. For $R>0$, we denote

$$\Gamma_R(t_0,t_1)=\{|x|>R+|t|: t\in[t_0,t_1]\}.$$

To lighten notation, we will denote

$$\Gamma_R(T)=\Gamma_R(0,T) \quad \text{and} \quad \Gamma_R=\Gamma_R(0,\infty).$$

We denote by $S_L(t)$ the linear wave group:

$$S_L(t)(u_0,u_1)=\cos(t\sqrt{-\Delta}\,)u_0+\frac{\sin(t\sqrt{-\Delta}\,)}{\sqrt{-\Delta}}u_1, \tag{2.1}$$

so that the general solution (in the Duhamel sense) of

$$\begin{cases} (\partial_t^2-\Delta)u=f \\ \vec{u}\restriction_{t=t_0}=(u_0,u_1)\in\mathcal{H}, \end{cases} \tag{2.2}$$

where I is an interval and $t_0\in I$, is

$$u(t)=S_L(t-t_0)(u_0,u_1)+\int_{t_0}^t S_L(t-s)(0,f(s))\,ds. \tag{2.3}$$

We note that, by finite speed of propagation, the restriction of u to $\Gamma_R(T)$ depends only on the restriction of f to $\Gamma_R(T)$ and the restriction of (u_0,u_1) to $\{r>R\}$.

2.2. Local and global Cauchy theory

We will denote by $\dot{W}^{s,p}(\mathbb{R}^N)$ the homogeneous Sobolev space defined as the closure of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm $\|\cdot\|_{\dot{W}^{s,p}}$ defined by

$$\|f\|_{\dot{W}^{s,p}}:=\|D^s f\|_{L^p},$$

where D^s is the Fourier multiplier of symbol $|\xi|^s$. We denote by $\dot{B}^s_{p,q}$ the standard homogeneous Besov space, which can be defined using Littlewood–Paley decomposition or the real interpolation method: $\dot{B}^s_{p,q}=[L^p,\dot{W}^{1,p}]_{s,q}$, $0<s<1$, $1\leqslant p,q\leqslant\infty$.

Following [4], we define

$$\begin{aligned} \mathsf{S}&:=L^{2(N+1)/(N-2)}(\mathbb{R}^{1+N}),\\ \mathsf{W}&:=L^{2(N+1)/(N-1)}(\mathbb{R},\dot{B}^{1/2}_{2(N+1)/(N-1),2}(\mathbb{R}^N)),\\ \mathsf{W}'&:=L^{2(N+1)/(N+3)}(\mathbb{R},\dot{B}^{1/2}_{2(N+1)/(N+3),2}(\mathbb{R}^N)). \end{aligned}$$

If I is an interval, we will denote by $\mathsf{S}(I)$, $\mathsf{W}(I)$ and $\mathsf{W}'(I)$ the restriction of these spaces, respectively, to $I\times\mathbb{R}^N$.

We will need the following Strichartz estimates (see [62] [31]): if $t_0\in I$, $f\in\mathsf{W}'(I)$ and $(u_0,u_1)\in\mathcal{H}$, then u (defined by (2.3)) is in $\mathsf{S}(I)\cap\mathsf{W}(I)$ and

$$\sup_{t\in\mathbb{R}}\|\vec{u}(t)\|_{\mathcal{H}}+\|u\|_{\mathsf{S}(I)}+\|u\|_{\mathsf{W}(I)}\lesssim\|(u_0,u_1)\|_{\mathcal{H}(I)}+\|f\|_{\mathsf{W}'(I)}. \tag{2.4}$$

We denote $F(u)=|u|^{4/(N-2)}u$.

Definition 2.1. Let I be an interval with $t_0\in I$, $(u_0,u_1)\in\mathcal{H}$. If $N\geqslant 6$, we call solution of (1.1) on $I\times\mathbb{R}^N$ with initial data

$$\vec{u}\restriction_{t=t_0}=(u_0,u_1) \tag{2.5}$$

a function $u\in C^0(I,\dot{H}^1)$ such that $\partial_t u\in C^0(I,L^2)$ and

$$u(t)=S_L(t-t_0)(u_0,u_1)+\int_{t_0}^{t}S_L(s-t_0)F(u(s))\,ds\quad\text{for all } t\in I. \tag{2.6}$$

If $N\in\{3,4,5\}$, a solution is defined in the same way, with the additional requirement that $u\in\mathsf{S}(J\times\mathbb{R}^N)$ for all compact intervals $J\subset I$.

It is known (see [31], [43] and [4]) that, for all initial data (u_0,u_1), there is a unique maximal solution u defined on a maximal interval (T_-,T_+) and that satisfies the following blow-up criterion:

$$T_+<\infty\quad\Longrightarrow\quad\|u\|_{\mathsf{S}([t_0,T_+))}=\infty.$$

We next recall from [24] the definition and some properties of solutions of (1.1) on the exterior $\Gamma_R(t_0,t_1)$ of wave cones. We will use the following continuity property of multiplication by characteristic functions on a Besov space (see [24, Lemma 2.3]).

LEMMA 2.2. *Let $R\geqslant 0$.*

- *The multiplication by the characteristic function $\mathbb{1}_{\{|x|>R\}}$ is a continuous function from $\dot{B}^{1/2}_{2(N+1)/(N+3),2}(\mathbb{R}^N)$ into itself, and from $\dot{W}^{2/N,2(N+1)/(N+3)}(\mathbb{R}^N)$ into itself. In both cases, the operator norm is independent of R.*
- *Let I be an interval. The multiplication by the characteristic function $\mathbb{1}_{\{|x|>R+|t|\}}$ is continuous from $\mathsf{W}'(I)$ into itself. The operator norm is independent of R and I.*

We also recall the following chain rule for fractional derivative outside wave cones:

$$\|\mathbb{1}_{\Gamma_R(T)}F(u)\|_{\mathsf{W}'((0,T))}\lesssim\|u\|^{4/(N-2)}_{\mathsf{S}(\Gamma_R(T))}\|u\|_{\mathsf{W}((0,T))}, \tag{2.7}$$

which is proved in [24] as a consequence of Lemma 2.2, Hölder's inequality and the usual chain rule for fractional derivative ([4, Lemma 2.10]).

Definition 2.3. Let $t_0<t_1$, $R\geqslant 0$. Let $(u_0,u_1)\in\mathcal{H}(R)$. A solution u of (1.1) on $\Gamma_R(t_0,t_1)$ with initial data (u_0,u_1) is the restriction to $\Gamma_R(t_0,t_1)$ of a solution $\tilde{u}\in C^0([t_0,t_1],\dot{H}^1)$, with $\partial_t\tilde{u}\in C^0([t_0,t_1],L^2)$, to the equation

$$\partial_t^2\tilde{u}-\Delta\tilde{u}=|\tilde{u}|^{4/(N-2)}\tilde{u}1\!\!1_{\{|x|>R+|t|\}}, \tag{2.8}$$

with an initial data

$$\vec{\tilde{u}}\restriction_{t=t_0}=(\tilde{u}_0,\tilde{u}_1), \tag{2.9}$$

where $(\tilde{u}_0,\tilde{u}_1)\in\mathcal{H}$ is an extension of (u_0,u_1)

Note that, by finite speed of propagation, the value of u on $\Gamma_R(t_0,t_1)$ does not depend on the choice of $(\tilde{u}_0,\tilde{u}_1)$, provided $(\tilde{u}_0,\tilde{u}_1)$ and (u_0,u_1) coincide for $r>R$.

Using Lemma 2.2 and finite speed of propagation, the Cauchy theory in [4] (or [43] for the case $N\in\{3,4,5\}$) adapts easily to the case of solutions outside wave cones. We give some of the statements, and omit the proofs that are the same as in [43], [4]. We refer to [24, §2] for a more complete exposition. The space $\mathsf{S}(\Gamma_R(T))$ in the following proposition is defined in §2.1.

PROPOSITION 2.4. (Local well-posedness) *Let $R\geqslant 0$, $(u_0,u_1)\in\mathcal{H}(R)$ and $T>0$. Assume*

$$\|(u_0,u_1)\|_{\mathcal{H}(R)}\leqslant A.$$

Then, there exists $\eta=\eta(A)$ such that, if

$$\|S_L(t)(u_0,u_1)\|_{\mathsf{S}(\Gamma_R(T))}<\eta,$$

then there exists a unique solution u to (1.1) on $\Gamma_R(T)$. Furthermore, for all $t\in[0,T]$,

$$\|\vec{u}(t)-\vec{S}_L(t)(u_0,u_1)\|_{\mathcal{H}(R+|t|)}\leqslant C\eta^{\theta_N}A^{1-\theta_N}$$

for some constant θ_N depending only on N.

(See [4, Theorem 3.3].) We have the following blow-up criterion (see [24, Lemma 2.8]): if $u\in S(\Gamma_R(T_R^+))$, then $T_R^+=+\infty$. Furthermore, u scatters to a linear solution for

$$\{|x|>R+|t|\},$$

that is, there exists a solution v_L of the linear wave equation on $\mathbb{R}\times\mathbb{R}^N$ such that

$$\lim_{t\to+\infty}\|\vec{u}(t)-\vec{v}_L(t)\|_{\mathcal{H}(R+|t|)}=0. \tag{2.10}$$

We also have the following long-time perturbation theory result (see [43, Theorem 2.20], [4, Theorem 3.6] and [57, Proposition A.1]).

PROPOSITION 2.5. *Let $A>0$. There exists $\eta_0=\eta_0(A)$ with the following property. Let $R>0$, $T\in(0,\infty]$, $(u_0,u_1)\in\mathcal{H}(R)$ and $(v_0,v_1)\in\mathcal{H}(R)$. Assume that v is a restriction to $\Gamma_R(0,T)$ of a function V such that $\vec{V}\in C^0([0,T],\mathcal{H})$ and*

$$\partial_t^2 V-\Delta V=\mathbb{1}_{\{|x|>R+|t|\}}(F(V)+e_1+e_2),$$

with

$$\sup_{0\leqslant t\leqslant T}\|V(t)\|_{\mathcal{H}(R+|t|)}+\|V\|_{\mathsf{W}(0,T)}\leqslant A$$

and

$$\|(u_0,u_1)-(v_0,v_1)\|_{\mathcal{H}(R)}+\|e_1\|_{\mathsf{W}'(0,T)}+\|e_2\|_{L^1((0,T),L^2)}=\eta\leqslant\eta_0.$$

Then, the solution with initial data (u_0,u_1) is defined on $\Gamma_R(T)$ and

$$\|v-u\|_{\mathsf{S}(\Gamma_R(T))}\leqslant C\eta^{c_N},$$

for some constant $c_N\in(0,1]$ depending only on $N\geqslant 3$.

Remark 2.6. In [43], [4] and [57] we have $e_2=0$, but the argument easily adapts to the setting of Proposition 2.5.

2.3. Profile decomposition

Let $\{(u_{0,n},u_{1,n})\}_n$ be a bounded sequence of radial functions in $\mathcal{H}$. We say that it admits a profile decomposition if for all $j\geqslant 1$, there exist a solution U_F^j to the free wave equation with initial data in $\mathcal{H}$ and sequences of parameters $\{\lambda_{j,n}\}_n\in(0,\infty)^{\mathbb{N}}$ and $\{t_{j,n}\}_n\in\mathbb{R}^{\mathbb{N}}$ such that, for $j\neq k$,

$$\lim_{n\to\infty}\left(\frac{\lambda_{j,n}}{\lambda_{k,n}}+\frac{\lambda_{k,n}}{\lambda_{j,n}}+\frac{|t_{j,n}-t_{k,n}|}{\lambda_{j,n}}\right)=+\infty, \tag{2.11}$$

and, denoting

$$U_{F,n}^j(t,r)=\frac{1}{\lambda_{j,n}^{N/2-1}}U_F^j\left(\frac{t-t_{j,n}}{\lambda_{j,n}},\frac{r}{\lambda_{j,n}}\right),\qquad j\geqslant 1 \tag{2.12}$$

$$w_n^J(t)=S_L(t)(u_{0,n},u_{1,n})-\sum_{j=1}^J U_{F,n}^j(t), \tag{2.13}$$

one has

$$\lim_{J\to\infty}\limsup_{n\to\infty}\|w_n^J\|_{\mathsf{S}(\mathbb{R})}=0. \tag{2.14}$$

We recall (see [1], [3]) that any bounded sequence in $\mathcal{H}$ has a subsequence that admits a profile decomposition. We recall also that the properties above imply that the following weak convergences hold, for $j\leqslant J$:

$$(\lambda_{j,n}^{N/2-1}w_n^J(t_{j,n},\lambda_{j,n}\cdot),\lambda_{j,n}^{N/2}\partial_t w_n^J(t_{j,n},\lambda_{j,n}\cdot))\xrightarrow[n\to\infty]{}0 \quad \text{in } \mathcal{H}. \tag{2.15}$$

If $\{(u_{0,n},u_{1,n})\}_n$ admits a profile decomposition, we may assume, extracting subsequences and time-translating the profiles if necessary, that the following limit exists:

$$\lim_{n\to\infty}\frac{-t_{j,n}}{\lambda_{j,n}}=\tau_j\in\{-\infty,0,+\infty\}.$$

Using the existence of wave operator for the equation (1.1) if $\tau_j\in\{\pm\infty\}$, or the local well-posedness if $\tau_j=0$, we define the non-linear profile U^j associated with

$$(U_F^j,\{\lambda_{j,n}\}_n,\{t_{j,n}\}_n)$$

as the unique solution to the non-linear wave equation (1.1) such that

$$\lim_{t\to\tau_j}\|\vec{U}^j(t)-\vec{U}_F^j(t)\|_{\mathcal{H}}=0.$$

We also denote by U_n^j the rescaled non-linear profile:

$$U_n^j(t,r)=\frac{1}{\lambda_{j,n}^{N/2-1}}U^j\left(\frac{t-t_{j,n}}{\lambda_{j,n}},\frac{r}{\lambda_{j,n}}\right).$$

Then, we have the following superposition principle outside the wave cone

$$\Gamma_0:=\{(t,x)\in\mathbb{R}\times\mathbb{R}^N:|x|>t>0\}.$$

PROPOSITION 2.7. *Let $\{(u_{0,n},u_{1,n})\}_n$ be a bounded sequence in $\mathcal{H}_{\mathrm{rad}}$. Assume that, for all j such that $\tau_j=0$, the non-linear profile U^j can be extended to a solution on Γ_0 (in the sense of Definition 2.3) such that $U^j\in\mathsf{S}(\Gamma_0)$. Then, for large n, there is a solution u_n defined on Γ_0 with initial data $\{(u_{0,n},u_{1,n})\}_n$ at $t=0$. Furthermore, setting, for $J\geqslant 1$ and $(t,r)\in\Gamma_0$,*

$$R_n^J(t,r)=u_n(t,r)-\sum_{j=1}^J U_n^j(t,r)-w_n^J(t,r),$$

we have

$$\lim_{J\to\infty}\lim_{n\to\infty}\left(\|R_n^J\|_{\mathsf{S}(\Gamma_0)}+\sup_{t\geqslant 0}\|\vec{R}_n^J(t)\|_{\mathcal{H}(t)}\right)=0.$$

We omit the proof, which is similar to the proof when the solution is not restricted to the exterior of a wave cone (see [57, Proposition 2.3]). Let us emphasize the fact that, under the assumptions of Proposition 2.7, the profiles $U_n^j(t,r)$ are well defined on Γ_0, so that the conclusion of the proposition makes sense. If $\tau_j=0$, this follows from the assumption that U^j is defined on Γ_0 and, if $\tau_j=+\infty$, from the fact that U^j is globally defined in the future. Finally, if $\tau_j=-\infty$, it follows from the fact that U^j is globally defined in the past, and also, using small data theory, defined on a cone $\Gamma_R(T,+\infty)$ where T is fixed in the interval of existence of U^j and R is large (see also the discussion after Proposition 2.11 in [24]).

2.4. Wave equation with a potential outside a wave cone

LEMMA 2.8. *Let $N\geqslant 3$ and $M\in(0,\infty)$. There exists $C_M>0$ such that, for all*

$$V\in L^{2(N+1)/(N+4)}_{\mathrm{loc}}(\mathbb{R},L^{2(N+1)/3}(\mathbb{R}^N))$$

with

$$\|\mathbb{1}_{\{|x|\geqslant|t|\}}V\|_{L^{2(N+1)/(N+4)}(\mathbb{R},L^{2(N+1)/3}(\mathbb{R}^N))}\leqslant M, \tag{2.16}$$

and for all solutions u of

$$\partial_t^2u-\Delta u+Vu=f_1+f_2,\quad \vec{u}\restriction_{t=0}=(u_0,u_1)\in\mathcal{H}, \tag{2.17}$$

where $f_1\in L^1(\mathbb{R},L^2(\mathbb{R}^N))$ and $f_2\in\mathsf{W}'$, one has

$$\begin{aligned}&\|u\mathbb{1}_{\{|x|\geqslant|t|\}}\|_{L^{2(N+1)/(N-2)}(\mathbb{R}\times\mathbb{R}^N)}+\sup_{t\in\mathbb{R}}\|\mathbb{1}_{\{|x|\geqslant|t|\}}\nabla_{t,x}u(t)\|_{L^2}\\&\qquad\leqslant C_M(\|(u_0,u_1)\|_{\mathcal{H}}+\|\mathbb{1}_{\{|x|\geqslant|t|\}}f_1\|_{L^1(\mathbb{R},L^2)}+\|\mathbb{1}_{\{|x|\geqslant|t|\}}f_2\|_{\mathsf{W}'}).\end{aligned} \tag{2.18}$$

If $N\in\{3,4,5\}$, one also has

$$\begin{aligned}&\|u\mathbb{1}_{\{|x|\geqslant|t|\}}\|_{L^{(N+2)/(N-2)}(\mathbb{R},L^{2(N+2)/(N-2)})}\\&\qquad\leqslant C_M(\|(u_0,u_1)\|_{\mathcal{H}}+\|\mathbb{1}_{\{|x|\geqslant|t|\}}f_1\|_{L^1(\mathbb{R},L^2)}+\|\mathbb{1}_{\{|x|\geqslant|t|\}}f_2\|_{\mathsf{W}'}).\end{aligned} \tag{2.19}$$

Finally, there exists $g\in L^2([0,+\infty))$ such that

$$\lim_{t\to\infty}\int_t^{+\infty}|r^{(N-1)/2}\partial_ru(t,r)-g(r-t)|^2\,dr=0. \tag{2.20}$$

Proof. By Strichartz inequality, for all $T>0$,

$$\|1\!\!1_{\{|x|\geqslant|t|\}}u\|_{L^{2(N+1)/(N-2)}([0,T]\times\mathbb{R}^N)} \lesssim \|(u_0,u_1)\|_{\mathcal{H}}+\|1\!\!1_{\{|x|\geqslant|t|\}}f_1\|_{L^1(\mathbb{R},L^2)}+\|1\!\!1_{\{|x|\geqslant|t|\}}f_2\|_{\mathsf{W}'}+\|1\!\!1_{\{|x|\geqslant|t|\}}Vu\|_{L^1((0,T),L^2)}.$$

Using Hölder inequality in the space variable, we deduce that

$$\begin{aligned}\|1\!\!1_{\{|x|\geqslant|t|\}}u\|_{L^{2(N+1)/(N-2)}([0,T]\times\mathbb{R}^N)} &\lesssim \|(u_0,u_1)\|_{\mathcal{H}}+\|1\!\!1_{\{|x|\geqslant|t|\}}f_1\|_{L^1(\mathbb{R},L^2)}+\|1\!\!1_{\{|x|\geqslant|t|\}}f_2\|_{\mathsf{W}'}\\ &\quad+\int_0^T\|1\!\!1_{\{|x|\geqslant|t|\}}V\|_{L^{2(N+1)/3}}\,\|1\!\!1_{\{|x|\geqslant|t|\}}u\|_{L^{2(N+1)/(N-2)}}\,dt,\end{aligned}$$

and thus, using a Grönwall-type lemma ([27, Lemma 8.1]), we obtain

$$\|1\!\!1_{\{|x|\geqslant|t|\}}u\|_{L^{2(N+1)/(N-2)}([0,T]\times\mathbb{R}^N)} \leqslant C_M(\|(u_0,u_1)\|_{\mathcal{H}}+\|1\!\!1_{\{|x|\geqslant|t|\}}f_1\|_{L^1(\mathbb{R},L^2)}+\|1\!\!1_{\{|x|\geqslant|t|\}}f_2\|_{\mathsf{W}'}).$$

Using Strichartz and Hölder's inequalities again, we deduce the rest of (2.18) and (2.19).

By an argument similar to the one in the proof of [24, Lemma 2.8], one can prove that there exists a solution u_F of the free wave equation such that

$$\lim_{t\to+\infty}\int_{|x|>|t|}|\nabla_{t,x}(u-u_F)|^2\,dx=0.$$

Since there exists $g\in L^2(\mathbb{R})$ such that

$$\lim_{t\to+\infty}\int_0^{+\infty}|r^{(N-1)/2}\partial_r u_F(t,r)-g(r-t)|^2\,dr=0,$$

(see e.g. the appendix of [22]), property (2.20) follows. □

3. Channels of energy for the linearized operator close to a multi-soliton

This section is devoted to the proof of an exterior energy bound, stated in §3.2, for the equation (1.1) linearized around a multi-soliton. We start (see §3.1) by recalling previous results obtained in [20] and [23], on exterior energy bounds for the free wave equation and the linearized equation around a single soliton.

3.1. Channels of energy for the free and the linearized wave equations

In [20], we have obtained the following exterior energy lower bound for solutions of the free wave equation.

THEOREM 3.1. *Assume N is odd. Let u_F be a solution of the free wave equation*

$$\partial_t^2 u_F - \Delta u_F = 0 \tag{3.1}$$

with initial data in $(u_0, u_1) \in \mathcal{H}$. Then,

$$\sum_{\pm} \lim_{t\to\pm\infty} \int_{|x|>|t|} |\nabla_{t,x} u_F|^2 \, dx = \|(u_0, u_1)\|_{\mathcal{H}}^2.$$

Let W be the ground-state stationary solution of (1.1), given by (1.5). Consider the linearized equation:

$$\partial_t^2 u + L_W u = 0, \tag{3.2}$$

where L_W is the linearized operator:

$$L_W = -\Delta - \frac{N+2}{N-2} W^{4/(N-2)}. \tag{3.3}$$

The existence and uniqueness of solutions of (3.2) with initial data in $\mathcal{H}$ can be easily proved by standard semi-group theory. In [23], we have proved an analogue of Theorem 3.1 for solutions of (3.2) that we will now describe. To lighten notation, we will restrict to radial functions in space dimension $N \geqslant 5$. Let

$$\Lambda W := x \cdot \nabla W + \left(\tfrac{1}{2} N - 1\right) W.$$

Then,

$$\operatorname{span}\{\Lambda W\} = \{Z \in \dot{H}^1_{\mathrm{rad}} : L_W Z = 0\}.$$

Indeed, the inclusion $\subset$ is due to the fact that (1.1) is invariant by scaling. The other inclusion is a well-known non-degeneracy property of W (see e.g. [56]). Note that $\Lambda W \in L^2$ since $N \geqslant 5$. Let

$$\boldsymbol{\mathcal{Z}} := \operatorname{span}\{\Lambda W\} \times \operatorname{span}\{\Lambda W\}.$$

If $(u_0, u_1) \in \boldsymbol{\mathcal{Z}}$, then the solution u of (3.2) with initial data (u_0, u_1) is given by

$$u(t, x) = u_0(x) + t u_1(x),$$

and in particular

$$\sum_{\pm} \lim_{t\to\pm\infty} \int_{|x|\geqslant|t|} |\nabla_{t,x} u(t, x)|^2 = 0.$$

If V is a closed subspace of $\mathcal{H}$, we denote by $V^\perp$ its orthogonal in $\mathcal{H}$, and π_V the orthogonal projection on V. Theorem 1 of [23] states that the solutions with initial data in $\boldsymbol{\mathcal{Z}}$ are the only solutions that do not satisfy an exterior energy lower bound.

THEOREM 3.2. *Assume $N\geqslant 5$ is odd. Then, there exists a constant $C>0$ such that, for all $(u_0,u_1)\in\mathcal{H}_{\text{rad}}$,*

$$\|\pi_{\mathcal{Z}^\perp}(u_0,u_1)\|^2\leqslant C\sum_{\pm}\lim_{t\to\pm\infty}\int_{|x|\geqslant|t|}|\nabla_{t,x}u(t,x)|^2, \tag{3.4}$$

where u is the solution of (3.2) with initial data (u_0,u_1).

Theorem 1 of [23] is indeed more general: it holds without the assumption that (u_0,u_1) is radial, and also in space dimension $N=3$, with a suitable definition of $\mathcal{Z}$. We refer to [23] for the details.

3.2. Bound from below of the exterior energy close to a multi-soliton

As a corollary of Theorem 3.2, we will prove an exterior energy lower bound for the linearized operator close to an approximate radial multi-soliton solution. We will consider only radial solutions, and fix $J\geqslant 2$.

We denote by G_J the following subset of $(0,\infty)^J$:

$$G_J=\{\boldsymbol{\lambda}=(\lambda_j)_{1\leqslant j\leqslant J},\quad 0<\lambda_J<\lambda_{J-1}<\ldots<\lambda_1\}. \tag{3.5}$$

If $\boldsymbol{\lambda}\in G_J$, we set

$$\gamma(\boldsymbol{\lambda})=\max_{2\leqslant j\leqslant J}\frac{\lambda_j}{\lambda_{j-1}}\in(0,1), \tag{3.6}$$

$$L_{\boldsymbol{\lambda}}=-\Delta-\sum_{j=1}^J\frac{N+2}{N-2}W_{(\lambda_j)}^{4/(N-2)}, \tag{3.7}$$

$$Z_{\boldsymbol{\lambda}}=\operatorname{span}\{((\Lambda W)_{(\lambda_j)},0),(0,(\Lambda W)_{[\lambda_j]})\} \tag{3.8}$$

(see §2.1 for the notation $(\Lambda W)_{(\lambda_j)}$ and $(\Lambda W)_{[\lambda_j]}$). Then, we have the following.

COROLLARY 3.3. *Assume $N\geqslant 5$ is odd. For any $J\geqslant 2$, there exist $\gamma_*>0$ and $C>0$ with the following property. For any $\boldsymbol{\lambda}$ with $\gamma(\boldsymbol{\lambda})\leqslant\gamma_*$ and for any solution u of*

$$\partial_t^2u+L_{\boldsymbol{\lambda}}u=0,\quad \vec{u}\restriction_{t=0}=(u_0,u_1)\in\mathcal{H}, \tag{3.9}$$

one has

$$\|\pi_{Z_{\boldsymbol{\lambda}}^\perp}(u_0,u_1)\|_{\mathcal{H}}^2 \leqslant C\biggl(\sum_{\pm}\lim_{t\to\pm\infty}\int_{|x|\geqslant|t|}|\nabla_{t,x}u(t,x)|^2\,dx+\gamma(\boldsymbol{\lambda})^{2\theta_N}\|\pi_{Z_{\boldsymbol{\lambda}}}(u_0,u_1)\|_{\mathcal{H}}^2\biggr), \tag{3.10}$$

where $\theta_5=\frac{1}{2}$, $\theta_7=\frac{3}{2}$ and $\theta_N=2$ if $N\geqslant 9$.

Corollary 3.3 also has a version for $N=3$, that we will not need here. We skip it for the sake of simplicity. We prove Corollary 3.3 in the next subsection. In §4.2, we will apply this corollary to a solution of (1.1) close to a multi-soliton manifold.

3.3. Proof of the exterior energy lower bound for the linearized equation

We prove Corollary 3.3 by contradiction. For this, we assume that there exists a sequence $\{\boldsymbol{\lambda}_n\}_n$ with

$$\lim_{n\to\infty} \gamma(\boldsymbol{\lambda}_n)=0, \tag{3.11}$$

and a sequence $\{(u_{0,n},u_{1,n})\}_n$ in $\mathcal{H}$ such that, denoting by u_n the solution of

$$\partial_t^2 u_n + L_{\boldsymbol{\lambda}_n} u_n = 0, \quad \vec{u}_n \restriction_{t=0} = (u_{0,n},u_{1,n}), \tag{3.12}$$

one has

$$\lim_{n\to\infty} \sum_{\pm} \lim_{t\to\pm\infty} \int_{|x|\geqslant |t|} |\nabla_{t,x} u_n(t,x)|^2\,dx + \gamma(\boldsymbol{\lambda}_n)^{2\theta_N} \|\pi_{Z_{\boldsymbol{\lambda}_n}}(u_{0,n},u_{1,n})\|_{\mathcal{H}}^2 = 0 \tag{3.13}$$

and

$$\|\pi_{Z^{\perp}_{\boldsymbol{\lambda}_n}}(u_{0,n},u_{1,n})\|_{\mathcal{H}} = 1. \tag{3.14}$$

Step 1. (Projection on the orthogonal of the singular directions) Let v_n be the solution of

$$\partial_t^2 v_n + L_{\boldsymbol{\lambda}_n} v_n = 0, \quad \vec{v}_n \restriction_{t=0} = \pi_{Z^{\perp}_{\boldsymbol{\lambda}_n}}(u_{0,n},u_{1,n}). \tag{3.15}$$

We claim that

$$\lim_{n\to\infty} \sum_{\pm} \lim_{t\to\pm\infty} \int_{|x|\geqslant |t|} |\nabla_{t,x} v_n(t,x)|^2\,dx = 0. \tag{3.16}$$

In view of (3.13), it is sufficient to prove that

$$\lim_{n\to\infty} \sum_{\pm} \lim_{t\to\pm\infty} \int_{|x|\geqslant |t|} |\nabla_{t,x} (u_n - v_n)(t,x)|^2\,dx = 0. \tag{3.17}$$

By definition of $Z_{\boldsymbol{\lambda}_n}$, we can write

$$\pi_{Z_{\boldsymbol{\lambda}_n}}(u_{0,n},u_{1,n}) = \sum_{j=1}^{J} (\alpha_{j,n} (\Lambda W)_{(\lambda_{j,n})}, \beta_{j,n} (\Lambda W)_{[\lambda_{j,n}]}), \tag{3.18}$$

where, by (3.13),

$$\lim_{n\to\infty} (|\alpha_{j,n}| + |\beta_{j,n}|) \gamma(\boldsymbol{\lambda}_n)^{\theta_N} = 0 \quad \text{for all } j \in [\![1,J]\!]. \tag{3.19}$$

We consider

$$w_n = \sum_{j=1}^{J} (\alpha_{j,n}(\Lambda W)_{(\lambda_{j,n})} + t\beta_{j,n}(\Lambda W)_{[\lambda_{j,n}]}), \tag{3.20}$$

and prove that w_n is, outside the wave cone, an approximate solution of the linearized equation around the multi-soliton in the following sense:

$$\lim_{n\to\infty} \|\mathbb{1}_{\{|x|\geqslant|t|\}}(\partial_t^2 + L_{\boldsymbol{\lambda}_n})w_n\|_{L^1(\mathbb{R},L^2)} = 0. \tag{3.21}$$

Indeed,

$$\begin{aligned}
&\|\mathbb{1}_{\{|x|\geqslant|t|\}}(\partial_t^2 + L_{\boldsymbol{\lambda}_n})w_n\|_{L^1(\mathbb{R},L^2)} \\
&\quad\lesssim \sum_{\substack{1\leqslant j\leqslant J\\ k\neq j}} |\alpha_{j,n}|\, \|\mathbb{1}_{\{|x|\geqslant|t|\}} W_{(\lambda_{k,n})}^{4/(N-2)} (\Lambda W)_{(\lambda_{j,n})}\|_{L^1(\mathbb{R},L^2)} \\
&\qquad + \sum_{\substack{1\leqslant j\leqslant J\\ k\neq j}} |\beta_{j,n}|\, \|t\mathbb{1}_{\{|x|\geqslant|t|\}} W_{(\lambda_{k,n})}^{4/(N-2)} (\Lambda W)_{[\lambda_{j,n}]}\|_{L^1(\mathbb{R},L^2)}.
\end{aligned}$$

By Claim A.2 in the appendix, we have

$$\begin{aligned}
&\|\mathbb{1}_{\{|x|\geqslant|t|\}} W_{(\lambda_{k,n})}^{4/(N-2)} (\Lambda W)_{(\lambda_{j,n})}\|_{L^1(\mathbb{R},L^2)} \\
&\quad + \|t\mathbb{1}_{\{|x|\geqslant|t|\}} W_{(\lambda_{k,n})}^{4/(N-2)} (\Lambda W)_{[\lambda_{j,n}]}\|_{L^1(\mathbb{R},L^2)} \lesssim (\gamma(\boldsymbol{\lambda}_n))^{\theta_N},
\end{aligned} \tag{3.22}$$

which yields (3.21) in view of (3.19). To conclude Step 1, we see that (3.21) implies

$$\lim_{n\to\infty} \|\mathbb{1}_{\{|x|\geqslant|t|\}}(\partial_t^2 + L_{\boldsymbol{\lambda}_n})(u_n - v_n - w_n)\|_{L^1(\mathbb{R},L^2)} = 0,$$

and, since $(\vec{u}_n - \vec{v}_n - \vec{w}_n)\restriction_{t=0} = 0$, (3.17) follows from Lemma 2.8, the fact that w_n satisfies

$$\lim_{n\to\infty} \sum_{\pm} \lim_{t\to\pm\infty} \int_{|x|\geqslant|t|} |\nabla_{t,x} w_n(t,x)|^2\, dx = 0$$

and the following bounds:

$$\mathbb{1}_{|x|\geqslant|t|} W^{4/(N-2)} \in L^{2(N+1)/(N+4)}(\mathbb{R}, L^{2(N+1)/3}) \tag{3.23}$$

and

$$\begin{aligned}
&\left\| \mathbb{1}_{|x|\geqslant|t|} \sum_{j=1}^{J} W_{(\lambda_{j,n})}^{4/(N-2)} \right\|_{L^{2(N+1)/(N+4)}(\mathbb{R},L^{2(N+1)/3})} \\
&\quad \leqslant J \|\mathbb{1}_{|x|\geqslant|t|} W^{4/(N-2)}\|_{L^{2(N+1)/(N+4)}(\mathbb{R},L^{2(N+1)/3})}.
\end{aligned} \tag{3.24}$$

The bound (3.24) follows from (3.23) and scaling invariance. To prove (3.23), we use the bound

$$|W(x)|^{4/(N-2)} \lesssim \min\left(1, \frac{1}{|x|^4}\right).$$

This proves that $W^{4/(N-2)} \in L^{2(N+1)/3}(\mathbb{R}^N)$ and

$$\|1\!\!1_{\{|x|\geqslant|t|\}} W^{4/(N-2)}\|^{2(N+1)/3}_{L_x^{2(N+1)/3}} \lesssim \int_t^\infty \frac{1}{r^{8(N+1)/3}} r^{N-1}\, dr \lesssim \frac{1}{t^{(5N+8)/3}}.$$

Hence,

$$\|1\!\!1_{\{|x|\geqslant|t|\}} W^{4/(N-2)}\|^{2(N+1)/(N+4)}_{L_x^{2(N+1)/3}} \lesssim \frac{1}{t^{(5N+8)/(N+4)}},$$

which yields (3.23) and concludes this step.

Step 2. (Profile decomposition) As it is recalled in §2.3, extracting subsequences, we may assume that the sequence $\{(v_{0,n}, v_{1,n}\}$ has a profile decomposition with profiles $\{U_F^j\}_{j\geqslant 1}$, and parameters $\{\lambda_{j,n}\}_n \in (0,\infty)^{\mathbb{N}}$ and $\{t_{j,n}\}_n \in \mathbb{R}^{\mathbb{N}}$. We denote by $U^j_{F,n}$ the rescaled linear profiles, defined in (2.12), and by

$$w_n^K = S_L(t)(v_{0,n}, v_{1,n}) - \sum_{j=1}^K U^j_{F,n}$$

the remainder, so that

$$\lim_{K\to\infty} \limsup_{n\to\infty} \|w_n^K\|_{\mathsf{S}(\mathbb{R})} = 0. \tag{3.25}$$

Reordering the profiles, we may assume

$$t_{j,n} = 0 \quad \text{for } 1 \leqslant j \leqslant J,$$

(where J is the number of solitons), and that, for $1\leqslant j\leqslant J$, the parameters $\lambda_{j,n}$ are the same $\lambda_{j,n}$ as in the beginning of the proof. Indeed, one can define the J first profiles by

$$\vec{U}_F^j(0) = \operatorname*{w-lim}_{n\to\infty}(\lambda_{j,n}^{N/2-1} v_{0,n}(\lambda_{j,n}\cdot), \lambda_{j,n}^{N/2} v^J_{1,n}(\lambda_{j,n}\cdot)) \quad \text{in } \mathcal{H},$$

then carry on with the profile decomposition to extract all the other profiles. Of course, in doing so, we do not exclude the fact that some of the profiles U_F^j, $1\leqslant j\leqslant J$, might be identically zero.

We will approximate v_n as follows. If $1\leqslant j\leqslant J$, we let U^j be the solution of

$$(\partial_t^2 + L_W)U^j = 0, \quad \vec{U}^j(0) = \vec{U}_F^j(0). \tag{3.26}$$

If $j\geqslant J+1$, we let $U^j=U^j_F$. We define

$$U^j_n(t,x)=\frac{1}{\lambda_{j,n}^{N/2-1}}U^j\left(\frac{t-t_{j,n}}{\lambda_{j,n}},\frac{x}{\lambda_{j,n}}\right)\quad\text{and}\quad v_n^K(t,x)=\sum_{j=1}^K U_n^j+w_n^K(t,x).\tag{3.27}$$

In this step we prove:

$$\lim_{K\to\infty}\limsup_{n\to\infty}\Big(\sup_{t\in\mathbb{R}}\|1\!\!1_{\{|x|\geqslant|t|\}}\nabla_{t,x}(v_n(t,x)-v_n^K(t,x))\|_{L^2}\Big)=0.\tag{3.28}$$

Denote by $r_n^K=v_n-v_n^K$. Then,

$$\partial_t^2 r_n^K+L_{\boldsymbol{\lambda}_n}r_n^K=-\sum_{j=1}^K(\partial_t^2+L_{\boldsymbol{\lambda}_n})U_n^j-(\partial_t^2+L_{\boldsymbol{\lambda}_n})w_n^K\quad\text{and}\quad \vec{r}_n^K\restriction_{t=0}=(0,0).$$

If $1\leqslant j\leqslant J$, we have, by (3.26),

$$(\partial_t^2+L_{\boldsymbol{\lambda}_n})U_n^j=-\frac{N+2}{N-2}\sum_{\substack{1\leqslant k\leqslant J\\ k\neq j}}W_{(\lambda_{k,n})}^{4/(N-2)}U_n^j.$$

If $j\geqslant J+1$, then

$$(\partial_t^2+L_{\boldsymbol{\lambda}_n})U_n^j=-\frac{N+2}{N-2}\sum_{1\leqslant k\leqslant J}W_{(\lambda_{k,n})}^{4/(N-2)}U_n^j.$$

Finally, for all $K\geqslant 1$,

$$(\partial_t^2+L_{\boldsymbol{\lambda}_n})w_n^K=-\frac{N+2}{N-2}\sum_{1\leqslant k\leqslant J}W_{(\lambda_{k,n})}^{4/(N-2)}w_n^K.$$

Using the pseudo-orthogonality (2.11) of the parameters and the property (3.25) of w_n^K, we obtain

$$\lim_{K\to\infty}\lim_{n\to\infty}\|(\partial_t^2+L_{\boldsymbol{\lambda}_n})r_n^K1\!\!1_{\{|x|\geqslant|t|\}}\|_{L^1(\mathbb{R},L^2)}=0.$$

By (3.24) and the approximation Lemma 2.8, we deduce (3.28).

Step 3. (End of the proof) Using the profile decomposition of the preceding step, we prove the corollary. We claim that

$$\|\vec{U}_n^j(0)\|_{\mathcal{H}}^2\lesssim\sum_{\pm}\lim_{t\to\pm\infty}\int_{|x|>|t|}|\nabla_{t,x}U_n^j(t,x)|^2\,dx\quad\text{for all } j\geqslant 1,\tag{3.29}$$

$$\|w_n^K(0)\|_{\mathcal{H}}^2\lesssim\sum_{\pm}\lim_{t\to\pm\infty}\int_{|x|\geqslant|t|}|\nabla_{t,x}w_n^K(t,x)|^2\,dx\quad\text{for all } K\geqslant 1\tag{3.30}$$

(where the implicit constants are independent of j, K and n), and

$$\lim_{n\to\infty}\lim_{t\to\pm\infty}\int_{|x|>|t|}|\nabla_{t,x}U_n^j(t,x)|^2\,dx=0\quad\text{for all } j\geqslant 1, \tag{3.31}$$

$$\sum_{\pm}\lim_{t\to\pm\infty}\int_{|x|\geqslant|t|}|\nabla_{t,x}w_n^K(t,x)|^2\,dx=0\quad\text{for all } K\geqslant 1. \tag{3.32}$$

Of course, combining inequalities (3.29)–(3.32) and Step 2, we would obtain

$$\lim_{n\to\infty}\|\vec{v}_n(0)\|_{\mathcal{H}}=0,$$

a contradiction with (3.14). It remains to prove these four assertions.

Recall that U^j (for $j\geqslant J+1$) and w_n^K (for any $K\geqslant 1$) are solutions of the free wave equation. The inequalities (3.29) for $j\geqslant J+1$, and (3.30) thus follow from the exterior energy bound in odd dimension proved in [19] (recalled in Theorem 3.1 above). We next prove (3.29) when j satisfies $1\leqslant j\leqslant J$. According to the channels of energy for the linearized equation at W (Theorem 3.2), it is sufficient to prove that

$$\int\nabla U^j(0,x)\cdot\nabla\Lambda W(x)\,dx=\int\partial_t U^j(0,x)\Lambda W(x)=0. \tag{3.33}$$

To prove (3.33), notice that, by weak convergence,

$$\begin{aligned}\int\nabla U^j(0,x)\cdot\nabla\Lambda W(x)\,dx&=\lim_{n\to\infty}\int\lambda_{j,n}^{N/2}\nabla v_{0,n}(\lambda_{j,n}x)\Lambda W(x)\,dx\\&=\lim_{n\to\infty}\int\nabla v_{0,n}\nabla(\Lambda W)_{(\lambda_{j,n})}=0,\end{aligned}$$

since $(v_{0,n},v_{1,n})\in Z^{\perp}_{\boldsymbol{\lambda}_n}$. By the same proof,

$$\int\partial_t U^j(0,x)\Lambda W(x)\,dx=0,$$

concluding the proof of (3.33), and thus of (3.29).

We next prove (3.31) and (3.32). We will use the pseudo-orthogonality of the parameters (2.11). We focus on the limits as $t\to+\infty$, the proof for the limits as $t\to-\infty$ being the same. Using the radiation term for the free wave equation (see appendix of [22]) if $j\geqslant J+1$, or for the linearized wave equation (see (2.20)) if $1\leqslant j\leqslant J$, we obtain that, for all $j\geqslant 1$, there exists $g^j\in L^2(\mathbb{R})$ such that

$$\lim_{t\to\infty}\int_t^{+\infty}|r^{(N-1)/2}\partial_r U^j(t,r)-g^j(r-t)|^2\,dr=0, \tag{3.34}$$

$$\lim_{t\to\infty}\int_t^{+\infty}|r^{(N-1)/2}\partial_t U^j(t,r)+g^j(r-t)|^2\,dr=0, \tag{3.35}$$

$$\lim_{t\to\infty}\int_t^{+\infty}\frac{1}{r^2}|U^j(t,r)|^2r^{N-1}\,dr=0. \tag{3.36}$$

If $j\geqslant J+1$, the preceding limits hold true with $\int_t^\infty$ replaced by $\int_0^\infty$. Also, for all $K\geqslant 1$ and n, there exist $G_n^K\in L^2(\mathbb{R})$ such that

$$\lim_{t\to\infty}\int_0^{+\infty}|r^{(N-1)/2}\partial_r w_n^K(t,r)-G_n^K(r-t)|^2\,dr=0, \tag{3.37}$$

$$\lim_{t\to\infty}\int_0^{+\infty}|r^{(N-1)/2}\partial_t w_n^K(t,r)+G_n^K(r-t)|^2\,dr=0. \tag{3.38}$$

Fix $j\geqslant 1$ and $\varepsilon>0$. Then, there exists $K\gg 1$ such that $K>j$ and (by (3.28) and (3.16))

$$\lim_{t\to+\infty}\int_{|x|>t}|\partial_{t,x}v_n^K(t,x)|^2\,dx\leqslant\varepsilon. \tag{3.39}$$

Using the definition (3.27) of v_n^K, we obtain

$$\begin{aligned}\int_{|x|>t}\nabla_{t,x}v_n^K(t)\cdot\nabla_{t,x}U_n^j(t)=\int_{|x|>t}&|\nabla_{t,x}U_n^j(t)|^2\\&+\sum_{\substack{1\leqslant k\leqslant K\\ j\neq k}}\int_{|x|>|t|}\nabla_{t,x}U_n^j(t)\cdot\nabla_{t,x}U_n^k(t)\\&+\int_{|x|>|t|}\nabla_{t,x}U_n^j(t)\nabla_{t,x}w_n^K(t),\end{aligned}$$

where U_n^j is as usual the modulated profile (see (3.27)).

If $j\neq k$, we have, in view of (3.34) and (3.35),

$$\begin{aligned}&\lim_{t\to+\infty}\int_{|x|>t}\nabla_{t,x}U_n^j(t)\cdot\nabla_{t,x}U_n^k(t)\\&\quad=2\lim_{t\to\infty}\int_t^{+\infty}\frac{1}{\lambda_{j,n}^{1/2}}g^j\left(\frac{r-|t-t_{j,n}|}{\lambda_{j,n}}\right)\frac{1}{\lambda_{k,n}^{1/2}}g^k\left(\frac{r-|t-t_{k,n}|}{\lambda_{k,n}}\right)dr\\&\quad=2\int_0^{+\infty}\frac{1}{\lambda_{j,n}^{1/2}}g^j\left(\frac{r+t_{j,n}}{\lambda_{j,n}}\right)\frac{1}{\lambda_{k,n}^{1/2}}g^k\left(\frac{r+t_{k,n}}{\lambda_{k,n}}\right)dr.\end{aligned}$$

In view of the pseudo-orthogonality (2.11) of the parameters, we deduce that

$$\lim_{n\to\infty}\lim_{t\to+\infty}\int_{|x|>t}\nabla_{t,x}U_n^j(t)\cdot\nabla_{t,x}U_n^k(t)=0. \tag{3.40}$$

Next, we consider

$$\begin{aligned}&\lim_{t\to+\infty}\int_{|x|>t}\nabla_{t,x}U_n^j(t)\cdot\nabla_{t,x}w_n^K(t)\\&\quad=2\lim_{t\to\infty}\int_t^{+\infty}\frac{1}{\lambda_{j,n}^{1/2}}g^j\left(\frac{r-|t-t_{j,n}|}{\lambda_{j,n}}\right)G_n^K(r-t)\,dr\\&\quad=2\int_0^{+\infty}\frac{1}{\lambda_{j,n}^{1/2}}g^j\left(\frac{r+t_{j,n}}{\lambda_{j,n}}\right)G_n^K(r)\,dr.\end{aligned}$$

If $t_{j,n}=0$ for all n, we obtain

$$\lim_{t\to+\infty}\int_{|x|>t}\nabla_{t,x}U_n^j(t)\cdot\nabla_{t,x}w_n^K(t)=2\int_0^{+\infty}\frac{1}{\lambda_{j,n}^{1/2}}g^j\left(\frac{r}{\lambda_{j,n}}\right)G_n^K(r)\,dr,$$

and the right-hand side goes to zero as n goes to infinity, since the condition

$$\operatorname*{w-lim}_{n\to\infty}(\lambda_{j,n}^{N/2-1}w_n^K(0,\lambda_{j,n}\cdot),\lambda_{j,n}^{N/2}\partial_t w_n^K(0,\lambda_{j,n}\cdot))=0$$

(see (2.15)) implies

$$\lim_{n\to\infty}\int_{\mathbb{R}}\frac{1}{\lambda_{j,n}^{1/2}}g\left(\frac{r}{\lambda_{j,n}}\right)G_n^K(r)\,dr=0\quad\text{for all } g\in L^2(\mathbb{R}). \tag{3.41}$$

If $\lim_n t_{j,n}/\lambda_{j,n}=+\infty$, then we have

$$\left|\int_0^{+\infty}\frac{1}{\lambda_{j,n}^{1/2}}g^j\left(\frac{r+t_{j,n}}{\lambda_{j,n}}\right)G_n^K(r)\,dr\right|\lesssim\|g^j\|_{L^2(r\geqslant t_{j,n}/\lambda_{j,n})}\xrightarrow[n\to\infty]{}0$$

Finally, if

$$\lim_{n\to\infty}\frac{t_{j,n}}{\lambda_{j,n}}=-\infty,$$

we have

$$\lim_{n\to\infty}\int_{-\infty}^0\left|\frac{1}{\lambda_{j,n}^{1/2}}g^j\left(\frac{t+t_{j,n}}{\lambda_{j,n}}\right)\right|^2dr=0,$$

and thus

$$\lim_{n\to\infty}\int_0^{+\infty}\frac{1}{\lambda_{j,n}^{1/2}}g^j\left(\frac{r+t_{j,n}}{\lambda_{j,n}}\right)G_n^K(r)\,dr=\lim_{n\to\infty}\int_{\mathbb{R}}\frac{1}{\lambda_{j,n}^{1/2}}g^j\left(\frac{r+t_{j,n}}{\lambda_{j,n}}\right)G_n^K(r)\,dr=0,$$

where we have used that, by the weak limit property (2.15) of w_n^K,

$$\lim_{n\to\infty}\int\nabla_{t,x}U_n^j(0,x)\cdot\nabla_{t,x}w_n^K(0,x)\,dx=0.$$

Combining the properties above, we obtain

$$\lim_{n\to\infty}\lim_{t\to+\infty}\int_{|x|\geqslant|t|}|\nabla_{t,x}U_n^j(t,x)|^2\,dx=\lim_{n\to\infty}\lim_{|x|>|t|}\int\nabla_{t,x}v_n^K\cdot\nabla_{t,x}U_n^j=0.$$

This yields (3.31). By a similar proof, we obtain (3.32), concluding this step.

4. Non-radiative solutions close to a multi-soliton

4.1. Preliminaries

Definition 4.1. Let $t_0 \in \mathbb{R}$, and let u be a solution of the non-linear wave equation (1.1) (or another wave equation considered in this paper). We say that u is *non-radiative* at $t=t_0$ if u is defined on $\{|x|>|t-t_0|\}$ and

$$\sum_{\pm} \lim_{t\to\pm\infty} \int_{|x|\geqslant |t-t_0|} |\nabla_{t,x} u(t,x)|^2 \, dx = 0.$$

We say that u is *weakly non-radiative* if, for large $R>0$, u is defined on $\{|x|>|t|+R\}$ and

$$\sum_{\pm} \lim_{t\to\pm\infty} \int_{|x|\geqslant |t|+R} |\nabla_{t,x} u(t,x)|^2 \, dx = 0.$$

If $N\geqslant 3$ is odd, according to the equirepartition property recalled in Theorem 3.1, the only non-radiative solution of (3.1) is zero. This fact persists if $N\geqslant 4$ is even (see [24, Proposition 1]), as a consequence of the asymptotic formula in [10].

In odd space dimension $N\geqslant 5$, the non-radiative solutions for the linearized wave equation around the stationary solutions W are also known, as a consequence of the main result of [23] (recalled in Theorem 3.2 above in the radial case).

Radial *weakly non-radiative* solutions of the free wave equation were explicited in [40]. Let

$$\mathcal{P} = \left\{ \left(\frac{1}{r^{N-2k_1}}, 0 \right) : 1 \leqslant k_1 \leqslant \left\lfloor \frac{N+2}{4} \right\rfloor \right\} \cup \left\{ \left(0, \frac{1}{r^{N-2k_2}} \right) : 1 \leqslant k_2 \leqslant \left\lfloor \frac{N}{4} \right\rfloor \right\},$$

and, for $R>0$, let $P(R)$ be the subspace of $\mathcal{H}(R)$ spanned by $\mathcal{P}$. According to [40], if $N\geqslant 3$ is odd and v is a radial solution $\partial_t^2 v - \Delta v = 0$, then

$$\sum_{\pm} \lim_{t\to\pm\infty} \int_{R+|t|} |\nabla_{t,x} u(t,x)|^2 \, dx = 0$$

if and only if $\vec{v}(0) \in P(R)$.

From [21], if $N=3$, the only radial, non-radiative solutions of (1.1) are the stationary solutions. The proof is specific to dimension 3, and the results available in higher dimension are less precise. We next recall from [24] some of these results, that will be needed in the sequel. Let $m=\frac{1}{2}(N-1)$ be the number of elements of $\mathcal{P}$. As in [24], we denote by $\mathcal{P}=\{\Xi_k\}_{k\in [\![1,m]\!]}$, choosing Ξ_k so that

$$\|\Xi_k\|_{\mathcal{H}(R)} = \frac{c_k}{R^{k-1/2}}, \tag{4.1}$$

for some constant $c_k \neq 0$. In particular, we choose $\Xi_m(r) = (r^{2-N}, 0)$. By scaling, one can check that, if $U \in P(R)$ and $(\theta_k(R))_{k \in [\![1,m]\!]}$ are its coordinates in $(\Xi_1, ..., \Xi_m)$, then

$$\|U\|_{\mathcal{H}(R)} \approx \sum_{k=1}^{m} \frac{|\theta_k(R)|}{R^{k-1/2}},$$

where the implicit constant is independent of $R>0$ (see [24, Claim 3.2]). Then, we have the following.

THEOREM 4.2. *Assume $N \geqslant 5$ is odd. There exists $\varepsilon_0 > 0$ with the following property. Let u be a radial weakly non-radiative solution of* (1.1)*. Then, there exist $k_0 \in [\![1, m]\!]$ and $\ell \in \mathbb{R}$ (with $\ell \neq 0$ if $k_0 < m$) such that, for all $t_0 \in \mathbb{R}$ and $R_0 > 0$, if u is defined on*

$$\{(t,r) : r > |t-t_0| + R_0\},$$

$\|\vec{u}(t_0)\|_{\mathcal{H}(R_0)} < \varepsilon_0$ and

$$\sum_{\pm} \lim_{t \to \pm\infty} \int_{|x| > |t-t_0| + R_0} |\nabla_{t,x} u(t,r)|^2 \, dx = 0,$$

then, for all $R > R_0$,

$$\|\vec{u}(t_0) - \ell \Xi_{k_0}\|_{\mathcal{H}(R)} \leqslant C \max\left\{ \left(\frac{R_0}{R}\right)^{(k_0-1/2)(N+2)/(N-2)}, \left(\frac{R_0}{R}\right)^{k_0+1/2} \right\}.$$

See [24, Theorem 2 and Remark 3.4]. If $k_0 = m$, the theorem implies that u is close, for large r, to one of the stationary solutions 0 (if $\ell = 0$) or $\pm W_{(\lambda)}$ for some λ depending on ℓ (if $\ell \neq 0$). Under the stronger assumption that u is non-radiative, [24, Theorem 3] gives a uniqueness result in this case.

THEOREM 4.3. *Assume $N \geqslant 5$ is odd. Let u be a radial non-radiative solution of* (1.1)*. Let k_0 be as in Theorem* 4.2*. Assume that $k_0 = m$. Then, u is a stationary solution.*

In the remainder of this section, we will assume $N \geqslant 5$ is odd and consider a non-radiative solution close to a multi-soliton. In §4.2, we will use the exterior energy bound for the linearized equation proved in §3 to give a first-order expansion of the solution. In §4.3 we will use Theorems 4.2 and 4.3 to give a lower bound of the exterior scaling parameter of the multi-soliton. These properties will be crucial in the proofs of the soliton resolution in §5 and §6.

4.2. Estimates on the coefficients

In this subsection, we assume as before that $N\geqslant 5$ is odd. We fix $J\geqslant 1$, $(\iota_j)\in\{\pm 1\}^J$, and consider a radial solution u of (1.1), defined on $\{(t,x)\in\mathbb{R}^N:|x|>t\}$, which is *non-radiative* at $t=0$ (see Definition 4.1). We assume that there exists $\boldsymbol{\lambda}=(\lambda_j)^J\in G_J$ such that

$$\|\vec{u}(0)-(M,0)\|_{\mathcal{H}}=:\delta\leqslant\varepsilon_J\ll 1,\quad \text{where } M=\sum_{j=1}^J \iota_j W_{(\lambda_j)}, \tag{4.2}$$

$$\gamma\leqslant\varepsilon_J\ll 1, \tag{4.3}$$

where, as before, $\gamma:=\gamma(\boldsymbol{\lambda})=\max_{1\leqslant j\leqslant J-1}\lambda_{j+1}/\lambda_j$. Denote

$$h_0=u_0-M.$$

By the implicit function theorem (see Lemma B.1), we can change the scaling parameters $(\lambda_j)_j$ so that the following orthogonality relations hold:

$$\int \nabla_x h_0 \nabla_x (\Lambda W)_{(\lambda_j)}=0 \quad \text{for all } j\in[\![1,J]\!]. \tag{4.4}$$

We expand $u_1=\partial_t u(0)$ as follows:

$$u_1=\sum_{j=1}^J \alpha_j(\Lambda W)_{[\lambda_j]}+g_1, \tag{4.5}$$

where

$$\int g_1(\Lambda W)_{[\lambda_j]}=0 \quad \text{for all } j\in[\![1,J]\!]. \tag{4.6}$$

We will prove the following result.

PROPOSITION 4.4. *We have*

$$\|(h_0,g_1)\|_{\mathcal{H}}\lesssim\gamma^{N/4}+\delta^{N/(N-2)}, \tag{4.7}$$

$$\left|\delta^2-\sum_{j=1}^J \alpha_j^2\|\Lambda W\|_{L^2}^2\right|\lesssim\gamma^{(N-1)/2}+\delta^{2(N-1)/(N-2)}. \tag{4.8}$$

We start by proving the following lemma.

LEMMA 4.5. *Let u be as above. Then,*

$$\|\pi_{Z_{\boldsymbol{\lambda}}^{\perp}}((u_0,u_1)-(M,0))\|_{\mathcal{H}}\lesssim\gamma^{N/4}+\delta^{N/(N-2)}.$$

Proof. We let $h(t)=u(t)-M$. Then, $\|\vec{h}(0)\|_{\mathcal{H}}=\delta$ and

$$\partial_t^2 h+L_{\boldsymbol{\lambda}}h=F(h)+\mathcal{N}(h), \tag{4.9}$$

where

$$\mathcal{N}(h)=F(M+h)-\sum_{j=1}^{J}F(\iota_j W_{(\lambda_j)})-F(h)-\frac{N+2}{N-2}\sum_{j=1}^{J}W_{(\lambda_j)}^{4/(N-2)}h.$$

By finite speed of propagation, h coincide, for $|x|>|t|$, with the solution $\tilde{h}$ of

$$\partial_t^2\tilde{h}+L_{\boldsymbol{\lambda}}\tilde{h}=(F(h)+\mathcal{N}(h))\mathbb{1}_{\{|x|\geqslant|t|\}}. \tag{4.10}$$

Let $T>0$ and set

$$\Gamma(T)=\{(t,x):|t|\leqslant\min\{|x|,T\}\}.$$

By the fractional chain rule (2.7), we have+++

$$\|F(h)\mathbb{1}_{\{|x|\geqslant|t|\}}\|_{\mathsf{W}'((0,T))}=\|F(\tilde{h})\mathbb{1}_{\{|x|\geqslant|t|\}}\|_{\mathsf{W}'((0,T))}\lesssim\|\tilde{h}\|_{\mathsf{W}((0,T))}\|\tilde{h}\|_{\mathsf{S}(\Gamma(T))}^{4/(N-2)}. \tag{4.11}$$

We first assume that $N\geqslant 7$. We use the inequality

$$\begin{aligned}&\left|F\left(\sum_{j=1}^{J}y_j+h\right)-\sum_{j=1}^{J}F(y_j)-F(h)-\frac{N+2}{N-2}\sum_{j=1}^{J}|y_j|^{4/(N-2)}h\right|\\&\quad\lesssim\sum_{j\neq k}\min\{|y_j|^{4/(N-2)}|y_k|,|y_k|^{4/(N-2)}|y_j|\}+\sum_{j=1}^{J}|y_j|\,|h|^{(N+1)/(N-2)},\end{aligned} \tag{4.12}$$

proved in the appendix (see Claim A.5). We obtain

$$|\mathcal{N}(h)|\lesssim\sum_{j\neq k}\min(W_{(\lambda_j)}^{4/(N-2)}W_{(\lambda_k)})+\sum_{j=1}^{J}W_{(\lambda_j)}^{1/(N-2)}|h|^{(N+1)/(N-2)}.$$

If $j\neq k$, we have, by Claim A.3 in the appendix,

$$\|\mathbb{1}_{\{|x|\geqslant|t|\}}\min\{W_{(\lambda_j)}^{4/(N-2)}W_{(\lambda_k)},W_{(\lambda_j)}^{4/(N-2)}W_{(\lambda_k)}\}\|_{L_t^1L_x^2}\lesssim\gamma^{(N+2)/4}.$$

Furthermore,

$$\begin{aligned}&\|\mathbb{1}_{\Gamma(T)}W_{(\lambda_j)}^{1/(N-2)}|h|^{(N+1)/(N-2)}\|_{L_t^1L^2}\\&\quad\leqslant\|\mathbb{1}_{\Gamma(T)}W_{(\lambda_j)}^{1/(N-2)}\|_{L_t^2L_x^\infty}\,\|\mathbb{1}_{\Gamma(T)}|h|^{(N+1)/(N-2)}\|_{L_{t,x}^2}\\&\quad\lesssim\|\tilde{h}\|_{\mathsf{S}(\Gamma(T))}^{(N+1)/(N-2)},\end{aligned}$$

where we have used that, since

$$W^{1/(N-2)} \lesssim \frac{1}{1+|x|},$$

we have $W^{1/(N-2)} 1\!\!1_{\{|x|\geqslant t\}} \in L^2_t(\mathbb{R}, L^\infty_x(\mathbb{R}^N))$. We let $h_L(t)$ be the solution of

$$\partial_t^2 h_L + L_{\boldsymbol{\lambda}} h_L = 0 \quad \text{and} \quad \vec{h}_L \restriction_{t=0} = (u_0, u_1) - (M, 0).$$

In view of the estimate (3.24) on the potential $\sum_j W_{(\lambda_j)}^{(N+2)/(N-2)}$, we can use the approximation lemma (Lemma 2.8). Due to the estimates above, we obtain

$$\|\tilde{h} - h_L\|_{\mathsf{S}(\Gamma_T)} \lesssim \gamma^{(N+2)/4} + \|\tilde{h}\|_{\mathsf{S}(\Gamma_T)}^{(N+1)/(N-2)} + \|\tilde{h}\|_{\mathsf{S}(\Gamma_T)}^{4/(N-2)} \|\tilde{h}\|_{\mathsf{W}((0,T)}.$$

Using again Strichartz estimates, we deduce that

$$\begin{aligned} \sup_{-T\leqslant t\leqslant T} \|\vec{\tilde{h}}(t) - \vec{h}_L(t)\|_{\mathcal{H}} &+ \|\tilde{h} - h_L\|_{\mathsf{W}((0,T))\cap \mathsf{S}((0,T))} \\ &\lesssim \gamma^{(N+2)/4} + \|\tilde{h}\|_{\mathsf{S}(\Gamma_T)}^{(N+1)/(N-2)} + \|\tilde{h}\|_{\mathsf{S}(\Gamma_T)}^{4/(N-2)} \|\tilde{h}\|_{\mathsf{W}((0,T)}, \end{aligned} \tag{4.13}$$

and thus, since $\|h_L\|_{\mathsf{W}((0,T))\cap \mathsf{S}((0,T))} \lesssim \delta$,

$$\|\tilde{h}\|_{\mathsf{W}((0,T))\cap \mathsf{S}((0,T))} \lesssim \gamma^{(N+2)/4} + \delta.$$

Going back to (4.13), we obtain

$$\sup_{-T\leqslant t\leqslant T} \|\vec{\tilde{h}}(t) - \vec{h}_L(t)\|_{\mathcal{H}} \lesssim \gamma^{(N+2)/4} + \delta^{(N+1)/(N-2)}.$$

This estimate is uniform in T. Hence,

$$\sup_{t\in\mathbb{R}} \|\vec{\tilde{h}}(t) - \vec{h}_L(t)\|_{\mathcal{H}} \lesssim \gamma^{(N+2)/4} + \delta^{(N+1)/(N-2)}.$$

Using that u is non-radiative, we deduce that

$$\sum_{\pm} \left(\lim_{t\to\pm\infty} \int_{\{|x|>|t|\}} |\nabla_{t,x} h_L(t,x)|^2 \, dx \right)^{1/2} \lesssim \gamma^{(N+2)/4} + \delta^{(N+1)/(N-2)}.$$

By Corollary 3.3,

$$\|\pi_{Z_{\boldsymbol{\lambda}}^\perp}(h_0, h_1)\|_{\mathcal{H}} \lesssim \gamma^{\theta_N} \delta + \gamma^{(N+2)/4} + \delta^{(N+1)/(N-2)},$$

where $\theta_7 = \frac{3}{2}$ and $\theta_N = 2$ if $N \geqslant 9$. The conclusion of the proposition follows, noting that

$$\gamma^{\theta_N} \delta \lesssim \gamma^{(N+2)/4} + \delta^{(N+1)/(N-2)}$$

if $N\geqslant 7$. Note that, in this case, the bound is slightly stronger, but we will not need this in the sequel.

The proof is almost the same when $N=5$, but we must replace the inequality (4.12) by

$$\left|F\left(\sum_{j=1}^{J} y_j+h\right)-\sum_{j=1}^{J} F(y_j)-\frac{7}{3}\sum_{j=1}^{J}|y_j|^{4/3}h\right| \lesssim \sum_{j\neq k}|y_j|^{4/3}|y_k|+\sum_{j=1}^{J}|y_j|^{1/3}|h|^2+F(h),$$

and use that, by Claim A.2, if $j\neq k$,

$$\|\mathbb{1}_{\{|x|\geqslant|t|\}}W_{(\lambda_j)}^{4/3}W_{(\lambda_k)}\|_{L^1(\mathbb{R},L^2)}\lesssim\gamma^{3/2}.$$

We omit the details. □

Proof of Proposition 4.4. According to Lemma 4.5, we have

$$\|\pi_{Z_{\boldsymbol{\lambda}}^{\perp}}(h_0,u_1)\|_{\mathcal{H}}\lesssim\delta^{N/(N-2)}+\gamma^{N/4}. \tag{4.14}$$

In view of the orthogonality condition (4.4) and the expansion (4.5) of u_1, we deduce (4.7). Since

$$\delta^2=\|h_0\|_{L^2}^2+\|g_1\|_{L^2}^2+\left\|\sum_{j=1}^{J}\alpha_j(\Lambda W)_{[\lambda_j]}\right\|_{L^2}^2,$$

and, by Claim A.1,

$$\int|(\Lambda W)_{[\lambda_j]}(\Lambda W)_{[\lambda_k]}|\lesssim\gamma^{N/2-2},$$

we obtain

$$\left|\delta^2-\sum_{j=1}^{J}\alpha_j^2\|\Lambda W\|_{L^2}^2\right|\lesssim\gamma^{N/2-2}\sum_{j=1}^{J}\alpha_j^2+\delta^{2N/(N-2)}+\gamma^{N/2}. \tag{4.15}$$

Noting that the previous inequality implies easily that

$$\sum_{j=1}^{J}\alpha_j^2\lesssim\delta^2+\gamma^{N/2},$$

and thus

$$\gamma^{N/2-2}\sum_{j=1}^{J}\alpha_j^2\lesssim\gamma^{N-2}+\delta^2\gamma^{N/2-2}\lesssim\gamma^{N-2}+\delta^{2(N-1)/(N-2)}+\gamma^{(N-4)(N-1)/2},$$

we deduce (4.8). □

4.3. Lower bound for the exterior scaling parameter

Let u be as in §4.2, and denote by ℓ and k_0 the parameters defined by Theorem 4.2. Assume that u is non-stationary so that, by Theorem 4.3, $k_0 \leqslant m-1$.

PROPOSITION 4.6. *There exists a constant* $C_0>0$ *such that, if* u *is as above, we have*

$$|\ell| \leqslant C_0 \delta^{2/N} \lambda_1^{k_0-1/2}.$$

Proof. Step 1. We note that, for $R \geqslant \lambda_1$, we have

$$\|\vec{u}(0)\|_{\mathcal{H}(R)} \lesssim \delta + \left(\frac{\lambda_1}{R}\right)^{m-1/2}.$$

Indeed, $\|\vec{u}(0)-(M,0)\|_{\mathcal{H}(R)} \lesssim \delta$ and

$$\|W_{(\lambda_j)}\|_{\dot{H}^1(R)} = \|W\|_{\dot{H}^1(R/\lambda_j)} \lesssim \left(\frac{\lambda_j}{R}\right)^{m-1/2},$$

which yields the announced estimate.

Step 2. Let ε_0 be as in Theorem 4.2. Fixing $B>0$ large enough, and using the smallness assumption (4.2) on δ, we see that

$$\|\vec{u}(0)\|_{\mathcal{H}(B\lambda_1)} \leqslant \varepsilon_0.$$

In view of Theorem 4.2, we see that, for all $R \geqslant B\lambda_1$,

$$\left| \|\vec{u}(0)\|_{\mathcal{H}(R)} - \frac{c_{k_0}\ell}{R^{k_0-1/2}} \right| \lesssim \max\left\{ \left(\frac{B\lambda_1}{R}\right)^{k_0+1/2}, \left(\frac{B\lambda_1}{R}\right)^{(k_0-1/2)(N+2)/(N-2)} \right\}.$$

Combining with the estimate of Step 1, and using that $k_0 \leqslant m-1$, we deduce that, for all $R \geqslant B\lambda_1$,

$$\frac{|\ell|}{R^{k_0-1/2}} \lesssim \delta + \left(\frac{\lambda_1}{R}\right)^{a_{k_0}} \quad \text{and} \quad a_{k_0} := \min\left\{ k_0+\frac{1}{2}, \left(k_0-\frac{1}{2}\right)\frac{N+2}{N-2} \right\}.$$

Choosing R such that $(\lambda_1/R)^{a_{k_0}} = \delta$, that is $R = \lambda_1 \delta^{-1/a_{k_0}}$, we obtain

$$|\ell| \lesssim \lambda_1^{k_0-1/2} \delta^{1-(k_0-1/2)/a_{k_0}},$$

which yields the conclusion of the proposition, since

$$\min_{1 \leqslant k_0 \leqslant m-1} 1 - \frac{k_0-1/2}{a_{k_0}} = \frac{2}{N}. \qquad \square$$

5. Reduction to a system of differential inequalities

The proof of Theorem 1 is by contradiction. Consider a global solution that does not satisfy the soliton resolution conjecture. Then by the work of Rodriguez [57], it is close, for a sequence of times $\{t_n\}_n$ going to infinity, to a sum of rescaled solitary waves.

Using the study on non-radiative solutions carried out in §4 and the equation (1.1), linearized around a sum of solitary waves, we will obtain an approximate differential system satisfied around t_n by the scaling parameters modulating the stationary solutions. We then deduce a contradiction from this system and differential inequality arguments. We divide the proof into two sections. In this section we will set up the contradiction argument and obtain some differential inequalities. In the next section we will restrict the time interval to prove a crucial lower bound (consequence of Proposition 4.6 above) on one of the scaling parameters, and prove that the differential system, together with this lower bound leads to a contradiction.

This section and §6 concern the case of global solution. We omit the very close proof for finite-time blow-up solutions (see e.g. [21, §4]).

5.1. Setting of the proof of the soliton resolution

Let u be a solution of (1.1) such that $T_+(u)=+\infty$ and

$$\limsup_{t\to+\infty} \|\vec{u}(t)\|_{\mathcal{H}} < \infty. \tag{5.1}$$

Let v_L be the unique solution of the free wave equation (3.1) such that, for all $A\in\mathbb{R}$,

$$\lim_{t\to+\infty} \int_{|x|\geqslant A+|t|} |\nabla_{t,x}(u-v_L)(t,x)|^2\,dx = 0 \tag{5.2}$$

(see [57, Proposition 4.1]). For $J\geqslant 1$, $\boldsymbol{\iota}\in\{\pm1\}^J$ and $(f,g)\in\mathcal{H}$, we set

$$d_{J,\boldsymbol{\iota}}(f,g) = \inf_{\boldsymbol{\lambda}\in G_J} \left\{ \left\| (f,g) - \sum_{j=1}^{J} \iota_j (W_{(\lambda_j)},0) \right\|_{\mathcal{H}} + \gamma(\boldsymbol{\lambda}) \right\}, \tag{5.3}$$

where, as before,

$$G_J = \{(\lambda_j)_j \in (0,\infty)^J : 0<\lambda_J<\ldots<\lambda_2<\lambda_1\} \quad \text{and} \quad \gamma(\boldsymbol{\lambda}) = \max_{2\leqslant j\leqslant J} \frac{\lambda_j}{\lambda_{j-1}} \in (0,1).$$

Assume that u does not scatter forward in time. By [57], we know that there exists $J\geqslant 1$, $\boldsymbol{\iota}\in\{\pm1\}^J$ and a sequence $\{t_n\}_n\to+\infty$ such that

$$\lim_{n\to\infty} d_{J,\boldsymbol{\iota}}(\vec{u}(t_n)-\vec{v}_L(t_n)) = 0. \tag{5.4}$$

We will prove by contradiction that

$$\lim_{t\to\infty} d_{J,\boldsymbol{\iota}}(\vec{u}(t)-\vec{v}_L(t))=0.$$

We thus assume that there exists a small $\varepsilon_0>0$ and a sequence $\{\tilde{t}_n\}_n\to+\infty$ such that

$$\tilde{t}_n<t_n \quad \text{for all } n, \tag{5.5}$$

$$d_{J,\boldsymbol{\iota}}(\vec{u}(t)-\vec{v}_L(t))<\varepsilon_0 \quad \text{for all } n \text{ and all } t\in(\tilde{t}_n,t_n], \tag{5.6}$$

$$d_{J,\boldsymbol{\iota}}(\vec{u}(\tilde{t}_n)-\vec{v}_L(\tilde{t}_n))=\varepsilon_0 \quad \text{for all } n. \tag{5.7}$$

We will denote $U=u-v_L$ and use notation that is analogous to the one of §4.2, although the setting is a bit different.

The implicit function theorem (see Lemma B.1 in the appendix) implies that, for all $t\in[\tilde{t}_n,t_n]$, we can choose $\boldsymbol{\lambda}(t)=(\lambda_1(t),...,\lambda_J(t))\in G_J$ such that, for all $j\in[\![1,J]\!]$,

$$\int \nabla(u(t)-v_L(t)-M(t))\cdot\nabla(\Lambda W)_{(\lambda_j(t))}=0, \tag{5.8}$$

where

$$M(t)=\sum_{j=1}^J \iota_j W_{(\lambda_j(t))}$$

and, in view of Remark B.2,

$$\|\vec{u}(t)-\vec{v}_L(t)-(M(t),0)\|_{\mathcal{H}}+\gamma(\boldsymbol{\lambda})\approx d_{J,\boldsymbol{\iota}}(\vec{u}(t)-v_L(t)). \tag{5.9}$$

In the sequel, we will denote

$$\begin{aligned}
&h(t)=u(t)-v_L(t)-M(t)=U(t)-M(t),\\
&\gamma(t)=\gamma(\boldsymbol{\lambda}(t)),\\
&\delta(t)=\sqrt{\|h(t)\|^2_{\dot{H}^1}+\|\partial_t(u-v_L)(t)\|^2_{L^2}}.
\end{aligned}$$

We will expand $\partial_t U=\partial_t u-\partial_t v_L$ as follows:

$$\partial_t U(t)=\sum_{j=1}^J \alpha_j(t)\iota_j\Lambda W_{[\lambda_j(t)]}+g_1(t), \tag{5.10}$$

where, for all $j\in[\![1,J]\!]$,

$$\int g_1(t)\Lambda W_{[\lambda_j(t)]}=0. \tag{5.11}$$

We also define

$$\beta_j(t)=-\iota_j\int(\Lambda W)_{[\lambda_j(t)]}\partial_t U(t)\,dx. \tag{5.12}$$

In this section we prove the following result.

PROPOSITION 5.1. *For all large n, for all $t\in[\tilde{t}_n, t_n]$,*

$$\delta \lesssim \gamma^{(N-2)/4} + o_n(1) \tag{5.13}$$

$$\left|\beta_j - \|\Lambda W\|_{L^2}^2 \lambda_j'\right| \leqslant C\gamma^{N/4} + o_n(1) \quad \text{for all } j\in[\![1,J]\!], \tag{5.14}$$

$$\left|\frac{1}{2}\sum_{j=1}^{J}\beta_j^2 - \kappa_1 \sum_{1\leqslant j\leqslant J-1} \iota_j\iota_{j+1}\left(\frac{\lambda_{j+1}}{\lambda_j}\right)^{(N-2)/2}\right| \leqslant C\gamma^{(N-1)/2} + o_n(1), \tag{5.15}$$

$$\left|\lambda_j\beta_j' + \kappa_0\left(\iota_j\iota_{j+1}\left(\frac{\lambda_{j+1}}{\lambda_j}\right)^{(N-2)/2} - \iota_j\iota_{j-1}\left(\frac{\lambda_j}{\lambda_{j-1}}\right)^{(N-2)/2}\right)\right| \leqslant C\gamma^{(N-1)/2} + o_n(1) \quad \text{for all } j\in[\![1,J]\!], \tag{5.16}$$

where $o_n(1)$ goes to zero as $n\to\infty$, uniformly with respect to n and $t\in[\tilde{t}_n, t_n]$, and

$$\kappa_0 = \frac{N^{N/2-1}(N-2)^{N/2}}{2}\int \frac{1}{|x|^{N-2}} W^{(N+2)/(N-2)}dx,$$

$$\kappa_1 = \|\Lambda W\|_{L^2}^2 \int \frac{(N(N-2))^{N/2-1}}{|x|^{N-2}} W^{(N+2)/(N-2)}\, dx.$$

Let us mention that the constants κ_0 and κ_1 can be computed explicitly. However, we will not need their exact values in the sequel.

The proof of Proposition 5.1 is based on the results of §4.2 on non-radiative solutions. We will first consider, in §5.2, a profile decomposition for a sequence of

$$\vec{U}(s_n) = \vec{u}(s_n) - \vec{v}_L(s_n),$$

(where $s_n\to\infty$), observing that any non-linear profile in this decomposition is non-radiative. In §5.3, we will use this observation and an expansion of the energy to deduce estimates on λ_j, β_j, γ and δ. In §5.4, we will obtain estimates on λ_j' and β_j' using equation (1.1) and the orthogonality conditions, and conclude the proof of (5.13)–(5.16).

We refer to [38, Proposition 3.8] for modulation equations similar to the ones of Proposition 5.1, in the context of equivariant wave maps, when $J=2$, at the threshold energy (so that in this work the analogue of the radiation term v_L is 0). One important novelty here compared to [38, Proposition 3.8] and its proof, is the proof that the non-linear profiles associated with a sequence $\{\vec{U}(s_n)\}$ are non-radiative solutions of (1.1) (see Lemma 5.2 below), yielding a crucial additional information.

5.2. Expansion along a sequence of times and renormalization

Consider a sequence of times $\{s_n\}_n$, with $s_n\in[\tilde{t}_n, t_n]$ for all n. Extracting subsequences, we define a partition of the interval $[\![1,J]\!]$ as follows. We let $1=j_1<j_2<\ldots<j_{K+1}=J+1$,

so that

$$[\![1,J]\!]=\bigcup_{k=1}^{K}[\![j_k,j_{k+1}-1]\!],$$

with

$$\lim_{n\to\infty}\frac{\lambda_{j_{k+1}}(s_n)}{\lambda_{j_k}(s_n)}=0\quad\text{for all } k\in[\![1,K-1]\!], \tag{5.17}$$

and,

$$\nu_j=\lim_{n\to\infty}\frac{\lambda_j(s_n)}{\lambda_{j_k}(s_n)}>0\quad\text{for all } k\in[\![1,K]\!]\text{ and all } j\in[\![j_k,j_{k+1}-1]\!]. \tag{5.18}$$

We note that $\nu_{j_k}=1$. In this subsection, we prove the following result.

Lemma 5.2. *Under the above assumptions, for all* $k\in[\![1,K]\!]$, *there exists* (V_0^k,V_1^k) *in* $\mathcal{H}$ *such that, denoting by* V^k *the solution of* (1.1) *with initial data* (V_0^k,V_1^k), *then* V^k *is defined on* $\{|x|>|t|\}$ *and is non-radiative. Furthermore, letting* $J^k=j_{k+1}-j_k$, $\boldsymbol{\iota}^k=(\iota_{j_k},\dots,\iota_{j_{k+1}-1})$ *and*

$$V_n^k(t,x)=\frac{1}{\lambda_{j_k}^{(N-2)/2}(s_n)}V^k\left(\frac{t}{\lambda_{j_k}(s_n)},\frac{x}{\lambda_{j_k}(s_n)}\right),$$

we have (*extracting subsequences if necessary*),

$$\lim_{n\to\infty}\left\|\vec{u}(s_n)-\vec{v}_L(s_n)-\sum_{k=1}^{K}\vec{V}_n^k(0)\right\|_{\mathcal{H}}=0 \tag{5.19}$$

and

$$d_{J^k,\boldsymbol{\iota}^k}(V_0^k,V_1^k)\leqslant C\varepsilon_0. \tag{5.20}$$

More precisely, after extraction,

$$\begin{cases} V_0^k=\displaystyle\sum_{j=j_k}^{j_{k+1}-1}\iota_jW_{(\nu_j)}+\check{h}_0^k\\ V_1^k=\displaystyle\sum_{j=j_k}^{j_{k+1}-1}\iota_j\check{\alpha}_j(\Lambda W)_{[\nu_j]}+\check{g}_1^k,\end{cases} \tag{5.21}$$

where

$$\check{h}_0^k=\operatorname*{w-lim}_{n\to\infty}\lambda_{j_k}^{(N-2)/2}(s_n)h(s_n,\lambda_{j_k}(s_n)\,\cdot\,), \tag{5.22}$$

$$\check{\alpha}_j=\lim_{n\to\infty}\alpha_j(s_n), \tag{5.23}$$

$$\check{g}_1^k=\operatorname*{w-lim}_{n\to\infty}\lambda_{j_k}^{N/2}(s_n)g_1(s_n,\lambda_{j_k}(s_n)\,\cdot\,). \tag{5.24}$$

Furthermore, we have

$$JE(W,0)=\sum_{k=1}^{K}E(\vec{V}^k(0)) \tag{5.25}$$

Note that the limits (5.18), (5.22), (5.23) and (5.24) imply the orthogonality conditions

$$\int \nabla \check{h}_0^k \cdot \nabla (\Lambda W)_{(\nu_j)} = \int \check{g}_1^k \cdot \nabla (\Lambda W)_{[\nu_j]} = 0 \quad \text{for all } j \in [\![j_k, j_{k+1}-1]\!]. \tag{5.26}$$

Proof of Lemma 5.2. In all the proof, we will denote

$$\mu_{k,n} = \lambda_{j_k}(s_n).$$

By (5.18),

$$\lim_{n\to\infty} \frac{\lambda_j(s_n)}{\mu_{k,n}} = \nu_j \quad \text{for all } j \in [\![j_k, j_{k+1}-1]\!].$$

Step 1. Denoting by

$$M(t) = \sum_{j=1}^{J} \iota_j W_{(\lambda_j(t))},$$

we see that, for $k \in [\![1, K]\!]$,

$$\mu_{k,n}^{N/2-1} M(s_n, \mu_{k,n} \cdot) \xrightarrow[n\to\infty]{} \sum_{j=j_k}^{j_{k+1}-1} \iota_j W_{(\nu_j)}.$$

Extracting subsequences, so that the limits (5.22)–(5.24) exist, we obtain

$$(\mu_{k,n}^{N/2-1}((u-v_L)(s_n,\mu_{k,n}\cdot)), \mu_{k,n}^{N/2}\partial_t(u-v_L)(s_n,\mu_{k,n}\cdot))) \xrightarrow[n\to\infty]{} (V_0^k, V_1^k), \tag{5.27}$$

where (V_0^k, V_1^k) is defined by (5.21). Note that, by (5.6),

$$\left\| (V_0^k, V_1^k) - \sum_{j=j_k}^{j_{k+1}-1} \iota_j (W_{(\nu_j)}, 0) \right\|_{\mathcal{H}} \lesssim \varepsilon_0, \tag{5.28}$$

and that, for $j \in [\![j_k, j_{k+1}-2]\!]$,

$$\frac{\nu_{j+1}}{\nu_j} \leqslant \lim_{n\to\infty} \gamma(\lambda(s_n)) \leqslant \varepsilon_0. \tag{5.29}$$

In particular, (5.20) is satisfied. The bounds (5.28) and (5.29) also imply, setting

$$\boldsymbol{\nu}^k = (\nu_{j_k}, \ldots, \nu_{j_{k+1}-1}),$$

that

$$\left| \|(V_0^k, V_1^k)\|_{\mathcal{H}}^2 - J_k \|\nabla W\|_{L^2}^2 \right| \lesssim \varepsilon_0 + \gamma(\boldsymbol{\nu}^k)^{N/2-2} = o_{\varepsilon_0}(1), \tag{5.30}$$

where $o_{\varepsilon_0}(1)$ goes to zero as ε_0 goes to zero. We have used the bound

$$\left|\int \nabla W_{(\nu_j)} \cdot \nabla W_{(\nu_\ell)}\right| \lesssim \gamma(\boldsymbol{\nu}^k)^{N/2-2}.$$

(see Claim A.1 in the appendix). Notice also that

$$\left|\|M(s_n)\|^2_{\dot{H}^1} - J\|\nabla W\|^2_{L^2}\right| = o_{\varepsilon_0}(1). \tag{5.31}$$

As a consequence of the weak limit (5.27), we see that the sequence $\{\vec{u}(s_n)-\vec{v}_L(s_n)\}_n$ has (after extraction of subsequences) a profile decomposition with profiles $(V_F^k)_{k\geqslant 1}$ and parameters $\{\mu_{k,n}, s_{k,n}\}_n$, $k\geqslant 1$, where, for $k\in[\![1,K]\!]$, $\mu_{k,n}=\lambda_{j_k}(s_n)$, $s_{k,n}=0$ and V_F^k is the solution of the free linear wave equation with initial data (V_0^k, V_1^k) (defined by (5.21)). Combining (5.27), (5.28), (5.30), (5.31) and the Pythagorean expansion of the profile decomposition, we obtain

$$\sum_{k\geqslant K+1} \|\vec{V}_F^k(0)\|^2_{\mathcal{H}} = o_{\varepsilon_0}(1). \tag{5.32}$$

As usual, we will denote by V^k the non-linear profile associated with V_F^k, $\{s_{k,n}\}_n$ and $\{\mu_{k,n}\}_n$.

Step 2. (Approximation for $\{|x|\geqslant|t|\}$ and lack of radiation) We claim that, for all $k\in[\![1,K]\!]$, the non-linear profile V^k is defined on $\{|x|\geqslant|t|\}$ and

$$V^k \in \mathsf{S}(\{|x|>|t|\}). \tag{5.33}$$

This follows from (5.28) and long-time perturbation theory. By Minkowski's inequality and scaling arguments,

$$\left\|\sum_{j=j_k}^{j_{k+1}-1} \iota_j W_{(\nu_j)}\right\|_{\mathsf{W}(\{|x|\geqslant|t|\})} \leqslant J\|W\|_{\mathsf{W}(\{|x|\geqslant|t|\})}.$$

One can check easily that the right-hand side of the preceding inequality is finite. This can be done directly. One can also use that W coincides for $|x|>|t|$ with the solution $\widetilde{W}$ of

$$(\partial_t^2-\Delta)\widetilde{W} = W^{(N+2)/(N-2)} \mathbb{1}_{\{|x|\geqslant|t|\}},$$

with

$$W^{(N+2)/(N-2)} \mathbb{1}_{\{|x|>|t|\}} \in L^1 L^2,$$

so that, by the Strichartz inequality (2.4),

$$\widetilde{W} \in \mathsf{W}(\{|x|>|t|\}).$$

By (5.29) and the Claim A.5 (with $h=0$) in the appendix, denoting

$$r_k=(\partial_t^2-\Delta)\left(\sum_{j=j_k}^{j_{k+1}-1}\iota_j W_{(\nu_j)}\right)-F\left(\sum_{j=j_k}^{j_{k+1}-1}\iota_j W_{(\nu_j)}\right),$$

where $F(z)=|z|^{4/(N-2)}z$, we have

$$\|1\!\!1_{\{|x|\geqslant|t|\}}r_k\|_{L^1L^2}=o_{\varepsilon_0}(1).$$

Taking $\varepsilon_0>0$ small enough, we deduce, by long time perturbation theory (Proposition 2.5), that V^k is defined on $\{|x|\geqslant|t|\}$ and satisfies (5.33).

In view of (5.32), all the non-linear profiles V^k, $k\geqslant K+1$, are globally defined and scatter. From the profile approximation property (Proposition 2.7), we obtain that, for large n, the solution $(\tau,x)\mapsto u(s_n+\tau,x)$ is defined on $\{|x|\geqslant|\tau|\}$, and that, for all $\ell\gg 1$,

$$u(s_n+\tau)=v_L(s_n+\tau)+\sum_{k=1}^{\ell}V_n^k(\tau)+w_n^\ell(\tau)+r_n^\ell(\tau), \tag{5.34}$$

where w_n^ℓ is the solution of the free wave equation (3.1) with initial data

$$\vec{u}(s_n)-\vec{v}_L(s_n)-\sum_{k=1}^{\ell}\vec{V}_n^k(0),$$

and r_n^ℓ satisfies

$$\lim_{\ell\to\infty}\limsup_{n\to\infty}\sup_{\tau\in\mathbb{R}}\|1\!\!1_{\{|x|\geqslant|\tau|\}}\nabla_{\tau,x}r_n^\ell(\tau)\|_{L^2}=0. \tag{5.35}$$

We next prove that, for all $k\geqslant 1$,

$$\lim_{n\to\infty}\sum_{\pm}\lim_{\tau\to\pm\infty}\int_{|x|\geqslant|\tau|}|\nabla_{\tau,x}V_n^k(\tau,x)|^2\,dx=0. \tag{5.36}$$

and that, for all $\ell\geqslant 1$,

$$\lim_{n\to\infty}\sum_{\pm}\lim_{\tau\to\pm\infty}\int_{|x|\geqslant|\tau|}|\nabla_{\tau,x}w_n^\ell(\tau,x)|^2\,dx=0. \tag{5.37}$$

The proof is similar to that of the analogous properties (3.31) and (3.32) in the proof of the exterior energy bound for the linearized equation close to a soliton. Using (5.34) and fixing $1\leqslant k<\ell$, we see that, for all τ,

$$\begin{aligned}\int_{|x|\geqslant|\tau|} &\nabla_{\tau,x}(u(s_n+\tau,x)-v_L(s_n+\tau,x))\cdot\nabla_{\tau,y}V_n^k(\tau,x)\,dx\\ &=\int_{|x|\geqslant|\tau|}|\nabla_{\tau,x}V_n^k(\tau,x)|^2\,dx\\ &\quad+\sum_{\substack{0\leqslant j\leqslant\ell\\ j\neq k}}\int_{|x|\geqslant|\tau|}\nabla_{\tau,x}V_n^j(\tau,x)\cdot\nabla_{\tau,x}V_n^k(\tau,x)\,dx\\ &\quad+\int_{|x|\geqslant|\tau|}\nabla_{\tau,x}w_n^\ell(\tau,x)\cdot\nabla_{\tau,x}V_n^k(\tau,x)\,dx\\ &\quad+\int_{|x|\geqslant|\tau|}\nabla_{\tau,x}^\ell r_n^\ell(\tau,x)\nabla_{\tau,x}V_n^k(\tau,x)\,dx.\end{aligned}\tag{5.38}$$

Since, for all j, V_n^j scatters in both time directions in $\{|x|\geqslant|\tau|\}$ (in the sense that it satisfies (2.10)), we have, using the pseudo orthogonality of the parameters as in the proof mentioned above,

$$\lim_{n\to\infty}\sum_{\pm}\lim_{\tau\to\pm\infty}\left|\int_{|x|\geqslant|\tau|}\nabla_{\tau,x}V_n^j(\tau,x)\cdot\nabla_{\tau,x}V_n^k(\tau,x)\,dx\right|=0\quad\text{for all } j\neq k,\tag{5.39}$$

and also, if $k\leqslant\ell$,

$$\lim_{n\to\infty}\sum_{\pm}\lim_{\tau\to\pm\infty}\left|\int_{|x|\geqslant|\tau|}\nabla_{\tau,x}w_n^\ell(\tau,x)\cdot\nabla_{\tau,x}V_n^k(\tau,x)\,dx\right|=0.$$

Furthermore, by (5.35),

$$\lim_{\ell\to\infty}\limsup_{n\to\infty}\sum_{\pm}\lim_{\tau\to\pm\infty}\left|\int_{|x|\geqslant|\tau|}\nabla_{\tau,x}r_n^\ell(\tau,x)\cdot\nabla_{\tau,x}V_n^k(\tau,x)\,dx\right|=0.$$

By the definition of v_L, we have that, for all n,

$$\begin{aligned}\lim_{\tau\to+\infty}&\int_{|x|\geqslant\tau}|\nabla_{t,x}(u(s_n+\tau,x)-v_L(s_n+\tau,x))|^2\,dx\\ &=\lim_{\sigma\to+\infty}\int_{|x|\geqslant\sigma-s_n}|\nabla_{t,x}(u(\sigma,x)-v_L(\sigma,x))|^2\,dx=0.\end{aligned}\tag{5.40}$$

On the other hand,

$$\begin{aligned}\lim_{\tau\to-\infty}&\int_{|x|\geqslant|\tau|}|\nabla_{t,x}(u(s_n+\tau,x)-v_L(s_n+\tau,x))|^2\,dx\\ &=\lim_{\sigma\to-\infty}\int_{|x|\geqslant s_n-\sigma}|\nabla_{t,x}(u(\sigma,x)-v_L(\sigma,x))|^2\,dx.\end{aligned}$$

By the small data theory, there exists a solution u_F of the free linear equation such that, if $A \gg 1$,

$$\lim_{t\to-\infty} \int_{|x|>A-t} |\nabla_{t,x}(u-u_F)(t,x)|^2\,dx = 0.$$

Combining with the large time asymptotics for linear wave equation, we deduce that there exists $g \in L^2([A,+\infty)$ (for a fixed $A \gg 1$), such that

$$\lim_{\tau\to-\infty} \int_{|x|\geqslant|\tau|} |\nabla_{t,x}(u(s_n+\tau,x)-v_L(s_n+\tau,x))|^2\,dx = \int_{\eta\geqslant s_n} |g(\eta)|^2\,d\eta. \tag{5.41}$$

Note that the right-hand side of (5.41) goes to zero as n goes to infinity. Combining (5.38)–(5.41), we obtain the desired estimate (5.36). A similar proof yields (5.37).

Step 3. (Consequence of the equirepartition of the energy) Let $k \geqslant K+1$. Then, by (5.32),

$$\sup_n \|V_n^k(0)\|_{\mathcal{H}}^2 = o_{\varepsilon_0}(1).$$

Using the small data theory and the equirepartition of the energy outside the wave cone for the free wave equation proved in [19] (see Theorem 3.1 above), we deduce, if ε_0 is small enough, that

$$\sum_{\pm} \lim_{t\to\pm\infty} \int_{|x|\geqslant|t|} |\nabla_{t,x}V_n^k(t,x)|^2\,dx \geqslant \frac{1}{2}\int |\nabla_{t,x}V_n^k(0,x)|^2\,dx \geqslant \frac{1}{4}\|V^k(0)\|_{\mathcal{H}}^2.$$

From (5.36), we deduce that $V^k \equiv 0$ for $k \geqslant K+1$. As a consequence, w_n^ℓ does not depend on ℓ if $\ell \geqslant K$. Setting $w_n = w_n^\ell$, we have, by (5.37),

$$\lim_{n\to\infty} \sum_{\pm} \lim_{\tau\to\pm\infty} \int_{|x|\geqslant|\tau|} |\nabla_{\tau,x} w_n(\tau,x)|^2\,dx = 0.$$

Since w_n is a solution of the free wave equation, we deduce (using Theorem 3.1 again), that

$$\lim_{n\to\infty} \int |\nabla_{t,x} w_n(0,x)|^2\,dx = 0,$$

and hence (5.19). It remains to observe that, if $1 \leqslant k \leqslant K$, the property (5.36) implies, since the time parameter $s_{k,n}$ is identically zero, that

$$\sum_{\pm} \lim_{t\to\pm\infty} \int_{|x|\geqslant|t|} |\nabla_{t,x} V^k(t,x)|^2\,dx = 0,$$

i.e. that V^k is non-radiative at $t=0$. □

For further use, we state the following important consequence of the proof of Lemma 5.2 (see (5.34)).

CLAIM 5.3. (Exterior expansion for all time) *We have, for* $|x| \geqslant |\tau|$,

$$u(s_n+\tau) = v_L(s_n+\tau) + \sum_{k=1}^{K} V_n^k(\tau) + r_n(\tau), \tag{5.42}$$

where

$$\lim_{n\to\infty} \sup_{\tau} \int_{|x|\geqslant|\tau|} |\nabla_{\tau,x} r_n|^2 \, dx = 0.$$

5.3. Estimates on λ_j and β_j

Recall from the introduction of this section the definitions of $h(t)$, $g_1(t)$, $\alpha_j(t)$, $\beta_j(t)$, $\gamma(t)$ and $\delta(t)$.

LEMMA 5.4. *There exists a constant* $C>0$, *depending only on* J *and* N, *such that, under the preceding assumptions, for all* $t\in[\tilde{t}_n, t_n]$, *we have*

$$\|(h, g_1)\|_{\mathcal{H}} \leqslant o_n(1) + C(\gamma^{N/4} + \delta^{N/(N-2)}), \tag{5.43}$$

$$\left|\delta^2 - \sum_{j=1}^{J} \alpha_j^2 \|\Lambda W\|_{L^2}^2\right| \leqslant o_n(1) + C(\gamma^{(N-1)/2} + \delta^{2(N-1)/(N-2)}), \tag{5.44}$$

$$\left|\beta_j + \alpha_j \|\Lambda W\|_{L^2}^2\right| \leqslant o_n(1) + C(\gamma^{N/4} + \delta^{N/(N-2)}), \tag{5.45}$$

where, in all inequalities, $o_n(1)$ *goes to zero as* n *goes to infinity* uniformly with respect to $t\in[\tilde{t}_n, t_n]$.

Proof. Note that (5.43) and (5.44) are time-dependent version of the estimates (4.7) and (4.8) for non-radiative solution. We will prove (5.44) as a consequence of (4.8). The proof of (5.43) using (4.7) is very similar and we omit it. We argue by contradiction. If (5.44) does not hold, there exists, after extraction, a sequence of times $\{s_n\}_n$ with $s_n\in[\tilde{t}_n, t_n]$, and an $\varepsilon_1>0$ such that, for all n,

$$\left|\delta^2(s_n) - \sum_{j=1}^{J} \alpha_j^2(s_n) \|\Lambda W\|_{L^2}^2\right| \geqslant C(\delta^{2(N-1)/(N-2)}(s_n) + \gamma^{(N-1)/2}(s_n)) + \varepsilon_1. \tag{5.46}$$

Using Lemma 5.2, we have

$$\lim_{n\to\infty} \left\| \vec{u}(s_n) - \vec{v}_L(s_n) - \sum_{k=1}^{K} \vec{V}_n^k(0) \right\|_{\mathcal{H}} = 0, \tag{5.47}$$

where the rescaled profiles V_n^k are defined as in Lemma 5.2:

$$V_n^k(t,x) = \frac{1}{\lambda_{j_k}^{N/2-1}(s_n)} V^k\left(\frac{t}{\lambda_{j_k(s_n)}}, \frac{x}{\lambda_{j_k(s_n)}}\right),$$

and V^k is a *non-radiative* solution to the non-linear wave equation with initial data (V_0^k, V_1^k), and

$$\begin{cases} V_0^k = \sum_{j=j_k}^{j_{k+1}-1} \iota_j W_{(\nu_j)} + \check{h}_0^k, \\ V_1^k = \sum_{j=j_k}^{j_{k+1}-1} \iota_j \check{\alpha}_j \Lambda W_{[\nu_j]} + \check{g}_1^k, \end{cases}$$

where $\check{h}_0^k$ and $\check{g}_1^k$ are defined as weak limits of $h(s_n)$ and $g_1(s_n)$, respectively, after an appropriate rescaling (see (5.22) and (5.24)), and

$$\check{\alpha}_j = \lim_{n\to\infty} \alpha_j(s_n).$$

Since, for all $k \in [\![1, K]\!]$, V^k is non-radiative, we can use the estimate (4.8), which writes

$$\left| \delta_k^2 - \sum_{j=j_k}^{j_{k+1}-1} \check{\alpha}_j^2 \|\Lambda W\|_{L^2}^2 \right| \leqslant C(\delta_k^{2(N-1)/(N-2)} + \gamma_k^{(N-1)/2}), \tag{5.48}$$

where

$$\delta_k^2 = \|\partial_t V^k(0)\|_{L^2}^2 + \|h_0^k\|_{\dot{H}^1}^2$$

and

$$\gamma_k = \max_{j_k \leqslant j \leqslant j_{k+1}-2} \frac{\nu_j}{\nu_{j+1}}$$

(as usual, if $j_{k+1} = 1 + j_k$, we let $\gamma_k = 0$).

Observe that

$$\lim_{n\to\infty} \gamma(s_n) = \max_{1\leqslant k\leqslant K} \gamma_k$$

and, by the expansion (5.47),

$$\lim_{n\to\infty} \delta^2(s_n) = \sum_{k=1}^{K} \delta_k^2.$$

Summing up (5.48), we deduce that

$$\left| \delta^2(s_n) - \sum_{j=1}^{J} \alpha_j^2(s_n) \|\Lambda W\|_{L^2}^2 \right| \leqslant C_J(\delta^{2(N-1)/(N-2)}(s_n) + \gamma^{(N-1)/2}(s_n) + o_n(1)),$$

where the constant C_J depends only on J. This contradicts (5.46) for large n. The proof is complete.

We next compare α_j and β_j and prove (5.45). We have, expanding $\partial_t U$ by (5.10),

$$\begin{aligned}\beta_j(t) &= -\iota_j \int (\Lambda W)_{[\lambda_j]} \partial_t U\\ &= -\iota_j \underbrace{\int (\Lambda W)_{[\lambda_j]} g_1}_{0} - \alpha_j \|\Lambda W\|_{L^2}^2 - \iota_j \iota_k \sum_{k\neq j} \alpha_k \int (\Lambda W)_{[\lambda_j]} (\Lambda W)_{[\lambda_k]}.\end{aligned}$$

By (5.44) and Claim A.1 in the appendix,

$$\begin{aligned}\left|\alpha_k \int (\Lambda W)_{[\lambda_j]} (\Lambda W)_{[\lambda_k]}\right| &\lesssim (\delta + \gamma^{(N-1)/2} + o_n(1))\gamma^{N/2-2}\\ &\lesssim \delta^{N/(N-2)} + \gamma^{N(N-4)/4}\gamma^{N-5/2} + o_n(1),\end{aligned}$$

which yields (5.45). □

We next prove the following result.

LEMMA 5.5. (Expansion of the energy)

$$\left|\frac{1}{2}\delta^2 - \kappa_1' \sum_{1\leqslant j\leqslant J-1} \iota_j \iota_{j+1} \left(\frac{\lambda_{j+1}}{\lambda_j}\right)^{(N-2)/2}\right| \lesssim o_n(1) + \gamma^{(N-1)/2}, \tag{5.49}$$

$$\delta \lesssim \gamma^{(N-2)/4} + o_n(1), \tag{5.50}$$

$$\|(h(t), g_1(t))\|_{\mathcal{H}} \lesssim o_n(1) + \gamma^{N/4}, \tag{5.51}$$

where $o_n(1)\to 0$ as $n\to\infty$, uniformly with respect to $t\in[\tilde{t}_n, t_n]$, and

$$\kappa_1' = \int \frac{(N(N-2))^{N/2-1}}{|x|^{N-2}} W^{(N+2)/(N-2)}\, dx = \frac{1}{\|\Lambda W\|_{L^2}^2}\kappa_1.$$

Proof. Note that (5.51) follows from (5.50) and (5.43).

We are thus left with proving (5.50) and (5.49). Recall that

$$\lim_{t\to\infty} E(\vec{u}(t) - \vec{v}_L(t)) = JE(W, 0).$$

Expanding the energy

$$E(\vec{u} - \vec{v}_L) = E\left(\sum_{j=1}^J \iota_j W_{(\lambda_j)} + h, \sum_{j=1}^J \alpha_j (\Lambda W)_{[\lambda_j]} + g_1\right),$$

we obtain, in view of the inequality

$$\begin{aligned}\Bigg|\frac{N-2}{2N}\Big|\sum_{j=1}^{J} y_j+h\Big|^{2N/(N-2)}-\frac{N-2}{2N}\sum_{j=1}^{J}|y_j|^{2N/(N-2)}\\ -\sum_{j=1}^{J}|y_j|^{4/(N-2)}y_jh-\sum_{\substack{1\leqslant j,k\leqslant J\\ j\neq k}}|y_j|^{4/(N-2)}y_jy_k\Bigg|\\ \lesssim |h|^{2N/(N-2)}+\sum_{j=1}^{J}|y_j|^{4/(N-2)}|h|^2\\ +\sum_{1\leqslant j<k\leqslant J}\big(\min\{|y_j|^{4/(N-2)}y_k^2,|y_k|^{4/(N-2)}y_j^2\}\\ +\min\{|y_j|^{(N+2)/(N-2)}|y_k|,|y_k|^{(N+2)/(N-2)}|y_j|\}\big),\end{aligned}$$

proved in Appendix A.3, and the estimate

$$\int|\Lambda W_{[\lambda_j]}\Lambda W_{[\lambda_k]}|\lesssim\gamma^{N/2-2}$$

(see Claim A.1 in the appendix),

$$\begin{aligned}\Bigg|\frac{J}{2}\|\nabla W\|_{L^2}^2+\sum_{1\leqslant j<k\leqslant J}\iota_j\iota_k\int\nabla W_{(\lambda_j)}\cdot\nabla W_{(\lambda_k)}+\sum_{1\leqslant j\leqslant J}\iota_j\int\nabla W_{(\lambda_j)}\cdot\nabla h\\ -\frac{N-2}{2N}J\int W^{2N/(N-2)}-\sum_{j=1}^{J}\iota_j\int W_{(\lambda_j)}^{(N+2)/(N-2)}h\\ -\sum_{\substack{1\leqslant j,k\leqslant J\\ k\neq j}}\iota_j\iota_k\int W_{(\lambda_j)}^{(N+2)/(N-2)}W_{(\lambda_k)}+\frac{1}{2}\sum_{j=1}^{J}\alpha_j^2\|\Lambda W\|_{L^2}^2-JE(W,0)\Bigg|\\ \lesssim\sum_{1\leqslant j<k\leqslant J}\int\big(\min\{W_{(\lambda_j)}^{4/(N-2)}W_{(\lambda_k)}^2,W_{(\lambda_k)}^{4/(N-2)}W_{(\lambda_j)}^2\}\\ +\min\{W_{(\lambda_j)}^{(N+2)/(N-2)}W_{(\lambda_k)},W_{(\lambda_k)}^{(N+2)/(N-2)}W_{(\lambda_j)}\}\big)\\ +\|g_1\|_{L^2}^2+\|\nabla h\|_{L^2}^2+\gamma^{N/2-2}\delta^2+o_n(1).\end{aligned}$$

We note that, for all j and k,

$$\int\nabla W_{(\lambda_j)}\cdot\nabla W_{(\lambda_k)}=\int W_{(\lambda_j)}^{(N+2)/(N-2)}W_{(\lambda_k)},$$
$$\int\nabla W_{(\lambda_j)}\cdot\nabla h=\int W_{(\lambda_j)}^{(N+2)/(N-2)}h.$$

Furthermore, we have

$$\begin{aligned}\int \min\{W_{(\lambda_j)}^{(N+2)/(N-2)} W_{(\lambda_k)}, W_{(\lambda_k)}^{(N+2)/(N-2)} W_{(\lambda_j)}\} \\ + \int \min\{W_{(\lambda_j)}^{4/(N-2)} W_{(\lambda_k)}^2, W_{(\lambda_k)}^{4/(N-2)} W_{(\lambda_j)}^2\} \\ \lesssim \int W_{(\lambda_j)}^{N/(N-2)} W_{(\lambda_k)}^{N/(N-2)} \lesssim \gamma^{N/2},\end{aligned}$$

by Claim A.1.

As a consequence, using also the estimate (5.43) on h and g_1, we have

$$\begin{aligned}\left| \frac{1}{2} \sum_{j=1}^{J} \alpha_j^2 \|\Lambda W\|_{L^2}^2 - \sum_{1\leqslant j<k\leqslant J} \iota_j \iota_k \int W_{(\lambda_j)}^{(N+2)/(N-2)} W_{(\lambda_k)} \right| \\ \lesssim \gamma^{N/2} + \gamma^{N/2-2} \delta^2 + \delta^{2N/(N-2)} + o_n(1).\end{aligned} \tag{5.52}$$

We next estimate, for $j<k$,

$$\begin{aligned}\int W_{(\lambda_j)}^{(N+2)/(N-2)} W_{(\lambda_k)} &= \left(\frac{\lambda_j}{\lambda_k}\right)^{(N-2)/2} \int W^{(N+2)/(N-2)}(x) W\left(\frac{\lambda_j x}{\lambda_k}\right) dx \\ &= \left(\frac{\lambda_k}{\lambda_j}\right)^{(N-2)/2} \int \frac{(N(N-2))^{(N-2)/2}}{|x|^{N-2}} W^{(N+2)/(N-2)}\, dx \\ &\quad + \mathcal{O}\left(\left(\frac{\lambda_k}{\lambda_j}\right)^{(N+1)/2} \int W^{(N+2)/(N-2)} \frac{1}{|x|^{N-1/2}}\, dx\right),\end{aligned}$$

where we have used

$$\left| W(x) - \frac{((N-2)N)^{(N-2)/2}}{|x|^{N-2}} \right| \lesssim \frac{1}{|x|^{N-1/2}}.$$

In particular, if $j<k-1$, we see that

$$\int W_{(\lambda_j)}^{(N+2)/(N-2)} W_{(\lambda_k)} \lesssim \gamma^{N-2}.$$

Combining with (5.44) and (5.52), we obtain

$$\delta^2 \lesssim \gamma^{(N-2)/2} + \gamma^{N/2-2} \delta^2 + \delta^{2N/(N-2)} + o_n(1),$$

which yields $\delta \lesssim \gamma^{(N-2)/4} + o_n(1)$, i.e. (5.50). Going back to (5.52), we deduce (5.49). □

5.4. System of equations and estimates on the derivatives

Under the above assumptions, we let as before $U(t)=u(t)-v_L(t)$, so that

$$h(t)=U(t)-\sum_{j=1}^{J}\iota_j W_{(\lambda_j)}=U(t)-M(t).$$

Expanding the non-linear wave equation (1.1), we see that $(h(t),\partial_t U(t))$ satisfy the following system of equations for $t\in[\tilde{t}_n,t_n]$:

$$\begin{cases} \dfrac{\partial h}{\partial t}=\dfrac{\partial U}{\partial t}+\displaystyle\sum_{j=1}^{J}\iota_j\lambda_j'(t)(\Lambda W)_{[\lambda_j(t)]}, \\ \dfrac{\partial}{\partial t}\left(\dfrac{\partial U}{\partial t}\right)-\Delta h=F(U)-\displaystyle\sum_{j=1}^{J}F(\iota_j W_{(\lambda_j)})+\sigma(h,v_L), \end{cases} \tag{5.53}$$

where

$$\sigma(h,v_L):=F(M+h+v_L)-F(M+h) \tag{5.54}$$

satisfies

$$|\sigma(h,v_L)|\lesssim|v_L(t)|^{(N+2)/(N-2)}+\sum_{j=1}^{J}(W_{(\lambda_j)}^{4/(N-2)}+|h(t)|^{4/(N-2)})|v_L(t)|. \tag{5.55}$$

We next estimate $\lambda_j'(t)$, using the orthogonality condition (5.8) and the first equation in (5.53). More precisely, we will prove the following.

LEMMA 5.6. (Derivative of the scaling parameters)

$$|\lambda_j'+\alpha_j|\lesssim\gamma^{N/4}+o_n(1), \tag{5.56}$$

where $o_n(1)$ goes to zero as $n\to\infty$, uniformly with respect to $t\in[\tilde{t}_n,t_n]$.

Proof. According to (5.8),

$$\int h(t)\frac{1}{\lambda_j^{N/2}}(\Delta\Lambda W)\left(\frac{x}{\lambda_j(t)}\right)dx=0 \quad \text{for all } t\in I.$$

Differentiating with respect to t, we obtain

$$0=\int\frac{\partial h}{\partial t}\frac{1}{\lambda_j^{N/2}}(\Delta\Lambda W)\left(\frac{x}{\lambda_j}\right)dx-\frac{N}{2}\lambda_j'\int h\frac{1}{\lambda_j^{1+N/2}}(\Lambda_0\Delta\Lambda W)\left(\frac{x}{\lambda_j}\right)dx,$$

where $\Lambda_0 = \frac{1}{2}N + x\cdot\nabla$. By the first equation in (5.53),

$$\begin{aligned}0 &= \int \frac{\partial U}{\partial t}\frac{1}{\lambda_j^{N/2}}(\Delta\Lambda W)\left(\frac{x}{\lambda_j}\right)dx \\ &\quad + \sum_{k=1}^{J}\iota_k\lambda_k'\int \frac{1}{\lambda_k^{N/2}}(\Lambda W)\left(\frac{x}{\lambda_k}\right)\frac{1}{\lambda_j^{N/2}}(\Delta\Lambda W)\left(\frac{x}{\lambda_j}\right)dx \\ &\quad - \frac{N}{2}\lambda_j'\int h\frac{1}{\lambda_j^{1+N/2}}(\Lambda_0\Delta\Lambda W)\left(\frac{x}{\lambda_j}\right)dx.\end{aligned}$$

In view of the definition (5.10) of g_1, we have

$$\partial_t U = \sum_k \iota_k\alpha_k(\Lambda W)_{[\lambda_k]} + g_1.$$

By the estimate (5.51) on g_1, the bound (5.50) on δ and the estimate

$$\left|\int (\Lambda W)_{[\lambda_j]}(\Delta\Lambda W)_{[\lambda_k]}\right| \lesssim \gamma^{N/2-2}, \quad j\neq k$$

(see (A.5) in the appendix), we obtain

$$\begin{aligned}&\int \frac{\partial U}{\partial t}\frac{1}{\lambda_j^{N/2}}(\Delta\Lambda W)\left(\frac{x}{\lambda_j}\right) \\ &\quad = \int g_1\frac{1}{\lambda_j^{N/2}}(\Delta\Lambda W)\left(\frac{x}{\lambda_j}\right) - \alpha_j\iota_j\|\Lambda W\|^2_{\dot H^1} + \sum_{k\neq j}\iota_k\alpha_k\int (\Lambda W)_{[\lambda_j]}(\Delta\Lambda W)_{[\lambda_k]} \\ &\quad = -\alpha_j\iota_j\|\Lambda W\|^2_{\dot H^1} + O(\gamma^{N/4}).\end{aligned}$$

By the estimate (5.51) on h,

$$\left|\lambda_j'\int h\frac{1}{\lambda_j^{1+N/2}}(\Lambda_0\Delta\Lambda W)\left(\frac{x}{\lambda_j}\right)dx\right| \lesssim |\lambda_j'|\,\|\nabla h\|_{L^2} \lesssim (\gamma^{N/4}+o_n(1))|\lambda_j'|.$$

Combining, we obtain, for all j,

$$\bigl|\alpha_j\|\Lambda W\|^2_{\dot H^1} + \lambda_j'\|\Lambda W\|^2_{\dot H^1}\bigr| \lesssim \gamma^{N/4}(|\lambda_j'|+1) + \gamma^{1/2}\sum_{k\neq j}|\lambda_k'| + o_n(1),$$

and thus, letting $\boldsymbol{\alpha} = (\alpha_1, ..., \alpha_J)$,

$$|\boldsymbol{\lambda}' + \boldsymbol{\alpha}| \lesssim |\boldsymbol{\lambda}'|\gamma^{1/2} + \gamma^{N/4} + o_n(1).$$

This implies, recalling that, by (5.50), $\delta \lesssim \gamma^{(N-2)/4}$,

$$|\boldsymbol{\lambda}'| \lesssim |\boldsymbol{\alpha}| + \gamma^{N/4} \lesssim \delta + \gamma^{N/4} + o_n(1) \lesssim \gamma^{(N-2)/4} + o_n(1).$$

The desired estimate (5.56) follows immediately from the two bounds above. □

LEMMA 5.7. (Second derivative of the scaling parameter) *For all* $j \in [\![1, J]\!]$,

$$\left|\lambda_j\beta_j' + \kappa_0\left(\iota_j\iota_{j+1}\left(\frac{\lambda_{j+1}}{\lambda_j}\right)^{N/2-1} - \iota_j\iota_{j-1}\left(\frac{\lambda_j}{\lambda_{j-1}}\right)^{N/2-1}\right)\right| \lesssim \gamma^{(N-1)/2} + o_n(1), \tag{5.57}$$

where κ_0 *is defined in Proposition* 5.1, *and, by definition,* $\iota_0 = \iota_{J+1} = 0$.

Note that β_j is, according to (5.45), (5.50) and (5.56), proportionate to λ_j' up to lower-order terms, so that (5.57) can be interpreted as an estimate on the second derivative of λ_j.

Proof. Differentiating the definition (5.12) of β_j, we obtain

$$\lambda_j\beta_j'(t) = \iota_j\lambda_j' \int (\Lambda_0\Lambda W)_{[\lambda_j]}\partial_t U - \iota_j\lambda_j \int (\Lambda W)_{[\lambda_j]}\partial_t^2 U.$$

We first prove that the first term of this sum is small. Using the expansion (5.10) of $\partial_t U$, we obtain

$$\int (\Lambda_0\Lambda W)_{[\lambda_j]}\partial_t U = \int (\Lambda_0\Lambda W)_{[\lambda_j]} g_1 + \iota_j\alpha_j \underbrace{\int (\Lambda_0\Lambda W)_{[\lambda_j]}(\Lambda W)_{[\lambda_j]}}_{0} + \sum_{k\neq j}\int \iota_k\alpha_k(\Lambda_0\Lambda W)_{[\lambda_j]}(\Lambda W)_{[\lambda_k]}.$$

Hence, by (5.50), (5.51), (5.44), (5.56) and the estimate

$$\left|\int (\Lambda_0\Lambda W)_{[\lambda_j]}(\Lambda W)_{[\lambda_k]}\right| \lesssim \gamma^{N/2-2}$$

(see (A.2) in the appendix), we obtain

$$\left|\lambda_j' \int (\Lambda_0\Lambda W)_{[\lambda_j]}\partial_t U\right| \lesssim \gamma^{(N-1)/2} + o_n(1). \tag{5.58}$$

By the second equation in (5.53), we have

$$\lambda_j \int (\Lambda W)_{[\lambda_j]}\partial_t^2 U$$

$$= -\int (\Lambda W)_{(\lambda_j)} L_{W_{(\lambda_j)}} h \tag{5.59}$$

$$+ \int (\Lambda W)_{(\lambda_j)}\sigma(h, v_L) \tag{5.60}$$

$$+ \int (\Lambda W)_{(\lambda_j)}\left(F(\iota_j W_{(\lambda_j)} + h) - F(\iota_j W_{(\lambda_j)}) - \frac{N+2}{N-2} W_{(\lambda_j)}^{4/(N-2)} h\right) \tag{5.61}$$

$$+ \int (\Lambda W)_{\lambda_j}(F(M+h) - F(M) + F(\iota_j W_{(\lambda_j)}) - F(\iota_j W_{(\lambda_j)} + h)) \tag{5.62}$$

$$+ \int (\Lambda W)_{(\lambda_j)}\left(F(M) - \sum_{k=1}^{J} F(\iota_k W_{(\lambda_k)})\right), \tag{5.63}$$

where

$$L_{W_{(\lambda_j)}} = -\Delta - \frac{N+2}{N-2} W_{(\lambda_j)}^{4/(N-2)}.$$

The leading term in this equality is (5.63). We first prove that the other terms are of order $\mathcal{O}(\gamma^{N/2}) + o_n(1)$.

Estimates on lower-order terms. We first note that, by integration by parts,

$$\int (\Lambda W)_{(\lambda_j)} L_{W_{(\lambda_j)}} h = \int L_{W_{(\lambda_j)}} (\Lambda W)_{(\lambda_j)} h = 0,$$

so that the term on the right-hand side of (5.59) is zero.

By Hölder inequality, and the estimate (5.55) on $\sigma(h, v_L)$, we have

$$|(5.60)| \lesssim \|v_L\|_{L^{2N/(N-2)}} + \|v_L\|_{L^{2N/(N-2)}}^{(N+2)/(N-2)}.$$

Since v_L is a solution to the linear wave equation, we have

$$\lim_{n\to\infty} \|v_L(t)\|_{L^{2N/(N-2)}} = 0,$$

which proves that the term (5.60) is $o_n(1)$.

To bound (5.61), we use the inequality

$$|F(a+b) - F(a) - F'(a)b| \lesssim b^2 \mathbb{1}_{\{|b|\leqslant|a|\}} a^{(6-N)/(N-2)} + b^{(N+2)/(N-2)} \mathbb{1}_{\{|b|\geqslant|a|\}}$$

proved in the appendix (see Claim A.6). We obtain

$$|(5.61)| \lesssim \int |(\Lambda W)_{(\lambda_j)}| \, h^2 \mathbb{1}_{\{|h|\leqslant W_{(\lambda_j)}\}} W_{(\lambda_j)}^{(6-N)/(N-2)} + \int |(\Lambda W)_{(\lambda_j)}| \, h^{(N+2)/(N-2)} \mathbb{1}_{\{|h|\geqslant W_{(\lambda_j)}\}}.$$

Since $|(\Lambda W)_{(\lambda_j)}| \lesssim W_{(\lambda_j)}$, we deduce

$$|(5.61)| \lesssim \int h^2 W_{(\lambda_j)}^{4/(N-2)} + \int |h|^{2N/(N-2)} \lesssim \|h\|_{L^{2N/(N-2)}}^2 + \|h\|_{L^{2N/(N-2)}}^{2N/(N-2)} \lesssim \gamma^{N/2} + o_n(1),$$

where we have used the estimate (5.51) on h.

To bound (5.62), we distinguish between the case $N \geqslant 7$ and the case $N=5$. If $N \geqslant 7$, we use the inequality

$$|F(a+b+c) - F(a+b) - F(a+c) + F(a)| \lesssim |c| \, |b|^{(N+2)/2(N-2)} |a|^{(6-N)/2(N-2)}$$

(see again Claim A.6), with $a=\iota_j W_{(\lambda_j)}$, $b=\sum_{k\neq j}\iota_k W_{(\lambda_k)}$ and $c=h$. We obtain

$$|(5.62)|\lesssim\int |\Lambda W_{(\lambda_j)}|\, W_{(\lambda_j)}^{(6-N)/2(N-2)}\,|h|\left|\sum_{j\neq k}\iota_k W_{(\lambda_k)}\right|^{(N+2)/2(N-2)}.$$

Since $|(\Lambda W)_{(\lambda_j)}|\lesssim W_{(\lambda_j)}$, we deduce that

$$\begin{aligned}|(5.62)|&\lesssim\sum_{k\neq j}\int |h| W_{(\lambda_k)}^{(N+2)/2(N-2)} W_{(\lambda_j)}^{(N+2)/2(N-2)}\\ &\lesssim \|h\|_{L^{2N/(N-2)}}\sum_{k\neq j}\left(\int W_{(\lambda_k)}^{N/(N-2)} W_{(\lambda_j)}^{N/(N-2)}\right)^{(N+2)/2N}.\end{aligned}$$

By the estimate (5.51) on h and the bound

$$\int W_{(\lambda_k)}^{N/(N-2)} W_{(\lambda_j)}^{N/(N-2)}\lesssim\gamma^{N/2}$$

(see Claim A.1), we deduce that, if $N\geqslant 7$,

$$|(5.62)|\lesssim\gamma^{(N+1)/2}+o_n(1).$$

If $N=5$, the inequality

$$|F(a+b+c)-F(a+b)-F(a+c)+F(a)|\lesssim|c|\,|b|\,(|a|+|b|+|c|)^{1/3}$$

proved in Claim A.6, with $a=\iota_j W_{(\lambda_j)}$, $b=\sum_{k\neq j}\iota_k W_{(\lambda_k)}$ and $c=h$, yields

$$|(5.62)|\lesssim\sum_{k\neq j}\left(\int W_{(\lambda_k)}^{4/3} W_{(\lambda_j)}|h|+\int W_{(\lambda_k)} W_{(\lambda_j)}^{4/3}|h|+\int |h|^{4/3} W_{(\lambda_j)} W_{(\lambda_k)}\right).$$

By Hölder's inequality, we deduce that

$$|(5.62)|\lesssim\|h\|_{L^{10/3}}\sum_{\substack{1\leqslant k,\ell\leqslant J\\ k\neq\ell}}\|W_{(\lambda_\ell)} W_{(\lambda_k)}^{4/3}\|_{L^{10/7}}+\|h\|_{L^{10/3}}^{4/3}\sum_{\substack{1\leqslant k,\ell\leqslant J\\ k\neq\ell}}\left(\int W_{(\lambda_\ell)}^{5/3} W_{(\lambda_k)}^{5/3}\right)^{3/5}.$$

Together with the estimates (A.3) and (A.4) of Claim A.1 in the appendix, and the bound (5.51) of h, we deduce that, when $N=5$,

$$|(5.62)|\lesssim\gamma^{5/4}\gamma^{3/2}+\gamma^{5/3}\gamma^{3/2}+o_n(1)\lesssim\gamma^{11/4}+o_n(1).$$

As a conclusion,

$$|(5.59)|+|(5.60)|+|(5.61)|+|(5.62)|\lesssim\gamma^{N/2}+o_n(1).$$

Estimate on the leading term. To conclude the proof, we will prove that

$$\begin{aligned}\Bigg|\int_{\mathbb{R}^N}\Big(F(M)-\sum_{k=1}^{J}F(\iota_k W_{(\lambda_k)})\Big)(\Lambda W)_{(\lambda_j)}\,dx\\ -\kappa_0\bigg(\iota_{j+1}\Big(\frac{\lambda_{j+1}}{\lambda_j}\Big)^{N/2-1}-\iota_{j-1}\Big(\frac{\lambda_j}{\lambda_{j-1}}\Big)^{N/2-1}\bigg)\Bigg|\lesssim\gamma^{N/2}.\end{aligned}\tag{5.64}$$

We will prove (5.64) as a consequence of the following inequalities:

$$\begin{aligned}\Bigg|\int_{\mathbb{R}^N}\Big(F(M)-\sum_{k=1}^{J}F(\iota_k W_{(\lambda_k)})\Big)(\Lambda W)_{(\lambda_j)}\,dx\\ -\frac{N+2}{N-2}\int_{\mathbb{R}^N}(\iota_{j+1}W_{(\lambda_j)}^{4/(N-2)}W_{(\lambda_{j+1})}+\iota_{j-1}W_{(\lambda_j)}^{4/(N-2)}W_{(\lambda_{j-1})})(\Lambda W)_{(\lambda_j)}dx\Bigg|\lesssim\gamma^{N/2},\end{aligned}\tag{5.65}$$

(where, by convention, $\iota_0=\iota_{J+1}=0$),

$$\begin{aligned}\Bigg|\int_{\mathbb{R}^N}W_{(\lambda_j)}^{4/(N-2)}W_{(\lambda_{j+1})}(\Lambda W)_{(\lambda_j)}\\ -\Big(\frac{\lambda_{j+1}}{\lambda_j}\Big)^{N/2-1}\frac{N^{N/2-1}(N-2)^{N/2+1}}{2(N+2)}\int\frac{1}{|x|^{N-2}}W^{(N+2)/(N-2)}dx\Bigg|\lesssim\gamma^{N/2}\end{aligned}\tag{5.66}$$

for $1\leqslant j\leqslant J-1$, and

$$\begin{aligned}\Bigg|\int_{\mathbb{R}^N}W_{(\lambda_j)}^{4/(N-2)}W_{(\lambda_{j-1})}(\Lambda W)_{(\lambda_j)}\\ +\Big(\frac{\lambda_j}{\lambda_{j-1}}\Big)^{N/2-1}\frac{N^{N/2-1}(N-2)^{N/2+1}}{2(N+2)}\int\frac{1}{|x|^{N-2}}W^{(N+2)/(N-2)}dx\Bigg|\lesssim\gamma^{N/2}\end{aligned}\tag{5.67}$$

for $2\leqslant j\leqslant J$.

Proof of (5.65). We adopt the convention $\lambda_0=+\infty$ and $\lambda_{J+1}=0$. We first notice that there exists a constant C such that, for any $k,\ell\in[\![1,J]\!]$, we have

$$\sqrt{\lambda_{k+1}\lambda_k}\leqslant|x|\leqslant\sqrt{\lambda_{k-1}\lambda_k}\quad\Longrightarrow\quad W_{(\lambda_\ell)}\lesssim W_{(\lambda_k)}\tag{5.68}$$

(this follows easily from the facts that W is positive and $W(x)\approx 1/|x|^{N-2}$ for large $|x|$). To prove (5.65), we write

$$\int_{\mathbb{R}^N}P_j(x)\,dx=\sum_{k=1}^{J}\int_{\sqrt{\lambda_k\lambda_{k+1}}\leqslant|x|\leqslant\sqrt{\lambda_k\lambda_{k-1}}}P_j(x)\,dx,\tag{5.69}$$

where

$$P_j(x) := (\Lambda W)_{(\lambda_j)} \times \Big(F(M) - \sum_{\ell=1}^{J} F(\iota_\ell W_{(\lambda_\ell)}) - \frac{N+2}{N-2}\iota_{j+1} W_{(\lambda_j)}^{4/(N-2)} W_{(\lambda_{j+1})} - \frac{N+2}{N-2}\iota_{j-1} W_{(\lambda_j)}^{4/(N-2)} W_{(\lambda_{j-1})}\Big).$$

If $j \neq k$, using that, if $\sqrt{\lambda_k\lambda_{k+1}} \leqslant |x| \leqslant \sqrt{\lambda_k\lambda_{k-1}}$,

$$|F(M) - F(\iota_k W_{(\lambda_k)})| \lesssim W_{(\lambda_k)}^{4/(N-2)} \sum_{\ell\neq k} W_{(\lambda_\ell)},$$

and that $|(\Lambda W)_{(\lambda_j)}| \lesssim W_{(\lambda_j)}$, we obtain

$$\begin{aligned}
&\int_{\sqrt{\lambda_k\lambda_{k+1}} \leqslant |x| \leqslant \sqrt{\lambda_k\lambda_{k-1}}} |P_j(x)|\, dx \\
&\lesssim \sum_{\ell\neq k} \int_{\sqrt{\lambda_k\lambda_{k+1}} \leqslant |x| \leqslant \sqrt{\lambda_k\lambda_{k-1}}} W_{(\lambda_k)}^{4/(N-2)} W_{(\lambda_\ell)} W_{(\lambda_j)} \\
&\quad + \sum_{\ell\neq k} \int_{\sqrt{\lambda_k\lambda_{k+1}} \leqslant |x| \leqslant \sqrt{\lambda_k\lambda_{k-1}}} W_{(\lambda_\ell)}^{(N+2)/(N-2)} W_{(\lambda_j)} \\
&\quad + \int_{\sqrt{\lambda_k\lambda_{k+1}} \leqslant |x| \leqslant \sqrt{\lambda_k\lambda_{k-1}}} W_{\lambda_j}^{(N+2)/(N-2)} (W_{(\lambda_{j+1})} + W_{(\lambda_{j-1})})\, dx
\end{aligned}$$

Since $j \neq k$, and, by (5.68), $\ell \neq k$ implies $W_{(\lambda_\ell)} \lesssim W_{(\lambda_k)}$ on the domain of integration, we can bound all the terms of the right-hand side of the preceding inequality by

$$\begin{aligned}
&\sum_{\ell\neq k} \int_{\sqrt{\lambda_k\lambda_{k+1}} \leqslant |x| \leqslant \sqrt{\lambda_k\lambda_{k-1}}} W_{(\lambda_k)} W_{(\lambda_\ell)}^{(N+2)/(N-2)} \\
&\quad \lesssim \sum_{\ell\neq k} \int_{\sqrt{\lambda_k\lambda_{k+1}} \leqslant |x| \leqslant \sqrt{\lambda_k\lambda_{k-1}}} W_{(\lambda_k)}^{N/(N-2)} W_{(\lambda_\ell)}^{N/(N-2)} \lesssim \gamma^{N/2},
\end{aligned}$$

by Claim A.1 in the appendix. Thus, we have proved that, for $k \neq j$,

$$\int_{\sqrt{\lambda_k\lambda_{k+1}} \leqslant |x| \leqslant \sqrt{\lambda_k\lambda_{k-1}}} |P_j(x)|\, dx \lesssim \gamma^{N/2}. \tag{5.70}$$

Next, by Claim A.6, we observe, setting

$$E_j := \Big\{ x \in \mathbb{R}^N : \Big| \sum_{\ell\neq j} \iota_\ell W_{(\lambda_\ell)}(x) \Big| \leqslant W_{(\lambda_j)}(x) \Big\},$$

that, for all x such that $\sqrt{\lambda_j\lambda_{j+1}}\leqslant|x|\leqslant\sqrt{\lambda_j\lambda_{j-1}}$, we have

$$\begin{aligned}\Bigl|F(M)-F(\iota_j W_{(\lambda_j)})-\frac{N+2}{N-2}W_{(\lambda_j)}^{4/(N-2)}\sum_{\ell\neq j}\iota_\ell W_{(\lambda_\ell)}\Bigr| &\lesssim \mathbb{1}_{E_j}\sum_{\ell\neq j}W_{(\lambda_\ell)}^2 W_{(\lambda_j)}^{(6-N)/(N-2)}+\mathbb{1}_{\mathbb{R}^N\setminus E_j}\Bigl(\sum_{\ell\neq j}\iota_\ell W_{(\lambda_\ell)}\Bigr)^{(N+2)/(N-2)}\\ &\lesssim \sum_{\ell\neq j}W_{(\lambda_\ell)}^2 W_{(\lambda_j)}^{(6-N)/(N-2)},\end{aligned}$$

where we have used the following facts.

- If $N\geqslant 7$, we have used that $(N+2)/(N-2)<2$ and that, on $\mathbb{R}^N\setminus E_j$, one has
$$W_{(\lambda_j)}^2\lesssim\sum_{\ell\neq j}W_{(\lambda_\ell)}^2.$$
- If $N=5$, $(N+2)/(N-2)=\frac{7}{3}>2$. However, the preceding inequality holds for
$$\sqrt{\lambda_j\lambda_{j+1}}\leqslant|x|\leqslant\sqrt{\lambda_j\lambda_{j-1}},$$

since in this set one has
$$\Bigl|\sum_{\ell\neq j}\iota_\ell W_{(\lambda_\ell)}(x)\Bigr|\lesssim W_{(\lambda_j)}(x)$$
and $6-N>0$.

As a consequence, using also $|(\Lambda W)_{(\lambda_j)}|\lesssim W_{(\lambda_j)}$,

$$\begin{aligned}\int_{\sqrt{\lambda_j\lambda_{j+1}}\leqslant|x|\leqslant\sqrt{\lambda_j\lambda_{j-1}}}|P_j(x)|\,dx \lesssim{}& \sum_{\ell\neq j}\int_{\sqrt{\lambda_j\lambda_{j+1}}\leqslant|x|\leqslant\sqrt{\lambda_j\lambda_{j-1}}}W_{(\lambda_\ell)}^2 W_{(\lambda_j)}^{4/(N-2)}\\ &+\sum_{\ell\notin\{j-1,j,j+1\}}\int_{\mathbb{R}^N}W_{(\lambda_\ell)}W_{(\lambda_j)}^{(N+2)/(N-2)}.\end{aligned}$$

Using again (5.68), we obtain, by (A.4) in the appendix,

$$\sum_{\ell\neq j}\int_{\sqrt{\lambda_j\lambda_{j+1}}\leqslant|x|\leqslant\sqrt{\lambda_j\lambda_{j-1}}}W_{(\lambda_\ell)}^2 W_{(\lambda_j)}^{4/(N-2)}\lesssim\sum_{\ell\neq j}\int W_{(\lambda_\ell)}^{N/(N-2)}W_{(\lambda_j)}^{N/(N-2)}\lesssim\gamma^{N/2}.$$

Furthermore, if $\ell\notin\{j-1,j,j+1\}$, by estimate (A.1) in the appendix,

$$\begin{aligned}\int W_{(\lambda_\ell)}W_{(\lambda_j)}^{(N+2)/(N-2)}&=\int\nabla W_{(\lambda_\ell)}\cdot\nabla W_{(\lambda_j)}\\ &\lesssim\min\biggl\{\Bigl(\frac{\lambda_j}{\lambda_\ell}\Bigr)^{(N-2)/2},\Bigl(\frac{\lambda_\ell}{\lambda_j}\Bigr)^{(N-2)/2}\biggr\}\lesssim\gamma^{N-2}.\end{aligned}$$

Combining, we obtain

$$\int_{\sqrt{\lambda_j\lambda_{j+1}}\leqslant|x|\leqslant\sqrt{\lambda_j\lambda_{j-1}}}|P_j(x)|\,dx\lesssim\gamma^{N/2},\tag{5.71}$$

which, together with (5.70), yields the desired inequality (5.65). □

Proof of (5.66). Recall that

$$W(x)=\left(1+\frac{|x|^2}{N(N-2)}\right)^{(2-N)/2} \quad \text{and} \quad \Lambda W(x)=x\cdot\nabla W(x)+\frac{N-2}{2}W.$$

Thus,

$$|W(x)-1|+\left|\Lambda W(x)-\frac{N-2}{2}\right|\lesssim |x| \tag{5.72}$$

and

$$\left|W(x)-\frac{(N(N-2))^{N/2-1}}{|x|^{N-2}}\right|+\left|\Lambda W(x)+\frac{(N(N-2))^{N/2}}{2N|x|^{N-2}}\right|\lesssim\frac{1}{|x|^{N-1}}. \tag{5.73}$$

By (5.73),

$$\begin{aligned}&\int W_{(\lambda_j)}^{4/(N-2)}W_{(\lambda_{j+1})}(\Lambda W)_{(\lambda_j)}\\&=\left(\frac{\lambda_j}{\lambda_{j+1}}\right)^{(N-2)/2}\int W^{4/(N-2)}(x)\Lambda W(x)W\left(\frac{\lambda_j x}{\lambda_{j+1}}\right)dx\\&=\left(\frac{\lambda_{j+1}}{\lambda_j}\right)^{(N-2)/2}\int W^{4/(N-2)}(x)\Lambda W(x)\frac{(N(N-2))^{N/2-1}}{|x|^{N-2}}\,dx+\mathcal{O}(\gamma^{N/2}),\end{aligned}$$

where we have used that the integral

$$\int W^{4/(N-2)}\Lambda W\frac{1}{|x|^{N-1}}\,dx$$

converges. Furthemore, by an easy integration by parts,

$$\int W^{4/(N-2)}x\cdot\nabla W\frac{1}{|x|^{N-2}}\,dx=\frac{-2(N-2)}{N+2}\int\frac{1}{|x|^{N-2}}W^{(N+2)/(N-2)}\,dx,$$

and thus

$$\int\frac{1}{|x|^{N-2}}W^{4/(N-2)}\Lambda W\,dx=\frac{(N-2)^2}{2(N+2)}\int\frac{1}{|x|^{N-2}}W^{(N+2)/(N-2)}\,dx.$$

Combining, we obtain (5.66). □

Proof of (5.67). By (5.72),

$$\begin{aligned}&\int W_{(\lambda_j)}^{4/(N-2)}W_{(\lambda_{j-1})}(\Lambda W)_{(\lambda_j)}\\&=\left(\frac{\lambda_j}{\lambda_{j-1}}\right)^{(N-2)/2}\int W^{4/(N-2)}\Lambda W(x)W\left(\frac{\lambda_j x}{\lambda_{j-1}}\right)dx\\&=\left(\frac{\lambda_j}{\lambda_{j-1}}\right)^{(N-2)/2}\int W^{4/(N-2)}\Lambda W(x)\left(1+\mathcal{O}\left(\frac{\lambda_j|x|}{\lambda_{j-1}}\right)\right)dx\\&=\left(\frac{\lambda_j}{\lambda_{j-1}}\right)^{(N-2)/2}\int W^{4/(N-2)}\Lambda W(x)\,dx+\mathcal{O}(\gamma^{N/2}).\end{aligned}$$

By a straightforward integration by parts, we obtain

$$\int W^{4/(N-2)}\Lambda W\,dx = -\frac{(N-2)^2}{2(N+2)}\int W^{(N+2)/(N-2)}\,dx,$$

and thus

$$\begin{aligned}&\int W_{(\lambda_j)}^{4/(N-2)} W_{(\lambda_{j-1})}(\Lambda W)_{(\lambda_j)}\\ &\qquad = -\frac{(N-2)^2}{2(N+2)}\left(\frac{\lambda_j}{\lambda_{j-1}}\right)^{(N-2)/2}\int W^{(N+2)/(N-2)} + \mathcal{O}(\gamma^{N/2}).\end{aligned} \tag{5.74}$$

Finally, we observe that, for $\sigma>0$,

$$W^{(N+2)/(N-2)}\left(\frac{N(N-2)}{\sigma}\right) = \frac{\sigma^{N+2}}{(N(N-2))^{(N+2)/2}} W^{(N+2)/(N-2)}(N(N-2)\sigma),$$

and thus, by the change of variable $r=N(N-2)/\sigma$,

$$\begin{aligned}&\int_0^\infty \frac{1}{r^{N-2}} W^{(N+2)/(N-2)}(r) r^{N-1}\,dr\\ &\qquad = \frac{1}{(N(N-2))^{(N-2)/2}}\int_0^\infty W^{(N+2)/(N-2)}(\sigma)\sigma^{N-1}\,d\sigma.\end{aligned}$$

Combining with (5.74), we obtain (5.67). □

The proof of (5.64) is now complete. □

End of the proof of Proposition 5.1. We next gather the results of the preceding Lemmas to conclude the proof of Proposition 5.1.

The estimate (5.13) is exactly (5.50) in Lemma 5.5.

By (5.45), (5.56) and (5.50),

$$\left|\beta_j - \lambda_j'\|\Lambda W\|_{L^2}^2\right| \lesssim \left|\beta_j + \alpha_j\|\Lambda W\|_{L^2}^2\right| + \left|\lambda_j' + \alpha_j\|\Lambda W\|_{L^2}^2\right| \lesssim \gamma^{N/4} + o_n(1),$$

hence (5.14).

Combining (5.44), (5.45) and (5.49), we have

$$\begin{aligned}&\left|\frac{1}{2}\sum_{j=1}^J \beta_j^2 - \kappa_1\sum_{j=1}^{J-1}\iota_j\iota_{j+1}\left(\frac{\lambda_{j+1}}{\lambda_j}\right)^{(N-2)/2}\right|\\ &\qquad \lesssim \frac{1}{2}\left|\|\Lambda W\|_{L^2}^4\sum_{j=1}^J\alpha_j^2 - \sum_{j=1}^J\beta_j^2\right|\\ &\qquad\qquad + \frac{1}{2}\left|\delta^2 - \|\Lambda W\|_{L^2}^2\sum_{j=1}^J\alpha_j^2\right|\,\|\Lambda W\|_{L^2}^2\\ &\qquad\qquad + \left|\frac{1}{2}\delta^2 - \kappa_1'\sum_{j=1}^J\iota_j\iota_{j+1}\left(\frac{\lambda_{j+1}}{\lambda_j}\right)^{(N-2)/2}\right|\,\|\Lambda W\|_{L^2}^2\\ &\qquad \lesssim \gamma^{(N-1)/2} + o_n(1).\end{aligned}$$

Hence (5.15). We have used that (5.45) and the estimate

$$|\beta_j|+|\alpha_j|\lesssim\gamma^{(N-2)/4}$$

(consequence of (5.44), (5.45) and (5.50)) implies that

$$\left|\beta_j^2-\|\Lambda W\|_{L^2}^4\alpha_j^2\right|=\left|(\beta_j-\|\Lambda W\|_{L^2}^2\alpha_j)(\beta_j+\|\Lambda W\|_{L^2}^2\alpha_j)\right|\lesssim\gamma^{N/4}\gamma^{(N-2)/4}+o_n(1).$$

Finally, (5.16) is exactly (5.57) in Lemma 5.7. □

6. End of the proof

6.1. Exit time for a system of differential inequalities

Using Proposition 5.1 and a lower bound on one of the scaling parameters λ_j (consequence of Proposition 4.6), we will reduce the proof to the following proposition.

Proposition 6.1. *Let $C>0$, $J_0\geqslant 2$, $a>0$. There exists $\varepsilon=\varepsilon(C,J_0,a)>0$ such that, for all $L>0$, there exists $T^*=T^*(L,C,J_0,a)$ with the following property. For all $T>0$ and for all C^1 functions*

$$\boldsymbol{\lambda}=(\lambda_j)_j\colon[0,T]P\longrightarrow G_{J_0}\quad\textit{and}\quad\boldsymbol{\beta}=(\beta_j)_j\colon[0,T]\longrightarrow\mathbb{R}^{J_0}$$

satisfying, for all $t\in[0,T]$,

$$\gamma(\boldsymbol{\lambda})=:\gamma\leqslant\varepsilon \tag{6.1}$$

$$\left|\beta_j-\|\Lambda W\|_{L^2}^2\lambda_j'\right|\leqslant C\gamma^{N/4}\quad\textit{for all } j, \tag{6.2}$$

$$\left|\frac{1}{2}\sum_{j=1}^{J_0}\beta_j^2-\kappa_1\sum_{1\leqslant j\leqslant J_0-1}\iota_j\iota_{j+1}\left(\frac{\lambda_{j+1}}{\lambda_j}\right)^{(N-2)/2}\right|\leqslant C\gamma^{(N-1)/2} \tag{6.3}$$

$$\left|\lambda_j\beta_j'+\kappa_0\left(\iota_j\iota_{j+1}\left(\frac{\lambda_{j+1}}{\lambda_j}\right)^{(N-2)/2}-\iota_j\iota_{j-1}\left(\frac{\lambda_j}{\lambda_{j-1}}\right)^{(N-2)/2}\right)\right|\leqslant C\gamma^{(N-1)/2}\quad\textit{for all } j, \tag{6.4}$$

$$L\leqslant C\gamma^{(N-2)/2}\left(\frac{\lambda_1}{\lambda_1(0)}\right)^a, \tag{6.5}$$

we have

$$T\leqslant T^*\lambda_1(0).$$

Remark 6.2. Let us emphasize that T^* is independent of $\varepsilon>0$ if it is chosen small enough, and that ε does not depend on L.

We postpone the proof of Proposition 6.1 to §6.4, and conclude the proof of Theorem 1 in the two next subsections. In view of Proposition 5.1, $(\lambda_j)_{1\leqslant j\leqslant J}$ and $(\beta_j)_{1\leqslant j\leqslant J}$ satisfy the assumptions of Proposition 6.1, except for the lower bound (6.5) and up to terms that are $o_n(1)$. In §6.2, we will eliminate the $o_n(1)$ terms. In order to do this, we will ignore all the exterior profiles that are equal to $\pm W$, restricting to indices $j\in[\![\tilde{J},J]\!]$ for an appropriate index $\tilde{J}$ and to a time interval $[\tilde{t}_n,t'_n]$ strictly included in $[\tilde{t}_n,t_n]$. In §6.3 we will show that the new exterior scaling parameter $\lambda_{\tilde{J}}$ satisfies the lower bound (6.5) and conclude the proof of Theorem 1 assuming Proposition 6.1. Finally, in §6.4, we prove Proposition 6.1.

6.2. Restriction of the indices and of the time interval

Recall from §5.1 the definitions of t_n, $\tilde{t}_n$, J and, for $j\in[\![1,J]\!]$, ι_j, $\alpha_j(t)$, $\beta_j(t)$ and $\lambda_j(t)$.

After extraction of subsequences, the following weak limits exits in $\mathcal{H}$:

$$(\widetilde{U}_0^j,\widetilde{U}_1^j)=\operatorname*{w-lim}_{n\to\infty}(\lambda_j(\tilde{t}_n)^{N/2-1}U(\tilde{t}_n,\lambda_j(\tilde{t}_n)\,\cdot),\lambda_j(\tilde{t}_n)^{N/2}\partial_tU(\tilde{t}_n,\lambda_j(\tilde{t}_n)\,\cdot)), \tag{6.6}$$

where, as before, $U=u-v_L$. We note that there exists $j\in[\![1,J]\!]$ such that

$$(\widetilde{U}_0^j,\widetilde{U}_1^j)\neq(\iota_jW,0).$$

If not, for all $k\in[\![1,K-1]\!]$, by (5.27)–(5.29), $j_{k+1}=j_k+1$ and, by the definition (5.17) of j_k, we see that, for all $j\in[\![1,J]\!]$,

$$\lim_{n\to+\infty}\frac{\lambda_{j+1}(\tilde{t}_n)}{\lambda_j(\tilde{t}_n)}=0.$$

This implies $\lim_{n\to\infty}\gamma(\tilde{t}_n)=0$ yielding, by Proposition 5.1, $\lim_{n\to\infty}\delta(\tilde{t}_n)=0$, a contradiction with the definition of $\tilde{t}_n$. We define $\tilde{J}$ as the unique index in $[\![1,J]\!]$ such that

$$(\widetilde{U}_0^j,\widetilde{U}_1^j)=(\iota_jW,0)\quad\text{for all } j\in[\![1,\tilde{J}-1]\!], \tag{6.7}$$

$$(\widetilde{U}_0^{\tilde{J}},\widetilde{U}_1^{\tilde{J}})\neq(\iota_{\tilde{J}}W,0). \tag{6.8}$$

If $(\widetilde{U}_0^1,\widetilde{U}_1^1)\neq(\iota_1W,0)$, we let $\tilde{J}=1$. By the argument above,

$$\lim_{n\to+\infty}\frac{\lambda_{j+1}(\tilde{t}_n)}{\lambda_j(\tilde{t}_n)}=0\quad\text{for all } j\in[\![1,\tilde{J}-1]\!]. \tag{6.9}$$

We set

$$\lambda_{\tilde{J},n}:=\lambda_{\tilde{J}}(\tilde{t}_n)\quad\text{and}\quad\widetilde{\gamma}(t):=\gamma((\lambda_j(t))_{\tilde{J}\leqslant j\leqslant J})=\max_{\tilde{J}\leqslant j\leqslant J-1}\frac{\lambda_{j+1}(t)}{\lambda_j(t)}. \tag{6.10}$$

In this subsection, we prove the following lemmas.

LEMMA 6.3. *We have*

$$\lim_{n\to\infty}\frac{t_n-\tilde{t}_n}{\lambda_{\tilde{J},n}}=+\infty. \tag{6.11}$$

LEMMA 6.4. *Let* $T>0$ *and*

$$t_n'=\tilde{t}_n+T\lambda_{\tilde{J},n}.$$

Then, for large n, *for all* $t\in[\tilde{t}_n,t_n']$ *and for all* $j\in[\![\tilde{J},J]\!]$,

$$\big|\beta_j-\|\Lambda W\|_{L^2}^2\lambda_j'\big|\leqslant C\widetilde{\gamma}^{N/4}, \tag{6.12}$$

$$\left|\lambda_j\beta_j'+\kappa_0\left(\iota_j\iota_{j+1}\left(\frac{\lambda_{j+1}}{\lambda_j}\right)^{(N-2)/2}-\iota_j\iota_{j-1}\left(\frac{\lambda_j}{\lambda_{j-1}}\right)^{(N-2)/2}\right)\right|\leqslant C\widetilde{\gamma}^{(N-1)/2}, \tag{6.13}$$

and, for all $t\in[\tilde{t}_n,t_n']$,

$$\left|\frac{1}{2}\sum_{j=\tilde{J}}^{J}\beta_j^2-\kappa_1\sum_{j=\tilde{J}}^{J-1}\iota_j\iota_{j+1}\left(\frac{\lambda_{j+1}}{\lambda_j}\right)^{(N-2)/2}\right|\leqslant C\widetilde{\gamma}^{(N-1)/2}. \tag{6.14}$$

Proof of Lemma 6.3. *Step* 1. (Expansion of the solution along the sequence $\{\tilde{t}_n\}_n$) Extracting subsequences if necessary, we introduce, as in the beginning of §5.2, the following partition of $[\![1,J]\!]$:

$$[\![1,J]\!]=\bigcup_{k=1}^{\widetilde{K}}[\![\tilde{\jmath}_k,\tilde{\jmath}_{k+1}-1]\!],$$

with

$$1=\tilde{\jmath}_1<\tilde{\jmath}_2<...<\tilde{\jmath}_{\widetilde{K}+1}=J+1$$

and letting

$$\lambda_{k,n}=\lambda_{\tilde{\jmath}_k}(\tilde{t}_n)\quad\text{for all } k\in[\![1,\widetilde{K}]\!],$$

we have

$$\lim_{n\to\infty}\frac{\lambda_j(\tilde{t}_n)}{\lambda_{k,n}}>0\quad\text{for all } k\in[\![1,\widetilde{K}]\!] \text{ and all } j\in[\![\tilde{\jmath}_k,\tilde{\jmath}_{k+1}-1]\!], \tag{6.15}$$

$$\lim_{n\to\infty}\frac{\lambda_{k+1,n}}{\lambda_{k,n}}=0\quad\text{for all } k\in[\![1,\widetilde{K}-1]\!]. \tag{6.16}$$

We set

$$(U_0^k,U_1^k):=\operatorname*{w-lim}_{n\to\infty}(\lambda_{k,n}^{(N-2)/2}U(\tilde{t}_n,\lambda_{k,n}\cdot),\lambda_{k,n}^{N/2}\partial_tU(\tilde{t}_n,\lambda_{k,n}\cdot)). \tag{6.17}$$

Note that, by (6.9),

$$(U_0^k,U_1^k)=(\iota_kW,0)\ \text{for all } k\in[\![1,\tilde{J}-1]\!]\quad\text{and}\quad \tilde{\jmath}_k=k\ \text{for all } k\in[\![1,\tilde{J}]\!]. \tag{6.18}$$

We let U^k be the solution of the non-linear wave equation (1.1) with initial data (U_0^k, U_1^k). According to Lemma 5.2, U^k is defined for $|x|>|t|$, non-radiative and, denoting

$$U_n^k(t,x)=\frac{1}{\lambda_{k,n}^{(N-2)/2}}U^k\left(\frac{t}{\lambda_{k,n}},\frac{x}{\lambda_{k,n}}\right), \tag{6.19}$$

we can expand $\vec{u}(\tilde{t}_n)$ as follows:

$$\lim_{n\to\infty}\left\|\vec{u}(\tilde{t}_n)-\vec{v}_L(\tilde{t}_n)-\sum_{k=1}^{\widetilde{K}}\vec{U}_n^k(0)\right\|_{\mathcal{H}}=0. \tag{6.20}$$

We now make a crucial observation on $(U_0^{\tilde{J}}, U_1^{\tilde{J}})$. Since, by the definition of $\tilde{J}$,

$$(U_0^{\tilde{J}}, U_1^{\tilde{J}})\neq(\iota_{\tilde{J}}W,0),$$

we see, by the analogue of the expansion (5.21) (where $k=\tilde{J}$ and (V_0^k,V_1^k) has to be replaced by $(U_0^{\tilde{J}}, U_1^{\tilde{J}})$), the orthogonality relations (5.26) and the uniqueness in Lemma B.1, that $(U_0^{\tilde{J}}, U_1^{\tilde{J}})$ is not the initial data of a stationary solution. Since the corresponding solution is non-radiative, we deduce from Theorems 4.2 and 4.3 that there exist $p_0\in[\![1,\tfrac{1}{2}(N-1)]\!]$ and $\ell\in\mathbb{R}\setminus\{0\}$ such that, for all $t\in\mathbb{R}$ and all R large (depending on t),

$$\|\vec{U}^{\tilde{J}}(t)-\ell\,\Xi_{p_0}\|_{\mathcal{H}(R)}\lesssim\max\left(\frac{1}{R^{(p_0-1/2)(N+2)/(N-2)}},\frac{1}{R^{p_0+1/2}}\right), \tag{6.21}$$

where the implicit constant might depend on R (but of course not on t).

Step 2. (Contradiction argument) Assuming that (6.11) does not hold, we have, extracting subsequences if necessary,

$$\lim_{n\to\infty}\frac{t_n-\tilde{t}_n}{\lambda_{\tilde{J},n}}=T\in[0,\infty). \tag{6.22}$$

By the expansion (5.42), with $\tau=t_n-\tilde{t}_n$ and $s_n=\tilde{t}_n$, we have

$$\vec{u}(t_n)=\vec{v}_L(t_n)+\sum_{k=1}^{\widetilde{K}}\vec{U}_n^k(t_n-\tilde{t}_n)+\vec{r}_n(t_n-\tilde{t}_n),\quad |x|>|t_n-\tilde{t}_n|, \tag{6.23}$$

where

$$\lim_{n\to\infty}\int_{|x|\geqslant|t_n-\tilde{t}_n|}|\nabla_{t,x}r_n(t_n-\tilde{t}_n)|^2\,dx=0. \tag{6.24}$$

Let

$$(A_0^{\tilde{J}}, A_1^{\tilde{J}})=\operatorname*{w-lim}_{n\to\infty}(\lambda_{\tilde{J},n}^{(N-2)/2}u(t_n,\lambda_{\tilde{J},n}\cdot),\lambda_{\tilde{J},n}^{N/2}\partial_t u(t_n,\lambda_{\tilde{J},n}\cdot)). \tag{6.25}$$

We claim that

$$(A_0^{\tilde{J}}, A_1^{\tilde{J}})(x) = \vec{U}^{\tilde{J}}(T, x), \quad |x| > |T|, \tag{6.26}$$

where T is defined by (6.22). Indeed, let $\varphi \in C_0^\infty(\{x \in \mathbb{R}^N : |x| > T\})$. Then, by the definition (6.19) of $U_n^{\tilde{J}}$, we have

$$\int \lambda_{\tilde{J},n}^{(N-2)/2} U_n^{\tilde{J}}(t_n - \tilde{t}_n, \lambda_{\tilde{J},n} x)\varphi(x)\, dx$$
$$= \int U^{\tilde{J}}\left(\frac{t_n - \tilde{t}_n}{\lambda_{\tilde{J},n}}, x\right)\varphi(x)\, dx \xrightarrow[n\to\infty]{} \int U^{\tilde{J}}(T, x)\varphi(x)\, dx,$$

where we have used (6.22) and the fact that $U^{\tilde{J}}\restriction_{\{|x|>|t|\}}$ is the restriction to $\{|x|>|t|\}$ of an element of $C^0(\mathbb{R}, \dot{H}^1)$ (see Definition 2.3). Furthermore, if $k \in [\![1, \widetilde{K}]\!] \setminus \{\tilde{J}\}$,

$$\int \lambda_{\tilde{J},n}^{(N-2)/2} U_n^k(t_n - \tilde{t}_n, \lambda_{\tilde{J},n} x)\varphi(x)\, dx \tag{6.27}$$

$$= \int \left(\frac{\lambda_{k,n}}{\lambda_{\tilde{J},n}}\right)^{(N+2)/2} U^k\left(\frac{t_n - \tilde{t}_n}{\lambda_{k,n}}, y\right)\varphi\left(\frac{\lambda_{k,n}}{\lambda_{\tilde{J},n}} y\right) dy. \tag{6.28}$$

Using that there exists $T' > T$ such that $|x| \geqslant T'$ in the support of φ, we see by (6.22) that $|y| > (t_n - \tilde{t}_n)/\lambda_{k,n}$ for large n in the support of the integrand in (6.28). Thus, for large n, (6.28) (or equivalently (6.27)) does not depend on the values of $U^k(t, x)$ for $|x| \leqslant |t|$. Recall that, after extraction,

$$\lim_{n\to\infty} \frac{\lambda_{k,n}}{\lambda_{\tilde{J},n}} \in \{0, +\infty\}. \tag{6.29}$$

If (after extraction)

$$\lim_n \frac{t_n - \tilde{t}_n}{\lambda_{k,n}} = \sigma \in [0, \infty),$$

then, using that $U^k\restriction_{\{|x|>|t|\}}$ is the restriction to $\{|x|>|t|\}$ of an element of $C^0(\mathbb{R}, \dot{H}^1)$,

$$\lim_{n\to\infty} U^k\left(\frac{t_n - \tilde{t}_n}{\lambda_{k,n}}\right) = U^k(\sigma) \quad \text{in } \dot{H}^1(\{|x| > \sigma\}),$$

and, since

$$\left(\frac{\lambda_{k,n}}{\lambda_{\tilde{J},n}}\right)^{(N+2)/2} \varphi\left(\frac{\lambda_{k,n}}{\lambda_{\tilde{J},n}} \cdot\right)$$

converges weakly to zero in $\dot{H}^{-1}$, we have that (6.28) (and hence (6.27)) goes to zero as n goes to infinity. On the other hand, if

$$\lim_n \frac{t_n - \tilde{t}_n}{\lambda_{k,n}} = +\infty,$$

using that, by Lemma 5.2, U^k is non-radiative, we obtain again that (6.27) goes to zero as n goes to infinity.

Finally, we have

$$\lim_{n\to\infty}\int \lambda_{\tilde{J},n}^{(N-2)/2} r_n(t_n-\tilde{t}_n,\lambda_{\tilde{J},n}x)\varphi(x)\,dx=0,$$

by (6.24) and (6.22), and

$$\lim_{n\to\infty}\int \lambda_{\tilde{J},n}^{(N-2)/2} v_L(t_n,\lambda_{\tilde{J},n}x)\varphi(x)\,dx=0$$

by the standard asymptotics of the linear wave equation. This yields

$$\int A_0^{\tilde{J}}(x)\varphi(x)\,dx=\int U_0^{\tilde{J}}(T,x)\varphi(x)\,dx \quad \text{for all } \varphi\in C_0^\infty(\{|x|>T\}),$$

and thus, arguing similarly on $A_1^{\tilde{J}}$ and $U_1^{\tilde{J}}$, the desired equality (6.26) follows.

Since $\lim_{n\to\infty} d_{J,\boldsymbol{\iota}}(t_n)=0$, we obtain that $(A_0^{\tilde{J}},A_1^{\tilde{J}})=(0,0)$ or $(A_0^{\tilde{J}},A_1^{\tilde{J}})=(\pm W_{(\mu)},0)$ for some sign $\pm$ and scaling parameter $\mu>0$, contradicting (6.21), and concluding the proof of Lemma 6.3. □

Remark 6.5. The same proof yields the following: let $\{s_n\}$ be a sequence of times with $s_n\in[\tilde{t}_n,t_n]$ such that

$$\lim_{n\to\infty} d_{J,\boldsymbol{\iota}}(s_n)=0.$$

Then,

$$\lim_{n\to\infty}\frac{s_n-\tilde{t}_n}{\lambda_{\tilde{J},n}}=+\infty.$$

Proof of Lemma 6.4. We will show that

$$\lim_{n\to\infty}\max_{\tilde{t}_n\leqslant t\leqslant t_n'}\left(|\beta_j(t)|+\frac{\lambda_{j+1}(t)}{\lambda_j(t)}\right)=0 \quad \text{for all } j\in[\![1,\tilde{J}-1]\!], \tag{6.30}$$

$$\liminf_{n\to\infty}\min_{\tilde{t}_n\leqslant t\leqslant t_n'}\widetilde{\gamma}(t)>0. \tag{6.31}$$

Assuming (6.30) and (6.31), the conclusion of Lemma 6.4 follows easily from Proposition 5.1. Indeed, by (6.30) and (6.31), restricting t to $[\tilde{t}_n,t_n']$, we can replace $\gamma(t)+o_n(1)$ by $\widetilde{\gamma}(t)$ in the right-hand sides of (5.14)–(5.16). Similarly, using again (6.30) and (6.31), we can restrict the indices in the sums in the left-hand side of (5.15) to $\tilde{J}\leqslant j$. This yields that (6.12), (6.13) (for $j\in[\tilde{J},J]$) and (6.14) hold for all large n and $t\in[\tilde{t}_n,t_n']$.

Proof of (6.30). We first claim that

$$\lim_{n\to\infty}\sup_{\tilde{t}_n\leqslant t\leqslant t'_n}\left|\frac{\lambda_j(t)}{\lambda_j(\tilde{t}_n)}-1\right|=0 \quad \text{for all } j\in[\![1,\tilde{J}-1]\!]. \tag{6.32}$$

Indeed, by the estimates (5.14) and (5.15), which imply $|\lambda'|\leqslant C\gamma^{(N-2)/4}+o_n(1)$, we see that, for all $t\in[\tilde{t}_n,t'_n]$,

$$|\lambda_j(t)-\lambda_j(\tilde{t}_n)|\lesssim \sup_{t\in[\tilde{t}_n,t'_n]}(\gamma^{(N-2)/4}(t)+o_n(1))(t'_n-\tilde{t}_n),$$

and thus, for large n,

$$\left|1-\frac{\lambda_j(t)}{\lambda_j(\tilde{t}_n)}\right|\lesssim\frac{t'_n-\tilde{t}_n}{\lambda_j(\tilde{t}_n)}=T\frac{\lambda_{\tilde{J},n}}{\lambda_j(\tilde{t}_n)}=T\frac{\lambda_{\tilde{J}}(\tilde{t}_n)}{\lambda_j(\tilde{t}_n)}, \tag{6.33}$$

and (6.32) follows in view of the fact that, by (6.9) and the definition of $\tilde{J}$,

$$\lim_{n\to\infty}\frac{\lambda_j(\tilde{t}_n)}{\lambda_{j+1}(\tilde{t}_n)}=+\infty \quad \text{for all } j\in[\![1,\tilde{J}-1]\!]. \tag{6.34}$$

Combining (6.32) with (6.34), we obtain

$$\sup_{\tilde{t}_n\leqslant t\leqslant t'_n}\frac{\lambda_{j+1}(t)}{\lambda_j(t)}=0 \quad \text{for all } j\in[\![1,\tilde{J}-1]\!]. \tag{6.35}$$

By the definition (5.12) of β_j and the definition of $\tilde{J}$, we also have

$$\lim_{n\to\infty}|\beta_j(\tilde{t}_n)|=0 \quad \text{for all } j\in[\![1,\tilde{J}-1]\!]. \tag{6.36}$$

By Lemma 5.7,

$$|\beta'_j(t)|\lesssim\frac{\gamma^{N/2-1}(t)+o_n(1)}{\lambda_j(t)}\lesssim\frac{1}{\lambda_j(t)} \quad \text{for all } t\in[\tilde{t}_n,t'_n] \text{ and all } j\in[\![1,\tilde{J}-1]\!].$$

Integrating in time, we obtain

$$|\beta_j(t)-\beta_j(\tilde{t}_n)|\lesssim\frac{\lambda_{\tilde{J},n}}{\lambda_j(t)}\lesssim\sup_{s\in[\tilde{t}_n,t'_n]}\frac{\lambda_{\tilde{J}}(\tilde{t}_n)}{\lambda_j(s)},$$

where the implicit constants depend on T. By (6.32) and (6.35), the right-hand side of the preceding inequality goes to zero as n tends to ∞. Combining with (6.35) and (6.36), we obtain (6.30). □

We finally prove (6.31). We argue by contradiction, assuming, after extraction of a subsequence, that there exists a sequence of times $\{s_n\}_n$ with $s_n\in[\tilde{t}_n,t'_n]$ such that $\lim_{n\to\infty}\widetilde{\gamma}(s_n)=0$. By (6.30), $\lim_{n\to\infty}\gamma(s_n)=0$. By Lemma 5.5, $\lim_{n\to\infty}\delta(s_n)=0$. Thus, $\lim_{n\to\infty}d_{J,\boldsymbol{\iota}}(s_n)=0$, a contradiction with the conclusion of Remark 6.5 since

$$\tilde{t}_n\leqslant s_n\leqslant\tilde{t}_n+T\lambda_{\tilde{J},n}.$$

□

6.3. Lower bound for the exterior scaling parameter and end of the proof

In this subsection we conclude the contradiction argument started in §5, using the same notation as in §5 and §6.2. We recall in particular that $t'_n=\tilde{t}_n+T\lambda_{\tilde{J},n}$, where the sequence $\{\tilde{t}_n\}_n$ is defined by (5.5)–(5.7), $\lambda_{\tilde{J},n}=\lambda_{\tilde{J}}(\tilde{t}_n)$, and T is a large positive parameter that will be chosen at the end of the proof in order to obtain a contradiction with Proposition 6.1. The small parameter ε_0 appearing in (5.7) does not depend on T. The constants are independent of ε_0 and T.

LEMMA 6.6. *Let ℓ and p_0 be defined by* (6.21). *Then, if ε_0 is chosen small enough* (*independently of T*), *then, for large n,*

$$|\ell|\leqslant 2C_0\left(\frac{\lambda_{\tilde{J}}(t)}{\lambda_{\tilde{J}}(\tilde{t}_n)}\right)^{p_0-1/2}\delta(t)^{2/N}\quad \text{for all } t\in[\tilde{t}_n,t'_n],$$

-where C_0 is the constant of Proposition 4.6.

End of the proof of Theorem 1. We first assume Lemma 6.6 and conclude the proof of Theorem 1. By (5.13), for large n,

$$\delta(t)^{2/N}\lesssim\gamma(t)^{(N-2)/(2N)}+o_n(1)\lesssim\gamma(t)^{(N-2)/2}+o_n(1)\quad \text{for all } t\in[\tilde{t}_n,t'_n].$$

Combining with (6.30) and (6.31), we see that

$$\delta(t)^{2/N}\lesssim\widetilde{\gamma}(t)^{(N-2)/2}\quad \text{for all } t\in[\tilde{t}_n,t'_n].$$

Thus, Lemma 6.6 implies that there exists a constant $C>0$ such that

$$|\ell|\leqslant C\left(\frac{\lambda_{\tilde{J}}(t)}{\lambda_{\tilde{J}}(\tilde{t}_n)}\right)^{p_0-1/2}\widetilde{\gamma}(t)^{(N-2)/2}\quad \text{for all } t\in[\tilde{t}_n,t'_n]. \tag{6.37}$$

By (6.37) and the estimates (6.12)–(6.14) of the preceding subsection, the parameters $(\beta_j)_{\tilde{J}\leqslant j\leqslant J}$ and $(\lambda_j)_{\tilde{J}\leqslant j\leqslant J}$ satisfy the assumptions of Proposition 6.1 for times $t\in[\tilde{t}_n,t'_n]$. The conclusion of the proposition yields $t'_n-\tilde{t}_n\leqslant T_*\lambda_{\tilde{J}}(\tilde{t}_n)$ for large n, and thus $T\leqslant T_*$, for a constant T_* depending only on the solution u and the parameters ℓ and p_0. Since we can take T arbitrarily large, we obtain a contradiction, concluding the proof of the theorem except for the fact that

$$\lim_{t\to\infty}\frac{\lambda_1(t)}{t}=0,$$

which follows from finite speed of propagation and the small data theory (see e.g. the proof of (3.53) in [21]). □

Proof of Lemma 6.6. We argue by contradiction, assuming (after extraction) that either

$$|\ell| \geqslant 2C_0\delta(\tilde{t}_n)^{2/N} \quad \text{for all } n, \tag{6.38}$$

or $|\ell| < 2C_0\delta(\tilde{t}_n)^{2/N}$ for all n and there exists a sequence $\{\tilde{s}_n\}_n$, with $\tilde{s}_n \in [\tilde{t}_n, t'_n]$, such that

$$|\ell| = 2C_0\left(\frac{\lambda_{\tilde{J}}(\tilde{s}_n)}{\lambda_{\tilde{J}}(\tilde{t}_n)}\right)^{p_0-1/2}\delta(\tilde{s}_n)^{2/N} \quad \text{for all } n. \tag{6.39}$$

If (6.38) holds, we will let $\tilde{s}_n = \tilde{t}_n$.

We use the expansion (5.42), as in the proof of Lemma 6.3 (see (6.23)) at $s_n = \tilde{t}_n$ and $\tau = \tilde{s}_n - \tilde{t}_n$. This yields

$$\vec{u}(\tilde{s}_n) = \vec{v}_L(\tilde{s}_n) + \sum_{k=1}^{\widetilde{K}} \vec{U}_n^k(\tilde{s}_n - \tilde{t}_n) + \vec{r}_n(\tilde{s}_n - \tilde{t}_n), \quad |x| > \tilde{s}_n - \tilde{t}_n,$$

where the U_n^k are defined in (6.17) and (6.19), and

$$\lim_{n\to\infty} \int_{\{|x|>|\tilde{s}_n - \tilde{t}_n|\}} |\nabla_{t,x} r_n(\tilde{s}_n - \tilde{t}_n)|^2 \, dx = 0.$$

Let

$$(B_0^{\tilde{J}}, B_1^{\tilde{J}}) = \operatorname*{w-lim}_{n\to\infty}(\lambda_{\tilde{J},n}^{(N-2)/2} U(\tilde{s}_n, \lambda_{\tilde{J},n}\cdot), \lambda_{\tilde{J},n}^{N/2}\partial_t U(\tilde{s}_n, \lambda_{\tilde{J},n}\cdot)), \tag{6.40}$$

$$\sigma = \lim_{n\to\infty} \frac{\tilde{s}_n - \tilde{t}_n}{\lambda_{\tilde{J},n}}. \tag{6.41}$$

Note that $\sigma \in [0, \infty)$, since

$$\frac{t'_n - \tilde{t}_n}{\lambda_{\tilde{J},n}} = T.$$

As in the proof of Lemma 6.3 (see (6.26)), we obtain

$$(B_0^{\tilde{J}}(x), B_1^{\tilde{J}}(x)) = \vec{U}^{\tilde{J}}(\sigma, x), \quad |x| > \sigma. \tag{6.42}$$

We will next use Lemma 5.2 along the sequence $\{\tilde{s}_n\}$. For this, we recall (see (6.32))

$$\lim_{n\to\infty} \frac{\lambda_j(\tilde{s}_n)}{\lambda_j(\tilde{t}_n)} = 1 \quad \text{for all } j \in [\![1, \tilde{J}-1]\!]. \tag{6.43}$$

On the other hand, after extraction,

$$\lim_{n\to\infty} \frac{\lambda_{\tilde{J}}(\tilde{t}_n)}{\lambda_{\tilde{J}}(\tilde{s}_n)} =: \tilde{\lambda} \in (0, \infty). \tag{6.44}$$

Indeed, if (6.38) holds, then $\tilde{t}_n=\tilde{s}_n$ for all n and (6.44) is trivial, with $\tilde{\lambda}=1$. If not, then (6.39) holds, and (6.44) follows since $\delta(\tilde{s}_n)$ is small and bounded from below by Remark 6.5 and the fact that $\tilde{s}_n\leqslant t'_n$, which implies that

$$\lim_{n\to\infty}\frac{\tilde{s}_n-\tilde{t}_n}{\lambda_{\tilde{J},n}}<\infty.$$

By Lemma 5.2, letting

$$(V_0^{\tilde{J}},V_1^{\tilde{J}})=\operatorname*{w-lim}_{n\to\infty}(\lambda_{\tilde{J}}^{(N-2)/2}(\tilde{s}_n)U(\tilde{s}_n,\lambda_{\tilde{J}}(\tilde{s}_n)\cdot),\lambda_{\tilde{J}}^{N/2}(\tilde{s}_n)\partial_tU(\tilde{s}_n,\lambda_{\tilde{J}}(\tilde{s}_n)\cdot)), \tag{6.45}$$

we have

$$V_0^{\tilde{J}}=\sum_{j=\tilde{J}}^{j_{\tilde{J}+1}-1}\iota_jW_{(\nu_j)}+\tilde{h}_0^{\tilde{J}}\quad\text{and}\quad V_1^{\tilde{J}}=\sum_{j=\tilde{J}}^{j_{\tilde{J}+1}-1}\tilde{\alpha}_j(\Lambda W)_{[\nu_j]}+\tilde{g}_1^{\tilde{J}}, \tag{6.46}$$

where

$$\nu_j=\lim_{n\to\infty}\frac{\lambda_j(\tilde{s}_n)}{\lambda_{\tilde{J}}(\tilde{s}_n)}>0,\quad j\in[\![\tilde{J},j_{\tilde{J}+1}-1]\!], \tag{6.47}$$

and $j_{\tilde{J}+1}$ is the first index $j>\tilde{J}$ such that

$$\lim_{n\to\infty}\frac{\lambda_j(\tilde{s}_n)}{\lambda_{\tilde{J}}(\tilde{s}_n)}=0 \tag{6.48}$$

(as usual, if (6.48) is not satisfied for any $j>\tilde{J}$, we take $j_{\tilde{J}+1}=J+1$), and $\tilde{h}_0^{\tilde{J}}$ and $\tilde{g}_1^{\tilde{J}}$ satisfy the orthogonality conditions

$$\int\nabla\tilde{h}_0^k\nabla(\Lambda W)_{(\tilde{\nu}_j)}=\int\tilde{g}_1^k(\Lambda W)_{[\tilde{\nu}_j]}=0,\quad j\in[\![\tilde{\jmath}_k,\tilde{\jmath}_{k+1}-1]\!], \tag{6.49}$$

By (6.44), (6.45) and the definition (6.40) of $(B_0^{\tilde{J}},B_1^{\tilde{J}})$, we see that

$$B_0^{\tilde{J}}=\tilde{\lambda}^{(N-2)/2}V_0^{\tilde{J}}(\tilde{\lambda}\cdot)\quad\text{and}\quad B_1^{\tilde{J}}=\tilde{\lambda}^{N/2}V_1^{\tilde{J}}(\tilde{\lambda}\cdot).$$

Using (6.21) and (6.42), we deduce that, for large R,

$$\|(\tilde{\lambda}^{(N-2)/2}V_0^{\tilde{J}}(\tilde{\lambda}\cdot),\tilde{\lambda}^{N/2}V_1^{\tilde{J}}(\tilde{\lambda}\cdot))-\ell\Xi_{p_0}\|_{\mathcal{H}(R)}\lesssim\max\biggl\{\frac{1}{R^{(p_0-1/2)(N+2)/(N-2)}},\frac{1}{R^{p_0+1/2}}\biggr\},$$

and, after rescaling, for large R,

$$\|(V_0^{\tilde{J}},V_1^{\tilde{J}})-\tilde{\lambda}^{p_0-1/2}\ell\Xi_{p_0}\|_{\mathcal{H}(R)}\lesssim\max\biggl\{\frac{1}{R^{(p_0-1/2)(N+2)/(N-2)}},\frac{1}{R^{p_0+1/2}}\biggr\}. \tag{6.50}$$

We next note that $(V_0^{\tilde{J}}, V_1^{\tilde{J}})$ satisfies the assumptions of §4.2 and §4.3. Indeed, it is close to the multi-soliton

$$\widetilde{M} = \sum_{j=\tilde{J}}^{j_{\tilde{J}+1}-1} \iota_j W_{\nu_j}$$

(see (5.28)), and the solution with initial data $(V_0^{\tilde{J}}, V_1^{\tilde{J}})$ is, by Lemma 5.2, non-radiative. We use Proposition 4.6 on $(V_0^{\tilde{J}}, V_1^{\tilde{J}})$. In view of (6.46) and the orthogonality conditions (6.49) satisfied by $\tilde{h}_0^{\tilde{J}}$ and $\tilde{h}_1^{\tilde{J}}$, the exterior scaling parameter (denoted by λ_1 in Proposition 4.6) is $\nu_{\tilde{J}}$ who is, by definition, equal to 1 (see (6.47)). By (6.50), we must replace, in the conclusion of Proposition 4.6, $|\ell|$ by $\tilde{\lambda}^{p_0-1/2}|\ell|$. This yields

$$\tilde{\lambda}^{p_0-1/2}|\ell| \leqslant C_0\delta(\tilde{s}_n)^{2/N}. \tag{6.51}$$

Using (6.44), we obtain

$$\left(\frac{\lambda_{\tilde{J}}(\tilde{t}_n)}{\lambda_{\tilde{J}}(\tilde{s}_n)}\right)^{p_0-1/2}|\ell| \leqslant \frac{3}{2}C_0\delta(\tilde{s}_n)^{2/N}$$

for large n. By (6.39), we get

$$2C_0\delta(\tilde{s}_n)^{2/N} \leqslant \tfrac{3}{2}C_0\delta(\tilde{s}_n)^{2/N}$$

for large n, which is a contradiction, since $\delta(\tilde{s}_n)\neq 0$. □

6.4. Study of a system of differential inequalities

In this subsection, we prove Proposition 6.1.

We first observe that, by the following change of variable of unknwown functions:

$$\tau = \frac{t}{\lambda_1(0)}, \quad \check{\lambda}_j(\tau) = \frac{\lambda_j(t)}{\lambda_1(0)} \quad \text{and} \quad \check{\beta}_j(\tau) = \beta_j(t),$$

we may assume

$$\lambda_1(0) = 1.$$

The proof is by contradiction and relies on a monotonicity formula that follows from the modulation equations (6.2)–(6.5). In all the proof, the estimates are uniform in $t\in[0,T]$, and we must follow the dependence of the constants with respect to L. The implicit constants implied by the symbols $\lesssim$, $\ll$, ... will thus never depend on L and $t\in[0,T]$. We will introduce in the course of the proof two constants m and M depending only on L and the parameters of the system. To simplify notation, we will let

$$\kappa_2 := \frac{1}{\|\Lambda W\|_{L^2}^2}.$$

We note that (6.3) and the smallness assumption (6.1) imply

$$\sum_{j=1}^{J_0} |\beta_j| \lesssim \gamma^{(N-2)/4}. \tag{6.52}$$

Together with (6.2), we obtain

$$\sum_{j=1}^{J_0} |\lambda_j'| \lesssim C\gamma^{(N-2)/4}. \tag{6.53}$$

The idea of the proof is to construct a function V which is of the same order as λ_1^2 and is convex. We first introduce a positive quantity $B(t)$ which will appear in the computation of $V''(t)$.

Step 1. (Introduction of a positive quantity) We define $\theta_1,...,\theta_{J_0}$ as follows: $\theta_1=1$ and, for all $j\in[\![2,J_0]\!]$,

$$\theta_j = \begin{cases} 2\theta_{j-1}, & \text{if } \iota_j\iota_{j-1}=1, \\ \frac{1}{2}\theta_{j-1}, & \text{if } \iota_j\iota_{j-1}=-1. \end{cases}$$

We let

$$B(t) = \sum_{j=1}^{J_0} \theta_j \lambda_j(t)\beta_j'(t).$$

In this step, we prove

$$B(t) \geqslant \frac{\kappa_0}{2^{J_0+1}}\gamma(t)^{(N-2)/2} \quad \text{for all } t\in[0,T], \tag{6.54}$$

where κ_0 is the constant appearing in the formula (6.4) for β_j'. Indeed, by (6.4), setting $\iota_0=\iota_{J_0+1}=0$, we have

$$\begin{aligned} B(t) &= \kappa_0 \sum_{j=1}^{J_0} \theta_j \left(\iota_j\iota_{j-1}\left(\frac{\lambda_j}{\lambda_{j-1}}\right)^{(N-2)/2} - \iota_j\iota_{j+1}\left(\frac{\lambda_{j+1}}{\lambda_j}\right)^{(N-2)/2}\right) + \mathcal{O}(\gamma^{(N-1)/2}) \\ &= \kappa_0 \sum_{j=2}^{J_0} \iota_j\iota_{j-1}\left(\frac{\lambda_j}{\lambda_{j-1}}\right)^{(N-2)/2} (\theta_j-\theta_{j-1}) + \mathcal{O}(\gamma^{(N-1)/2}). \end{aligned}$$

By the definition of θ_j, for $2\leqslant j\leqslant J_0$, we have

$$\iota_{j-1}\iota_j(\theta_j-\theta_{j-1}) = \begin{cases} \theta_{j-1}, & \text{if } \iota_{j-1}\iota_j=1, \\ \frac{1}{2}\theta_{j-1}, & \text{if } \iota_{j-1}\iota_j=-1. \end{cases}$$

Hence, $\iota_{j-1}\iota_j(\theta_j-\theta_{j-1})\geqslant 1/2^{J_0}$, and (6.54) follows, since γ is small.

Step 2. (Approximate first derivative of V) We let

$$A(t) = \sum_{j=1}^{J_0} \theta_j \lambda_j(t) \beta_j(t). \tag{6.55}$$

In this step, we prove

$$A'(t) \gtrsim \gamma^{(N-2)/2}(t) \tag{6.56}$$

and that there exists $C>0$ such that

$$A'(t) \geqslant (\kappa_2^{-1} + C^{-1}) \sum_{j=1}^{J_0} \theta_j {\lambda_j'}^2(t), \tag{6.57}$$

$$A'(t) \geqslant (\kappa_2 + C^{-1}) \sum_{j=1}^{J_0} \theta_j \beta_j^2(t). \tag{6.58}$$

Indeed,

$$A'(t) = \sum_{j=1}^{J_0} \theta_j \lambda_j'(t) \beta_j(t) + B(t) = \kappa_2 \sum_{j=1}^{J_0} \theta_j \beta_j^2(t) + B(t) + \mathcal{O}(\gamma^{(N-1)/2}),$$

where we have used (6.2) and (6.52). By (6.54), we deduce that

$$A'(t) \geqslant \kappa_2 \sum_{j=1}^{J_0} \theta_j \beta_j^2(t) + \frac{1}{C} \gamma(t)^{(N-2)/2},$$

and hence (6.56). The estimate (6.58) follows also immediately, since

$$\sum_j \beta_j^2 \lesssim \gamma^{(N-2)/2}.$$

Together with (6.2), we also obtain (6.57).

Step 3. (Choice of an intermediate time) In this step, we will show that $A(1/L)$ is bounded from below by a constant depending only on L. By (6.5) and (6.56), recalling that $\lambda_1(0)=1$, we have

$$L \lesssim C A'(t) \lambda_1^a(t).$$

Integrating between 0 and $\tau \leqslant T$, we obtain

$$L\tau \lesssim \int_0^\tau A'(t) \lambda_1^a(t)\, dt. \tag{6.59}$$

Furthermore, by integration by parts, using again that $\lambda_1(0)=1$,

$$\int_0^\tau A'(t)\lambda_1^a(t)\,dt = A(\tau)\lambda_1^a(\tau)-A(0)-a\int_0^\tau A(t)\lambda_1'(t)\lambda_1^{a-1}(t)\,dt$$
$$= A(\tau)\lambda_1^a(\tau)-A(0)-\frac{a}{\kappa_2}\int_0^\tau (\lambda_1')^2\lambda_1^a+\mathcal{O}\biggl(\int_0^\tau |\lambda_1'|\gamma^{N/4}\lambda_1^a\biggr).$$

For this, we have used that, by (6.2) and the bound $|\beta_j|\lambda_j \lesssim \gamma^{(N+2)/4}\lambda_1$ (for $j\geqslant 2$) which follows from (6.52) and the definition of γ, we have

$$\begin{aligned} A(t) = \sum_{j=1}^{J}\theta_j\lambda_j\beta_j &= \kappa_2^{-1}\lambda_1\lambda_1' + (\beta_1-\kappa_2^{-1}\lambda_1')\lambda_1 + \sum_{j=2}^{J_0}\theta_j\lambda_j\beta_j \\ &= \kappa_2^{-1}\lambda_1\lambda_1' + \mathcal{O}(\lambda_1\gamma^{N/4}). \end{aligned} \tag{6.60}$$

Combining with the bound (6.53) on $|\lambda_1'|$, we deduce that

$$\int_0^\tau A'(t)\lambda_1^a(t)\,dt = A(\tau)\lambda_1^a(\tau)-A(0)-\frac{a}{\kappa_2}\int_0^\tau (\lambda_1'(t))^2\lambda_1^a(t)\,dt+\mathcal{O}\biggl(\int_0^\tau \gamma^{(N-1)/2}\lambda_1^a\biggr).$$

Using (6.59) and the fact that, by (6.56),

$$\int_0^\tau A'(t)\lambda_1^a(t)\,dt \gtrsim \int_0^\tau \gamma^{(N-2)/2}\lambda_1^a \gtrsim \frac{1}{\sqrt{\varepsilon}}\int_0^\tau \gamma^{(N-1)/2}\lambda_1^a,$$

we deduce, if ε is small enough, that, for all $\tau\in(0,T]$,

$$\frac{1}{2}L\tau+\frac{a}{\kappa_2}\int_0^\tau (\lambda_1'(t))^2\lambda_1^a(t)\,dt + A(0) \leqslant A(\tau)\lambda_1^a(\tau). \tag{6.61}$$

Assume in all the sequel that $T\geqslant 1/L$. Using that $|\lambda_1'|\leqslant 1$ (see (6.53)), we obtain

$$\lambda_1\Bigl(\frac{1}{L}\Bigr)\leqslant 1+\frac{1}{L}.$$

Noting that, by (6.5) at $t=0$, and since ε is small, $1/L\geqslant 1$, we deduce, taking ε small enough, that

$$\lambda_1\Bigl(\frac{1}{L}\Bigr)\leqslant \frac{2}{L} \tag{6.62}$$

Going back to (6.61), using that $|A(0)|\leqslant\frac{1}{100}$, we obtain

$$A\Bigl(\frac{1}{L}\Bigr)\geqslant \frac{1}{2}\frac{1}{\lambda_1^a(1/L)}\geqslant \frac{L^a}{2^{a+1}}. \tag{6.63}$$

Note that, since $A'(t)\geqslant 0$ by Step 2, (6.63) implies that $A(t)>0$ for all $t\in[1/L,T]$.

Step 4. (Bound from above of λ_1) We let

$$V(t)=\sum_{j=1}^{J_0}\theta_j\lambda_j^2,$$

and note that, using that $\theta_1=1$ and that $\lambda_j\leqslant\varepsilon\lambda_1$ for $j\geqslant 2$,

$$|V(t)-\lambda_1^2(t)|\lesssim\varepsilon^2\lambda_1^2(t).$$

In this step, we show that there exists $C>0$ and $M=M(L)>0$ such that

$$V(t)\leqslant M\varepsilon^{1/C}\quad\text{for all } t\in\left[\frac{1}{L},T\right]. \tag{6.64}$$

Indeed,

$$V'(t)=2\sum_{j=1}^{J_0}\theta_j\lambda_j\lambda_j',$$

and thus

$$V'(t)A(t)=2\sum_{j=1}^{J_0}\theta_j\lambda_j\lambda_j'\sum_{j=1}^{J_0}\theta_j\lambda_j\beta_j\leqslant 2V(t)\sqrt{\sum_{j=1}^{J_0}\theta_j{\lambda_j'}^2}\sqrt{\sum_{j=1}^{J_0}\theta_j\beta_j^2}\leqslant 2^-V(t)A'(t),$$

by Step 2. Here,

$$2^-=\frac{2}{(\kappa_2^{-1}+C^{-1})(\kappa_2+C^{-1})}$$

is a fixed positive constant, smaller than 2, and independent of L. Recall that, for $t\geqslant 1/L$, we have $A(t)>0$. For such t, we deduce that

$$2^-\frac{A'(t)}{A(t)}\geqslant\frac{V'(t)}{V(t)}, \tag{6.65}$$

and thus, for $t\geqslant 1/L$,

$$\frac{d}{dt}\left(\frac{A^{2^-}(t)}{V(t)}\right)\geqslant 0. \tag{6.66}$$

Hence, by (6.62) (which implies $V(1/L)\lesssim 1/L^2$) and (6.63), for $t\geqslant 1/L$,

$$\frac{A^{2^-}(t)}{V(t)}\geqslant\frac{A^{2-}(1/L)}{V(1/L)}\gtrsim L^{2+2^-a}. \tag{6.67}$$

Using the inequality

$$A(t)\lesssim\sqrt{V(t)}\gamma^{(N-2)/4}(t),$$

we deduce that, for all $t \leqslant 1/L$,

$$(\gamma^{(N-2)/2})^{2^-/2} \gtrsim \frac{A(t)^{2^-}}{V(t)^{2^-/2}} \gtrsim V(t)^{(2-2^-)/2} \frac{A^{2^-}(t)}{V(t)} \gtrsim L^{2+2^- a} V(t)^{(2-2^-)/2}.$$

This yields (6.64), with

$$M = \frac{1}{L^{2(2+2^- a)/(2-2^-)}}.$$

Step 5. (Bound from below of λ_1) By (6.65), we have, for $t \geqslant 1/L$,

$$\frac{d}{dt}\left(\frac{A(t)}{\sqrt{V(t)}}\right) \geqslant 0.$$

As a consequence, for $t \geqslant 1/L$,

$$\frac{A(t)}{\sqrt{V(t)}} \geqslant \frac{A(1/L)}{\sqrt{V(1/L)}}.$$

By the bound (6.62) on λ_1, the fact that $V(t) \approx \lambda_1^2(t)$ and the bound (6.63) on $A(1/L)$, we have

$$\frac{A(t)}{\sqrt{V(t)}} \geqslant \frac{A(1/L)}{\sqrt{V(1/L)}} \gtrsim L^{a+1} =: m. \tag{6.68}$$

Since $|V'(t) - 2\kappa_2 A(t)| \lesssim \gamma^{N/4}(t)\lambda_1(t)$ (see (6.60)), we deduce from (6.68) that

$$\frac{V'(t)}{\sqrt{V(t)}} \gtrsim m + \mathcal{O}(\gamma^{N/4}(t)) \quad \text{for all } t \geqslant \frac{1}{L}.$$

Integrating, we obtain

$$\int_{1/L}^{t} \gamma^{N/4}(s)\, ds + \sqrt{V(t)} - \sqrt{V\left(\frac{1}{L}\right)} \gtrsim \left(t - \frac{1}{L}\right) m,$$

and thus, by the bound (6.64) on $V \approx \lambda_1^2$,

$$\sqrt{M}\varepsilon^{1/(2C)} + \int_{1/L}^{t} \gamma^{N/4}(s)\, ds \gtrsim \left(t - \frac{1}{L}\right) m \quad \text{for all } t \in \left[\frac{1}{L}, T\right]. \tag{6.69}$$

Notice that, by Step 2, for $t \in [1/L, T]$,

$$\begin{aligned}\int_{1/L}^{t} \gamma^{N/4} &\leqslant \sqrt{\varepsilon} \int_{1/L}^{t} \gamma^{(N-2)/4}(s)\, ds \lesssim \sqrt{\varepsilon}\sqrt{t}\sqrt{\int_{1/L}^{t} A'(s)\, ds} \\ &\lesssim \sqrt{\varepsilon}\sqrt{t}\sqrt{A(t)} \lesssim \sqrt{\varepsilon}\sqrt{t}\,\varepsilon^{(N-2)/8}\sqrt{\lambda_1(t)} \lesssim \varepsilon^{(N+2)/8}\sqrt{t}M^{1/4},\end{aligned}$$

where we have used the bound $|A(t)|\lesssim\lambda_1(t)\gamma^{(N-2)/4}$ and, to get the last inequality, the bound (6.64) on V. Going back to (6.69), we obtain

$$m\left(t-\frac{1}{L}\right)\lesssim\varepsilon^{1/(2C)}\sqrt{M}+\varepsilon^{(N+2)/8}M^{1/4}\sqrt{t}. \tag{6.70}$$

Taking ε small, we deduce

$$m\left(T-\frac{1}{L}\right)\leqslant\sqrt{M}+M^{1/4}\sqrt{T}, \tag{6.71}$$

i.e.

$$\left(\sqrt{T}-\frac{M^{1/4}}{2m}\right)^2\leqslant\frac{1}{L}+\frac{\sqrt{M}}{m}+\frac{\sqrt{M}}{4m^2},$$

which implies

$$T\leqslant T^*:=\left(\sqrt{\frac{1}{L}+\frac{\sqrt{M}}{m}+\frac{\sqrt{M}}{4m^2}}+\frac{M^{1/4}}{2m}\right)^2,$$

concluding the proof, since the constants m and M depend only on L and the parameters of the system.

7. Inelastic collision

This section is dedicated to the proof of Theorem 2. The proof is almost contained in the proof of Theorem 1 and we only sketch it. Let u satisfy the assumptions of Theorem 2. If u scatters forward in time, then

$$\lim_{t\to+\infty}\|\vec{u}(t)\|_{\mathcal{H}}=0,$$

and, by the small data theory, u is identically zero. Thus, u does not scatter as $t\to\infty$, and according to Theorem 1, there exist $J\geqslant1$, signs $\{\iota_j\}_{1\leqslant j\leqslant J}$ and parameters $\lambda_j(t)$ defined for large t such that

$$0<\lambda_J(t)<\lambda_{J-1}(t)<...<\lambda_1(t),\quad \lim_{t\to\infty}\gamma(t)=0,\quad \lim_{t\to\infty}\frac{\lambda_1(t)}{t}=0$$

(where $\gamma(t)=\gamma(\boldsymbol{\lambda}(t))=\max_{j\in[\![1,J-1]\!]}\lambda_{j+1}(t)/\lambda_j(t)$) and

$$\lim_{t\to\infty}\left\|\vec{u}(t)-\sum_{j=1}^{J}(\iota_jW_{(\lambda_j(t))},0)\right\|_{\mathcal{H}}=0.$$

Note that (1.10) implies that $v_L\equiv0$. We will use the notation of §5. Using Lemma B.1 in the appendix, we can choose the $\lambda_j(t)$, for large t, such that

$$\int\nabla(u(t)-M(t))\cdot\nabla(\Lambda W_{(\lambda_j(t))})=0\quad\text{for all } j\in[\![1,J]\!], \tag{7.1}$$

where

$$M(t):=\sum_{j=1}^{J}\iota_j W_{(\lambda_j(t))}.$$

For $t\geqslant T$, T large, we expand

$$u(t)=M(t)+h(t)\quad\text{and}\quad \partial_t u(t)=\sum_{j=1}^{J}\alpha_j(t)\iota_j(\Lambda W)_{[\lambda_j(t)]}+g_1(t),$$

and denote

$$\delta(t):=\sqrt{\|h(t)\|_{\dot H^1}^2+\|\partial_t u(t)\|_{L^2}^2}\quad\text{and}\quad \beta_j(t):=-\iota_j\int(\Lambda W)_{[\lambda_j(t)]}\partial_t u(t).$$

Observe that the expansions above are valid for all large times, as opposed to the analogous expansions in §5 that are made on intervals of the form $[\tilde t_n,t_n]$, where $\tilde t_n,t_n\to\infty$ as $n\to\infty$.

Then, we have the following variant of Proposition 5.1:

Proposition 7.1. *There exists $C>0$ such that, for all $t\geqslant T$,*

$$\delta\leqslant C\gamma^{(N-2)/4} \tag{7.2}$$

$$\big|\beta_j-\|\Lambda W\|_{L^2}^2\lambda_j'\big|\leqslant C\gamma^{N/4}\qquad \text{for all } j\in[\![1,J]\!], \tag{7.3}$$

$$\left|\frac{1}{2}\sum_{j=1}^{J}\beta_j^2-\kappa_1\sum_{1\leqslant j\leqslant J-1}\iota_j\iota_{j+1}\left(\frac{\lambda_{j+1}}{\lambda_j}\right)^{(N-2)/2}\right|\leqslant C\gamma^{(N-1)/2} \tag{7.4}$$

$$\left|\lambda_j\beta_j'+\kappa_0\left(\iota_j\iota_{j+1}\left(\frac{\lambda_{j+1}}{\lambda_j}\right)^{(N-2)/2}-\iota_j\iota_{j-1}\left(\frac{\lambda_j}{\lambda_{j-1}}\right)^{(N-2)/2}\right)\right|\leqslant C\gamma^{(N-1)/2}\qquad \text{for all } j\in[\![1,J]\!], \tag{7.5}$$

where the positive constants κ_0 and κ_1 are as in Proposition 5.1.

Sketch of proof. The proof is the same as the proof of Proposition 5.1 in §5.3 and §5.4, observing that in the context of Proposition 7.1, we do not need §5.2 and we can remove all the $o_n(1)$ in the estimates. More precisely, we have the following.

- The estimates of Lemma 5.4 hold for $t\geqslant T$, without the $o_n(1)$ terms, as a direct consequence of Proposition 4.4.
- The estimates of Lemma 5.5 can be proved for all $t\geqslant T$, without the $o_n(1)$ terms, expanding the equality

$$E(\vec u(t))=JE(W,0),$$

and with the same proof. In Lemma 5.5, the terms $o_n(1)$ came from the fact that the preceding equality was replaced by the weaker statement

$$\lim_{t\to\infty} E(\vec{u}(t)-\vec{v}_L(t)) = JE(W,0).$$

- Similarly, one can prove Lemmas 5.6 and 5.7 for $t\geqslant T$ without $o_n(1)$ with the same proofs, observing that in the proofs of these lemmas, the $o_n(1)$ terms come either from the $o_n(1)$ terms of §5.3, or from the term $\sigma(h, v_L)$ defined in (5.54) and which is zero in our setting.

Assuming that u is not stationary, it is now easy to obtain a contradiction: by Proposition 4.6,

$$|\ell| \leqslant C\delta(t)^{2/N}\lambda_1^{k_0-1/2} \quad \text{for all } t\geqslant T.$$

Combining with the estimates of Proposition 7.1, we see that this contradicts Proposition 6.1, concluding the proof. □

Appendix A. Proof of some estimates

In this appendix, we gather a few purely computational proofs.

A.1. Estimates on integrals in the space variable

Claim A.1. *Let $0<\lambda<\mu$. Assume $N\geqslant 5$. Then,*

$$\int_{\mathbb{R}^N} |\nabla(\Lambda W_{(\lambda)})\cdot\nabla(\Lambda W_{(\mu)})| + \int_{\mathbb{R}^N} |\nabla W_{(\lambda)}\cdot\nabla W_{(\mu)}| \lesssim \left(\frac{\lambda}{\mu}\right)^{N/2-1}, \tag{A.1}$$

$$\int_{\mathbb{R}^N} |(\Lambda W)_{[\lambda]}(\Lambda W)_{[\mu]}| + \int_{\mathbb{R}^N} |(\Lambda W)_{[\lambda]}(\Lambda_0\Lambda W)_{[\mu]}| \lesssim \left(\frac{\lambda}{\mu}\right)^{N/2-2}, \tag{A.2}$$

$$\begin{aligned} \|W_{(\lambda)}W_{(\mu)}^{4/(N-2)}\|_{L^{2N/(N+2)}} &\lesssim \left(\frac{\lambda}{\mu}\right)^{(N-2)/2}, \\ \|W_{(\mu)}W_{(\lambda)}^{4/(N-2)}\|_{L^{2N/(N+2)}} &\lesssim \left(\frac{\lambda}{\mu}\right)^{2}, \end{aligned} \tag{A.3}$$

$$\left|\int_{\mathbb{R}^N} W_{(\lambda)}^{N/(N-2)} W_{(\mu)}^{N/(N-2)}\right| \lesssim \left(\frac{\lambda}{\mu}\right)^{N/2}, \tag{A.4}$$

$$\begin{aligned} \int |(\Lambda W)_{[\lambda]}(\Delta\Lambda W)_{[\mu]}| &\lesssim \left(\frac{\lambda}{\mu}\right)^{N/2-2}, \\ \int |(\Lambda W)_{[\mu]}(\Delta\Lambda W)_{[\lambda]}| &\lesssim \left(\frac{\lambda}{\mu}\right)^{N/2}. \end{aligned} \tag{A.5}$$

Proof. We have

$$|\Lambda_0\Lambda W(x)|+|\Lambda W(x)|+|W(x)|\lesssim \min\left\{1,\frac{1}{|x|^{N-2}}\right\} \tag{A.6}$$

$$\begin{aligned}|\nabla\Lambda W(x)|+|\nabla W(x)|&\lesssim \min\left\{1,\frac{1}{|x|^{N-1}}\right\},\\ |\Delta\Lambda W|&\lesssim \min\left\{1,\frac{1}{|x|^{N}}\right\}.\end{aligned} \tag{A.7}$$

In view of these bounds, the estimates (A.1)–(A.5) are consequences of the following inequality, which holds for any $a,b\in\mathbb{R}$ with $a+b>N$, and can be proved by integrating separately on $\{|x|<\lambda\}$, $\{\lambda<|x|<\mu\}$ and $\{|x|>\mu\}$:

$$\int_{\mathbb{R}^N}\min\left(1,\left\{\frac{\lambda}{|x|}\right\}^a\right)\min\left\{1,\left(\frac{\mu}{|x|}\right)^b\right\}dx\lesssim \lambda^a\mu^{N-a}. \tag{A.8}$$

We will prove (A.3). The proofs of (A.1), (A.2), (A.4) and (A.5) are very similar. By (A.6), we have

$$\begin{aligned}&\int W_{(\lambda)}^{2N/(N+2)}W_{(\mu)}^{8N/((N-2)(N+2))}\\ &\quad\lesssim \frac{1}{\lambda^{N(N-2)/(N+2)}\mu^{4N/(N+2)}}\\ &\qquad\times\int \min\left\{1,\left(\frac{\lambda}{|x|}\right)^{2N(N-2)/(N+2)}\right\}\min\left\{1,\left(\frac{\mu}{|x|}\right)^{8N/(N+2)}\right\},\end{aligned}$$

and the first estimate of (A.3) follows from (A.8) with $a=2N(N-2)/(N+2)$. Similarly,

$$\begin{aligned}&\int W_{(\mu)}^{2N/(N+2)}W_{(\lambda)}^{8N/((N-2)(N+2))}\\ &\quad\lesssim \frac{1}{\mu^{N(N-2)/(N+2)}\lambda^{4N/(N+2)}}\\ &\qquad\times\int \min\left\{1,\left(\frac{\mu}{|x|}\right)^{2N(N-2)/(N+2)}\right\}\min\left\{1,\left(\frac{\lambda}{|x|}\right)^{8N/(N+2)}\right\},\end{aligned}$$

and the second estimate of (A.3) follows from (A.8) with $a=8N/(N+2)$. □

A.2. Estimates on space time norms

Claim A.2. *Assume $N\geqslant 5$. Let $0<\lambda<\mu$. Then,*

$$\|\mathbb{1}_{\{|x|\geqslant|t|\}}W_{(\lambda)}^{4/(N-2)}W_{(\mu)}\|_{L^1(\mathbb{R},L^2)}\lesssim\begin{cases}\left(\dfrac{\lambda}{\mu}\right)^{3/2}, & \text{if } N=5,\\ \left(\dfrac{\lambda}{\mu}\right)^{2}, & \text{if } N\geqslant 7,\end{cases} \tag{A.9}$$

$$\|1\!\!1_{\{|x|\geqslant|t|\}}W_{(\mu)}^{4/(N-2)}W_{(\lambda)}\|_{L^1(\mathbb{R},L^2)}\lesssim\begin{cases}\left(\dfrac{\lambda}{\mu}\right)^{3/2}, & \textit{if } N=5,\\ \left(\dfrac{\lambda}{\mu}\right)^{2}, & \textit{if } N\geqslant 7,\end{cases}\tag{A.10}$$

$$\|1\!\!1_{\{|x|\geqslant|t|\}}tW_{(\lambda)}^{4/(N-2)}W_{[\mu]}\|_{L^1(\mathbb{R},L^2)}\lesssim\left(\frac{\lambda}{\mu}\right)^2\tag{A.11}$$

$$\|1\!\!1_{\{|x|\geqslant|t|\}}tW_{(\mu)}^{4/(N-2)}W_{[\lambda]}\|_{L^1(\mathbb{R},L^2)}\lesssim\begin{cases}\left(\dfrac{\lambda}{\mu}\right)^{1/2}, & \textit{if } N=5,\\ \left(\dfrac{\lambda}{\mu}\right)^{3/2}, & \textit{if } N=7,\\ \left(\dfrac{\lambda}{\mu}\right)^{2}, & \textit{if } N\geqslant 9.\end{cases}\tag{A.12}$$

The same inequalities remain valid when replacing W by ΛW anywhere in the preceding norms.

Proof. In all of the proof of the claim, we will use the bound

$$|W(x)|+|\Lambda W(x)|\lesssim\min\{1,|x|^{2-N}\}.\tag{A.13}$$

By scaling, we may assume that $\mu=1$. By symmetry, it is sufficient to bound the integrals for $t\geqslant 0$. The proofs for all four bounds are the same. We divide the domain of integration for r in three parts, $(0,\lambda)$, $(\lambda,1)$ and $(1,\infty)$, writing

$$\begin{aligned}\int_0^{+\infty}\left(\int_t^{+\infty}\ldots r^{N-1}\,dr\right)^{1/2}dt\lesssim&\int_0^{\lambda}\left(\int_t^{\lambda}\ldots r^{N-1}\,dr\right)^{1/2}dt\\&+\int_0^{1}\left(\int_{\max\{t,\lambda\}}^{1}\ldots r^{N-1}\,dr\right)^{1/2}dt\\&+\int_0^{+\infty}\left(\int_{\max\{t,1\}}^{+\infty}\ldots r^{N-1}\,dr\right)^{1/2}dt\\=&(1)+(2)+(3),\end{aligned}$$

where ... is either

$$(W_{(\lambda)}^{4/(N-2)}W)^2,\quad (W^{4/(N-2)}W_{(\lambda)})^2,\quad (tW_{(\lambda)}^{4/(N-2)}W)^2\quad\text{or}\quad (tW^{4/(N-2)}W_{[\lambda]})^2.$$

In the integrals (1), we use the bound $W(r/\lambda)+W(r)\lesssim 1$, in the integrals (2), we use the bounds $W(r/\lambda)\lesssim(\lambda/r)^{N-2}$ and $W(r)\lesssim 1$, and in the integrals (3), we use the bounds $W(r/\lambda)\lesssim(\lambda/r)^{N-2}$ and $W(r)\lesssim 1/r^{N-2}$.

We will detail the proof of (A.9) and sketch the proof of the other estimates.

Proof of (A.9). We have

$$(1)=\int_0^\lambda\left(\int_t^\lambda W^2\frac{1}{\lambda^4}W^{8/(N-2)}\left(\frac{r}{\lambda}\right)r^{N-1}\,dr\right)^{1/2}dt$$
$$\lesssim\frac{1}{\lambda^2}\int_0^\lambda\left(\int_t^\lambda r^{N-1}\,dr\right)^{1/2}dt\lesssim\lambda^{(N-2)/2}.$$

If $N\in\{5,7\}$, we have

$$(2)=\int_0^1\left(\int_{\max\{t,\lambda\}}^1 W^2\frac{1}{\lambda^4}W^{8/(N-2)}\left(\frac{r}{\lambda}\right)r^{N-1}\,dr\right)^{1/2}dt$$
$$\lesssim\lambda^2\int_0^\lambda\left(\int_\lambda^1 r^{N-9}\,dr\right)^{1/2}dt+\lambda^2\int_\lambda^1\left(\int_t^1 r^{N-9}\,dr\right)^{1/2}dt$$
$$\lesssim\lambda^2\int_0^\lambda\lambda^{(N-8)/2}\,dt+\lambda^2\int_\lambda^1 t^{(N-8)/2}\,dt$$
$$\lesssim\begin{cases}\lambda^2, & \text{if } N=7,\\ \lambda^{3/2}, & \text{if } N=5.\end{cases}$$

If $N\geqslant 9$, we obtain

$$(2)\lesssim\lambda^2\int_0^\lambda\left(\int_\lambda^1 r^{N-9}\,dr\right)^{1/2}dt+\lambda^2\int_\lambda^1\left(\int_t^1 r^{N-9}\,dr\right)^{1/2}dt\lesssim\lambda^2.$$

It remains to bound the third integral:

$$(3)=\int_0^{+\infty}\left(\int_{\max\{1,|t|\}}^{+\infty}W_{(\lambda)}^{8/(N-2)}W^2r^{N-1}\,dr\right)^2dt$$
$$\lesssim\int_0^{+\infty}\left(\int_{\max\{1,|t|\}}^{+\infty}\frac{\lambda^4}{r^{8+2(N-2)}}r^{N-1}\,dr\right)^2dt\lesssim\lambda^2.$$

Combining the preceding bounds, we obtain (A.9). □

Sketch of proof of (A.10)–(A.12). By analogous arguments, we obtain the following bounds.

- For (A.10),

$$(1)\lesssim\lambda^2,\quad (2)\lesssim\begin{cases}\lambda^{3/2}, & \text{if } N=5,\\ \lambda^2, & \text{if } N\geqslant 7,\end{cases}\quad\text{and}\quad (3)\lesssim\lambda^{(N-2)/2}.$$

- For (A.11),

$$(1)\lesssim\lambda^{N/2},\quad (2)\lesssim\lambda^2\quad\text{and}\quad(3)\lesssim\lambda^2.$$

- For (A.12),

$$(1)\lesssim\lambda^2,\quad (2)\lesssim\begin{cases}\lambda^{1/2}, & \text{if } N=5,\\ \lambda^{3/2}, & \text{if } N=7,\\ \lambda^2, & \text{if } N\geqslant 9,\end{cases}\quad\text{and}\quad (3)\lesssim\lambda^{(N-4)/2}.$$ □

This concludes the proof of Claim A.2. □

CLAIM A.3. *Assume $N\geqslant 5$. Let $0<\lambda<\mu$. Then,*

$$\|1\!\!1_{\{|x|\geqslant|t|\}}\min\{W_{(\lambda)}^{4/(N-2)}W_{(\mu)},W_{(\mu)}^{4/(N-2)}W_{(\lambda)}\}\|_{L^1(\mathbb{R},L^2)}\lesssim\left(\frac{\lambda}{\mu}\right)^{(N+2)/4}. \tag{A.14}$$

Proof. As before, we will use continuously the bound

$$|W(x)|\lesssim\min\{1,|x|^{2-N}\}.$$

By scaling, we may assume that $\mu=1$ (and thus $\lambda\leqslant 1$). We note that

$$\sqrt{\lambda}\lesssim r\implies W_{(\lambda)}\lesssim W\quad\text{and}\quad r\lesssim\sqrt{\lambda}\implies W_{(\lambda)}\gtrsim W.$$

We divide the space into four regions, writing

$$\begin{aligned}
&\frac{1}{2}\|1\!\!1_{\{|x|\geqslant|t|\}}\min\{W_{(\lambda)}^{4/(N-2)}W,W^{4/(N-2)}W_{(\lambda)}\}\|_{L^1(\mathbb{R},L^2)}\\
&\quad=\int_0^{+\infty}\left(\int_t^{+\infty}\min\{W_{(\lambda)}^{4/(N-2)}W,W^{4/(N-2)}W_{(\lambda)}\}r^{N-1}\,dr\right)^{1/2}dt\\
&\quad\lesssim\int_0^{\lambda}\left(\int_t^{\lambda}...\right)^{1/2}dt+\int_0^{\sqrt{\lambda}}\left(\int_{\max\{t,\lambda\}}^{\sqrt{\lambda}}...\right)^{1/2}dt\\
&\qquad+\int_0^{1}\left(\int_{\max\{t,\sqrt{\lambda}\}}^{1}...\right)^{1/2}dt+\int_0^{+\infty}\left(\int_{\max\{t,1\}}^{+\infty}...\right)^{1/2}dt\\
&\quad=A_1+A_2+A_3+A_4.
\end{aligned}$$

Case 1. ($N\geqslant 7$) In this case, $4/(N-2)<1$. We have

$$A_1\lesssim\int_0^{\lambda}\left(\int_{|t|}^{\lambda}W^2\frac{1}{\lambda^4}W^{8/(N-2)}\left(\frac{r}{\lambda}\right)r^{N-1}\,dr\right)^{1/2}dt.$$

Using that W is bounded, we obtain

$$A_1\lesssim\frac{1}{\lambda^2}\int_0^{\lambda}\left(\int_0^{\lambda}r^{N-1}\,dr\right)^{1/2}dt\lesssim\lambda^{(N-2)/2}. \tag{A.15}$$

We next consider A_2:

$$A_2\lesssim\int_0^{\sqrt{\lambda}}\left(\int_{\max\{t,\lambda\}}^{\sqrt{\lambda}}W^2\frac{1}{\lambda^4}W^{8/(N-2)}\left(\frac{r}{\lambda}\right)r^{N-1}\,dr\right)^{1/2}dt.$$

Using the bounds

$$W^2\lesssim 1\quad\text{and}\quad W^{8/(N-2)}\left(\frac{r}{\lambda}\right)\lesssim\frac{\lambda^8}{r^8},$$

we obtain

$$A_2 \lesssim \lambda^2 \int_0^{\sqrt{\lambda}} \left(\int_{\max\{t,\lambda\}}^{\sqrt{\lambda}} r^{N-9}\, dr \right)^{1/2} dt.$$

If $N=7$, this yields

$$A_2 \lesssim \lambda^2 \int_0^{\lambda} \frac{1}{\lambda^{1/2}}\, dt + \lambda^2 \int_{\lambda}^{\sqrt{\lambda}} \frac{1}{\sqrt{t}}\, dt \lesssim \lambda^{9/4}.$$

If $N\geqslant 9$, we deduce that

$$A_2 \lesssim \lambda^2 \int_0^{\sqrt{\lambda}} \lambda^{(N-8)/4}\, dt = \lambda^{(N+2)/4}.$$

In both cases, we have obtained

$$A_2 \lesssim \lambda^{(N+2)/4}. \tag{A.16}$$

We have

$$A_3 \lesssim \frac{1}{\lambda^{(N-2)/2}} \int_0^1 \left(\int_{\max\{t,\sqrt{\lambda}\}}^1 W^2\left(\frac{r}{\lambda}\right) W^{8/(N-2)}(r) r^{N-1}\, dr \right)^{1/2} dt.$$

Using the bounds

$$W^2\left(\frac{r}{\lambda}\right) \lesssim \left(\frac{\lambda}{r}\right)^{2(N-2)} \quad \text{and} \quad W^{8/(N-2)} \lesssim 1,$$

we deduce that

$$A_3 \lesssim \lambda^{(N-2)/2} \int_0^{\sqrt{\lambda}} \left(\int_{\sqrt{\lambda}}^1 r^{3-N}\, dr \right)^{1/2} dt + \lambda^{(N-2)/2} \int_{\sqrt{\lambda}}^1 \left(\int_t^1 r^{3-N}\, dr \right)^{1/2} dt,$$

which yields

$$A_3 \lesssim \lambda^{(N+2)/4}. \tag{A.17}$$

Finally, we bound A_4. We have

$$A_4 \lesssim \frac{1}{\lambda^{(N-2)/2}} \int_0^{\infty} \left(\int_{\max\{1,t\}}^{+\infty} W^2\left(\frac{r}{\lambda}\right) W^{8/(N-2)}(r) r^{N-1}\, dr \right)^{1/2} dt.$$

Using the bounds

$$W^2\left(\frac{r}{\lambda}\right) \lesssim \left(\frac{\lambda}{r}\right)^{2(N-2)} \quad \text{and} \quad W^{8/(N-2)} \lesssim \frac{1}{r^8},$$

we deduce that

$$A_4 \lesssim \lambda^{(N-2)/2}. \tag{A.18}$$

Combining (A.15)–(A.18), and noting that $\frac{1}{4}(N+2)\leqslant\frac{1}{2}(N-2)$ if $N\geqslant 6$, we deduce the bound (A.14) when $N\geqslant 7$.

Case 2. ($N=5$) In this case, we have

$$\frac{4}{N-2}=\frac{4}{3}>1.$$

The proof is the same as in the preceding case, except that

$$\min\{W_{(\lambda)}^{4/3}W, W^{4/3}W_{(\lambda)}\}(r)\approx\begin{cases} W^{4/3}(r)W_{(\lambda)}(r), & \text{if } r\leqslant\sqrt{\lambda},\\ W_{(\lambda)}^{4/3}(r)W(r), & \text{if } r\geqslant\sqrt{\lambda}.\end{cases}$$

By explicit computation, one obtains the bounds

$$A_1\lesssim\lambda^2,\quad A_2\lesssim\lambda^{7/4},\quad A_3\lesssim\lambda^{7/4}\quad\text{and}\quad A_4\lesssim\lambda^2,$$

which yields the bound (A.14). We leave the details to the reader. □

A.3. Pointwise bounds

Claim A.4. *Assume $N\geqslant 5$ and $J\geqslant 1$. For all $(y_1,...,y_J,h)\in\mathbb{R}^{J+1}$,*

$$\begin{aligned}\Bigg|\frac{N-2}{2N}\bigg|\sum_{j=1}^J y_j+h\bigg|^{2N/(N-2)}&-\frac{N-2}{2N}\sum_{j=1}^J|y_j|^{2N/(N-2)}\\ &-\sum_{j=1}^J|y_j|^{4/(N-2)}y_jh-\sum_{\substack{1\leqslant j,k\leqslant J\\ j\neq k}}|y_j|^{4/(N-2)}y_jy_k\Bigg|\\ \lesssim|h|^{2N/(N-2)}&+\sum_{j=1}^J|y_j|^{4/(N-2)}h^2\\ +\sum_{1\leqslant j<k\leqslant J}&(\min\{|y_j|^{4/(N-2)}y_k^2,|y_k|^{4/(N-2)}y_j^2\}\\ &+\min\{|y_j|^{(N+2)/(N-2)}|y_k|,|y_k|^{(N+2)/(N-2)}|y_j|\}),\end{aligned}\tag{A.19}$$

Proof. We fix $(y_1,...,y_J,h)\in\mathbb{R}^{J+1}$ and distinguish between two cases.

Case 1. ($|h|\geqslant\max_{1\leqslant j\leqslant J}|y_j|$) In this case, the inequality is trivial, since all terms of the left-hand side are bounded by $|h|^{2N/(N-2)}$ up to a constant.

Case 2. ($|h|\leqslant\max_{1\leqslant j\leqslant J}|y_j|$) We assume without loss of generality that

$$|y_1|=\max_{1\leqslant j\leqslant J}|y_j|.$$

We use the inequality

$$\left|\frac{N-2}{2N}|1+s|^{2N/(N-2)}-\frac{N-2}{2N}-s\right|\lesssim s^2+|s|^{2N/(N-2)},$$

with

$$s=\frac{1}{y_1}\left(h+\sum_{j=2}^J y_j\right).$$

Multiplying the resulting inequality by $|y_1|^{2N/(N-2)}$, we obtain

$$\begin{aligned}\left|\frac{N-2}{2N}\left|h+\sum_{j=1}^J y_j\right|^{2N/(N-2)}-\frac{N-2}{2N}|y_1|^{2N/(N-2)}-|y_1|^{4/(N-2)}y_1\left(h+\sum_{j=2}^J y_j\right)\right|\\ \lesssim |y_1|^{4/(N-2)}\left(h^2+\sum_{j=2}^J y_j^2\right)+|h|^{2N/(N-2)}+\sum_{j=2}^J |y_j|^{2N/(N-2)}.\end{aligned}\tag{A.20}$$

Since $|y_1|=\max_{1\leqslant j\leqslant J}|y_j|$ and $4/(N-2)<2$, we have that the right-hand side of (A.20) is clearly bounded by

$$|h|^{2N/(N-2)}+\sum_{j=1}^J |y_j|^{4/(N-2)}h^2+\sum_{1\leqslant j<k\leqslant J}\min\{|y_j|^{4/(N-2)}y_k^2,|y_k|^{4/(N-2)}y_j^2\}.$$

It remains to bound the terms that appear in the left-hand side of (A.19) but not on the left-hand side of (A.20). Using again that $|y_1|=\max_{1\leqslant j\leqslant J}|y_j|$, we have

$$\begin{aligned}\sum_{j=2}^J |y_j|^{2N/(N-2)}&\lesssim\sum_{j\neq k}\min\{|y_j|^{4/(N-2)}y_k^2,|y_k|^{4/(N-2)}y_j^2\},\\ \left|\sum_{j=2}^J |y_j|^{4/(N-2)}y_jh\right|&\lesssim\sum_{j=2}^J |y_j|^{2N/(N-2)}+\sum_{j=2}^J |y_j|^{4/(N-2)}h^2\\ &\lesssim\sum_{j\neq k}\min\{|y_j|^{4/(N-2)}y_k^2,|y_k|^{4/(N-2)}y_j^2\}+\sum_{j=2}^J |y_j|^{4/(N-2)}h^2,\\ \left|\sum_{\substack{2\leqslant j\leqslant J\\1\leqslant k\leqslant J\\ j\neq k}} |y_j|^{4/(N-2)}y_jy_k\right|&\lesssim\sum_{2\leqslant j\leqslant J}|y_1|\,|y_j|^{(N+2)/(N-2)}\\ &\lesssim\sum_{j\neq k}\min\{|y_j|^{(N+2)/(N-2)}\,|y_k|,|y_k|^{(N+2)/(N-2)}\,|y_j|\},\end{aligned}$$

which concludes the proof. □

Recall the notation $F(\sigma)=|\sigma|^{4/(N-2)}\sigma$.

CLAIM A.5. *Assume $N\geqslant 7$. Let $J\geqslant 1$. Then, for all $(y_1,\dots,y_J,h)\in\mathbb{R}^{J+1}$, if $N\geqslant 7$, then*

$$\begin{aligned}&\left|F\left(h+\sum_{j=1}^J y_j\right)-\sum_{j=1}^J F(y_j)-\frac{N+2}{N-2}\sum_{j=1}^J |y_j|^{4/(N-2)}h-F(h)\right|\\ &\qquad\lesssim \sum_{1\leqslant j<k\leqslant J}\min\{|y_j|^{4/(N-2)}|y_k|,|y_k|^{4/(N-2)}|y_j|\}\\ &\qquad\quad+|h|^{(N+1)/(N-2)}\sum_{j=1}^J |y_j|^{1/(N-2)},\end{aligned}\tag{A.21}$$

and, if $N=5$, then

$$\begin{aligned}&\left|F\left(\sum_{j=1}^J y_j+h\right)-\sum_{j=1}^J F(y_j)-\frac{7}{3}\sum_{j=1}^J |y_j|^{4/3}h-\frac{7}{3}\sum_{\substack{1\leqslant j,k\leqslant J\\ j\neq k}}|y_j|^{4/3}y_k-F(h)\right|\\ &\lesssim\sum_{j=1}^J |y_j|^{1/3}h^2+\sum_{1\leqslant j<k\leqslant J}\min\{|y_j|^{4/3}\,|y_k|,|y_k|^{4/3}\,|y_j|\}.\end{aligned}\tag{A.22}$$

Proof. Case 1. ($\max_j |y_j|\leqslant|h|$) We use the fact that

$$|s|\leqslant J\quad\Longrightarrow\quad\left|F(1+s)-1-\frac{N+2}{N-2}s\right|\lesssim s^2,\tag{A.23}$$

with

$$s=\frac{1}{h}\sum_{j=1}^J y_j.$$

Multiplying the resulting inequality by $|h|^{(N+2)/(N-2)}$, we obtain

$$\left|F\left(h+\sum_{j=1}^J y_j\right)-F(h)-\frac{N+2}{N-2}|h|^{4/(N-2)}\sum_{j=1}^J y_j\right|\lesssim|h|^{(N+2)/(N-2)-2}\sum_{j=1}^J y_j^2.$$

Using that $\max_j |y_j|\leqslant|h|$, we deduce (A.21) or (A.22).

Case 2. ($|h|\leqslant\max_j |y_j|$) We assume, without loss of generality, that

$$|y_1|=\max_{1\leqslant j\leqslant J}|y_j|.$$

We use (A.23) with

$$s=\frac{1}{y_1}\left(h+\sum_{j=2}^J y_j\right).$$

Multiplying the resulting inequality by $|y_1|^{(N+2)/(N-2)}$, we obtain

$$\left|F\left(h+\sum_{1\leqslant j\leqslant J} y_j\right)-F(y_1)-\frac{N+2}{N-2}|y_1|^{4/(N-2)}\left(h+\sum_{j=2}^J y_j\right)\right| \lesssim |y_1|^{(N+2)/(N-2)-2}\left(h^2+\sum_{j=2}^J y_j^2\right).$$

Let $j\geqslant 2$. Since $|y_j|\leqslant|y_1|$, we have

$$F(y_j)\lesssim \min\{|y_j|^{4/(N-2)}\,|y_1|, |y_1|^{4/(N-2)}\,|y_j|\}.$$

Furthermore,

$$|y_j|^{4/(N-2)}\,|h|\lesssim\begin{cases} |y_j|^{1/(N-2)}|h|^{(N+1)/(N-2)}, & \text{if } |y_j|<|h|,\\ \min\{|y_j|^{4/(N-2)}\,|y_1|, |y_1|^{4/(N-2)}\,|y_j|\}, & \text{if } |h|\leqslant|y_j|,\end{cases}$$

and also

$$|y_1|^{(N+2)/(N-2)-2}h^2+F(h)\lesssim |y_1|^{1/(N-2)}\,|h|^{(N+1)/(N-2)}$$
$$|y_1|^{(N+2)/(N-2)-2}y_j^2\leqslant \min\{|y_1|^{4/(N-2)}\,|y_j|, |y_j|^{4/(N-2)}\,|y_1|\}.$$

If $N\geqslant 7$, we have $4/(N-2)<1$, and thus

$$|y_1|^{4/(N-2)}|y_j|=\min\{|y_1|^{4/(N-2)}|y_j|, |y_j|^{4/(N-2)}|y_1|\}.$$

If $N=5$, we note that, if $2\leqslant j,k$ with $j\neq k$,

$$|y_j|^{4/3}\,|y_k|\lesssim \min\{|y_1|^{4/3}\,|y_j|, |y_j|^{4/3}\,|y_1|\}+\min\{|y_1|^{4/3}\,|y_k|, |y_k|^{4/3}\,|y_1|\}.$$

Combining the preceding inequalities, we obtain (A.21) or (A.22). □

CLAIM A.6. *Let $(a,b,c)\in\mathbb{R}^3$, with $a\neq 0$. We have*

$$|F(a+b)-F(a)-F'(a)b|\lesssim \mathbb{1}_{\{|b|\leqslant|a|\}}b^2a^{(6-N)/(N-2)}+\mathbb{1}_{\{|b|\geqslant|a|\}}b^{(N+2)/(N-2)}, \tag{A.24}$$

and

$$|F(a+b+c)-F(a+b)-F(a+c)+F(a)| \lesssim\begin{cases} |a|^{(6-N)/2(N-2)}\,|b|^{(N+2)/2(N-2)}\,|c|, & \text{if } N\geqslant 7,\\ |b|\,|c|(|a|+|b|+|c|)^{1/3}, & \text{if } N=5.\end{cases} \tag{A.25}$$

Proof of (A.24). By scaling, we may assume $a=1$. We are thus reduced to prove

$$|F(1+b)-F(1)-F'(1)b| \lesssim \mathbb{1}_{|b|\leqslant 1} b^2 + \mathbb{1}_{|b|\geqslant 1} b^{(N+2)/(N-2)}, \quad b\in\mathbb{R}, \tag{A.26}$$

which follows easily from the fact that $F(z)$ is C^2 outside $z=0$ and of order $|z|^{(N+2)/(N-2)}$ as $|z|\to\infty$. □

Proof of (A.25) *in the case* $N\geqslant 7$. Note that

$$1+\frac{N+2}{2(N-2)}+\frac{6-N}{2(N-2)}=\frac{N+2}{N-2}.$$

Thus, both sides of (A.25) are homogeneous of degree $(N+2)/(N-2)$, and we may assume, without loss of generality, $a=1$. We are thus reduced to prove (assuming $N\geqslant 7$)

$$|G(b,c)| \lesssim |b|^{(N+2)/2(N-2)}\,|c|, \tag{A.27}$$

where

$$G(b,c) := |F(1+b+c)-F(1+b)-F(1+c)+F(1)|.$$

We distinguish between two cases.

Case 1. ($|c|\leqslant|b|$) There exists b_1 and d_1 such that

$$F(1+b+c)-F(1+b)=F'(b_1)c, \quad |b+1-b_1|\leqslant|c|, \tag{A.28}$$

$$F(1+c)-F(1)=F'(d_1)c, \qquad |d_1-1|\leqslant|c|. \tag{A.29}$$

In particular,

$$|G(b,c)| \lesssim (1+|b|+|c|)^{4/(N-2)}|c|.$$

If $|b|\geqslant\frac{1}{10}$, this implies (A.27), since

$$\frac{4}{N-2}\leqslant\frac{N+2}{2(N-2)}.$$

If $|b|\leqslant\frac{1}{10}$, we use the fact that F is C^2 outside the origin. Therefore, there exists $d_2\in[b_1,d_1]$ (or $[d_1,b_1]$) such that

$$F'(b_1)-F'(d_1)=F''(d_2)(d_1-b_1).$$

Since $\frac{1}{2}\leqslant d_2\leqslant 2$, we have $|F''(d_2)|\lesssim 1$, and we obtain, by the triangle inequality,

$$|b_1-d_1|\leqslant|b_1-(1+b)|+|d_1-1|+|b|\leqslant|c|+|b|,$$

which yields

$$|G(b,c)| = |F''(d_2)(b_1-d_1)c| \leqslant |c|(|c|+|b|),$$

yielding (A.27), since $|c|\leqslant|b|\leqslant\frac{1}{10}$ and

$$\frac{N+2}{2(N-2)} < 1.$$

Case 2. ($|c|\geqslant|b|$) The same proof as in the preceding case, inverting b and c, yields

$$|G(b,c)| \lesssim (1+|b|+|c|)^{4/(N-2)}|b|,$$

and, if $|c|\leqslant\frac{1}{10}$,

$$|G(b,c)| \leqslant |b|(|c|+|b|).$$

Using that

$$\frac{4}{N-2} < \frac{N+2}{2(N-2)} < 1,$$

we obtain (A.27). □

Proof of (A.25) *in the case* $N=5$. By homogeneity, we may assume $a=1$, and we are thus reduced to prove, with the same notation $G(b,c)$ as before,

$$|G(b,c)| \lesssim |b|\,|c|(1+|b|+|c|)^{1/3}.$$

The inequality is symmetric in (b,c) and we may assume $|b|\geqslant|c|$. Again, we use that there exist b_1 and d_1 such that (A.28) and (A.29) hold. Since F is of class C^2, we also know that there exists $d_2\in[b_1,d_1]$ (or $[d_1,b_1]$) such that

$$F'(b_1)-F'(d_1) = F''(d_2)(d_1-b_1).$$

We have $|d_1-b_1|\lesssim|b|+|c|\lesssim|b|$ and $|F''(d_2)|\lesssim(1+|b|+|c|)^{1/3}$, and thus

$$|G(b,c)| = |F''(d_2)(d_1-b_1)c| \lesssim |c|\,|b|(1+|b|+|c|)^{1/3}.$$ □

Appendix B. Choice of the scaling parameters

Lemma B.1. *Let* $J\geqslant1$. *There exists a small constant* $\varepsilon_J>0$ *and a large constant* $C_J>0$, *with the following property. For all* $\varepsilon\in(0,\varepsilon_J)$, *for all* $\boldsymbol{\mu}=(\mu_j)_j\in(0,\infty)^J$ *with* $\mu_J<\mu_{J-1}<\ldots<\mu_1$ *such that* $\gamma(\boldsymbol{\mu})<\varepsilon$, *for all* $(\iota_j)_j\in\{\pm1\}^J$ *and for all* $f\in\dot H^1$ *such that*

$$\bigg\| f-\sum_{j=1}^J \iota_j W_{(\mu_j)}\bigg\|_{\dot H^1} \leqslant \varepsilon,$$

there exists a unique $\boldsymbol{\lambda}\in(0,\infty)^J$ *such that*

$$\max_{1\leqslant j\leqslant J}\left|\frac{\lambda_j}{\mu_j}-1\right|\leqslant C_J\varepsilon$$

and

$$\int\nabla\Big(f-\sum_{j=1}^J\iota_j W_{(\lambda_j)}\Big)\cdot\nabla(\Lambda W)_{(\lambda_j)}=0\quad \textit{for all } j\in[\![1,J]\!].$$

Furthermore, the map $f\mapsto\boldsymbol{\lambda}$ *is of class* C^1*.*

Remark B.2. Let us mention that

$$\Big\|\sum_{j=1}^J\iota_j W_{(\lambda_j)}-\sum_{j=1}^J\iota_j W_{(\mu_j)}\Big\|_{\dot H^1}\leqslant C_J\varepsilon$$

(see the computation in the proof below) and that $\gamma(\boldsymbol{\lambda})\approx\gamma(\boldsymbol{\mu})$.

Sketch of proof. This is standard and follows from the implicit function theorem. However, we have to check that the uniformity of the constant with respect to $\boldsymbol{\mu}$ stated in the lemma follows from the proof.

We fix $\boldsymbol{\mu}$ and f such that

$$\gamma(\boldsymbol{\mu})<\varepsilon\quad\text{and}\quad\Big\|f-\sum_{j=1}^J\iota_j W_{(\mu_j)}\Big\|_{\dot H^1}<\varepsilon.$$

We consider

$$\Phi\colon(0,\infty)^J\longrightarrow\mathbb{R}^J,\quad \Phi=(\phi_\ell)_{1\leqslant\ell\leqslant J},$$

defined by

$$\Phi_\ell(\boldsymbol{\lambda})=\lambda_\ell-\frac{1}{\int|\nabla\Lambda W|^2}\mu_\ell\iota_\ell\int\nabla\Big(f-\sum_{j=1}^J\iota_j W_{(\lambda_j)}\Big)\cdot\nabla(\Lambda W_{(\lambda_\ell)}).$$

We will prove that Φ is a contraction of the compact set:

$$B_\eta=\left\{(\lambda_j)_{1\leqslant j\leqslant J}:\max_\ell\left|1-\frac{\lambda_\ell}{\mu_\ell}\right|\leqslant\eta\right\},$$

where

$$\eta=M_J\varepsilon$$

for a large positive M_J to be specified. Choosing ε small enough, we have that $\boldsymbol{\lambda}\in B(\eta)$ implies that

$$\frac12\leqslant\frac{\lambda_j}{\mu_j}\leqslant\frac32\quad\text{for all } j.$$

If $j\neq\ell$, we have

$$\frac{\partial\Phi_\ell}{\partial\lambda_j}(\boldsymbol{\lambda})=-\iota_j\iota_\ell\frac{\mu_\ell}{\lambda_j}\frac{1}{\int|\nabla\Lambda W|^2}\int\nabla(\Lambda W_{(\lambda_j)})\cdot\nabla(\Lambda W_{(\lambda_\ell)}).$$

Since by Claim A.1, we have

$$\left|\int\nabla(\Lambda W_{(\lambda_j)})\cdot\nabla(\Lambda W_{(\lambda_\ell)})\right|\lesssim\max\left\{\left(\frac{\lambda_\ell}{\lambda_j}\right)^{3/2},\left(\frac{\lambda_j}{\lambda_\ell}\right)^{3/2}\right\},$$

we deduce that

$$\left|\frac{\partial\Phi_\ell}{\partial\lambda_j}(\boldsymbol{\lambda})\right|\lesssim\frac{\mu_\ell}{\mu_j}\varepsilon^{3/2}. \tag{B.1}$$

Moreover,

$$\frac{\partial\Phi_\ell}{\partial\lambda_\ell}(\boldsymbol{\lambda})=1-\frac{\mu_\ell}{\lambda_\ell}+\frac{1}{\lambda_\ell\int|\nabla\Lambda W|^2}\iota_\ell\int\nabla\left(f-\sum_{j=1}^J\iota_jW_{(\lambda_j)}\right)\cdot\nabla(\Lambda\Lambda W_{(\lambda_\ell)}),$$

and thus

$$\left|\frac{\partial\Phi_\ell}{\partial\lambda_\ell}(\boldsymbol{\lambda})\right|\lesssim\left|1-\frac{\mu_\ell}{\lambda_\ell}\right|+\left\|f-\sum_{j=1}^J\iota_jW_{(\mu_j)}\right\|_{\dot H^1}+\left\|\sum_{j=1}^J\iota_jW_{(\mu_j)}-\iota_jW_{(\lambda_j)}\right\|_{\dot H^1}\lesssim\eta+\varepsilon, \tag{B.2}$$

where we have used

$$\begin{aligned}\int|\nabla(W_{(\lambda_j)}-W_{(\mu_j)})|^2&=\int\left|\left(\frac{\mu_j}{\lambda_j}\right)^{N/2}\nabla W\left(\frac{\mu_j}{\lambda_j}x\right)-\nabla W(x)\right|^2dx\\&\lesssim\eta^2+\int\left|\nabla W\left(\frac{\mu_j}{\lambda_j}x\right)-\nabla W(x)\right|^2dx\\&\lesssim\eta^2+\int_0^{+\infty}\left(r-\frac{\mu_j}{\lambda_j}r\right)^2\frac{r^{N-1}}{(1+r^N)^2}dr\lesssim\eta^2,\end{aligned}$$

since

$$|\nabla W(r)-\nabla W(\rho)|\lesssim\frac{|r-\rho|}{1+r^N},\quad r\approx\rho.$$

Furthermore,

$$\left|\frac{1}{\mu_\ell}\Phi_\ell(\boldsymbol{\mu})-1\right|\lesssim\varepsilon. \tag{B.3}$$

Combining (B.1)–(B.3), we see that, if $\boldsymbol{\lambda}\in B_\eta$,

$$\left|\frac{1}{\mu_\ell}\Phi_\ell(\boldsymbol{\lambda})-1\right|\lesssim\varepsilon+(\eta+\varepsilon)\eta.$$

This proves that, if $\eta=M_J\varepsilon$ for some large constant M_J, and $\varepsilon\leqslant\varepsilon_J\ll M_J^{-1}$, Φ maps B_η into B_η. By (B.1) and (B.2), Φ is a contraction of B_η. By the Banach fixed point theorem, there exists a unique $\boldsymbol{\lambda}\in B_\eta$ such that $\Phi(\boldsymbol{\lambda})=\boldsymbol{\lambda}$, which exactly means that it satisfies the desired orthogonality conditions. The fact that $f\mapsto\boldsymbol{\lambda}$ is C^1 is classical and we omit the proof. □

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