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The Fuglede conjecture for convex domains is true in all dimensions

by

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1. Introduction

1.1. Let $\Omega\subset\mathbb{R}^d$ be a bounded, measurable set of positive measure. We say that Ω is *spectral* if there exists a countable set $\Lambda\subset\mathbb{R}^d$ such that the system of exponential functions

$$E(\Lambda)=\{e_\lambda\}_{\lambda\in\Lambda},\quad e_\lambda(x)=e^{2\pi i\langle\lambda,x\rangle}, \tag{1.1}$$

is orthogonal and complete in $L^2(\Omega)$, that is, the system constitutes an orthogonal basis in the space. Such a set Λ is called a *spectrum* for Ω. The classical example of a spectral set is the unit cube $\Omega=\left[-\frac{1}{2},\frac{1}{2}\right]^d$, for which the set $\Lambda=\mathbb{Z}^d$ serves as a spectrum.

Which other sets Ω can be spectral? The research on this problem has been influenced for many years by a famous paper [F] due to Fuglede (1974), who suggested that there should be a concrete, geometric way to characterize the spectral sets. We say that Ω *tiles the space by translations* if there exists a countable set $\Lambda\subset\mathbb{R}^d$ such that the collection of sets $\{\Omega+\lambda\}$, $\lambda\in\Lambda$, consisting of translated copies of Ω, constitutes a partition of $\mathbb{R}^d$ up to measure zero. In his paper, Fuglede stated the following conjecture: "*A set $\Omega\subset\mathbb{R}^d$ is spectral if and only if it can tile the space by translations*".

For example, Fuglede proved that a triangle and a disk in the plane are not spectral sets. He also proved that if Ω can tile with respect to a *lattice* translation set Λ then the dual lattice Λ^* is a spectrum for Ω, and conversely. Fuglede's conjecture inspired extensive research over the years, and a number of interesting results establishing connections between spectrality and tiling had since been obtained.

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The conjecture remained open for thirty years until a counterexample was discovered by Tao [T], who constructed in dimensions 5 and higher an example of a spectral set which cannot tile by translations. Since then, counterexamples to both directions of the conjecture were found in dimensions $d \geqslant 3$ (see [KM, §4]). These examples are composed of finitely many unit cubes in special arithmetic arrangements. The conjecture is still open in dimensions $d=1$ and 2 in both directions.

On the other hand, it was believed that Fuglede's conjecture should be true in all dimensions d if the set $\Omega \subset \mathbb{R}^d$ is assumed to be a *convex body* (that is, a compact convex set with non-empty interior(1)). Indeed, all the known counterexamples to the conjecture are highly non-convex sets, being the union of a finite number of unit cubes centered at points of the integer lattice $\mathbb{Z}^d$. Moreover, it has long been known [V], [McM] that a convex body which tiles by translations must be a polytope, and that it admits a face-to-face tiling by a lattice translation set Λ and therefore has a spectrum given by the dual lattice Λ^*. So, this implies that for a convex body $\Omega \subset \mathbb{R}^d$ the "tiling implies spectral" part of the conjecture is in fact true in any dimension d.

To the contrary, the "spectral implies tiling" direction of the conjecture for convex bodies was proved only in $\mathbb{R}^2$ [IKT2], and also in $\mathbb{R}^3$ under the a-priori assumption that Ω is a convex polytope [GL2]. In higher dimensions, this direction of the conjecture remained completely open (even in the case when Ω is a polytope) and could not be treated using the previously developed techniques.

It is our goal in the present paper to establish that the result in fact holds in all dimensions and for general convex bodies. We will prove the following theorem.

THEOREM 1.1. *Let Ω be a convex body in $\mathbb{R}^d$. If Ω is a spectral set, then Ω must be a convex polytope, and it tiles the space face-to-face by translations along a lattice.*

This fully settles the Fuglede conjecture for convex bodies affirmatively: we obtain that a convex body in $\mathbb{R}^d$ is a spectral set if and only if it can tile by translations.

1.2. There is a complete characterization due to Venkov [V], that was rediscovered by McMullen [McM], of the convex bodies which tile by translations.

A convex body $\Omega \subset \mathbb{R}^d$ can tile the space by translations if and only if it satisfies the following four conditions:

(i) *Ω is a convex polytope*;

(ii) *Ω is centrally symmetric*;

(1) The compactness and non-empty interior assumptions can be made with no loss of generality, as any convex set of positive and finite measure coincides with a convex body up to a set of measure zero.

(iii) *all the facets of* Ω *are centrally symmetric*;

(iv) *each belt of* Ω *consists of either four or six facets.*

Moreover, a convex body Ω *satisfying the four conditions* (i)–(iv) *admits a face-to-face tiling by translations along a certain lattice.*

We recall that a *facet* of a convex polytope $\Omega\subset\mathbb{R}^d$ is a $(d-1)$-dimensional face of Ω. If Ω has centrally symmetric facets, then a *belt* of Ω is a system of facets obtained in the following way: Let G be a *subfacet* of Ω, that is, a $(d-2)$-dimensional face. Then, G lies in exactly two adjacent facets of Ω, say F and F'. Since F' is centrally symmetric, there is another subfacet G' obtained by reflecting G through the center of F' (so in particular, G' is a translate of $-G$). In turn, G' is the intersection of F' with another facet F''. Continuing in this way, we obtain a system of facets $F, F', F'', ..., F^{(m)}=F$, called *the belt of* Ω *generated by the subfacet* G, such that the intersection $F^{(i-1)}\cap F^{(i)}$ of any pair of consecutive facets in the system is a translate of either G or $-G$.

Fuglede's conjecture for convex bodies can thus be equivalently stated by saying that for a convex body $\Omega\subset\mathbb{R}^d$ to be spectral, it is necessary and sufficient that the four conditions (i)–(iv) above hold.

In relation with the first condition (i), a result proved in [IKP] states that if Ω is a ball in $\mathbb{R}^d$ $(d\geqslant 2)$, then Ω is not a spectral set. In [IKT1] this result was extended to the class of convex bodies $\Omega\subset\mathbb{R}^d$ that have a smooth boundary.

As for (ii), Kolountzakis [K1] proved that if a convex body $\Omega\subset\mathbb{R}^d$ is spectral, then it must be centrally symmetric. If Ω is assumed a priori to be a polytope, then another approach to this result was given in [KP] (see also [GL2, §3]).

Recently, also the necessity of condition (iii) for spectrality was established. It was proved in [GL2, §4] that if a convex, centrally symmetric polytope $\Omega\subset\mathbb{R}^d$ is a spectral set, then all the facets of Ω must also be centrally symmetric. The proof is based on a development of the argument in [KP].

The last condition (iv) was addressed so far only in dimensions $d=2$ and $d=3$. Iosevich, Katz and Tao proved in [IKT2] that if a convex polygon $\Omega\subset\mathbb{R}^2$ is a spectral set, then it must be either a parallelogram or a (centrally symmetric) hexagon. In three dimensions, it was recently proved [GL1], [GL2] that if a convex polytope $\Omega\subset\mathbb{R}^3$ is spectral, then it can tile the space by translations (as a consequence, the condition (iv) must hold, although the proof does not establish it directly).

In this paper, we will show that the conditions (i) and (iv) are in fact necessary for the spectrality of a general convex body Ω in every dimension, thus obtaining a proof of the full Fuglede conjecture for convex bodies. Our main results are as follows.

THEOREM 1.2. *Let Ω be a convex body in $\mathbb{R}^d$. If Ω is a spectral set, then Ω must be a convex polytope.*

THEOREM 1.3. *Let $\Omega\subset\mathbb{R}^d$ be a convex polytope, which is centrally symmetric and has centrally symmetric facets. If Ω is a spectral set, then each belt of Ω must consist of either four or six facets.*

Theorem 1.1 above follows as a consequence of these two theorems, combined with the results in [K1] (or [KP]), [GL2, §4], and the Venkov–McMullen theorem.

1.3. In the above mentioned papers [IKP], [IKT1], [KP], [IKT2], [GL2] the approach relies on the asymptotic behavior of the Fourier transform of the indicator function of Ω, and involves an analysis of its set of zeros. In the present paper we introduce a new approach to the problem, based on establishing a link between the notion of spectrality and a geometric notion which we refer to as "weak tiling".

Definition 1.4. Let $\Omega\subset\mathbb{R}^d$ be a bounded, measurable set. We say that another measurable, possibly unbounded, set $\Sigma\subset\mathbb{R}^d$ admits a *weak tiling* by translates of Ω, if there exists a positive, locally finite (Borel) measure μ on $\mathbb{R}^d$ such that $\mathbb{1}_\Omega*\mu=\mathbb{1}_\Sigma$ a.e.

If the measure μ is the sum of unit masses at the points of a locally finite set Λ, that is, $\mu=\sum_{\lambda\in\Lambda}\delta_\lambda$, then the condition $\mathbb{1}_\Omega*\mu=\mathbb{1}_\Sigma$ a.e. means that the collection $\{\Omega+\lambda\}$, $\lambda\in\Lambda$, of translated copies of Ω, constitutes a partition of Σ up to measure zero. In this case, we say that the weak tiling is a *proper tiling*.

For example, one can check that any bounded set Ω of positive Lebesgue measure tiles the whole space $\mathbb{R}^d$ weakly by translates with respect to the measure $d\mu=m(A)^{-1}\,dx$ (where dx denotes the Lebesgue measure on $\mathbb{R}^d$). This is in sharp contrast to the obvious fact that not every set Ω can tile the space properly by translations.

We will prove the following theorem, which gives a necessary condition for spectrality in terms of weak tiling.

THEOREM 1.5. *Let Ω be a bounded, measurable set in $\mathbb{R}^d$. If Ω is spectral, then its complement $\Omega^{\complement}=\mathbb{R}^d\setminus\Omega$ admits a weak tiling by translates of Ω. That is, there exists a positive, locally finite measure μ such that $\mathbb{1}_\Omega*\mu=\mathbb{1}_{\Omega^{\complement}}$ a.e.*

Our proof of this result involves a construction due to Hof [H], that is often used in mathematical crystallography in order to describe the diffraction pattern of an atomic structure (see also [BaGr, Chapter 9]).

Notice that if the complement $\Omega^{\complement}$ has a *proper* tiling by translates of Ω, then it just means that Ω can tile the space by translations. Theorem 1.5 thus establishes a weak form of the "spectral implies tiling" part of Fuglede's conjecture, which is valid for all bounded, measurable sets $\Omega\subset\mathbb{R}^d$. We observe that the weak tiling conclusion cannot be strengthened to proper tiling without imposing extra assumptions on the set Ω, since there exist examples of spectral sets which cannot tile by translations.

We will prove that if Ω is a convex body in $\mathbb{R}^d$, and if it can tile its complement $\Omega^{\complement}$ weakly by translations, then Ω must in fact be a convex polytope (Theorem 4.1). We will also prove that if, in addition, Ω is centrally symmetric and has centrally symmetric facets, then each belt of Ω must have either four or six facets (Theorem 6.1). So, in the latter case, it follows that Ω can in fact tile its complement $\Omega^{\complement}$ not only weakly, but even properly, by translations.

The potential applications of Theorem 1.5 are not limited to the class of convex bodies in $\mathbb{R}^d$. As an example, we will use this theorem to give a simple geometric condition which is necessary for the spectrality of a bounded, measurable set $\Omega\subset\mathbb{R}^d$ (Theorem 3.5). Based on this condition we will prove that the boundary of a bounded, open spectral set must have Lebesgue measure zero (Theorem 3.6).

1.4. The rest of the paper is organized as follows.

In §2 we present some preliminary background. We fix notation that will be used in the paper and discuss basic results about measures and tempered distributions, spectral sets and weak tiling.

In §3 we prove that if a bounded, measurable set $\Omega\subset\mathbb{R}^d$ is spectral, then its complement $\Omega^{\complement}$ admits a weak tiling by translates of Ω (Theorem 1.5). As an application, we show that a connected spectral domain cannot have any "holes", and that the boundary of an open spectral domain must have Lebesgue measure zero.

In §4 we prove that if a convex body $K\subset\mathbb{R}^d$ is a spectral set, then K must be a convex polytope (Theorem 1.2).

In the last two §5 and §6 we establish that each belt of a spectral convex polytope $K\subset\mathbb{R}^d$ must consist of either four or six facets (Theorem 1.3). The proof is based on an analysis of the measure μ that provides a weak tiling of $K^{\complement}$ by translates of K.

2. Preliminaries

2.1. Notation

If $A\subset\mathbb{R}^d$ then $\operatorname{int}(A)$ will denote the interior of A, and $\operatorname{bd}(A)$ the boundary of A. We use $A^{\complement}$ to denote the complement $\mathbb{R}^d\setminus A$ of the set A. If $A,B\subset\mathbb{R}^d$ then $A\triangle B$ is the symmetric difference of A and B. We denote by $|A|$ the number of elements in A.

If $A\subset\mathbb{R}^d$, then for each $\tau\in\mathbb{R}^d$ we let $A+\tau=\{a+\tau:a\in A\}$ denote the image of A under translation by the vector τ. If $s\in\mathbb{R}$, then $sA=\{sa:a\in A\}$ will denote the image of A under dilation with ratio s. If A and B are two subsets of $\mathbb{R}^d$, then $A+B$ and $A-B$ denote respectively their set of sums and set of differences.

We use $\langle\cdot,\cdot\rangle$ and $|\cdot|$ to denote the Euclidean scalar product and norm in $\mathbb{R}^d$.

By a *lattice* in $\mathbb{R}^d$ we mean a set L which can be obtained as the image of $\mathbb{Z}^d$ under an invertible linear transformation. The *dual lattice* L^* is the set of all vectors $\lambda^*\in\mathbb{R}^d$ such that $\langle\lambda,\lambda^*\rangle\in\mathbb{Z}$ for every $\lambda\in L$.

We denote by $m(A)$ the Lebesgue measure of a set $A\subset\mathbb{R}^d$. We also use $m_k(A)$ to denote the k-dimensional volume measure of A (so, in particular, $m_d(A)=m(A)$).

If $A\subset\mathbb{R}^d$ is a bounded, measurable set, then we define

$$\Delta(A):=\{x\in\mathbb{R}^d:m(A\cap(A+x))>0\}.$$

Then, $\Delta(A)$ is a bounded open set, and we have $\Delta(A)=-\Delta(A)$ (which means that $\Delta(A)$ is symmetric with respect to the origin). One can think of the set $\Delta(A)$ as the measure-theoretic analog of the set of differences $A-A$. In particular, one can check that if A is an open set then $\Delta(A)=A-A$. In general we have $\Delta(A)\subset A-A$, but this inclusion can be strict.

The Fourier transform of a function $f\in L^1(\mathbb{R}^d)$ is defined by

$$\hat{f}(t)=\int_{\mathbb{R}^d} f(x)e^{-2\pi i\langle t,x\rangle}\,dx.$$

2.2. Measures and distributions

By a "measure" we will refer to a Borel (either positive, or complex) measure on $\mathbb{R}^d$. We use $\operatorname{supp}(\mu)$ to denote the closed support of a measure μ. We denote by δ_λ the Dirac measure consisting of a unit mass at the point λ. If $\Lambda\subset\mathbb{R}^d$ is a countable set, then we define $\delta_\Lambda:=\sum_{\lambda\in\Lambda}\delta_\lambda$.

If α is a tempered distribution on $\mathbb{R}^d$, and if φ is a Schwartz function on $\mathbb{R}^d$, then we use $\langle\alpha,\varphi\rangle$ to denote the action of α on φ. A tempered distribution α is *positive* if we have $\langle\alpha,\varphi\rangle\geqslant0$ for any Schwartz function $\varphi\geqslant0$. If a tempered distribution α is positive, then α is a positive measure. The Fourier transform $\hat{\alpha}$ of a tempered distribution α is defined by $\langle\hat{\alpha},\varphi\rangle=\langle\alpha,\widehat{\varphi}\rangle$. A tempered distribution α is said to be *positive-definite* if $\hat{\alpha}$ is a positive distribution. See [BaGr, §8.4], [R].

If μ is a measure on $\mathbb{R}^d$, then μ is said to be *locally finite* if we have $|\mu|(B)<\infty$ for every open ball B.

A measure μ on $\mathbb{R}^d$ is said to be *translation-bounded* if for every (or equivalently, for some) open ball B we have

$$\sup_{x\in\mathbb{R}^d} |\mu|(B+x) < \infty.$$

If a measure μ on $\mathbb{R}^d$ is translation-bounded, then it is a tempered distribution. If μ is a translation-bounded measure on $\mathbb{R}^d$, and if ν is a finite measure on $\mathbb{R}^d$, then the convolution $\mu*\nu$ is a translation-bounded measure.

LEMMA 2.1. *Let ν be a finite measure on $\mathbb{R}^d$, and suppose that μ is a translation-bounded measure on $\mathbb{R}^d$ whose Fourier transform $\hat{\mu}$ is a locally finite measure. Then, the Fourier transform of the convolution $\mu*\nu$ is the measure $\hat{\mu}\cdot\hat{\nu}$.*

See e.g. [BaGr, §8.6], [KL, §2.5].

A sequence of measures $\{\mu_n\}$ is said to be *uniformly translation-bounded* if, for every (or equivalently, for some) open ball B, one can find a constant C not depending on n such that $\sup_x |\mu_n|(B+x) \leqslant C$ for every n.

If $\{\mu_n\}$ is a uniformly translation-bounded sequence of measures, then we say that μ_n *converges vaguely* to a measure μ if, for every continuous, compactly supported function φ, we have $\int \varphi\, d\mu_n \to \int \varphi\, d\mu$. In this case, the vague limit μ must also be a translation-bounded measure. For a uniformly translation-bounded sequence of measures $\{\mu_n\}$ to converge vaguely, it is necessary and sufficient that $\{\mu_n\}$ converge in the space of tempered distributions. From any uniformly translation-bounded sequence of measures $\{\mu_n\}$ one can extract a vaguely convergent subsequence $\{\mu_{n_j}\}$.

LEMMA 2.2. *Let $f\in L^1(\mathbb{R}^d)$, and let $\{\mu_n\}$ be a uniformly translation-bounded sequence of measures on $\mathbb{R}^d$, such that $f*\mu_n=1$ a.e. for every n. If μ_n converges vaguely to a measure μ, then also $f*\mu=1$ a.e.*

Proof. Let φ be a continuous, compactly supported function on $\mathbb{R}^d$. Then, the sequence of functions $\mu_n*\varphi$ is uniformly bounded and converges pointwise to $\mu*\varphi$. Hence, by the dominated convergence theorem, we have $f*(\mu_n*\varphi)\to f*(\mu*\varphi)$ pointwise. In turn, this implies that $(f*\mu_n)*\varphi\to(f*\mu)*\varphi$ pointwise, since the convolution is associative (by Fubini's theorem). But $f*\mu_n=1$ a.e. for every n, so we conclude that $(f*\mu)*\varphi=\int\varphi$. Since this is true for an arbitrary φ, the assertion follows. □

2.3. Spectra

If Ω is a bounded, measurable set in $\mathbb{R}^d$ of positive measure, then by a *spectrum* for Ω we mean a countable set $\Lambda\subset\mathbb{R}^d$ such that the system of exponential functions $E(\Lambda)$ defined by (1.1) is orthogonal and complete in the space $L^2(\Omega)$.

For any two points λ and λ' in $\mathbb{R}^d$ we have $\langle e_\lambda, e_{\lambda'}\rangle_{L^2(\Omega)} = \hat{\mathbb{1}}_\Omega(\lambda'-\lambda)$, where $\hat{\mathbb{1}}_\Omega$ is the Fourier transform of the indicator function $\mathbb{1}_\Omega$ of the set Ω. The orthogonality of the system $E(\Lambda)$ in $L^2(\Omega)$ is therefore equivalent to the condition

$$\Lambda - \Lambda \subset \mathcal{Z}(\hat{\mathbb{1}}_\Omega) \cup \{0\}, \tag{2.1}$$

where $\mathcal{Z}(\hat{\mathbb{1}}_\Omega) := \{t \in \mathbb{R}^d : \hat{\mathbb{1}}_\Omega(t) = 0\}$ is the set of zeros of the function $\hat{\mathbb{1}}_\Omega$.

A set $\Lambda \subset \mathbb{R}^d$ is said to be *uniformly discrete* if there is $\delta > 0$ such that $|\lambda' - \lambda| \geqslant \delta$ for any two distinct points λ and λ' in Λ. The condition (2.1) implies that every spectrum Λ of Ω is a uniformly discrete set.

The set Λ is *relatively dense* if there is $R > 0$ such that every ball of radius R contains at least one point from Λ. It is well known that if Λ is a spectrum for Ω, then Λ must be a relatively dense set (see e.g. [GL2, §2C]).

The following lemma gives a frequently used characterization of the spectra of Ω.

LEMMA 2.3. (See [K2, §3.1]) *Let Ω be a bounded, measurable set in $\mathbb{R}^d$, and define the function $f := m(\Omega)^{-2} |\hat{\mathbb{1}}_\Omega|^2$. Then, a set $\Lambda \subset \mathbb{R}^d$ is a spectrum for Ω if and only if $f * \delta_\Lambda = 1$ a.e.*

2.4. Weak tiling

If Ω is a bounded, measurable set in $\mathbb{R}^d$, and if Σ is another, possibly unbounded, measurable set in $\mathbb{R}^d$, then we say that Σ admits a *weak tiling* by translates of Ω, if there exists a positive, locally finite measure μ on $\mathbb{R}^d$ such that $\mathbb{1}_\Omega * \mu = \mathbb{1}_\Sigma$ a.e.

The following lemma shows that the measure μ in any weak tiling is not only locally finite, but in fact must be translation-bounded.

LEMMA 2.4. *Let $\Omega \subset \mathbb{R}^d$ be a bounded, measurable set of positive Lebesgue measure, and suppose that μ is a positive, locally finite measure such that $\mathbb{1}_\Omega * \mu \leqslant 1$ a.e. Then, μ is a translation-bounded measure.*

Proof. Fix $r > 0$, and let B_r be the open ball of radius r centered at the origin. It will be enough to show that there is a constant $M > 0$ such that $(\mathbb{1}_{B_r} * \mu)(x) \leqslant M$ for every $x \in \mathbb{R}^d$. Since Ω is a bounded set, we can choose a sufficiently large number $s > 0$ such that we have $B_s + y \supset B_r$ for every $y \in \Omega$. It follows that $\mathbb{1}_{B_s} * \mathbb{1}_\Omega \geqslant m(\Omega)\mathbb{1}_{B_r}$. In turn, this implies that

$$m(\Omega)\mathbb{1}_{B_r} * \mu \leqslant (\mathbb{1}_{B_s} * \mathbb{1}_\Omega) * \mu = \mathbb{1}_{B_s} * (\mathbb{1}_\Omega * \mu) \leqslant \mathbb{1}_{B_s} * 1 = m(B_s).$$

We thus see that the constant $M := m(B_s)/m(\Omega)$ satisfies $\mathbb{1}_{B_r} * \mu \leqslant M$ as needed. $\square$

The next lemma implies that if Ω tiles Σ weakly by translations with respect to a measure μ, then we must have $m((\Omega+t)\cap\Sigma^{\complement})=0$ for every $t\in\operatorname{supp}(\mu)$.

LEMMA 2.5. *Let Ω be a bounded, measurable set in $\mathbb{R}^d$, and let μ be a positive, locally finite measure on $\mathbb{R}^d$. Suppose that we have $\mathbb{1}_\Omega*\mu=0$ a.e. on the complement $\Sigma^{\complement}$ of another measurable set $\Sigma\subset\mathbb{R}^d$. Then, $m((\Omega+t)\cap\Sigma^{\complement})=0$ for every $t\in\operatorname{supp}(\mu)$.*

Proof. Let $\varphi(t):=m((\Omega+t)\cap\Sigma^{\complement})$, $t\in\mathbb{R}^d$. Suppose to the contrary that $\varphi(t)>0$ for some $t\in\operatorname{supp}(\mu)$. Since φ is a continuous function, there is an open neighborhood U of t such that $\varphi(t')>0$ for all $t'\in U$. Since $t\in\operatorname{supp}(\mu)$, we must have $\mu(U)>0$, and it follows that $\int\varphi\,d\mu>0$. But on the other hand, using Fubini's theorem we have

$$\int\varphi\,d\mu=\int_{\Sigma^{\complement}}(\mathbb{1}_\Omega*\mu)\,dm=0,$$

since we have assumed that $\mathbb{1}_\Omega*\mu=0$ a.e. on $\Sigma^{\complement}$. We thus obtain a contradiction. □

COROLLARY 2.6. *Let Ω be a bounded, measurable set in $\mathbb{R}^d$. Assume that the complement $\Omega^{\complement}$ of Ω admits a weak tiling by translates of Ω, that is, there is a positive, locally finite measure μ such that $\mathbb{1}_\Omega*\mu=\mathbb{1}_{\Omega^{\complement}}$ a.e. Then,* $\operatorname{supp}(\mu)\subset\Delta(\Omega)^{\complement}$.

Indeed, this follows from Lemma 2.5 in the special case when $\Sigma=\Omega^{\complement}$.

3. Spectrality and weak tiling

In this section, our main goal is to prove Theorem 1.5, which states that if a bounded, measurable set $\Omega\subset\mathbb{R}^d$ is spectral, then its complement $\Omega^{\complement}$ admits a weak tiling by translates of Ω. The proof of this result involves a construction due to Hof [H], that is often used in mathematical crystallography in order to describe the diffraction pattern of an atomic structure.

We also present some simple applications of Theorem 1.5. As an example, we will use it to prove that if a bounded, open set $\Omega\subset\mathbb{R}^d$ is spectral, then its boundary must have Lebesgue measure zero. Some other examples will also be discussed.

3.1. In [H], Hof suggested a mathematical model for the diffraction experiment used in crystallography to analyze the atomic structure of a solid. In this model, the configuration of the atoms is represented by a uniformly discrete and relatively dense set $\Lambda\subset\mathbb{R}^d$. Hof used volume averaged convolution of the two measures δ_Λ and $\delta_{-\Lambda}$ in order to define the *autocorrelation measure* γ of the set Λ. He then argued that the diffraction by the atomic structure is described by the Fourier transform $\widehat{\gamma}$ of the autocorrelation, which is a positive measure called the *diffraction measure* of the set Λ (for more details, see [BaGr, Chapter 9]).

Here, we will apply this technique to a set Λ that constitutes a spectrum for a bounded, measurable set $\Omega\subset\mathbb{R}^d$. We will obtain the following result.

THEOREM 3.1. *Let Ω be a bounded, measurable set in $\mathbb{R}^d$. If Ω is a spectral set, then there exists a measure γ on $\mathbb{R}^d$ with the following properties*:

(a) *γ is a positive, translation-bounded measure*;

(b) *the support of γ is contained in the closed set $\mathcal{Z}(\hat{\mathbb{1}}_\Omega)\cup\{0\}$*;

(c) *$\gamma=\delta_0$ in some open neighborhood of the origin*;

(d) *$\widehat{\gamma}$ is also a positive, translation-bounded measure*;

(e) *$\widehat{\gamma}=m(\Omega)\delta_0$ in the open set $\Delta(\Omega)$.*

A similar result can be proved in the more general context of locally compact abelian groups, but in this paper we work in the euclidean setting only.

We remark that a minor difference in our proof compared to the construction in [H], is that we define the autocorrelation measure γ not as the volume averaged convolution of the two measures δ_Λ and $\delta_{-\Lambda}$, but instead as a convolution averaged with respect to the number of points in Λ. It turns out that using this normalization allows in our context to establish more directly the desired properties of the measure γ.

Proof. Suppose that Ω is a spectral set, and let Λ be a spectrum for Ω. Then, Λ is a uniformly discrete and relatively dense set in $\mathbb{R}^d$. For each $r>0$ we denote $\Lambda_r:=\Lambda\cap B_r$, where B_r is the open ball of radius r centered at the origin. Then, there is r_0 such that Λ_r is non-empty for every $r\geqslant r_0$. Consider a family of measures $\{\gamma_r\}$, $r\geqslant r_0$, defined by

$$\gamma_r:=|\Lambda_r|^{-1}\delta_{\Lambda_r}*\delta_{-\Lambda_r}.$$

Each γ_r is a positive, finite measure on $\mathbb{R}^d$, whose support is the finite set $\Lambda_r-\Lambda_r$. We have $\widehat{\gamma}_r(t)=|\Lambda_r|^{-1}\,|\hat{\delta}_{\Lambda_r}(t)|^2\geqslant 0$, hence γ_r is a positive-definite measure. Since Λ is uniformly discrete, the measures γ_r are uniformly translation-bounded, namely, for any open ball B we have $\sup_x\gamma_r(B+x)\leqslant C(\Lambda,B)$, where $C(\Lambda,B)$ is a constant which does not depend on r. It follows that we may choose a sequence $r_n\to\infty$ such that if we define $\mu_n:=\gamma_{r_n}$, then the sequence $\{\mu_n\}$ converges vaguely to some measure γ on $\mathbb{R}^d$.

The measure γ is translation-bounded, positive and positive-definite. It follows from the positive-definiteness of γ that its Fourier transform $\widehat{\gamma}$ is a positive measure. Given any open ball B, let φ_B be a Schwartz function such that $\varphi_B\geqslant\mathbb{1}_B$. Then, we have

$$\widehat{\gamma}(B+x)\leqslant\int\varphi_B(y-x)\,d\widehat{\gamma}(y)=\int e^{-2\pi i\langle x,t\rangle}\widehat{\varphi}_B(t)\,d\gamma(t),$$

and hence

$$\sup_{x\in\mathbb{R}^d} \widehat{\gamma}(B+x) \leqslant \int |\widehat{\varphi}_B(t)|\, d\gamma(t) < \infty$$

(the last integral is finite since $\widehat{\varphi}_B$ has fast decay, while γ is translation-bounded). We conclude that $\widehat{\gamma}$ is a translation-bounded measure.

Define now another sequence of measures $\{\nu_n\}$ by $\nu_n := |\Lambda_{r_n}|^{-1}\delta_\Lambda * \delta_{-\Lambda_{r_n}}$. The measures ν_n are uniformly translation-bounded, again due to the uniform discreteness of Λ. We claim that the sequence $\{\nu_n\}$ converges vaguely to the same limit as the sequence $\{\mu_n\}$, namely, to the measure γ. To see this, it will be enough to check that the sequence of differences $\{\nu_n - \mu_n\}$ converges vaguely to zero. Indeed, we have

$$\nu_n - \mu_n = |\Lambda_{r_n}|^{-1}\delta_{\Lambda\setminus\Lambda_{r_n}} * \delta_{-\Lambda_{r_n}}.$$

For any fixed $s>0$, the total mass of the measure $\nu_n - \mu_n$ in the open ball B_s is equal to $|\Lambda_{r_n}|^{-1}$ times the number of pairs $(\lambda, \lambda')\in\Lambda\times\Lambda$ such that $|\lambda|<r_n$, $|\lambda'|\geqslant r_n$ and $|\lambda'-\lambda|<s$. As $n\to\infty$, the number of such pairs is not greater than a constant multiple of r_n^{d-1}, since Λ is a uniformly discrete set, while the number of elements in the set Λ_{r_n} is not less than a constant multiple of r_n^d, since Λ is also relatively dense. Hence, $|\nu_n-\mu_n|(B_s)$ tends to zero as $n\to\infty$. Since this is true for any $s>0$, it follows that the sequence $\{\nu_n-\mu_n\}$ converges to zero, and so the sequence ν_n converges to γ.

Let $f := m(\Omega)^{-2}|\hat{\mathbb{1}}_\Omega|^2$. Since Λ is a spectrum for Ω, by Lemma 2.3 we have $f*\delta_\Lambda = 1$ a.e. As ν_n is an average of translates of the measure δ_Λ, we also have $f*\nu_n=1$ a.e. for every n. Since the measures ν_n are uniformly translation-bounded and converge vaguely to γ, it follows from Lemma 2.2 that $f*\gamma=1$ a.e. as well. In turn, using Lemma 2.1, this implies that $\hat{f}\cdot\widehat{\gamma}=\delta_0$. Since $\hat{f}(0)=\int f = m(\Omega)^{-1}$, we deduce that $\widehat{\gamma}=m(\Omega)\delta_0$ in the open set $\{t : \hat{f}(t)\neq 0\}$. But the Fourier transform of f is the function $\hat{f}=m(\Omega)^{-2}\mathbb{1}_\Omega * \mathbb{1}_{-\Omega}$, that is, we have $\hat{f}(t)=m(\Omega)^{-2}m(\Omega\cap(\Omega+t))$ for every $t\in\mathbb{R}^d$. Hence, $\hat{f}(t)\neq 0$ if and only if $t\in\Delta(\Omega)$, and we conclude that $\widehat{\gamma}=m(\Omega)\delta_0$ in the open set $\Delta(\Omega)$.

Since Λ is a uniformly discrete set, there exists $\varepsilon>0$ such that $(\Lambda-\lambda)\cap B_\varepsilon=\{0\}$ for every $\lambda\in\Lambda$. In other words, for every $\lambda\in\Lambda$ we have $\delta_\Lambda*\delta_{-\lambda}=\delta_0$ in the ball B_ε. The measure ν_n is an average of measures of the form $\delta_\Lambda*\delta_{-\lambda}$ $(\lambda\in\Lambda)$, so we also have $\nu_n=\delta_0$ in the ball B_ε. It follows that the same is true for the vague limit γ of the sequence ν_n, that is, we have $\gamma=\delta_0$ in the open ball B_ε.

The supports of all the measures μ_n are contained in the set of differences $\Lambda-\Lambda$. Since Λ is a spectrum for Ω, the set $\Lambda-\Lambda$ is contained in the closed set $\mathcal{Z}(\hat{\mathbb{1}}_\Omega)\cup\{0\}$. The measure γ is the vague limit of the sequence μ_n, so it follows that the closed support of γ is also contained in the set $\mathcal{Z}(\hat{\mathbb{1}}_\Omega)\cup\{0\}$. The theorem is thus proved. $\square$

3.2. We can now use Theorem 3.1 to deduce Theorem 1.5.

Proof of Theorem 1.5. Let Ω be a bounded, measurable set in $\mathbb{R}^d$, and assume that Ω is spectral. We must show that $\Omega^{\complement}$ admits a weak tiling by translates of Ω.

Indeed, let γ be the measure given by Theorem 3.1. We have $\hat{\mathbb{1}}_{-\Omega}(t)=\overline{\hat{\mathbb{1}}_{\Omega}(t)}$, and hence $\operatorname{supp}(\gamma)$ is contained in the set $\mathcal{Z}(\hat{\mathbb{1}}_{-\Omega})\cup\{0\}$. We also have $\gamma=\delta_0$ in some open neighborhood of the origin. Since $\hat{\mathbb{1}}_{-\Omega}(0)=m(\Omega)$, it follows that $\gamma\cdot\hat{\mathbb{1}}_{-\Omega}=m(\Omega)\delta_0$.

The measure γ is translation-bounded, and its Fourier transform $\widehat{\gamma}$ is a translation-bounded measure as well. Hence, by Lemma 2.1, the Fourier transform of the measure $\gamma\cdot\hat{\mathbb{1}}_{-\Omega}$ is the function $\widehat{\gamma}*\mathbb{1}_{\Omega}$. We thus conclude that $\widehat{\gamma}*\mathbb{1}_{\Omega}=m(\Omega)$ a.e.

The measure $\widehat{\gamma}$ is positive, and satisfies $\widehat{\gamma}=m(\Omega)\delta_0$ in the open set $\Delta(\Omega)$. We can therefore write $\widehat{\gamma}=m(\Omega)(\delta_0+\mu)$, where μ is a positive, translation-bounded (and hence locally finite) measure. The condition $\widehat{\gamma}*\mathbb{1}_{\Omega}=m(\Omega)$ a.e. is then equivalent to $\mathbb{1}_{\Omega}*\mu=\mathbb{1}_{\Omega^{\complement}}$ a.e., and we obtain that $\Omega^{\complement}$ has a weak tiling by translates of Ω. $\square$

3.3. To illustrate the construction of the autocorrelation measure γ in the proof of Theorem 3.1, and the corresponding weak tiling of $\Omega^{\complement}$ obtained in Theorem 1.5, we now look at a few examples.

Example 3.2. Assume that $\Omega\subset\mathbb{R}^d$ tiles the space with respect to a *lattice* translation set L. Let Λ be a spectrum of Ω given by the dual lattice, that is, $\Lambda=L^*$. Then, $\Lambda-\lambda=\Lambda$ for every $\lambda\in\Lambda$, and hence the measures $\nu_n:=|\Lambda_{r_n}|^{-1}\delta_\Lambda*\delta_{-\Lambda_{r_n}}$, whose vague limit is the autocorrelation measure γ, satisfy $\nu_n=\delta_\Lambda$ for every n. We conclude that the autocorrelation measure γ is given by $\gamma=\delta_\Lambda=\delta_{L^*}$. In turn, by the Poisson summation formula we get $\widehat{\gamma}=m(\Omega)\delta_L$, and so the weak tiling measure μ in Theorem 1.5 is given by $\mu=\delta_{L\setminus\{0\}}$. We thus see that the weak tiling of $\Omega^{\complement}$ is in this case a proper tiling.

Example 3.3. Let $\Omega=\left[0,\frac{1}{2}\right]\cup\left[1,\frac{3}{2}\right]$. Then, Ω tiles the real line $\mathbb{R}$ properly with respect to the translation set $\Lambda=2\mathbb{Z}\cup\left(2\mathbb{Z}+\frac{1}{2}\right)$. It is well known and not difficult to verify that the same set Λ is also a spectrum for Ω. The set Λ is periodic, but it is not a lattice. We calculate the autocorrelation measure γ as the vague limit of the measures $\nu_n:=|\Lambda_n|^{-1}\delta_\Lambda*\delta_{-\Lambda_n}$, where $\Lambda_n:=\Lambda\cap[-2n,2n)$. Then, all the measures ν_n, as well as the vague limit γ, coincide with the measure $\sum_{k\in\mathbb{Z}}\cos^2\left(\frac{1}{4}\pi k\right)\delta_{k/2}$ (so that, in fact, ν_n does not depend on n). In turn, one can check using the Poisson summation formula that $\widehat{\gamma}=\gamma$. We thus obtain that the weak tiling measure μ in Theorem 1.5 is a pure point measure (that is, μ has no continuous part), but the weak tiling of $\Omega^{\complement}$ with respect to this measure μ is not a proper tiling.

Example 3.4. Let $\Omega=[-\frac{1}{2},\frac{1}{2}]^2$ be the unit cube in $\mathbb{R}^2$, and let α be an irrational real number. Define Λ to be the set of all points of the form $(n,n^2\alpha+m)$ where n and m are integers. It is easy to see that Ω tiles the plane $\mathbb{R}^2$ with respect to the translation set Λ. It is known, see [JP], that the translation sets for tilings by the unit cube Ω coincide with the spectra of Ω. Hence, Λ is a spectrum for Ω.

In this example, we will calculate the autocorrelation measure γ as the vague limit of the measures $\nu_N:=|\Lambda_N|^{-1}\delta_\Lambda*\delta_{-\Lambda_N}$, where $\Lambda_N:=\Lambda\cap Q_N$ is the intersection of Λ with the cube $Q_N:=[-N,N)^2$, that is, we will be averaging not with respect to balls, but instead with respect to cubes with sides parallel to the axes. Notice that the proof of Theorem 3.1 remains valid if the averages are taken with respect to cubes instead of balls (in fact, Hof used cubes and not balls in [H]), although the autocorrelation measure γ could, in principle, depend on the shape over which the average is taken.

It follows from the fact that $\Lambda-\Lambda\subset\mathbb{Z}\times\mathbb{R}$ that all the measures ν_N, and hence also the autocorrelation measure γ, are supported on $\mathbb{Z}\times\mathbb{R}$. The restriction of the measure ν_N to a line of the form $\{h\}\times\mathbb{R}$, where $h\in\mathbb{Z}$, is given by

$$\frac{1}{2N}\sum_{-N\leqslant n<N}\sum_{k\in\mathbb{Z}}\delta_{(h,2hn\alpha+h^2\alpha+k)}.$$

Since α is irrational, it follows from Weyl's equidistribution theorem that ν_N converges vaguely to the 1-dimensional Lebesgue measure on each line $\{h\}\times\mathbb{R}$ such that $h\neq0$. On the other hand, on the line $\{0\}\times\mathbb{R}$ all the measures ν_N coincide with $\delta_{\{0\}\times\mathbb{Z}}$. We conclude that $\gamma=\delta_0\times\delta_\mathbb{Z}+\delta_{\mathbb{Z}\setminus\{0\}}\times m_1$, where m_1 denotes the 1-dimensional Lebesgue measure. In turn, the diffraction measure $\widehat{\gamma}$ is given by $\widehat{\gamma}=\delta_\mathbb{Z}\times\delta_0+m_1\times\delta_{\mathbb{Z}\setminus\{0\}}$. So, in this example, we obtain that the weak tiling of $\Omega^\complement$ given by Theorem 1.5 involves a measure μ that has both a pure point part and a (singular) continuous part.

3.4. Our main application of Theorem 1.5 will be for the proof of Fuglede's conjecture for convex bodies in $\mathbb{R}^d$. This will be done in the following sections. But before that, we mention some simple applications of the theorem to other classes of domains.

The following result, for instance, excludes many sets from being spectral.

THEOREM 3.5. *Let Ω be a bounded, measurable set in $\mathbb{R}^d$. Suppose that there exists a measurable set $S\subset\mathbb{R}^d$ with the following properties*:

(i) $m(S)>0$;

(ii) $m(S\cap\Omega)=0$;

(iii) *for every $x\in\mathbb{R}^d$, if $m((\Omega+x)\cap S)>0$, then $m((\Omega+x)\cap\Omega)>0$.*

Then, Ω cannot be a spectral set.

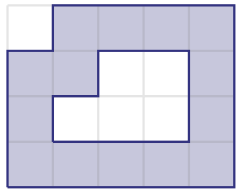

Figure 3.1. The illustration presents a planar domain that can be shown to be non-spectral using Theorem 3.5.

Informally speaking, the assumption in this theorem means that there exists a portion S of the complement $\Omega^{\complement}$ of Ω such that no translated copy $\Omega+x$ of Ω can even partly cover S unless this translated copy also covers a part of Ω as well. The theorem says that a set Ω satisfying this assumption cannot be spectral.

Proof of Theorem 3.5. Suppose to the contrary that Ω is spectral. Then, by Theorem 1.5, the complement $\Omega^{\complement}$ of Ω admits a weak tiling by translates of Ω, so there is a positive, locally finite measure μ such that $\mathbb{1}_\Omega * \mu = \mathbb{1}_{\Omega^{\complement}}$ a.e. Using condition (ii), we then have

$$m(S) = m(S \cap \Omega^{\complement}) = \int_S \mathbb{1}_{\Omega^{\complement}}\, dm = \int_S (\mathbb{1}_\Omega * \mu)\, dm = \int \varphi\, d\mu, \tag{3.1}$$

where $\varphi(x) := m((\Omega+x)\cap S)$. The measure μ is supported on $\Delta(\Omega)^{\complement}$ by Corollary 2.6, while the function φ vanishes on $\Delta(\Omega)^{\complement}$ due to condition (iii). Hence, the right-hand side of (3.1) must be zero. It follows that $m(S)=0$, a contradiction to condition (i). □

In Figure 3.1 we illustrate an example of a planar domain that can be shown to be non-spectral using Theorem 3.5. The essential feature of this domain is that it is connected but its complement is not connected, so that there is a "hole" inside the domain. We observe that if S is a set of positive measure contained in the hole, then S satisfies the conditions (i)–(iii) in Theorem 3.5, and therefore the domain cannot be spectral.

As another example, we can use Theorem 3.5 to obtain the following result.

Theorem 3.6. *Let Ω be a bounded, open set in $\mathbb{R}^d$. If Ω is spectral, then its boundary* $\mathrm{bd}(\Omega)$ *must be a set of Lebesgue measure zero.*

Proof. Suppose that $\mathrm{bd}(\Omega)$ has positive Lebesgue measure. We will show that the set $S := \mathrm{bd}(\Omega)$ satisfies the conditions (i)–(iii) in Theorem 3.5, and hence Ω cannot be spectral. Indeed, condition (i) holds by assumption. Since Ω is an open set, the two sets Ω and S are disjoint, hence condition (ii) holds as well. To verify condition (iii), we let $x \in \mathbb{R}^d$ be such that $m((\Omega+x)\cap S) > 0$. In particular, the set $(\Omega+x)\cap S$ is non-empty. The set $\Omega+x$ is open, while S is contained in the closure of Ω, so it follows that $(\Omega+x)\cap\Omega$

must be non-empty. But the set $(\Omega+x)\cap\Omega$ is open, since it is the intersection of two open sets, so it can be non-empty only if $m((\Omega+x)\cap\Omega)>0$. This confirms condition (iii), and thus the proof is concluded by Theorem 3.5. □

On the other hand, if Ω is a convex body in $\mathbb{R}^d$, then there exists no set S satisfying the conditions (i)–(iii) in Theorem 3.5. So, in order to study the spectrality problem for convex bodies, we must use the weak tiling condition $\mathbb{1}_\Omega*\mu=\mathbb{1}_{\Omega^\complement}$ a.e. in a stronger way. This will be done in the following sections.

4. Spectral convex bodies are polytopes

In this section we prove Theorem 1.2, which states that if a convex body $\Omega\subset\mathbb{R}^d$ is spectral, then Ω must be a convex polytope. This shows that the first condition (i) in the Venkov–McMullen theorem is necessary not only for tiling by translations, but also for the spectrality of a convex body Ω.

The fact that a convex body which tiles by translations must be a polytope, is a classical result that is due to Minkowski, see [McM] or [G, §32.2]. We will prove a stronger version of this result, which involves only a weak tiling assumption.

THEOREM 4.1. *Let K be a convex body in $\mathbb{R}^d$. If the complement $K^\complement$ of K has a weak tiling by translates of K, then K must be a convex polytope.*

Combining Theorem 4.1 with Theorem 1.5 yields that any spectral convex body in $\mathbb{R}^d$ must be a convex polytope, and Theorem 1.2 thus follows.

The rest of the section is devoted to the proof of Theorem 4.1.

4.1. We start by recalling some basic facts about convex bodies in $\mathbb{R}^d$. For more details, the reader is referred to [G, §4.1, §5.1 and §14.1].

Recall that a set $K\subset\mathbb{R}^d$ is called a *convex body* if K is a compact, convex set with non-empty interior. In what follows, we assume K to be a convex body in $\mathbb{R}^d$.

A hyperplane H is called a *support hyperplane* of K at a point $x\in\mathrm{bd}(K)$, if $x\in H$ and K is contained in one of the two closed half-spaces bounded by H. In this case, we denote the half-space containing K by H^-, and the other one by H^+. H^- is called a *support half-space* of K at the point x. It can be represented in the form

$$H^-=\{z\in\mathbb{R}^d:\langle z,\xi\rangle\leqslant\langle x,\xi\rangle\},$$

where ξ is a vector in $\mathbb{R}^d$ such that $|\xi|=1$. We call ξ an *exterior normal unit vector* of K, or of H, at the point x.

For each $x \in \mathrm{bd}(K)$, there exists at least one support hyperplane H of K at x. The support hyperplane at x need not, in general, be unique. If the support hyperplane is unique then x is called a *regular* boundary point of K, and otherwise it is called a *singular* boundary point.

For each vector $\xi \in \mathbb{R}^d$, $|\xi|=1$, there is a unique support hyperplane $H(K,\xi)$ of K with exterior normal vector ξ. The set $S(K,\xi):=K \cap H(K,\xi)$ is called the *support set* of K determined by ξ. The support set is a compact, convex subset of $\mathrm{bd}(K)$.

A hyperplane H is said to *separate* two convex bodies K and K', if K is contained in one of the two closed half-spaces bounded by H, while K' is contained in the other closed half-space. Any two convex bodies K and K' with disjoint interiors have at least one separating hyperplane H.

A *convex polytope* $P \subset \mathbb{R}^d$ is the convex hull of a finite number of points. Equivalently, a convex polytope is a bounded set P which can be represented as the intersection of finitely many closed half-spaces.

4.2. We now turn to the proof of Theorem 4.1. It is divided into several lemmas.

LEMMA 4.2. *Let K be a convex body in $\mathbb{R}^d$. If K is not a polytope, then there exists an infinite sequence x_n of regular boundary points of K such that the corresponding exterior normal unit vectors ξ_n are distinct, that is, $\xi_n \neq \xi_m$ whenever $n \neq m$.*

Proof. We will rely on the fact that if $K \subset \mathbb{R}^d$ is a convex body, then the set of regular boundary points of K constitutes a dense subset of $\mathrm{bd}(K)$ (see [G, §5.1, Theorem 5.1 or Theorem 5.2]).

We construct the sequence x_n by induction. Let x_1 be any regular boundary point of K, and ξ_1 be the exterior normal unit vector at the point x_1. Now suppose that the points $x_1, x_2, ..., x_{n-1}$ have already been chosen, that they are regular boundary points of K, and their corresponding exterior normal unit vectors $\xi_1, \xi_2, ..., \xi_{n-1}$ are distinct.

Let

$$F_n := \bigcup_{j=1}^{n-1} S(K, \xi_j),$$

where $S(K,\xi_j)$ is the support set of K determined by ξ_j. Then, F_n is a closed subset of $\mathrm{bd}(K)$. We claim that F_n must be a proper subset of $\mathrm{bd}(K)$. Indeed, suppose to the contrary that $\mathrm{bd}(K)=F_n$. Then, each point $y \in \mathrm{bd}(K)$ belongs to $S(K,\xi_{j(y)})$ for some $1 \leqslant j(y) \leqslant n-1$. Hence, the closed half-space

$$H_y^- := \{z : \langle z, \xi_{j(y)} \rangle \leqslant \langle x_{j(y)}, \xi_{j(y)} \rangle\}$$

is a support half-space of K at the point y. It is known (see [G, §4.1, Corollary 4.1]) that if, for each $y \in \mathrm{bd}(K)$, H_y^- is a support half-space of K at the point y, then

$$K = \bigcap_{y \in \mathrm{bd}(K)} H_y^- .$$

But the collection $\{H_y^- : y \in \mathrm{bd}(K)\}$ has at most $n-1$ distinct members. It follows that K is the intersection of finitely many closed half-spaces, hence K is a convex polytope. Since we have assumed that K is not a polytope, we arrive at a contradiction. This establishes our claim that F_n must be a proper subset of $\mathrm{bd}(K)$.

Since F_n is a closed set, it follows that its complement $\mathrm{bd}(K) \setminus F_n$ is a non-empty, relatively open subset of $\mathrm{bd}(K)$. The set of regular boundary points of K is dense in $\mathrm{bd}(K)$, hence we can choose a regular point x_n in $\mathrm{bd}(K) \setminus F_n$. The exterior normal unit vector ξ_n at the point x_n is then distinct from all the vectors $\xi_1, \xi_2, ..., \xi_{n-1}$. This completes the inductive construction, and thus concludes the proof of the lemma. □

LEMMA 4.3. *Let K be a convex body in $\mathbb{R}^d$, let x be a regular boundary point of K, and let ξ be the exterior normal unit vector at the point x. Suppose that U is an open neighborhood of the set $-S(K,-\xi)+x$, where $S(K,-\xi)$ is the support set of K determined by the vector $-\xi$. Then, there is an open neighborhood V of x such that, for any $t \in \Delta(K)^{\complement}$, the set $K+t$ cannot intersect V unless $t \in U$.*

Proof. Suppose to the contrary that the assertion is not true. Then, there is a sequence $t_j \in \Delta(K)^{\complement} \cap U^{\complement}$, and for each j there is a point $x_j \in K+t_j$ such that $x_j \to x$. Since t_j must remain bounded as $j \to \infty$, we may assume, by passing to a subsequence if needed, that $t_j \to t$.

Since the set $\Delta(K)^{\complement}$ is closed, the limit t of the sequence t_j belongs to $\Delta(K)^{\complement}$. Hence, K and $K+t$ are two convex bodies with disjoint interiors. It follows that there exists a hyperplane H separating K from $K+t$. The point x lies on both K and $K+t$, and therefore x must lie on H. It follows that x is a boundary point of both K and $K+t$, and H is a support hyperplane of both K and $K+t$ at the point x. Since x is a regular boundary point of K, H is the unique support hyperplane $H(K,\xi)$ of K at the point x.

Now $H(K,\xi)$ is a support hyperplane of $K+t$ at the point x, with exterior normal vector $-\xi$. It follows that $H(K,\xi)-t$ is a support hyperplane of K at the point $x-t$, with exterior normal vector $-\xi$. Since a support hyperplane of K with a given exterior normal vector is unique, we must have that $H(K,\xi)-t = H(K,-\xi)$, and hence the point $x-t$ lies on the support set $S(K,-\xi)$ determined by the vector $-\xi$. In other words, we have obtained that $t \in -S(K,-\xi)+x$. But since the latter set is contained in U, we conclude that $t \in U$. However this is not possible, since t is the limit of a sequence $\{t_j\} \subset U^{\complement}$ and the set U is open. This contradiction completes the proof. □

Lemma 4.4. *Let $K \subset \mathbb{R}^d$ be a convex body whose complement $K^{\complement}$ admits a weak tiling by translates of K, that is, there is a positive, locally finite measure μ such that $\mathbb{1}_K * \mu = \mathbb{1}_{K^{\complement}}$ a.e. Let x be a regular boundary point of K, and let ξ be the exterior normal unit vector at the point x. Then, we have*

$$\mu(-S(K,-\xi)+x) \geqslant 1,$$

where $S(K,-\xi)$ is the support set of K determined by the vector $-\xi$.

Proof. Suppose to the contrary that $\mu(-S(K,-\xi)+x)<1$. Then, there is an open neighborhood U of the set $-S(K,-\xi)+x$ such that also $\mu(U)<1$. By Lemma 4.3, there is an open neighborhood V of x such that, for any $t \in \Delta(K)^{\complement}$, the set $K+t$ cannot intersect V, unless $t \in U$.

We now decompose the measure μ into a sum $\mu=\mu'+\mu''$, where $\mu' := \mu \cdot \mathbb{1}_U$ and $\mu'' := \mu \cdot \mathbb{1}_{U^{\complement}}$. The support of the measure μ must be contained in $\Delta(K)^{\complement}$, due to the weak tiling assumption (Corollary 2.6), and therefore μ'' is supported on the closed set $\Delta(K)^{\complement} \cap U^{\complement}$. Hence, for any $t \in \operatorname{supp}(\mu'')$, the set $K+t$ does not intersect V, which implies that $\mathbb{1}_K * \mu'' = 0$ a.e. in V. It follows that

$$\mathbb{1}_{V \cap K^{\complement}} = \mathbb{1}_V \cdot (\mathbb{1}_K * \mu) = \mathbb{1}_V \cdot (\mathbb{1}_K * \mu') + \mathbb{1}_V \cdot (\mathbb{1}_K * \mu'') = \mathbb{1}_V \cdot (\mathbb{1}_K * \mu') \quad \text{a.e.},$$

and therefore

$$\|\mathbb{1}_{V \cap K^{\complement}}\|_{L^\infty(\mathbb{R}^d)} \leqslant \|\mathbb{1}_K * \mu'\|_{L^\infty(\mathbb{R}^d)} \leqslant \int d\mu' = \mu(U) < 1.$$

This implies that the set $V \cap K^{\complement}$ must have Lebesgue measure zero. But this is not possible, since V is an open neighborhood of the point $x \in \operatorname{bd}(K)$, and hence $V \cap K^{\complement}$ has non-empty interior. We thus arrive at a contradiction, which concludes the proof. □

Lemma 4.5. *Let K be a convex body in $\mathbb{R}^d$, and x and x' be two regular boundary points of K. Let ξ and ξ' be the exterior normal unit vectors at the points x and x', respectively, and let $S(K,-\xi)$ and $S(K,-\xi')$ be the support sets of K determined by $-\xi$ and $-\xi'$, respectively. If we have $\xi \neq \xi'$, then the two sets $-S(K,-\xi)+x$ and $-S(K,-\xi')+x'$ are disjoint.*

Proof. Suppose to the contrary that the two sets $-S(K,-\xi)+x$ and $-S(K,-\xi')+x'$ are not disjoint, so they have at least one point in common. Then, there exist two points $y \in S(K,-\xi)$ and $y' \in S(K,-\xi')$ such that $x-y=x'-y'$.

The support hyperplanes of K with exterior normal vectors ξ and $-\xi$ are respectively given by

$$H(K,\xi) = \{z : \langle z,\xi\rangle = \langle x,\xi\rangle\} \quad \text{and} \quad H(K,-\xi) = \{z : \langle z,\xi\rangle = \langle y,\xi\rangle\}. \tag{4.1}$$

Since K is contained in the closed slab bounded by these two hyperplanes, we have

$$K \subset \{z : \langle y, \xi\rangle \leqslant \langle z, \xi\rangle \leqslant \langle x, \xi\rangle\}. \tag{4.2}$$

The point x' is a regular boundary point of K, and so $H(K, \xi')$ is the unique support hyperplane of K at the point x'. Since $\xi \neq \xi'$, it follows that x' does not lie on $H(K, \xi)$. Using (4.1) and (4.2), this implies that $\langle x', \xi\rangle < \langle x, \xi\rangle$.

If we now denote $h := x - y = x' - y'$, then it follows that

$$\langle y', \xi\rangle = \langle x' - h, \xi\rangle < \langle x - h, \xi\rangle = \langle y, \xi\rangle.$$

But since $y' \in K$ this contradicts (4.2), and thus the lemma is proved. □

4.3. We can now prove Theorem 4.1 based on the previous lemmas.

Proof of Theorem 4.1. Assume that $K \subset \mathbb{R}^d$ is a convex body whose complement $K^{\complement}$ has a weak tiling by translates of K. We will prove that K is necessarily a polytope.

Suppose to the contrary that K is not a polytope. Then, by Lemma 4.2, there exists an infinite sequence x_n of regular boundary points of K such that the corresponding exterior normal unit vectors ξ_n are mutually distinct.

The set $K^{\complement}$ has a weak tiling by translates of K, so there is a positive, locally finite measure μ such that $\mathbb{1}_K * \mu = \mathbb{1}_{K^{\complement}}$ a.e. Let $S(K, -\xi_n)$ be the support set of K determined by the vector $-\xi_n$. Then, by Lemma 4.4, we have $\mu(-S(K, -\xi_n) + x_n) \geqslant 1$ for each n.

Since the exterior normal unit vectors ξ_n are distinct, we get from Lemma 4.5 that the sets $-S(K, -\xi_n) + x_n$ are pairwise disjoint. On the other hand, all these sets are contained in $K - K$. Hence, the set $K - K$ contains an infinite sequence of pairwise disjoint subsets such that the total mass of μ in each one of these sets is at least 1. It follows that we must have $\mu(K - K) = +\infty$. But this is not possible, as $K - K$ is a bounded set and μ is a locally finite measure. We thus arrive at a contradiction, which concludes the proof of the theorem. □

5. Spectral convex polytopes can tile by translations, I

So far, we have established that a spectral convex body $\Omega \subset \mathbb{R}^d$ must be a convex polytope (Theorem 1.2). We also know from the result in [K1] (or [KP]) that Ω must be centrally symmetric, and from the result in [GL2, §4] that all the facets of Ω must be centrally symmetric as well. It now remains to prove that each belt of Ω must consist of either four or six facets (Theorem 1.3). The proof will be given throughout the present section and the next one.

The key results of the present section are Lemmas 5.8 and 5.9.

5.1. We begin by recalling some basic facts about convex polytopes in $\mathbb{R}^d$. For more details, we refer to [BrGu, §1.A], [G, §14.1] and [S, §2.4].

A *convex polytope* $A\subset\mathbb{R}^d$ is the convex hull of a finite number of points. Equivalently, a convex polytope is a bounded set A which can be represented as the intersection of finitely many closed half-spaces.

We denote by $\operatorname{aff}(A)$ the *affine hull* of a convex polytope $A\subset\mathbb{R}^d$, that is, $\operatorname{aff}(A)$ is the smallest affine subspace containing A. By the *relative interior* and *relative boundary* of A, we refer respectively to the interior and boundary relative to $\operatorname{aff}(A)$. The relative interior of A will be denoted by $\operatorname{relint}(A)$.

A *face* of a convex polytope A is a support set of A, that is, the intersection of A with a support hyperplane of A. (There is also an alternative, equivalent definition, according to which a face of A is an *extreme subset* of A, that is, a convex subset $F\subset A$ such that, if $x,y\in A$ and $(1-\lambda)x+\lambda y\in F$, $0<\lambda<1$, then $x,y\in F$.)

If $A\subset\mathbb{R}^d$ is a convex polytope with non-empty interior, then a $(d-1)$-dimensional face of A is called a *facet* of A, while a $(d-2)$-dimensional face is called a *subfacet* of A. If G is a subfacet of A, then there exist exactly two facets F_1 and F_2 of A which contain G, and we have $G=F_1\cap F_2$ (see [BrGu, Proposition 1.12]).

A convex polytope $A\subset\mathbb{R}^d$ is said to be *centrally symmetric* if the set $-A$ is a translate of A. In this case, there is a unique point $x\in\mathbb{R}^d$ such that $-A+x=A-x$. The point x is called the *center of symmetry* of A, and A is then said to be *symmetric* with respect to x.

If X and Y are two convex subsets of $\mathbb{R}^d$, then we use $\operatorname{conv}\{X,Y\}$ to denote the convex hull of the union $X\cup Y$.

A *prism* in $\mathbb{R}^d$ is a convex polytope of the form $\operatorname{conv}\{F,F+\tau\}$, where F is a $(d-1)$-dimensional convex polytope, and τ is a vector such that F and $F+\tau$ do not lie on the same hyperplane. The two sets F and $F+\tau$ are called the *bases* of the prism.

A *slab* in $\mathbb{R}^d$ is the closed region between two parallel hyperplanes, that is, a set of the form $\{z:c_1\leqslant\langle z,\xi\rangle\leqslant c_2\}$, where ξ is a non-zero vector, and c_1 and c_2 are constants.

5.2. We now state and prove several lemmas.

Lemma 5.1. *Let $A\subset\mathbb{R}^d$ be a convex polytope with non-empty interior. Then,*

$$\operatorname{int}(A)-\operatorname{int}(A)=\operatorname{int}(A)-A=\operatorname{int}(A-A).$$

Proof. The fact that $\operatorname{int}(A)$ is a subset of A implies that $\operatorname{int}(A)-\operatorname{int}(A)\subset\operatorname{int}(A)-A$. In turn, $\operatorname{int}(A)-A$ is the union of all sets of the form $\operatorname{int}(A)-x$ where $x\in A$, and all these sets are open. Hence, $\operatorname{int}(A)-A$ is an open subset of $A-A$, and it follows that

$\mathrm{int}(A)-A\subset\mathrm{int}(A-A)$. It remains to prove that $\mathrm{int}(A-A)\subset\mathrm{int}(A)-\mathrm{int}(A)$. Indeed, let $h\in\mathrm{int}(A-A)$. Then, there is $\varepsilon>0$ such that $h+\varepsilon h$ is in $A-A$. Let $x,y\in A$ be such that $h+\varepsilon h=y-x$. Let a be an interior point of A, and define the points $x':=\lambda x+(1-\lambda)a$ and $y':=\lambda y+(1-\lambda)a$, where $\lambda:=(1+\varepsilon)^{-1}$. Then, $x',y'\in\mathrm{int}(A)$, and we have $y'-x'=h$, which shows that $h\in\mathrm{int}(A)-\mathrm{int}(A)$. □

LEMMA 5.2. *Let $A\subset\mathbb{R}^d$ be a convex polytope with non-empty interior, and let F be a facet of A. Then,*

$$\mathrm{relint}(F-F)\subset\mathrm{int}(A-A).$$

Proof. Let $h\in\mathrm{relint}(F-F)$. Then, there is an open set U such that

$$h\in U\cap H\subset F-F,$$

where H denotes the hyperplane through the origin parallel to F. It follows that if we choose $\varepsilon>0$ small enough, then $h+\varepsilon h\in F-F$. Let $x,y\in F$ be such that $h+\varepsilon h=y-x$. Let a be an interior point of A, and define the points

$$x'=\lambda x+(1-\lambda)a \quad\text{and}\quad y'=\lambda y+(1-\lambda)a,$$

where $\lambda=(1+\varepsilon)^{-1}$. Then, $x',y'\in\mathrm{int}(A)$ and we have $y'-x'=h$, which shows that

$$h\in\mathrm{int}(A)-\mathrm{int}(A).$$

Using Lemma 5.1 we conclude that $h\in\mathrm{int}(A-A)$ and the lemma is proved. □

LEMMA 5.3. *Let $A\subset\mathbb{R}^d$ be a convex polytope with non-empty interior. Let F be a facet of A, and let H^- be the support half-space of A at the facet F. Suppose that E is a compact subset of* $\mathrm{relint}(F)$. *Then, there is an open neighborhood V of E such that the set $V\cap H^-$ is contained in A.*

Proof. Let F_i, $i=1,2,\dots,m$, be all the facets of A, and suppose that $F=F_1$. For each i, let H_i be the hyperplane containing the facet F_i, and let H_i^- be the support half-space of A at the facet F_i. Then,

$$A=H_1^-\cap H_2^-\cap\dots\cap H_m^- \tag{5.1}$$

(see [G, Theorem 14.2]). In particular, the set E is contained in all the half-spaces H_i^-. On the other hand no point of E can lie on any one of the hyperplanes H_i, $i\neq1$, since E is a subset of $\mathrm{relint}(F)$. Hence, there is an open neighborhood V of E such that V is contained in H_i^- for all $i\neq1$. Using this together with (5.1) implies the claim. □

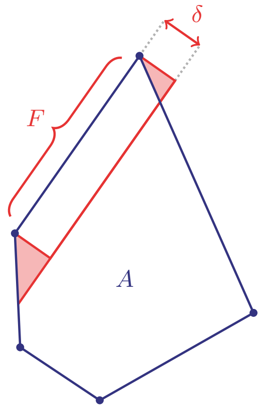

Figure 5.1. The shaded region in the illustration represents the symmetric difference $P_\delta \triangle Q_\delta$ of the sets P_δ and Q_δ in Lemma 5.5.

Lemma 5.4. *Let $A \subset \mathbb{R}^d$ be a convex polytope with non-empty interior, and let G be a subfacet of A. Let F_1 and F_2 be the two adjacent facets of A that meet at the subfacet G, and let H_1^- and H_2^- be the support half-spaces of A at the facets F_1 and F_2, respectively. Suppose that E is a compact subset of* relint(G). *Then, there is an open neighborhood V of E such that the set $V \cap H_1^- \cap H_2^-$ is contained in A.*

Proof. We continue to use the same notation as in the proof of Lemma 5.3. The set E is again contained in all the half-spaces H_i^-. However, no point of E can lie on any one of the hyperplanes H_i, $i \neq 1, 2$, since E is a subset of relint(G). Hence, there is an open neighborhood V of E such that V is contained in H_i^- for all $i \neq 1, 2$. Using this with (5.1), we obtain the assertion of the lemma. $\square$

Lemma 5.5. *Let $A \subset \mathbb{R}^d$ be a convex polytope with non-empty interior, and let F be a facet of A. Suppose that ξ is the exterior normal unit vector of A at the facet F, and that $H^- = \{z : \langle z, \xi \rangle \leqslant c\}$ is the support half-space of A at the facet F. For each $\delta > 0$ we let $P_\delta = A \cap S_\delta$ be the intersection of A with the slab*

$$S_\delta = \{z : c - \delta \leqslant \langle z, \xi \rangle \leqslant c\},$$

and we let $Q_\delta = \mathrm{conv}\{F, F - \delta\xi\}$ be the prism with bases F and $F - \delta\xi$. Then, we have $m(P_\delta \triangle Q_\delta) = o(\delta)$ as $\delta \to 0$ (see Figure 5.1).

Proof. Let H be the hyperplane containing the facet F. For each $\varepsilon > 0$, let $F_{+\varepsilon}$ be the set of all points of H whose distance from F is at most ε, and $F_{-\varepsilon}$ be the set of all points of F whose distance from the relative boundary of F is at least ε.

Let $I_\delta=\{s\xi:0\leqslant s\leqslant\delta\}$ be the closed line segment which connects the origin to the point $\delta\xi$. Then, there is $\delta_0=\delta_0(A,F,\varepsilon)>0$ such that, for every $\delta<\delta_0$, we have

$$F_{-\varepsilon}-I_\delta\subset P_\delta\subset F_{+\varepsilon}-I_\delta.$$

(the left inclusion may be deduced from Lemma 5.3). Observe that $Q_\delta=F-I_\delta$, hence

$$P_\delta\triangle Q_\delta\subset(F_{+\varepsilon}\setminus F_{-\varepsilon})-I_\delta.$$

This implies that

$$m_d(P_\delta\triangle Q_\delta)\leqslant\delta m_{d-1}(F_{+\varepsilon}\setminus F_{-\varepsilon}).$$

Since $m_{d-1}(F_{+\varepsilon}\setminus F_{-\varepsilon})$ tends to zero as $\varepsilon\to 0$, the assertion follows. □

LEMMA 5.6. *Let $A\subset\mathbb{R}^d$ be a convex polytope with non-empty interior, and let G be a subfacet of A. Let F_1 and F_2 be the two adjacent facets of A that meet at the subfacet G, and let H_1^- and H_2^- be the support half-spaces of A at the facets F_1 and F_2, respectively. For each $\delta>0$, we let P_δ be the set of all points of A whose distance from* $\operatorname{aff}(G)$ *is not greater than δ, and we let*

$$Q_\delta=(G+S_\delta)\cap H_1^-\cap H_2^-,$$

where S_δ is a closed 2-dimensional ball of radius δ centered at the origin and orthogonal to $\operatorname{aff}(G)$*. Then, we have $m(P_\delta\triangle Q_\delta)=o(\delta^2)$ as $\delta\to 0$.*

Proof. The proof is similar to that of Lemma 5.5. For each $\varepsilon>0$, let $G_{+\varepsilon}$ be the set of all points of $\operatorname{aff}(G)$ whose distance from G is at most ε, and $G_{-\varepsilon}$ be the set of all points of G whose distance from the relative boundary of G is at least ε. If $\delta_0=\delta_0(A,G,\varepsilon)>0$ is small enough, then for every $\delta<\delta_0$ we have

$$(G_{-\varepsilon}+S_\delta)\cap H_1^-\cap H_2^-\subset P_\delta\subset(G_{+\varepsilon}+S_\delta)\cap H_1^-\cap H_2^-$$

(here, the left inclusion may be inferred from Lemma 5.4), and hence

$$P_\delta\triangle Q_\delta\subset((G_{+\varepsilon}\setminus G_{-\varepsilon})+S_\delta)\cap H_1^-\cap H_2^-.$$

This implies that

$$m_d(P_\delta\triangle Q_\delta)\leqslant m_{d-2}(G_{+\varepsilon}\setminus G_{-\varepsilon})m_2(S_\delta).$$

But we have $m_2(S_\delta)=\pi\delta^2$, while $m_{d-2}(G_{+\varepsilon}\setminus G_{-\varepsilon})$ tends to zero as $\varepsilon\to 0$. □

Lemma 5.7. *Let A and B be two convex polytopes in $\mathbb{R}^d$ with non-empty interiors. Let L be one of the facets of B, let ξ be the exterior normal unit vector of B at the facet L, and let E be a compact subset of* $\operatorname{relint}(L)$. *Suppose that A has a facet F on which the exterior normal unit vector is $-\xi$, and let U be an open neighborhood of the set $E-F$. Then, there is an open neighborhood V of E such that, for any t, if $A+t$ and B have disjoint interiors and if $A+t$ intersects V, then $t\in U$.*

This can be proved in essentially the same way as Lemma 4.3 above.

5.3. The following two lemmas are the key results of this section.

Lemma 5.8. *Let A and B be two convex polytopes in $\mathbb{R}^d$ with non-empty, disjoint interiors, and suppose that A and B share a common facet F. Assume that μ is a positive, locally finite measure such that $\mathbb{1}_A*\mu\geqslant 1$ a.e. on A, while $\mathbb{1}_A*\mu=0$ a.e. on B. Let μ' denote the restriction of the measure μ to* $\operatorname{relint}(F-F)$. *Then, we have $\mathbb{1}_F*\mu'\geqslant 1$ a.e. with respect to the $(d-1)$-dimensional volume measure on the facet F.*

Notice that the convolution $\mathbb{1}_F*\mu'$ vanishes outside of $\operatorname{aff}(F)$, since $F+t$ lies on $\operatorname{aff}(F)$ for every $t\in\operatorname{supp}(\mu')$. An equivalent way to formulate the conclusion of the lemma is to say that, if σ_F denotes the $(d-1)$-dimensional volume measure restricted to the facet F, then we have $(\sigma_F*\mu')(E)\geqslant\sigma_F(E)$ for every Borel set E.

Proof. By applying a rotation and a translation, we may assume that the facet F is contained in the hyperplane $\{x:x_1=0\}$. Hence, F has the form $F=\{0\}\times\Omega$, where Ω is a convex polytope in $\mathbb{R}^{d-1}$ with non-empty interior. We may also suppose that

$$A\subset\{x:x_1\geqslant 0\}\quad\text{and}\quad B\subset\{x:x_1\leqslant 0\}.$$

Let $\eta>0$, and let Σ be a compact subset of $\operatorname{int}(\Omega)$. Then, the set $E:=\{0\}\times\Sigma$ is a compact subset of $\operatorname{relint}(F)=\{0\}\times\operatorname{int}(\Omega)$. Choose $\varepsilon>0$ such that the total mass of the measure μ in the set $((-\varepsilon,0)\cup(0,\varepsilon))\times\operatorname{int}(\Omega-\Omega)$ is less than η, and define

$$U:=(-\varepsilon,\varepsilon)\times\operatorname{int}(\Omega-\Omega).$$

Using Lemma 5.1, we see that U is an open neighborhood of the set $E-F$. By Lemma 5.7, there is an open neighborhood V of E such that, for any t, if $A+t$ and B have disjoint interiors and if $A+t$ intersects V, then $t\in U$.

Consider the following three subsets of $\mathbb{R}^d$:

(i) $Y':=\{0\}\times\operatorname{int}(\Omega-\Omega)=\operatorname{relint}(F-F)$;

(ii) $Y'':=((-\varepsilon,0)\cup(0,\varepsilon))\times\operatorname{int}(\Omega-\Omega)$;

(iii) $Y''':=(Y'\cup Y'')^{\complement}=U^{\complement}$.

Then, the sets Y', Y'' and Y''' are pairwise disjoint, and they cover the whole space. It follows that we may decompose the measure μ into the sum $\mu=\mu'+\mu''+\mu'''$, where the three measures μ', μ'' and μ''' are the restrictions of μ to the sets Y', Y'' and Y''', respectively.

The assumption that $\mathbb{1}_A*\mu=0$ a.e. on B implies that the sets $A+t$ and B must have disjoint interiors for every $t\in\operatorname{supp}(\mu)$ (Lemma 2.5). Since the support of the measure μ''' is contained in $\operatorname{supp}(\mu)\cap U^{\complement}$, it follows that if $t\in\operatorname{supp}(\mu''')$ then $A+t$ cannot intersect V. Hence, we have $\mathbb{1}_A*\mu'''=0$ a.e. in V. We also have

$$\|\mathbb{1}_A*\mu''\|_{L^\infty(\mathbb{R}^d)} \leqslant \int_{\mathbb{R}^d} d\mu'' = \mu(Y'') < \eta.$$

Combining this with the assumption that $\mathbb{1}_A*\mu\geqslant 1$ a.e. on A, this implies that

$$\mathbb{1}_A*\mu' = \mathbb{1}_A*\mu - \mathbb{1}_A*\mu'' - \mathbb{1}_A*\mu''' \geqslant 1-\eta \quad \text{a.e. on } A\cap V. \tag{5.2}$$

For each $\delta>0$ we let $P_\delta:=A\cap S_\delta$ be the intersection of A with the slab

$$S_\delta := [0,\delta]\times\mathbb{R}^{d-1},$$

and we also consider the prism $Q_\delta:=[0,\delta]\times\Omega$. Then, by Lemma 5.5, we can choose δ small enough such that

$$m(P_\delta\triangle Q_\delta)\mu(F-F) < \delta\eta^2 m_{d-1}(\Sigma). \tag{5.3}$$

We may also assume, by choosing δ small enough, that the set $D_\delta:=[0,\delta]\times\Sigma$ is contained in both V and A (the inclusion in A can be deduced from Lemma 5.3).

The support of the measure μ' is contained in the hyperplane $\{0\}\times\mathbb{R}^{d-1}$. For each $t\in\operatorname{supp}(\mu')$, we therefore have $(A+t)\cap S_\delta=P_\delta+t$. This implies that $\mathbb{1}_A*\mu'=\mathbb{1}_{P_\delta}*\mu'$ a.e. on the slab S_δ. In particular, it follows from (5.2) that

$$\mathbb{1}_{P_\delta}*\mu' \geqslant 1-\eta \quad \text{a.e. on } D_\delta. \tag{5.4}$$

Let D'_δ be the set of all points $x\in D_\delta$ such that $(\mathbb{1}_{P_\delta\triangle Q_\delta}*\mu')(x)<\eta$. Then, we have

$$\mathbb{1}_{Q_\delta}*\mu' \geqslant \mathbb{1}_{P_\delta}*\mu' - \mathbb{1}_{P_\delta\triangle Q_\delta}*\mu' \geqslant 1-2\eta \quad \text{a.e. on } D'_\delta,$$

which follows from (5.4). On the other hand, by (5.3) we have

$$\int_{\mathbb{R}^d} (\mathbb{1}_{P_\delta\triangle Q_\delta}*\mu')\, dm = m(P_\delta\triangle Q_\delta)\int_{\mathbb{R}^d} d\mu' < \delta\eta^2 m_{d-1}(\Sigma),$$

which in turn implies that

$$m(D_\delta \setminus D'_\delta) \leqslant \eta^{-1} \int_{\mathbb{R}^d} (\mathbb{1}_{P_\delta \triangle Q_\delta} * \mu')\, dm < \delta\eta m_{d-1}(\Sigma) = \eta m(D_\delta),$$

that is, we have $m(D'_\delta) > (1-\eta) m(D_\delta)$. We conclude that

$$\mathbb{1}_{Q_\delta} * \mu' \geqslant 1-2\eta \text{ a.e. on } D'_\delta, \quad D'_\delta \subset D_\delta \quad \text{and} \quad m(D'_\delta) > (1-\eta) m(D_\delta). \tag{5.5}$$

Now, recall that the support of the measure μ' is contained in $\{0\} \times \mathbb{R}^{d-1}$. This implies that the value of $(\mathbb{1}_{Q_\delta} * \mu')(x)$ does not depend, in the slab S_δ, on the first coordinate x_1 of the point x. Hence, it follows from (5.5) that $\mathbb{1}_F * \mu' \geqslant 1-2\eta$ a.e. with respect to the $(d-1)$-dimensional volume measure on some set of the form $\{0\} \times \Sigma'$, where Σ' is a subset of Σ (which depends on both Σ and η) such that

$$m_{d-1}(\Sigma') \geqslant (1-\eta) m_{d-1}(\Sigma).$$

However, as the number η was arbitrary, it follows that we must have $\mathbb{1}_F * \mu' \geqslant 1$ a.e. with respect to the $(d-1)$-dimensional volume measure on $\{0\} \times \Sigma$. In turn, Σ was an arbitrary compact subset of $\operatorname{int}(\Omega)$. Using the fact that $\operatorname{bd}(\Omega)$ is a set of measure zero in $\mathbb{R}^{d-1}$, this implies that $\mathbb{1}_F * \mu' \geqslant 1$ a.e. with respect to the $(d-1)$-dimensional volume measure on $\{0\} \times \Omega = F$, which is what we had to prove. □

LEMMA 5.9. *Let A be a convex polytope in $\mathbb{R}^d$ with non-empty interior, and let G be a subfacet of A. Let F_1 and F_2 be the two adjacent facets of A that meet at the subfacet G, and let H_1^- and H_2^- be the support half-spaces of A at the facets F_1 and F_2, respectively. For each $\delta > 0$ we let*

$$Q_\delta := (G + S_\delta) \cap H_1^- \cap H_2^-,$$

where S_δ is a closed 2-dimensional ball of radius δ centered at the origin and orthogonal to $\operatorname{aff}(G)$. Assume that μ is a positive, finite measure supported on $G - G$ and satisfying $\mathbb{1}_G * \mu \geqslant 1$ a.e. with respect to the $(d-2)$-dimensional volume measure on G. Then, for any $\eta > 0$ we have

$$m\{x \in Q_\delta : (\mathbb{1}_A * \mu)(x) < 1-\eta\} = o(m(Q_\delta)), \quad \delta \to 0.$$

Proof. Let $P_\delta = A \cap (\operatorname{aff}(G) + S_\delta)$ be the set of all points of A whose distance from $\operatorname{aff}(G)$ is not greater than δ. The assumption that $\operatorname{supp}(\mu) \subset G - G$ implies that we have $(A+t) \cap (\operatorname{aff}(G) + S_\delta) = P_\delta + t$ for every $t \in \operatorname{supp}(\mu)$, and hence $\mathbb{1}_A * \mu = \mathbb{1}_{P_\delta} * \mu$ a.e. on $\operatorname{aff}(G) + S_\delta$. In particular,

$$\mathbb{1}_A * \mu = \mathbb{1}_{P_\delta} * \mu \quad \text{a.e. on } Q_\delta. \tag{5.6}$$

We have also assumed that $\mathbb{1}_G * \mu \geqslant 1$ a.e. with respect to the $(d-2)$-dimensional volume measure on G. Using Fubini's theorem this implies that

$$\mathbb{1}_{Q_\delta} * \mu \geqslant 1 \quad \text{a.e. on } Q_\delta. \tag{5.7}$$

Denote by $D_{\delta,\eta}$ the set of all points $x \in Q_\delta$ such that $(\mathbb{1}_{P_\delta \triangle Q_\delta} * \mu)(x) < \eta$. Then,

$$\mathbb{1}_A * \mu = \mathbb{1}_{P_\delta} * \mu \geqslant \mathbb{1}_{Q_\delta} * \mu - \mathbb{1}_{P_\delta \triangle Q_\delta} * \mu > 1 - \eta \quad \text{a.e. on } D_{\delta,\eta},$$

which follows from (5.6) and (5.7). Hence, to prove the assertion of the lemma, it is enough to show that $m(Q_\delta \setminus D_{\delta,\eta}) = o(m(Q_\delta))$ as $\delta \to 0$. We observe that

$$m(Q_\delta) = m_{d-2}(G) \cdot \tfrac{1}{2} \theta \delta^2,$$

where θ is the dihedral angle of A at its subfacet G. Let $\varepsilon > 0$. Then, by Lemma 5.6, there is $\delta_0 = \delta_0(A, G, \eta, \varepsilon) > 0$ such that, for any $\delta < \delta_0$, we have

$$m(P_\delta \triangle Q_\delta) \mu(G - G) < \varepsilon \eta m(Q_\delta).$$

It follows that

$$\int_{\mathbb{R}^d} (\mathbb{1}_{P_\delta \triangle Q_\delta} * \mu)\, dm = m(P_\delta \triangle Q_\delta) \int_{\mathbb{R}^d} d\mu < \varepsilon \eta m(Q_\delta),$$

which in turn implies

$$m(Q_\delta \setminus D_{\delta,\eta}) \leqslant \eta^{-1} \int_{\mathbb{R}^d} (\mathbb{1}_{P_\delta \triangle Q_\delta} * \mu)\, dm < \varepsilon m(Q_\delta).$$

This confirms that indeed we have $m(Q_\delta \setminus D_{\delta,\eta}) = o(m(Q_\delta))$ as $\delta \to 0$. □

6. Spectral convex polytopes can tile by translations, II

In this section we prove the following theorem, which is the final result needed for the proof of Fuglede's conjecture for convex bodies.

THEOREM 6.1. *Let $A \subset \mathbb{R}^d$ be a convex polytope, which is centrally symmetric and has centrally symmetric facets. Assume that the complement $A^{\complement}$ of A admits a weak tiling by translates of A, that is, there exists a positive, locally finite measure μ such that $\mathbb{1}_A * \mu = \mathbb{1}_{A^{\complement}}$ a.e. Then, each belt of A consists of either four or six facets.*

Theorem 1.3 is then obtained as a consequence of Theorems 1.5 and 6.1.

The rest of the section is devoted to the proof of Theorem 6.1.

6.1. Let G be any one of the subfacets of A, and suppose that the belt of A generated by G has $2m$ facets. Let

$$F_0, F_1, F_2, ..., F_{2m-1}, F_{2m} = F_0$$

be an enumeration of the facets of the belt such that F_{i-1} and F_i are adjacent facets for each $1 \leqslant i \leqslant 2m$. The intersection $F_{i-1} \cap F_i$ of any pair of consecutive facets in the belt is then a translate of either G or $-G$. We shall suppose that G itself is given by $G = F_1 \cap F_2$.

Our goal is to show that under the assumptions in Theorem 6.1, the belt can have only four or six facets, that is, we must have $m \leqslant 3$.

The belt generated by G consists of m pairs of opposite facets $\{F_i, F_{i+m}\}$, with $0 \leqslant i \leqslant m-1$. We may assume, with no loss of generality, that A is symmetric with respect to the origin, that is, $A = -A$. Then, for each facet F_i in the belt, its opposite facet is given by $-F_i$. Since the facets of A are centrally symmetric, $-F_i$ is a translate of F_i, so there is a translation vector τ_i which carries $-F_i$ onto F_i, that is, $F_i = -F_i + \tau_i$.

6.2. Recall that we have assumed that the complement $A^{\complement}$ of A admits a weak tiling by translates of A. This means that there exists a positive, locally finite measure μ such that $\mathbb{1}_A * \mu = \mathbb{1}_{A^{\complement}}$ a.e. For each $0 \leqslant i \leqslant 2m$, we define

$$T_i := \operatorname{relint}(F_i - F_i) + \tau_i, \quad \mu_i' := \mu \cdot \mathbb{1}_{T_i}, \quad \nu_i' := \mu_i' * \delta_{-\tau_i}. \tag{6.1}$$

We observe that $\operatorname{supp}(\mu_i')$ is contained in the hyperplane passing through the point τ_i and which is parallel to the facet F_i, while $\operatorname{supp}(\nu_i')$ is contained in the hyperplane through the origin which is parallel to F_i.

We also notice that we have $-F_i = F_i - \tau_i$, and hence $T_i = 2 \operatorname{relint}(F_i)$. This implies that the sets T_i $(0 \leqslant i \leqslant 2m-1)$ are pairwise disjoint, because any two distinct facets of A have disjoint relative interiors.

Lemma 6.2. *For every $0 \leqslant i \leqslant 2m$, we have that $\mathbb{1}_{F_i} * \nu_i' \geqslant 1$ a.e. with respect to the $(d-1)$-dimensional volume measure on the facet F_i.*

Proof. Let $A' := A + \tau_i$ and $B' := A$. Then, A' and B' are two convex polytopes in $\mathbb{R}^d$ with non-empty, disjoint interiors, and they share F_i as a common facet. Let $\rho := \mu * \delta_{-\tau_i}$. Then, we have $\mathbb{1}_{A'} * \rho = \mathbb{1}_{A^{\complement}}$ a.e., and in particular $\mathbb{1}_{A'} * \rho \geqslant 1$ a.e. on A' and $\mathbb{1}_{A'} * \rho = 0$ a.e. on B'. Since ν_i' is the restriction of the measure ρ to $\operatorname{relint}(F_i - F_i)$, we may apply Lemma 5.8 and conclude that $\mathbb{1}_{F_i} * \nu_i' \geqslant 1$ a.e. with respect to the $(d-1)$-dimensional volume measure on the facet F_i, as we had to show. $\square$

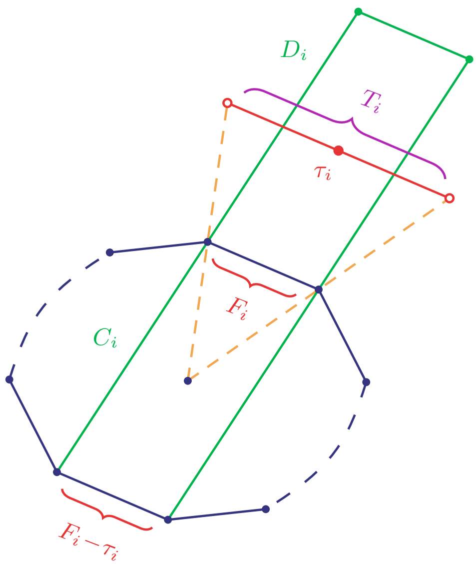

Figure 6.1. According to Lemma 6.3, the prism D_i is "weakly covered" by the translates of the prism C_i with respect to the measure μ_i'.

6.3. Let

$$C_i := \operatorname{conv}\{F_i, F_i - \tau_i\}$$

be the prism contained in A with bases F_i and $F_i - \tau_i = -F_i$. We also define the prism

$$D_i := C_i + \tau_i = \operatorname{conv}\{F_i, F_i + \tau_i\}.$$

LEMMA 6.3. *For each $0 \leqslant i \leqslant 2m$, we have $\mathbb{1}_{C_i} * \mu_i' \geqslant 1$ a.e. on D_i.*

Proof. By Lemma 6.2, the facet F_i has a subset E_i of full $(d-1)$-dimensional volume measure such that $(\mathbb{1}_{F_i} * \nu_i')(y) \geqslant 1$ for every $y \in E_i$. By Fubini's theorem, the set of all points x of the form $x = y + \lambda\tau_i$, with $y \in E_i$ and $0 \leqslant \lambda \leqslant 1$, constitutes a subset D_i' of D_i of full d-dimensional volume measure. Since $\operatorname{supp}(\nu_i')$ is contained in the hyperplane through the origin parallel to F_i, it follows that, for any point x of the above form, we have $(\mathbb{1}_{D_i} * \nu_i')(x) = (\mathbb{1}_{F_i} * \nu_i')(y) \geqslant 1$. We obtain that $\mathbb{1}_{D_i} * \nu_i' \geqslant 1$ a.e. on D_i. But this is equivalent to the assertion of the lemma. $\square$

In order to better understand the assertion of Lemma 6.3, observe that the prism D_i is contained (up to measure zero) in the complement $A^{\complement}$ of A, and hence we trivially have the "weak covering" property $\mathbb{1}_A * \mu \geqslant 1$ a.e. on D_i. However, the point of the lemma is that only the part of A which lies in the prism C_i, and only the part of the measure μ which lies in T_i, can in fact contribute to this covering (see Figure 6.1).

6.4. Let us recall that $F_0, F_1, F_2, ..., F_{2m-1}, F_{2m}=F_0$ is an enumeration of the facets of the belt generated by the subfacet G of the convex polytope A, and we have assumed that the subfacet G is given by $G=F_1\cap F_2$.

For $i\in\{1,2\}$ (and only for these two values of i) we denote

$$S_i := \operatorname{relint}(G-G)+\tau_i, \quad \mu_i'' := \mu_i'\cdot \mathbb{1}_{S_i} \quad \text{and} \quad \nu_i'' := \mu_i'' * \delta_{-\tau_i}. \tag{6.2}$$

We notice that S_i is a subset of the set T_i defined in (6.1). This follows from Lemma 5.2 applied relative to the affine hull of the facet F_i. In particular, this shows that S_1 and S_2 are disjoint sets, because the sets T_1 and T_2 are disjoint.

It also follows that we have $\mu_i''=\mu\cdot\mathbb{1}_{S_i}$, that is, in the definition of the measure μ_i'' it does not matter whether we restrict μ_i' or μ to the set S_i.

Lemma 6.4. *Assume that the belt of A generated by the subfacet G has six or more facets. Then, for each $i\in\{1,2\}$, we have $\mathbb{1}_G * \nu_i'' \geqslant 1$ a.e. with respect to the $(d-2)$-dimensional volume measure on the subfacet G.*

Proof. We suppose that i is an element of the set $\{1,2\}$, and we let j be the other element, so that $(i,j)=(1,2)$ or $(i,j)=(2,1)$. The proof is divided into several steps.

Step 1. We first claim that if $t\in\operatorname{supp}(\nu_i')$, then F_i+t cannot intersect the interior of the prism D_j. For suppose that $(F_i+t)\cap\operatorname{int}(D_j)$ is non-empty. Let $s:=t+\tau_i$. Then, $s\in\operatorname{supp}(\mu_i')$ and we have that $(F_i-\tau_i+s)\cap\operatorname{int}(D_j)$ is non-empty. Since $F_i-\tau_i+s$ is a facet of the prism C_i+s, it follows that $(C_i+s)\cap D_j$ has non-empty interior, and in particular we have $m((C_i+s)\cap D_j)>0$. By Lemma 2.5, this implies that $\mathbb{1}_{C_i}*\mu_i'$ cannot vanish a.e. on D_j, and hence there exist $\eta>0$ and a set $E\subset D_j$, with $m(E)>0$, such that $\mathbb{1}_{C_i}*\mu_i'\geqslant\eta$ a.e. on E. On the other hand, we have $\mathbb{1}_{C_j}*\mu_j'\geqslant\mathbb{1}_{D_j}$ a.e. by Lemma 6.3. Using the fact that C_i and C_j are both subsets of A, and that μ_i' and μ_j' are the restrictions of μ to the two disjoint sets T_i and T_j, respectively, we conclude that

$$\mathbb{1}_A*\mu \geqslant \mathbb{1}_{C_i}*\mu_i'+\mathbb{1}_{C_j}*\mu_j' \geqslant \eta\mathbb{1}_E+\mathbb{1}_{D_j} \geqslant (1+\eta)\mathbb{1}_E \quad \text{a.e.}$$

However, this contradicts the weak tiling assumption $\mathbb{1}_A*\mu=\mathbb{1}_{A^\complement}$ a.e. This establishes our claim that F_i+t cannot intersect the interior of the prism D_j.

Step 2. Let H_i be the hyperplane containing the facet F_i, and define

$$B_i := H_i\cap D_j.$$

We claim that B_i is a $(d-1)$-dimensional convex polytope, that $\operatorname{relint}(B_i)\subset\operatorname{int}(D_j)$, and that G is a $(d-2)$-dimensional face of B_i.

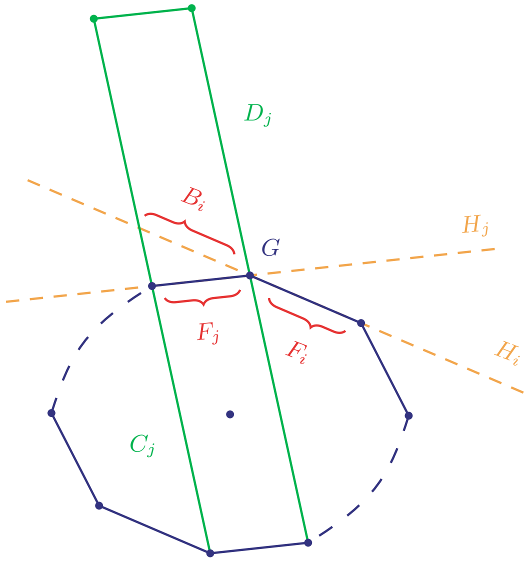

Figure 6.2. If the belt generated by the subfacet G has six or more facets, then the hyperplane H_i containing the facet F_i intersects the interior of the prism D_j, and $B_i = H_i \cap D_j$ is a $(d-1)$-dimensional convex polytope.

First, it is clear that B_i is a convex polytope, being the intersection of a convex polytope and a hyperplane.

Next, recall that we have assumed the belt of A generated by the subfacet G to have six or more facets. This implies that the facet $L_j := \mathrm{conv}\{G, G - \tau_j\}$ of the prism C_j divides the dihedral angle of A at its subfacet G into two (strictly positive) angles θ and φ, where θ is the dihedral angle between L_j and F_i, and φ is the dihedral angle between L_j and F_j. Hence, the hyperplane H_i divides the dihedral angle of the prism D_j at its subfacet G into two strictly positive angles θ and $\pi - \theta - \varphi$. It follows that H_i must intersect the interior of the prism D_j and so B_i is a $(d-1)$-dimensional convex polytope (see Figure 6.2).

Let H_j be the hyperplane containing the facet F_j, and H_j^+ be the closed half-space bounded by H_j which contains the prism D_j. Then, B_i is contained in H_j^+ and we have $B_i \cap H_j = G$. This shows that H_j is a support hyperplane of B_i and G is a face (of dimension $d-2$) of B_i.

Finally, we show that $\mathrm{relint}(B_i) \subset \mathrm{int}(D_j)$. Indeed, let $x \in \mathrm{relint}(B_i)$. As H_i is the affine hull of B_i, this means that there is an open set V such that $x \in V \cap H_i \subset B_i$. In particular, this implies that $x \in D_j$, so it is enough to prove that x cannot lie on $\mathrm{bd}(D_j)$. Indeed, as $x \in V \cap H_i \subset D_j$, the point x can lie on $\mathrm{bd}(D_j)$ only if H_i is a support hyperplane of D_j. But this is not the case, as H_i intersects $\mathrm{int}(D_j)$, so we must have $x \in \mathrm{int}(D_j)$.

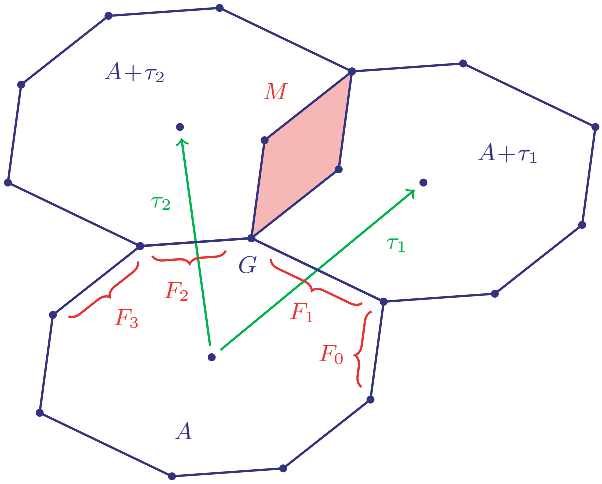

Figure 6.3. The shaded region represents the convex polytope $M=(A+\tau_1)\cap(A+\tau_2)$ in Lemma 6.5.

Step 3. It follows that if $t\in\operatorname{supp}(\nu_i')$, then F_i+t cannot intersect $\operatorname{relint}(B_i)$. Indeed, as we have shown above, $\operatorname{relint}(B_i)$ is contained in the interior of the prism D_j, while F_i+t cannot intersect the interior of D_j. Hence, F_i+t and $\operatorname{relint}(B_i)$ are disjoint sets for every $t\in\operatorname{supp}(\nu_i')$. From this, we conclude that $\mathbb{1}_{F_i}*\nu_i'=0$ a.e. with respect to the $(d-1)$-dimensional volume measure on B_i.

Step 4. The sets F_i and B_i are two $(d-1)$-dimensional convex polytopes contained in the same hyperplane H_i, they have disjoint relative interiors, and they share G as a common $(d-2)$-dimensional face. Recall that, by Lemma 6.2, we have $\mathbb{1}_{F_i}*\nu_i'\geqslant 1$ a.e. with respect to the $(d-1)$-dimensional volume measure on F_i, while we have just shown in Step 3 above that $\mathbb{1}_{F_i}*\nu_i'=0$ a.e. with respect to the $(d-1)$-dimensional volume measure on B_i. We may therefore apply Lemma 5.8 (relative to the hyperplane H_i containing F_i and B_i) and conclude that the measure ν_i'' obtained by restricting the measure ν_i' to $\operatorname{relint}(G-G)$, satisfies $\mathbb{1}_G*\nu_i''\geqslant 1$ a.e. with respect to the $(d-2)$-dimensional volume measure on G. So, we obtain the assertion of the lemma. □

6.5. For $i\in\{1,2\}$, we denote by N_i the hyperplane which contains the subfacet G and which is parallel to the facet F_0 if $i=1$, or parallel to the facet F_3 if $i=2$. Let N_i^- be the closed half-space bounded by N_i that has exterior normal unit vector which is opposite to the exterior normal vector of A at the facet F_0 (if $i=1$) or F_3 (if $i=2$). It is not difficult to verify that N_i^- is the support half-space of $A+\tau_i$ at its facet $F_0-\tau_0+\tau_1$ for $i=1$, or $F_3-\tau_3+\tau_2$ for $i=2$.

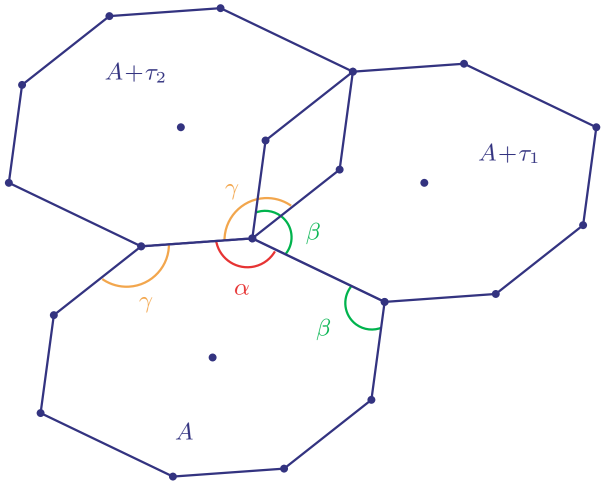

Figure 6.4. If the belt of A generated by the subfacet G has eight or more facets, then the sum of the dihedral angles α, β and γ at the subfacets G, $-G+\tau_1$ and $-G+\tau_2$, respectively, is strictly greater than 2π.

LEMMA 6.5. *Assume that the belt of A generated by the subfacet G has eight or more facets. Then, the set $M:=(A+\tau_1)\cap(A+\tau_2)$ is a convex polytope with non-empty interior, G is a subfacet of M, and N_1^- and N_2^- are the support half-spaces of M at its two facets which meet at the subfacet G (see Figure 6.3).*

Proof. Let α, β and γ denote the dihedral angles of A at the subfacets G, $-G+\tau_1$ and $-G+\tau_2$, respectively. If the belt of A generated by G has eight or more facets, then we must have $\alpha+\beta+\gamma>2\pi$ (see Figure 6.4). On the other hand, each one of α, β and γ is strictly smaller than π. Hence, the hyperplane N_1 divides the dihedral angle of $A+\tau_2$ at the subfacet G into two strictly positive angles $2\pi-\alpha-\beta$ and $\alpha+\beta+\gamma-2\pi$, while the hyperplane N_2 divides the dihedral angle of $A+\tau_1$ at G into two strictly positive angles $2\pi-\alpha-\gamma$ and $\alpha+\beta+\gamma-2\pi$. In particular, the two hyperplanes N_1 and N_2 are not parallel, and thus $N_1\cap N_2=\operatorname{aff}(G)$.

It is clear that

$$M:=(A+\tau_1)\cap(A+\tau_2)$$

is a convex polytope, being the intersection of two convex polytopes. Since F_1 is a facet of $A+\tau_1$ and F_2 is a facets of $A+\tau_2$, then G is a subfacet of both $A+\tau_1$ and $A+\tau_2$. Let E be a closed $(d-2)$-dimensional ball contained in $\operatorname{relint}(G)$. For $\delta>0$ we denote

$$D(E,\delta):=(E+S_\delta)\cap N_1^-\cap N_2^-,$$

where S_δ is a closed 2-dimensional ball of radius δ centered at the origin and orthogonal to $\operatorname{aff}(G)$. Using Lemma 5.4, we obtain that if $\delta=\delta(A,G,E)>0$ is small enough, then

$D(E,\delta)$ is contained both in $A+\tau_1$ and in $A+\tau_2$, and hence $D(E,\delta)\subset M$. It follows that M has non-empty interior, and that N_1 and N_2 are support hyperplanes of M such that the corresponding support sets $M\cap N_1$ and $M\cap N_2$ are $(d-1)$-dimensional, and hence these support sets are facets of M. We conclude that G is a subfacet of M, being the intersection of two adjacent facets $M\cap N_1$ and $M\cap N_2$ of M, and that N_1^- and N_2^- are the support half-spaces of M at its two facets which meet at the subfacet G. $\square$

6.6. We can now finish the proof of Theorem 6.1.

Indeed, suppose to the contrary that the belt of A generated by the subfacet G has eight or more facets. By Lemma 6.5, the set $M:=(A+\tau_1)\cap(A+\tau_2)$ is a convex polytope with non-empty interior, G is a subfacet of M, and N_1^-, N_2^- are the support half-spaces of M at its two facets which meet at the subfacet G. For each $i\in\{1,2\}$, ν_i'' is a finite measure supported on $G-G$, and by Lemma 6.4 we have $\mathbb{1}_G*\nu_i''\geqslant 1$ a.e. with respect to the $(d-2)$-dimensional volume measure on the subfacet G. Then, we can apply Lemma 5.9 to the convex polytope M. It follows from the lemma that if we denote

$$Q_\delta=(G+S_\delta)\cap N_1^-\cap N_2^-,$$

where S_δ is a closed 2-dimensional ball of radius δ centered at the origin and orthogonal to $\operatorname{aff}(G)$, then for any $\eta>0$ we have

$$m\{x\in Q_\delta:(\mathbb{1}_M*\nu_i'')(x)<1-\eta\}=o(m(Q_\delta)),\quad \delta\to 0,\quad i\in\{1,2\}. \tag{6.3}$$

Fix any number $0<\eta<\frac{1}{2}$, and let $D(\delta,\eta)$ denote the set of all points $x\in Q_\delta$ for which

$$(\mathbb{1}_M*\nu_i'')(x)\geqslant 1-\eta$$

for both $i=1$ and $i=2$. The set Q_δ has positive measure, and therefore it follows from (6.3) that if δ is small enough then also $D(\delta,\eta)$ has positive measure. On the other hand,

$$\mathbb{1}_A*\mu\geqslant\mathbb{1}_A*(\mu_1''+\mu_2'')=\mathbb{1}_{A+\tau_1}*\nu_1''+\mathbb{1}_{A+\tau_2}*\nu_2''\geqslant\mathbb{1}_M*(\nu_1''+\nu_2''),$$

where the first inequality is due to the fact that the measures μ_1'' and μ_2'' are obtained by restricting μ, respectively, to the disjoint sets S_1 and S_2 defined in (6.2), while the second inequality holds since M is a subset of both $A+\tau_1$ and $A+\tau_2$. This implies that we have $\mathbb{1}_A*\mu\geqslant 2(1-\eta)>1$ a.e. on $D(\delta,\eta)$. However, this contradicts the weak tiling assumption $\mathbb{1}_A*\mu=\mathbb{1}_{A^\complement}$ a.e. We conclude that the belt of A generated by the subfacet G can have only four or six facets, and this completes the proof of Theorem 6.1. $\square$

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