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Ancient low-entropy flows, mean-convex neighborhoods, and uniqueness

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Contents

1. Introduction

A family of surfaces $M_t \subset \mathbb{R}^3$ moves by mean-curvature flow if the normal velocity at each point is given by the mean-curvature vector,[1]

$$(\partial_t x)^\perp = \mathbf{H}(x), \quad x \in M_t. \tag{1.1}$$

Given any smooth initial surface $M \subset \mathbb{R}^3$, say closed and embedded, by classical theory [Hu1], [HP] there exists a unique smooth solution $\mathcal{M} = \{M_t\}_{t\in[0,T)}$ with initial condition M defined on a maximal time interval $[0, T)$. The first singular time $T < \infty$ is characterized by the fact that the curvature blows up, i.e.

$$\lim_{t \nearrow T} \max_{x \in M_t} |A|(x, t) = \infty,$$

where $|A|$ denotes the norm of the second fundamental form. The main task in the study of mean-curvature flow, both from the theoretical point of view and also in order to facilitate the most striking applications, is then to understand the structure of singularities, to find ways to continue the flow beyond the first singular time, and to analyze its properties.

In the mean-convex case, i.e. when the mean-curvature vector at every point on the surface points inwards, there is a highly developed theory. On the one hand, the flow can be continued smoothly as a surgical solution as constructed by Brendle–Huisken [BH] and Haslhofer–Kleiner [HK2]. This in turn facilitates topological and geometric

[1] The equation of course also makes sense in higher dimension and co-dimension and in other ambient manifolds. In this paper, however, we focus on evolving surfaces in $\mathbb{R}^3$.

applications; see e.g. [BHH], [HKe]. On the other hand, the flow can also be continued uniquely as a weak (generalized) solution. Weak solutions can be described either as level set solutions as in Evans–Spruck [ES] and Chen–Giga–Goto [CGG], or in the framework of geometric measure theory using Brakke solutions [Bra].([2]) By the deep structure theory of White [W3], [W5] (see also [HK1]) the space-time dimension of the singular set is at most one, and all blowup limits are smooth and convex. In fact, by a result of Colding–Minicozzi [CM3], the space-time singular set is contained in finitely many compact embedded Lipschitz curves together with a countable set of point singularities. Moreover, the recent work of Brendle–Choi [BC] and Angenent–Daskalopoulos–Sesum [ADS2] provides a short list of all potential blowup limits (singularity models) in the flow of mean-convex surfaces: the round shrinking sphere, the round shrinking cylinder, the translating bowl soliton [AW], and the ancient ovals [W3], [HH].

In stark contrast to the above, when the initial surface is not mean convex, the theory is much more rudimentary. This is, to some extent, an unavoidable feature of the equation. In particular, as already pointed out in the pioneering work of Brakke [Bra] and Evans–Spruck [ES], there is the phenomenon of non-uniqueness or fattening. Angenent–Ilmanen–Chopp [AIC] and Ilmanen–White [W4] gave examples of smooth embedded surfaces $M \subset \mathbb{R}^3$ whose level set flow $F_t(M)$ develops a non-empty interior at some positive time. In particular, $F_t(M)$ does not look at all like a 2-dimensional evolving surface. These examples also illustrate, in a striking way, the non-uniqueness of (enhanced) Brakke flows.

In the present paper, we make some progress towards decreasing the gap between the theory in the mean-convex case and the theory in the general case without curvature assumptions. Most importantly, in Theorem 1.7 (see also Theorem 1.6), we prove the mean-convex neighborhood conjecture. As an application, we prove the non-fattening conjecture for mean-curvature flow through cylindrical or spherical singularities (see Theorem 1.9). In particular, assuming Ilmanen's multiplicity-1 conjecture, we conclude that for embedded 2-spheres the mean-curvature flow through singularities is well-posed (see Theorem 1.12). The mean-convex neighborhood conjecture is in turn a consequence of a general classification result (Theorem 1.2) for ancient low-entropy flows that arise as potential limit flows near spherical or cylindrical singularities.

The rest of this introduction is organized as follows. In §1.1, we describe our general classification result for ancient low-entropy flows. In §1.2, we state our results establishing the mean-convex neighborhood conjecture and the uniqueness conjectures. In §1.3, we give an outline of the proofs.

([2]) In the mean-convex setting these notions of weak solutions are essentially equivalent [W3], [MS].

1.1. Classification of ancient low-entropy flows

Let $M \subset \mathbb{R}^3$ be a surface. The entropy, introduced by Colding–Minicozzi [CM1], is defined as the supremum of the Gaussian area over all centers and all scales, namely

$$\mathrm{Ent}[M] = \sup_{\substack{y \in \mathbb{R}^3 \\ \lambda > 0}} \int_M \frac{1}{4\pi\lambda} e^{-|x-y|^2/4\lambda} \, dA(x). \tag{1.2}$$

The entropy measures, in a certain sense, the complexity of the surface. For example, the values for a plane, sphere and cylinder are

$$\mathrm{Ent}[\mathbb{R}^2] = 1, \quad \mathrm{Ent}[\mathbb{S}^2] = \frac{4}{e} \sim 1.47, \quad \text{and} \quad \mathrm{Ent}[\mathbb{S}^1 \times \mathbb{R}] = \sqrt{\frac{2\pi}{e}} \sim 1.52.$$

If $\mathcal{M} = \{M_t\}_{t \in I}$ evolves by mean-curvature flow, then $t \mapsto \mathrm{Ent}[M_t]$ is non-increasing by Husiken's monotonicity formula [Hu2], and hence

$$\mathrm{Ent}[\mathcal{M}] := \sup_{t \in I} \mathrm{Ent}[M_t] = \lim_{t \to \inf(I)} \mathrm{Ent}[M_t].$$

For example, the entropy of a flat plane $\mathcal{P}$, a round shrinking sphere $\mathcal{S}$, a round shrinking cylinder $\mathcal{Z}$, a translating bowl soliton $\mathcal{B}$, and an ancient oval $\mathcal{O}$, are given by

$$\mathrm{Ent}[\mathcal{P}] = 1, \quad \mathrm{Ent}[\mathcal{S}] = \frac{4}{e} \sim 1.47$$
$$\mathrm{Ent}[\mathcal{Z}] = \mathrm{Ent}[\mathcal{B}] = \mathrm{Ent}[\mathcal{O}] = \sqrt{\frac{2\pi}{e}} \sim 1.52.$$

By a beautiful classification result of Bernstein–Wang [BW] any self-similarly shrinking mean-curvature flow in $\mathbb{R}^3$ with entropy at most $\sqrt{2\pi/e}$ is either a flat plane, round shrinking sphere, or round shrinking cylinder. By a recent uniqueness result of Hershkovits [He] any self-similarly translating flow in $\mathbb{R}^3$ with entropy at most $\sqrt{2\pi/e}$ is a bowl soliton. These results in turn build on a pioneering paper by Colding–Ilmanen–Minicozzi–White [CIMW] that was the genesis of the study of low-entropy flows.

For the study of singularities, it is important that these concepts are also available in the non-smooth setting. Singular surfaces in Euclidean space are described most easily by 2-rectifiable Radon measures, which generalize the area measure of smooth 2-dimensional surfaces; see e.g. [Si], [I1]. Recall that a 2-rectifiable Radon measure μ in $\mathbb{R}^3$ is a Radon measure that has a 2-dimensional tangent plane of positive multiplicity at almost every point. The entropy of μ is defined as

$$\mathrm{Ent}[\mu] = \sup_{y \in \mathbb{R}^3, \lambda > 0} \int \frac{1}{4\pi\lambda} e^{-|x-y|^2/4\lambda} \, d\mu(x).$$

As in [Bra], [I1] we consider Brakke flows $\mathcal{M}=\{\mu_t\}_{t\in I}$ that are given by a family of Radon measures in $\mathbb{R}^3$ that is 2-rectifiable for almost all times and satisfies

$$\frac{d}{dt}\int \varphi\, d\mu_t \leqslant \int (-\varphi \mathbf{H}^2 + (\nabla\varphi)^{\perp}\cdot\mathbf{H})\, d\mu_t \tag{1.3}$$

for all non-negative test functions φ; see §2.1 for details. The class of all Brakke flows, as originally defined in [Bra], is too large for most practical purposes. For example, by the very nature of the definition via the inequality (1.3), Brakke flows can suddenly vanish without any cause. Also, given that the initial surface is embedded, it is useful to keep track of its inside and outside. For these and other reasons the definition of Brakke flows has been refined over the years by Ilmanen [I1], White [W6], [W7], and their collaborators. For our purpose it is most appropriate to consider only Brakke flows that are integral (which prevents non-integer multiplicity), unit-regular (which to a certain extent prevents sudden vanishing), and cyclic (which partly keeps track of the inside and outside); see §2.1 for definitions and details. In particular, all Brakke flows starting at any closed embedded surface $M\subset\mathbb{R}^3$ that are constructed via Ilmanen's elliptic regularization [I1] are integral, unit regular, and cyclic, and all these properties are preserved under passing to limits of sequences of Brakke flows. The entropy of a Brakke flow is defined by

$$\mathrm{Ent}[\mathcal{M}] = \sup_{t\in I} \mathrm{Ent}[\mu_t].$$

Finally, let us recall that an *ancient* solution of a parabolic PDE, such as the mean-curvature flow, is a solution that is defined for all $t\in(-\infty, T)$, where $T\leqslant\infty$. Ancient solutions are on the one hand some of the most interesting solutions by themselves, and on the other hand crucial for the analysis of singularities. In particular, every blowup limit is an ancient solution. A systematic study of ancient solutions of geometric flows has been pursued over the last decade by Daskalopoulos, Sesum, and their collaborators; see [D] for a nice overview of the main problems and key results.

We consider the following class of Brakke flows.

Definition 1.1. (Ancient low-entropy flows) The class of *ancient low-entropy flows* consists of all ancient, unit-regular, cyclic, integral Brakke flows

$$\mathcal{M} = \{\mu_t\}_{t\in(-\infty, T_E(\mathcal{M})]}$$

in $\mathbb{R}^3$ with

$$\mathrm{Ent}[\mathcal{M}] \leqslant \sqrt{\frac{2\pi}{e}},$$

where $T_E(\mathcal{M})\leqslant\infty$ denotes the extinction time.

Our main theorem provides a complete classification.

THEOREM 1.2. (Classification of ancient low-entropy flows) *Any ancient low-entropy flow in* $\mathbb{R}^3$ *is either*

- *a flat plane, or*
- *a round shrinking sphere, or*
- *a round shrinking cylinder, or*
- *a translating bowl soliton, or*
- *an ancient oval.*

To explain the scope of our main classification result (Theorem 1.2), let us start by pointing out that the assumptions are essentially sharp.([3]) If one removes the assumption of being unit-regular the flow can suddenly disappear, e.g. one could have a solution that looks like a translating bowl soliton until some time, but is not eternal. If one removes the assumption of being cyclic, then one can have a static or quasistatic configuration of three half-planes meeting at 120-degree angles, which have entropy $\frac{3}{2}$. If one increases the entropy too much, then one can get other ancient solutions, e.g. the self-similarly shrinking Angenent torus [An2] has entropy bigger than $\sqrt{2\pi/e}$ but less than 2. Finally, removing the integer multipicity assumption would give rise to a zoo of solutions, in particular the infinite collection of shrinkers constructed by Kapouleas–Kleene–Moller and Nguyen [KKM], [N].

Let us discuss some of the most important prior classification results for solutions of the mean-curvature flow of surfaces. Wang proved uniqueness of the bowl soliton among entire convex translating graphs [WXJ]. This was improved recently by Spruck–Xiao [SX], who showed that every entire mean-convex translating graph is in fact convex.([4]) Without assuming that the translator is graphical, Martin, Savas-Halilaj, and Smoczyk proved a uniqueness result for the bowl soliton under rather strong asymptotic assumptions [MSHS]. This was improved by Haslhofer [Has], who showed uniqueness of the bowl among translators that are convex and non-collapsed, and more recently by Hershkovits [He], who proved uniqueness among translators, not necessarily convex, that are asymptotic to a cylinder. Most closely related to the present article are the recent result by Brendle–Choi [BC], which proves uniqueness of the bowl among non-compact ancient solutions that are non-collapsed and convex, and the recent result by Angenent–Daskalopoulos–Sesum [ADS2], which proves uniqueness of the ancient ovals among compact ancient solutions that are non-collapsed and convex. Other highly important results that play a direct role in the present paper are the classification of low-entropy shrinkers by Bernstein–Wang [BW], and the classification of genus-zero shrinkers by Brendle [Bre2].

([3]) The entropy assumption can be relaxed; see Corollary 1.4 below.

([4]) In the setting of non-collapsed solutions it was known before that every ancient non-collapsed mean-convex solution of the mean-curvature flow is in fact convex [HK1].

All prior classification results assume either convexity or self-similarity (or both), and the story is similar for other equations such as the Ricci flow; see e.g. [Bre1]. Our classification result seems to be the first one which assumes neither self-similarity nor convexity. To wit, convexity is not an assumption but a consequence:

COROLLARY 1.3. (Convexity) *Any ancient low-entropy flow in* $\mathbb{R}^3$ *is convex.*

More generally, a key feature of our classification theorem (Theorem 1.2) is that besides the entropy bound we assume almost nothing, essentially only that the solution is ancient. In particular, a priori, an ancient low-entropy flow could be quite singular and could have pathological behaviour caused by spatial infinity, e.g. there could be topological changes caused by "neckpinches at infinity", or there could be "contracting cusps" (see e.g. [T]), or the flow could "escape to spatial infinity" (see e.g. [IW]).

The entropy assumption in our main classification result (Theorem 1.2) can be relaxed. To state the sharp result, let

$$\Upsilon := \inf\{\mathrm{Ent}[S] : S \text{ is a smooth embedded shrinker in } \mathbb{R}^3, \\ \text{which is not a plane, sphere of cylinder}\}.$$

By Bernstein–Wang [BW], the difference $\delta := \Upsilon - \sqrt{2\pi/e} > 0$ is strictly positive. Theorem 1.2 immediately implies the following corollary.

COROLLARY 1.4. (Sharp ancient low-entropy classification) *Any ancient, cyclic, unit-regular, integral Brakke flow* $\mathcal{M}$ *in* $\mathbb{R}^3$ *with*

$$\mathrm{Ent}[\mathcal{M}] < \Upsilon$$

is either a flat plane, a round shrinking sphere, a round shrinking cylinder, a translating bowl soliton, or an ancient oval.

The proof of Theorem 1.2 involves, not surprisingly, many steps. For an overview of the main steps and ideas, please see §1.3.

1.2. Mean-convex neighborhoods and uniqueness

The main conjecture towards reducing the gap between the theory in the mean-convex case and the general case, is the following.

Conjecture 1.5. (Mean-convex neighborhood conjecture([5])) If the mean-curvature flow of closed embedded surfaces has a spherical or cylindrical singularity at (x, t), then there is a space-time neighborhood of (x, t) in which the flow is mean convex.

([5]) See e.g. Problem 4 on Ilmanen's problem list [I4], Conjecture 10.2 in the survey of Colding–Minicozzi–Pedersen [CMP], and the paragraph between Theorem 3.5 and Remark 3.6 in Hershkovits–White [HW].

Let us explain some background. Let $\mathcal{M}=\{M_t\}$ be a Brakke flow starting at a closed embedded surface $M\subset\mathbb{R}^3$, say constructed via Ilmanen's elliptic regularization [I1]. Given any space-time point $X=(x,t)\in\mathcal{M}$ and any factor $\lambda>0$, we denote by $\mathcal{M}_{X,\lambda}$ the Brakke flow which is obtained from $\mathcal{M}$ by translating X to the space-time origin and parabolically rescaling by λ^{-1}. By Huisken's monotonicity formula [Hu2] and Ilmanen's compactness theorem [I1], for any sequence $\lambda_i\to 0$, one can find a subsequence $\lambda_{i'}$ such that $\mathcal{M}_{X,\lambda_{i'}}$ converges in the sense of Brakke flows to a limit $\widehat{\mathcal{M}}_X$, called a tangent flow at X, and any tangent flow is backwardly self-similar. The only backwardly self-similar solutions of positive mean curvature are the round shrinking sphere and the round shrinking cylinder [Hu3], [W5], [CM1]. One precise formulation of the conjecture says that if some (and thus every [CIM]) tangent flow $\widehat{\mathcal{M}}_X$ is either a round shrinking sphere or a round shrinking cylinder of multiplicity 1, then there exists some $\varepsilon=\varepsilon(X)>0$ such that the flow $\mathcal{M}$ is, possibly after flipping the orientation, mean convex in the parabolic ball

$$P(X,\varepsilon)=\{(x',t')\in\mathbb{R}^3\times\mathbb{R}:|x'-x|<\varepsilon \text{ and } t-\varepsilon^2<t'\leqslant t\}.$$

The major challenge in proving the mean-convex neighborhood conjecture is to upgrade the infinitesimal information about the mean-convex tangent flow to a conclusion at some macroscopic scale $\varepsilon>0$. Specifically, there are certain singularities that are not fully captured by just looking at the tangent flows. Indeed, it has been clear since the work of White [W3], [W5], and reinforced in the work of Perelman [P1], [P2], Bamler–Kleiner [BK] and Hershkovits–White [HW], that in order to fully capture all singularities and regions of high curvature one needs to understand all limit flows and not just all tangent flows. Given $X\in\mathcal{M}$, a limit flow is any subsequential limit of $\mathcal{M}_{X_i,\lambda_i}$, where $X_i\to X$ and $\lambda_i\to 0$. For example, in the case of the degenerate neckpinch [AV], one needs to carefully choose a varying sequence of space-time points X_i that "follow the tip", in order to see the translating bowl soliton in the limit; see [Ham]. Potentially there could be very complex limit flows that are not self-similar; see e.g. [I4, Problem 6] or [W5, Conjecture 3].

The mean-convex neighborhood conjecture (Conjecture 1.5) has been known to be true in some important special cases. In the rotationally symmetric setting this is by the attracting axis theorem of Altschuler–Angenent–Giga [AAG]. More recently, Colding–Ilmanen–Minicozzi [CIM] proved the conjecture in the case of self-similarly shrinking limit flows, and Hershkovits [He] proved the conjecture in the case of self-similarly translating limit flows. Without assuming self-similarity, in a very interesting recent paper based on hard PDE techniques, Gang Zhou [G] verified the mean-convex neighborhood conjecture in certain regimes. However, these regimes exclude by assumption some of the main challenges of the conjecture, e.g. the potential scenario of a degenerate neckpinch with a

non-convex cap.

In Theorem 1.7, we establish the mean-convex neighborhood conjecture for the mean-curvature flow of surfaces in $\mathbb{R}^3$ at all times.([6]) Since it is somewhat easier to state, let us first give the result at the first singular time.

THEOREM 1.6. (Mean-convex neighborhoods at the first singular time) *Let*

$$\mathcal{M} = \{M_t\}_{t\in[0,T)}$$

be a mean-curvature flow of closed embedded surfaces in $\mathbb{R}^3$, *where* T *is the first singular time. If some tangent flow at* $X=(x,T)$ *is a sphere or cylinder with multilicity one, then there exists an* $\varepsilon=\varepsilon(X)>0$ *such that, possibly after flipping the orientation, the flow* $\mathcal{M}$ *is mean convex in the parabolic ball* $P(X,\varepsilon)$. *Moreover, any limit flow at* X *is either a round shrinking sphere, a round shrinking cylinder, or a translating bowl soliton.*([7])

Theorem 1.6 follows, via a short argument by contradiction, from our classification of ancient low-entropy flows (Theorem 1.2).

Before stating the solution of the mean-convex neighborhoods conjecture at subsequent singularities, let us explain better what it actually means to continue the mean-curvature flow through singularities. We start by recalling some facts that are explained in more detail in [I3], [HW]. For any closed set $K\subset\mathbb{R}^3$, its level set flow $F_t(K)$ is the maximal family of closed sets starting at K that satisfies the avoidance principle. Now, given any closed embedded surface $M\subset\mathbb{R}^3$ there are at least three quite reasonable ways to evolve it through singularities, namely the *level set flow* $F_t(M)$, the *outer flow* M_t and the *inner flow* M'_t. The latter two are defined as follows. Let K be the compact domain bounded by M, and let $K':=\overline{\mathbb{R}^3\setminus K}$. Denote the corresponding level set flows by

$$K_t = F_t(K) \quad \text{and} \quad K'_t = F_t(K').$$

Let $\mathcal{K}$ and $\mathcal{K}'$ be their space-time tracks, namely

$$\mathcal{K} = \{(x,t)\in\mathbb{R}^3\times\mathbb{R}_+ : x\in K_t\},$$
$$\mathcal{K}' = \{(x,t)\in\mathbb{R}^3\times\mathbb{R}_+ : x\in K'_t\}.$$

The outer flow and inner flow are then defined by

$$M_t = \{x\in\mathbb{R}^3 : (x,t)\in\partial\mathcal{K}\},$$
$$M'_t = \{x\in\mathbb{R}^3 : (x,t)\in\partial\mathcal{K}'\}.$$

([6]) Of course our results carry over immediately to other ambient manifolds.

([7]) It is easy to see that ancient ovals cannot arise as limit flows at the first singular time. It is unknown whether or not ancient ovals can arise as limit flows at subsequent singularities. This is related to potential accumulations of neckpinches; see e.g. [CM4].

As long as the evolution is smooth, all three flows are of course the same.

Let T be the first singular time, and for $x \in M_T$ and $\lambda > 0$, denote by $\mathcal{K}_{X,\lambda}$ and $\mathcal{K}'_{X,\lambda}$ the flows that are obained from $\mathcal{K}$ and $\mathcal{K}'$, respectively, by shifting $X = (x, T)$ to the origin, and parabolically rescaling by λ^{-1}. Observe that the assumption of Theorem 1.6 is equivalent to the assumption that for $\lambda \to 0$ either $\mathcal{K}_{X,\lambda}$ or $\mathcal{K}'_{X,\lambda}$ converges smoothly with multiplicity 1 to a round shrinking solid cylinder or round shrinking solid ball. Also observe that the conclusion of Theorem 1.6 is equivalent to the assertion that either

$$K_{t_2} \cap B(x, \varepsilon) \subseteq K_{t_1} \setminus M_{t_1} \quad \text{or} \quad K'_{t_2} \cap B(x, \varepsilon) \subseteq K'_{t_1} \setminus M'_{t_1}$$

for all $T - \varepsilon^2 < t_1 < t_2 \leqslant T$. This reformulation has the advantage that it generalizes not just beyond the first singular time, but even beyond any potential fattening-time (see equation (1.4) below). We prove the following.

THEOREM 1.7. (Mean-convex neighborhoods at all times) *Assume that $X = (x, t)$ is a space-time point such that $\mathcal{K}_{X,\lambda}$ converges for $\lambda \to 0$ smoothly with multiplicity 1 to a round shrinking solid cylinder or a round shrinking solid ball. Then, there exists an $\varepsilon = \varepsilon(X) > 0$ such that*

$$K_{t_2} \cap B(x, \varepsilon) \subseteq K_{t_1} \setminus M_{t_1}$$

for all $T - \varepsilon^2 < t_1 < t_2 < T + \varepsilon^2$. Similarly, if $\mathcal{K}'_{X,\lambda}$ converges for $\lambda \to 0$ smoothly with multiplicity 1 to a round shrinking solid cylinder or a round shrinking solid ball, then for some $\varepsilon = \varepsilon(X) > 0$ we have

$$K'_{t_2} \cap B(x, \varepsilon) \subseteq K'_{t_1} \setminus M'_{t_1}$$

for all $T - \varepsilon^2 < t_1 < t_2 < T + \varepsilon^2$. Furthermore, in both cases, any limit flow at X is either a round shrinking sphere, a round shrinking cylinder, a translating bowl soliton, or an ancient oval.

Note that Theorem 1.7 establishes the strongest version of the mean-convex neighborhood conjecture, where the parabolic neighborhood is backward and forward in time. The proof of Theorem 1.7 is again based on our classification of ancient low-entropy flows (Theorem 1.2). During the course of the proof, we also construct a unit-regular, cyclic, integral Brakke flow whose support is M_t or M'_t, respectively, and which has density less than 2 in $P(X, \delta)$, for some $\delta > 0$. This highlights another advantage of our formulation of Theorem 1.7: its applicability even without assuming a priori that a Brakke flow starting from M has multiplicity 1 almost everywhere.

Let us now discuss the applications concerning uniqueness of mean-curvature flow through singularities. We recall from Ilmanen–White [W4] that there are examples of singularities with curvature of mixed signs, which cause non-uniqueness, i.e. fattening.

The main conjecture regarding uniqueness at cylindrical or spherical singularities can be phrased as follows.

Conjecture 1.8. (Non-fattening conjecture([8])) If all singularities of the level set flow are either cylindrical or spherical then the level set flow does not fatten.

We recall that the *fattening time* is defined by

$$T_{\text{fat}} = \inf\{t > 0 : F_t(M) \text{ has non-empty interior}\}, \tag{1.4}$$

and the *discrepancy time* is defined by

$$T_{\text{disc}} = \inf\{t > 0 : M_t,\ M_t', \text{ and } F_t(M) \text{ are not all equal}\},$$

see [HW]. It follows directly from the definitions that

$$T_{\text{fat}} \geqslant T_{\text{disc}}.$$

It is unknown whether or not $T_{\text{fat}} = T_{\text{disc}}$ holds; see [HW, Example 2.5]. In any case, if one proves the (potentially) stronger result that there is no discrepancy, then one obtains non-fattening as a consequence. We prove:

THEOREM 1.9. (Non-fattening) *Suppose that* $0 < T \leqslant T_{disc}$*, and suppose that all the backward singularities of the outer flow* $\{M_t\}$ *at time* T *are cylindrical or spherical. Then* $T < T_{disc}$*. In particular, the level set flow does not fatten as long as all singularities are cylindrical or spherical.*

Theorem 1.9 follows immediately by combining the main theorem of Hershkovits–White [HW], which establishes non-discrepancy assuming the existence of mean-convex neighborhoods around all singularities a priori, and Theorem 1.7, which proves the existence of such mean-convex neighborhoods.

Our final application in this paper concerns the well-posedness problem for the mean-curvature flow of embedded 2-spheres through singularities. The main conjecture in this regard is the following.

Conjecture 1.10. (Evolution of embedded 2-spheres([9])) The mean-curvature flow of embedded 2-spheres through singularities is unique.

([8]) The conjecture has been indicated first in White's ICM lecture [W4]; see also the discussion in the recent work of Hershkovits–White [HW] (indeed, the non-fattening conjecture is a combination of [HW, Conjecture 1.2] and the "very interesting open problem" described in the paragraph between Theorem 3.5 and Remark 3.6 in [HW]).

([9]) See e.g. White's ICM lecture [W4], and the introduction of Bamler–Kleiner [BK].

One reason why Conjecture 1.10 is highly intriguing, is that the mean-curvature flow of embedded 2-spheres can be viewed as (somewhat more approachable) cousin of the Ricci flow of 4-spheres, whose analysis seems to be out of reach with current technology.([10]) A solution of Conjecture 1.10 would also facilitate several topological and geometric applications. E.g. there is a proposal of Yau for a flow proof of the Smale conjecture [R] (see also the work of Bamler–Kleiner [BK] and Buzano–Haslhofer–Hershkovits [BHH]). Moreover, a well-posed flow of embedded-2 spheres would also be useful for the Lusternik-Schnirelman problem of finding four embedded minimal 2-spheres in 3-spheres equipped with an arbitrary Riemannian metric (the first such minimal 2-spheres has been found by Simon-Smith [Sm] in 1983, and a second one has been found recently by Haslhofer-Ketover [HKe] using combined efforts of mean-curvature flow and min-max theory).

Some very important recent progress towards Conjecture 1.10 has been made by Brendle [Bre2], who proved that the only non-trivial shrinkers of genus zero are the round shrinking sphere and the round shrinking cylinder. Combining Brendle's uniqueness result with our solution of the non-fattening conjecture (Theorem 1.9), we can prove Conjecture 1.10 modulo (a special case of) Ilmanen's multiplicity-1 conjecture.

Conjecture 1.11. (Multiplicity-1 conjecture for 2-spheres([11])) For mean-curvature flow starting at any closed embedded 2-sphere M,([12]) all tangent flows have multiplicity 1.

THEOREM 1.12. (Evolution of embedded 2-spheres) *Assuming the multiplicity-1 conjecture* (*Conjecture* 1.11), *mean-curvature flow of embedded 2-spheres through singularities is unique.*

Let us conclude this section by sketching a heuristic picture for the mean-curvature flow of embedded surfaces of higher genus: By Ilmanen's strict genus-reduction conjecture [I4, Conjecture 13] one expects only a controlled number (bounded by the genus of the initial surface) of singularities modelled on asymptotically conical shrinkers of higher genus; see in particular the important work of Brendle [Bre2] and Wang [WL2]. At these

([10]) When comparing mean-curvature flow and Ricci flow one has to multiply the dimension by 2. For example, convergence to a round limit holds for curve-shortening flow and 2-dimensional Ricci flow, respectively. Also, Huisken's monotonicity formula combined with Gauss–Bonnet gives local bounds for the L^2-norm of the second fundamental form, see [I2], which is scale invariant in dimension 2, while on the other hand Perelman's monotonicity formula combined with Gauss–Bonnet gives local bounds for the L^2-norm of the Riemann tensor, see [HM], which is scale invariant in dimension 4.

([11]) Multiplicity-1 was called an hypothesis in Brakke's pioneering monograph [Bra], and promoted to a conjecture in Ilmanen's paper [I2], where he proved that for surfaces one can always find smooth tangent flows; see also the approach by Ecker [E].

([12]) To be concrete, we mean by that a unit-regular, cyclic, integral Brakke flow, whose support is the outer flow, and whose initial data is $\mathcal{H}^n \lfloor M$. A posteriori, this will turn out to imply that the (non-fattening) matching motion has only multiplicity-1 tangents.

singularities the evolution can be non-unique, and one has to decide how to flow out of the conical singularities, either by hand or via stability or via a stochastic selection principle. Our solution of the non-fattening conjecture (Theorem 1.9) suggests that the evolution is determined completely by these finitely many choices.

1.3. Outline of the proofs

For the outline of the proof of the classification theorem for ancient low-entropy flows (Theorem 1.2), let $\mathcal{M}$ be an ancient low-entropy flow (see Definition 1.1) which is not a flat plane, round shrinking sphere, or round shrinking cylinder. The task is to prove that $\mathcal{M}$ is either a translating bowl soliton or an ancient oval.

In §3, we start by establishing several coarse properties:

First, we prove a partial regularity result (Theorem 3.2), which shows that $\mathcal{M}$ is smooth except for at most countably many spherical singularities. This is mostly based on ideas from Bernstein–Wang [BW], and uses the assumptions that our Brakke flow is integral, unit-regular and cyclic.

Next, given any space-time point $X=(x,t)\in\mathcal{M}$ we consider the flow on backwards parabolic annuli around X with radii $r_j=2^j$. We prove a rough neck theorem (Theorem 3.5), which shows that there exists a scale $Z(X)=2^{J(X)}$, the *cylindrical scale*, at which the flow starts to look ε-cylindrical, and a controlled number $N=N(\varepsilon)<\infty$, such that the flow looks ε-cylindrical at all scales $j\geqslant J(X)+N$. The proof uses the Bernstein–Wang classification of low-entropy shrinkers [BW], Huisken's monotonicity formula [Hu2], and ideas from quantitative differentiation; see e.g. [CHN].

Third, we prove the vanishing asymptotic slope theorem (Theorem 3.9), which shows that at spatial infinity, the surfaces M_t open up slower than any cone of positive angle. The proof is based on a blowdown argument, which shows that the cylindrical scale grows sublinearly. The result facilitates barrier arguments in later sections.

In §4, which is the longest section of the proof, we carry out a fine neck analysis. Given any point $X=(x_0,t_0)\in\mathcal{M}$, we consider the rescaled flow

$$\overline{M}_\tau^X = e^{\tau/2}(M_{-e^{-\tau}}-x_0),$$

where $\tau=-\log(t_0-t)$. The surfaces $\overline{M}_\tau^X$ can be written as the graph of a function $u_X(\cdot,\tau)$ with small norm over a cylindrical domain of length $\rho(\tau)$, where $\rho(\tau)\to\infty$ as $\tau\to-\infty$. By Colding–Minicozzi [CM2], we may assume that the axis of each cylinder is in z-direction. The goal is to derive very sharp asymptotic estimates for the function $u_X(\cdot,\tau)$.

To get started, in §4.1 we set up the fine neck analysis similarly as in Angenent–Daskalopoulos–Sesum [ADS1] and Brendle–Choi [BC]. The analysis is governed by the linear operator

$$\mathcal{L} = \Delta - \tfrac{1}{2} x^{\tan} \cdot \nabla + 1$$

on the round cylinder. The $\mathcal{L}$-operator already played a fundamental role in the work on generic mean-curvature flow by Colding–Minicozzi [CM1]. It has four positive eigenfunctions $(1, z, \sin\theta, \cos\theta)$, three zero-eigenfunctions $(z^2-2, z\cos\theta, z\sin\theta)$, and countably many negative eigenfunctions. Using the ODE-lemma from Merle–Zaag [MZ], we see that for $\tau \to -\infty$ either the plus mode is dominant or the neutral mode is dominant.

In §4.2, we analyze the case where the plus mode is dominant. Our key result in that section is the fine neck theorem (Theorem 4.15). It says that there exists a constant $a = a(\mathcal{M}) \neq 0$, independent of the base point X, such that, after re-centering, we have the asymptotic expansion

$$u_X(x,\tau) = a z e^{\tau/2} + o(e^{\tau/2}), \tag{1.5}$$

whenever $|x| \leqslant 100$ and $\tau \ll \log Z(X)$. Assuming $a = 1/\sqrt{2}$ without loss of generality, the estimate (1.5) says that the area of the cross sections of the fine necks increases at unit rate if one moves in positive z-direction.([13]) To prove the fine neck theorem, we first use barrier arguments and the results from §3 to show that the cylindrical radius $\rho(\tau)$ grows exponentially as $\tau \to -\infty$. We then project onto the plus mode (after multiplying by a cutoff function) and consider the evolution equations for the coefficients of

$$P_+ \hat{u}_X = a_X z + b_X \cos\theta + c_X \sin\theta + d_X.$$

Carefully analyzing these evolution equations and estimating all the error terms, after re-centering, we obtain the estimate (1.5).

In §4.3, we analyze the case where the neutral mode is dominant. Our main result in that section is the inwards quadratic neck theorem (Theorem 4.28), which gives an inwards quadratic bending of the central neck. The result is related to the main result of Angenent–Daskalopoulos–Sesum [ADS1], but we assume neither convexity nor rotational symmetry. Roughly speaking, the method of our proof of the inwards quadratic neck theorem is as follows. We consider the expansion

$$\hat{u}_X = \alpha_1 (z^2 - 2) + \alpha_2 z \cos\theta + \alpha_3 z \sin\theta + o(|\vec{\alpha}|).$$

We show that α_2 and α_3 are rapidly decaying, by using the Lojasiewicz inequality from Colding–Minicozzi and the Brendle–Choi neck improvement theorem. We then derive a

([13]) It is an instructive exercise to check that the translating bowl soliton indeed satisfies the fine neck estimate, where a is proportional to the reciprocal of the speed of the translating soliton.

differential inequality for α_1. Analyzing this differential inequality we show that there is some positive constant $A>0$ such that for $\tau\to-\infty$ we have

$$\alpha_1(\tau)=\frac{-A+o(1)}{|\tau|}.$$

The proof is quite technical, since we have to estimate the optimal graphical radius, estimate the error terms, and analyze the differential inequality, and all those steps are interrelated. Finally, combining the inwards quadratic neck theorem with a barrier argument, we see that in the neutral mode case the solution must be compact (Corollary 4.30).

In §5, we assume that the plus mode is dominant. In §5.1, we prove a global curvature estimate (Theorem 5.6), which says that the flow is eternal with uniformly bounded curvature. In particular, this rules out potential pathologies caused by spatial infinity. Roughly speaking, the idea is that by the fine neck estimate (1.5), the necks open up at spatial infinity, and thus the curvature decays to zero at spatial infinity. The actual proof is somewhat more involved, since we have to relate the cylindrical scale and the regularity scale. This gives a curvature bound on compact time intervals. Together with a local type-I estimate, which follows from a simple blowdown argument, this can be upgraded to a global curvature bound. In particular, the solution is eternal.

In §5.2, we estimate the cap size and analyze the asymptotics. To begin with, we consider the "height of the tip" function

$$\psi(t):=\inf_{x\in M_t} x_3.$$

This is a strictly increasing function and the infimum is attained at some point $p_t\in M_t$. In the cap size control theorem (Theorem 5.9), we prove that there is a uniform constant $C=C(\mathcal{M})<\infty$ such that every point in $M_t\setminus B_C(p_t)$ lies on a fine neck. We also show that $M_t\setminus B_C(p_t)$ is the graph of a function r in cylindrical coordinates around the z-axis satisfying

$$r(t,z,\theta)=\sqrt{2(z-\psi(t))}+o(\sqrt{z-\psi(t)}), \tag{1.6}$$

and that the height of the tip function ψ satisfies

$$\psi(t)=t+o(|t|).$$

Roughly speaking, the idea is that the fine neck estimate (1.5) can be integrated to get the shape of a parabola. In order to do this, we first prove, via a blowdown argument, that the tip does not slow down too much, which gives us enough "fast points" in space-time around which the fine neck estimate can be applied.

In §6, we still assume that the plus mode is dominant.

First, in the fine asymptotics theorem (Theorem 6.6) we prove that, after shifting $\mathcal{M}$ in space-time, the function r from (1.6) becomes rotationally symmetric at a very fast rate, namely

$$|\partial_\theta r| = O(1/r^{100})$$

uniformly in time. The proof is based on the Brendle–Choi neck improvement theorem [BC] and some careful estimates controlling how the fine necks align with each other. In particular, the estimate uniformly bounds the motion of the tip in the xy-plane. Combining the fine asymptotics theorem with a parabolic version of the moving plane method, we show that the solution is rotationally symmetric (Theorem 6.14).

In §7, we complete the classification of ancient low-entropy flows.

In Theorem 7.1, we show that if the plus mode is dominant, then the solution must be a bowl soliton. To this end, we analyze the rotationally symmetric solution from §6. Using elementary geometric arguments, we show that the height function $z: M_t \to \mathbb{R}$ does not have local maxima, and similarly that the radius function $r = r(z,t)$ does not have local maxima. Thus, the function $f = \langle \nu, e_3 \rangle$ is positive. Together with the fine asymptotic theorem and the maximum principle, we conclude that $H = f$. Hence, the solution is a mean-convex non-collapsed translating soliton, which by the uniqueness result for translators from Haslhofer [Has] yields that the solution is a bowl soliton.

Finally, in Theorem 7.3 we show that if the neutral mode is dominant, then the solution must be an ancient oval. The idea is that, via blowup around the tips, we can find eternal low-entropy flows, which by the above must be bowl solitons. We also have a central cylinder. Using the maximum principle, we show that the solution is mean-convex and non-collapsed also in the region in between. Using the result of Angenent–Daskalopoulos–Sesum [ADS2], we can then conclude that the solution is an ancient oval.

Finally, let us outline the proof of the mean-convex neighborhood conjecture (Theorem 1.7). Under the assumptions of Theorem 1.7, we first construct a unit-regular, cyclic, integral Brakke flow $\mathcal{M} = \{\mu_t\}_{t \geqslant t_0 - \delta}$ whose support is equal to the outer flow $\{M_t\}_{t \geqslant t_0 - \delta}$. Moreover, we can arrange that $\mathcal{M}$ has a multiplicity-1 cylindrical tangent flow at X_0, and that $\mathcal{M}$ is smooth in $P(X_0, \delta)$ at almost every time, with only spherical and cylindrical singularities. The latter properties follow from upper semicontinuity of Huisken's density and the basic regularity theory from §3.

We then show that, possibly after decreasing δ, the mean curvature does not vanish at any regular point in $P(X_0, \delta)$. If this did not hold, then we could pass to a limit flow which on the one hand had a point with vanishing mean curvature, but on the other hand had strictly positive mean curvature by our classification result of ancient low-entropy flows (Theorem 1.2).

Next, we show that, after decreasing δ again, the space-time connected component

that reaches X_0 has positive mean curvature at all regular points. The proof is based on a continuity argument in time and uses that the singular set is small.

Finally, arguing as in [HW] we show that the mean curvature is positive also on the non-smooth points, in the sense that the surface moves everywhere strictly inwards. This concludes the outline of the proof.

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2. Preliminaries

2.1. Brakke flows

As in Ilmanen [I1, Definitions 6.2 and 6.3] a 2-dimensional *Brakke flow* in $\mathbb{R}^3$ is a family of Radon measures $\mathcal{M}=\{\mu_t\}_{t\in I}$ in $\mathbb{R}^3$ that is 2-rectifiable for almost all times and satisfies

$$\frac{d}{dt}\int \varphi\, d\mu_t \leqslant \int (-\varphi \mathbf{H}^2+(\nabla\varphi)^\perp\cdot\mathbf{H})\, d\mu_t$$

for all test functions $\varphi\in C^1_c(\mathbb{R}^3,\mathbb{R}_+)$. Here, d/dt denotes the limsup of difference quotients, $\perp$ denotes the normal projection, and $\mathbf{H}$ denotes the mean-curvature vector of the associated varifold V_{μ_t}, which is defined via the first variation formula and exists almost everywhere at almost all times. The integral on the right-hand side is interpreted as $-\infty$ whenever it does not make sense literally.

All Brakke flows $\mathcal{M}=\{\mu_t\}_{t\in I}$ that we encounter in the present paper have the following three additional good properties of being:

- *integral* (cf. [Bra], [I1]): μ_t is integer 2-rectifiable for almost all t;
- *unit-regular* (cf. [W6], [SW]): every spacetime point of Gaussian density one is a regular point, i.e. for all $X=(x,t)\in\mathcal{M}$ with $\Theta_X=1$, there exists an $\varepsilon=\varepsilon(X)>0$ such that

$$t'\longmapsto \mathrm{spt}(\mu_{t'})\cap B(x,\varepsilon)$$

is a smooth mean-curvature flow for $t'\in[t-\varepsilon^2,t+\varepsilon^2]$;

- *cyclic* (cf. [W7]): for almost all t, the associated $\mathbb{Z}_2$ flat chain $[V_{\mu_t}]$ is cyclic. Here, the $\mathbb{Z}_2$ flat chain $[V]$ associated with an integral varifold V is called cyclic if $\partial[V]=0$.

By Allard's closure theorem [Al, Theorem 6.4] and a result of White [W7, Theorem 3.3], respectively, being integral and cyclic is preserved under varifold convergence with locally bounded first variation. Limits of sequences of Brakke flows can be taken via Ilmanen's compactness theorem [I1, §7]. As a consequence of the above quoted results (see also White [W7, Theorem 4.2 and Remark 4.4]), being integral and cyclic is preserved under limits of Brakke flows. By the local regularity theorem [W6] (see also Schulze–White [SW, Theorem 4.2]), being unit-regular is also preserved under limits of Brakke flows. In particular, Brakke flows starting at any closed embedded surface $M\subset\mathbb{R}^3$ that are constructed via Ilmanen's elliptic regularization [I1] are integral, unit-regular, and cyclic. We also recall from Ilmanen [I1, §10] that $M_t:=\mathrm{spt}(\mu_t)$ satisfies the avoidance principle.

2.2. Monotonicity formula and tangent flows

Let $\mathcal{M}=\{\mu_t\}_{t\in I}$ be a 2-dimensional unit-regular, integral Brakke flow in $\mathbb{R}^3$, say with bounded area ratios.(14) Given a space-time point $X_0=(x_0,t_0)\in\mathcal{M}$, let

$$\rho_{X_0}(x,t)=\frac{1}{4\pi(t_0-t)}e^{-|x-x_0|^2/4(t_0-t)},\quad t<t_0.$$

By Huisken's monotonicity formula [Hu2] (see also [I2]), we have

$$\frac{d}{dt}\int\rho_{X_0}\,d\mu_t\leqslant-\int\left|\mathbf{H}-\frac{(x-x_0)^\perp}{2(t-t_0)}\right|^2\rho_{X_0}\,d\mu_t. \tag{2.1}$$

The Gaussian density of $\mathcal{M}$ at X_0 is defined by

$$\Theta_{X_0}(\mathcal{M})=\lim_{t\nearrow t_0}\int\rho_{X_0}(x,t)\,d\mu_t(x).$$

It follows from the monotonicity formula that the Gaussian density is a well-defined real number $\geqslant 1$ and that the map

$$(x,t)\longmapsto\Theta_{(x,t)}(\mathcal{M})$$

is upper-semicontinous. Moreover, by the local regularity theorem for the mean-curvature flow [Bra], [W6], there exists a universal constant $\varepsilon_0>0$ such that any $X\in\mathcal{M}$ with $\Theta_X\leqslant 1+\varepsilon_0$ is a regular point.

Given $X\in\mathcal{M}$ and $\lambda_i\to 0$, let $\mathcal{M}^i$ be the Brakke flow which is obtained from $\mathcal{M}$ by translating X to the space-time origin and parabolically rescaling by λ_i^{-1}. By the

(14) In particular, any flow with finite entropy has bounded area ratios.

compactness theorem for Brakke flows [I1], one can pass to a subsequential limit $\widehat{\mathcal{M}}_X$, which is called a tangent flow at X. By the monotonicity formula every tangent flow is backwardly self-similar, i.e. $\widehat{\mathcal{M}}_X \cap \{t \leqslant 0\}$ is invariant under parabolic dilation

$$\mathcal{D}_\lambda(x,t) = (\lambda x, \lambda^2 t).$$

If $\mathcal{M}$ is ancient, then for any $\lambda_i \to \infty$ one can also pass along a subsequence to a backwardly self-similar limit $\widetilde{\mathcal{M}}$, which is called a tangent flow at infinity.

3. Coarse properties of ancient low-entropy flows

3.1. Partial regularity

The goal of this section is to prove a partial regularity result for ancient low-entropy flows. To this end, we start with the following lemma.

LEMMA 3.1. (Low-entropy cyclic minimal cones) *Let μ be an integer 2-rectifiable Radon measure in $\mathbb{R}^3$ with $\mathrm{Ent}[\mu]<2$. If the associated varifold V_μ is a cyclic minimal cone, then $\mu = \mathcal{H}^2 \lfloor P$ for some flat plane P.*

Proof. The proof is along the lines of [BW, Lemma 4.1]. Let o be the vertex of the cone. If $x \in \mathrm{spt}(\mu) \setminus \{o\}$, since being cyclic is preserved under weak limits, any tangent cone C at x is a cyclic minimal cone which splits off a line. Such a minimal cone consists of a static configuration of even number of half-planes meeting along a common line. The entropy assumption implies that there are only two of those half-planes, and since the configuration is static, it follows that C is a multiplicity-1 plane. Hence, by Allard's regularity theorem [Al], x is a regular point, and so the only potential singularity is at the vertex o. As the link of $\mathrm{spt}(\mu)$ is a smooth multiplicity-1 geodesic in S^2, it is a great circle, and so μ is indeed the Hausdorff measure of a flat multiplicity-1 plane. □

THEOREM 3.2. (Partial regularity) *Let $\mathcal{M}$ be an ancient low-entropy flow. Then there is either* (1) *a cylindrical singularity in which case $\mathcal{M}$ must be a round shrinking cylinder, or* (2) *no singularity, or* (3) *an at most countable number of spherical singularities.*([15])

Proof. Let $\widetilde{\mathcal{M}}$ be a tangent flow at infinity (see §2.2). By preservation under weak limits, $\widetilde{\mathcal{M}}$ is integral, unit regular, and cyclic. Using Lemma 3.1 it follows that any tangent cone to $\widetilde{M}_{-1}$ is a flat multiplicity-1 plane. Thus, by Allard's regularity theorem [Al],

([15]) At this point of the paper we cannot exclude the possibility of more than one spherical singularity, since we cannot exclude yet the potential scenario that several compact connected components form via "neckpinches at infinity" and "contracting cusps at infinity".

$\widetilde{M}_{-1}$ is smooth. Hence, by the classification of low-entropy shrinkers by Bernstein–Wang [BW], $\widetilde{\mathcal{M}}$ is either a flat plane, a round shrinking sphere, or a round shrinking cylinder (all of multiplicity 1). The same reasoning implies that any tangent flow $\widehat{\mathcal{M}}_X$ at any space-time point $X \in \mathcal{M}$ is either a flat plane, a round shrinking sphere, or a round shrinking cylinder (all of multiplicity 1). If a tangent flow at X is a flat multiplicity-1 plane, then X is a smooth point by the local regularity theorem [Bra], [W6]. If there is some $X \in \mathcal{M}$ with a cylindrical tangent flow, then $\mathcal{M}$ is a round shrinking cylinder by the equality case of Huisken's monotonicity formula (see §2.2). Finally, spherical tangent flows can happen at most countably many times since the set of strict local maxima of the map $X \mapsto \Theta_X(\mathcal{M})$ is countable; see e.g. [W2]. This proves the theorem. □

By Theorem 3.2, any ancient low-entropy flow $\mathcal{M} = \{\mu_t\}_{t \in (-\infty, T_E(\mathcal{M})]}$ is smooth, except for at most countably many spherical singularities until it becomes extinct. Hence, recalling also that higher multiplicities are ruled out by the low-entropy assumption, it is safe from now on to conflate the compact sets $M_t = \mathrm{spt}(\mu_t)$ and the Radon-measures μ_t in the notation.

COROLLARY 3.3. (Extinction time) *If $\mathcal{M}$ is an ancient low-entropy flow then exactly one of the following happens*:[16]

(i) $T_E(\mathcal{M}) = \infty$, *i.e. the flow $\mathcal{M}$ is eternal, or*

(ii) $T_E(\mathcal{M}) < \infty$ *and $\mathcal{M}$ is a round shrinking cylinder, or*

(iii) $T_E(\mathcal{M}) < \infty$ *and $\mathcal{M}$ becomes extinct in at most countably many round points, or*

(iv) $T_E(\mathcal{M}) < \infty$ *and for every $R < \infty$ there exist $T(R) < T_E(\mathcal{M})$ such that*

$$B(0,R) \cap M_t = \varnothing \quad \text{for every } t \in (T(R), T_E(\mathcal{M})].$$

Proof. Assume $T_E(\mathcal{M}) < \infty$, and suppose (iv) does not hold. Then, by upper semi-continuity of Huisken's density (see §2.2), there exist a point $X = (x_0, T_E(\mathcal{M}))$ such that $\Theta_X \geqslant 1$. If the tangent flow at X is a plane, then this contradicts the definition of $T_E(\mathcal{M})$ by unit regularity. If the tangent flow at X is a cylinder, then we are in case (ii). If the tangent flow at X is a round shrinking sphere, then we are in case (iii). This proves the corollary. □

[16] At this stage of the paper we cannot yet exclude the scenarios that the solution "becomes extinct in more than one point" or that the solution "escapes to infinity".

3.2. Finding necks back in time

Let $\mathcal{M}$ be an ancient low-entropy flow. Given a point $X=(x,t)\in\mathcal{M}$ and a scale $r>0$, we consider the flow

$$\mathcal{M}_{X,r}=\mathcal{D}_{1/r}(\mathcal{M}-X),$$

which is obtained from $\mathcal{M}$ by translating X to the space-time origin and parabolically rescaling by $1/r$. Here, $\mathcal{D}_\lambda(x,t)=(\lambda x,\lambda^2 t)$.

Definition 3.4. Fix $\varepsilon>0$. We say that $\mathcal{M}$ is *ε-cylindrical around X at scale r*, if $\mathcal{M}_{X,r}$ is ε-close in $C^{\lfloor 1/\varepsilon\rfloor}$ in $B(0,1/\varepsilon)\times[-1,-2]$ to the evolution of a round shrinking cylinder with radius $r(t)=\sqrt{-2t}$ and axis through the origin. The notions *ε-planar* and *ε-spherical* are defined similarly.

Given any point $X=(x,t)\in\mathcal{M}$, we analyze the solution around X at the diadic scales $r_j=2^j$, where $j\in\mathbb{Z}$.

THEOREM 3.5. *For any small enough $\varepsilon>0$, there is a positive integer $N=N(\varepsilon)<\infty$ with the following significance. If $\mathcal{M}$ is an ancient low-entropy flow, which is not a round shrinking sphere, round shrinking cylinder or a flat plane, then for every $X\in\mathcal{M}$ there exists an integer $J(X)\in\mathbb{Z}$ such that*

$$\mathcal{M}\text{ is not }\varepsilon\text{-cylindrical around }X\text{ at scale }r_j\text{ for all }j<J(X), \tag{3.1}$$

and

$$\mathcal{M}\text{ is }\tfrac{1}{2}\varepsilon\text{-cylindrical around }X\text{ at scale }r_j\text{ for all }j\geqslant J(X)+N. \tag{3.2}$$

Proof. In the following argument we will frequently use the local regularity theorem for the mean-curvature flow (see [Bra], [W6]) without explicitly mentioning it. Given $\varepsilon>0$, for any $X=(x,t)\in\mathcal{M}$ we define

$$J(X):=\inf\{j\in\mathbb{Z}:\mathcal{M}\text{ is }\varepsilon\text{-cylindrical around }X\text{ at scale }r_j\}. \tag{3.3}$$

Recall that, by Huisken's monotonicity formula [Hu2], [I2], the quantity

$$\Theta_{(x,t)}(r)=\int_{M_{t-r^2}}\frac{1}{4\pi r^2}e^{-|x-y|^2/4r^2}\,dA(y)$$

is monotone, with equality only in the self-similarly shrinking case. Recall also that by Bernstein–Wang [BW], the only self-similarly shrinking solutions with low entropy are the flat plane, the round shrinking sphere and the round shrinking cylinder.

Since $\mathcal{M}$ is non-flat and not a round shrinking sphere, its tangent flow at infinity $\widetilde{\mathcal{M}}$ (see §2.2) must be a round shrinking cylinder. Hence, $J(X)<\infty$. Similarly, if $\mathcal{M}$ had a

cylindrical tangent flow $\widehat{\mathcal{M}}_X$, then by the equality case of Huisken's monotonicity formula $\mathcal{M}$ would be a round shrinking cylinder. Hence, $J(X)>-\infty$. Therefore, $J(X)\in\mathbb{Z}$ and the statement (3.1) holds true by the definition from equation (3.3).

To prove (3.2), note first that (recalling that our fixed ε is small) we have

$$\Theta_X(r_{J(X)}) \geqslant \mathrm{Ent}[S^1]-\tfrac{1}{100}, \tag{3.4}$$

and

$$\lim_{j\to\infty}\Theta_X(r_j)=\mathrm{Ent}[S^1]. \tag{3.5}$$

Next, observe that by the equality case of Huisken's monotonicity formula, if

$$\Theta_X(r_{j+1})-\Theta_X(r_{j-1})=0,$$

then $\mathcal{M}$ is zero-self-similar around X at scale r_j. This can be made quantitative (cf. Cheeger–Haslhofer–Naber [CHN]). Namely, there exists a $\delta=\delta(\varepsilon)>0$ such that, if

$$\Theta_X(r_{j+1})-\Theta_X(r_{j-1})\leqslant\delta, \tag{3.6}$$

then

$$\mathcal{M} \text{ is } \tfrac{1}{2}\varepsilon\text{-self-similar around } X \text{ at scale } r_j.$$

For $j\geqslant J(X)$ in our context $\frac{1}{2}\varepsilon$-self-similar simply means $\frac{1}{2}\varepsilon$-cylindrical.

Finally, using again monotonicity and quantitative rigidity we see that after going from scale $J(X)$ to scale $J(X)+N$, where $N=N(\varepsilon)<\infty$, we have

$$\Theta_X(r_{J(X)+N(\varepsilon)}) \geqslant \mathrm{Ent}[S^1]-\delta. \tag{3.7}$$

Combining the above facts, we conclude that (3.2) holds. This finishes the proof of the theorem. □

We fix a small enough parameter $\varepsilon>0$ quantifying the quality of the necks for the rest of the paper.

Definition 3.6. The *cylindrical scale* of $X\in\mathcal{M}$ is defined by

$$Z(X)=2^{J(X)}.$$

Remark 3.7. Recall that the regularity scale $R(X)$ is defined as the maximal radius r such that $|A|\leqslant 1/r$ in the parabolic ball $P(X,r)$. It follows from the definition of regularity scale and the low-entropy assumption that there exists a positive integer $N=N(\varepsilon)<\infty$ such that

$$\mathcal{M} \text{ is } \varepsilon\text{-planar around } X \text{ at scale } r_j \text{ for all } j\leqslant \log_2 R(X)-N. \tag{3.8}$$

3.3. Asymptotic slope

Let $\mathcal{M}$ be an ancient low-entropy flow, which is not a round shrinking sphere, round shrinking cylinder or a flat plane. Recall from above that $\mathcal{M}$ has an asymptotic cylinder $\widetilde{\mathcal{M}}$ for $t\to-\infty$. By Colding–Minicozzi [CM2], the axis is unique. We may assume without loss of generality that the axis is in x_3-direction.

PROPOSITION 3.8. (Asymptotic cylindrical scale) *For every $\delta>0$ there exists*

$$\Lambda=\Lambda(\delta)<\infty$$

such that if $(p_0,t_0)\in\mathcal{M}$ then

$$Z(p,t_0)\leqslant\delta|p-p_0|$$

for all $p\in M_{t_0}$ with $|p-p_0|\geqslant\Lambda Z(p_0,t_0)$.

Proof. Assume without loss of generality that $(p_0,t_0)=(0,0)$ and $Z(0,0)=1$. If the assertion fails, then there is a sequence $p_i\in M_0$ with $|p_i|\geqslant i$, but

$$Z(p_i,0)\geqslant\delta|p_i| \tag{3.9}$$

for some $\delta>0$. Let $\mathcal{M}^i$ be the flow which is obtained by parabolically rescaling by $1/|p_i|$ around $(0,0)$ and pass to a limit $\mathcal{M}^\infty$. The limit $\mathcal{M}^\infty$ is an ancient low-entropy flow, which has a cylindrical singularity at $(0,0)$. Hence, by the equality case of Huisken's monotonicity formula $\mathcal{M}^\infty$ is a round shrinking cylinder that becomes extinct at time $T=0$, in contradiction with (3.9). This proves the proposition. □

We normalize $\mathcal{M}$ such that $X_0=(0,0)\in\mathcal{M}$ and $Z(X_0)\leqslant1$.

THEOREM 3.9. (Asymptotic slope) *For every $\delta>0$ there exists $\Lambda=\Lambda(\delta)<\infty$ such that*

$$\frac{\sup\{\sqrt{x_1^2+x_2^2}|(x_1,x_2,x_3)\in M_t\}}{|x_3|}<\delta$$

whenever $t\leqslant-10$ and $|(x_1,x_2,x_3)|\geqslant\Lambda\sqrt{-t}$.

Proof. We first claim that for every $\delta>0$ there exist

$$\Lambda_1=\Lambda_1(\delta)<\infty\quad\text{and}\quad T_1=T_1(\delta)>-\infty$$

such that every $(p,t)\in\mathcal{M}$ with $t\leqslant T_1$ and $|p|/\sqrt{-t}\geqslant\Lambda_1$ satisfies

$$x_1^2+x_2^2\leqslant\tfrac{1}{2}\delta^2x_3^2. \tag{3.10}$$

Indeed, if $(p,t)\in\mathcal{M}$ is any point with $t\leqslant -10$ and $|p|/\sqrt{-t}\gg 1$, then by Proposition 3.8 it has cylindrical scale $Z(p,t)\ll|p|$. If (3.10) was violated, then going back further in time this neck would intersect the central neck of our asymptotic cylinder; a contradiction.

Consequently, for every $\delta>0$ there exist $T_2=T_2(\delta)>-\infty$ and $L(\delta)<\infty$ such that

$$\frac{1}{\sqrt{-t}}M_t\subset\{|z|\leqslant L, x_1^2+x_2^2\leqslant 3\}\cup\big\{|x_3|\geqslant L, x_1^2+x_2^2\leqslant \tfrac{1}{2}\delta^2 x_3^2\big\}$$

for all $t\leqslant T_2$. Using suitable large spheres as barriers, this implies the assertion. □

Corollary 3.10. (Barrier for the rescaled flow) *There exists an even smooth function $\varphi\colon\mathbb{R}\to\mathbb{R}_+$ with*

$$\lim_{z\to\pm\infty}\varphi'(z)=0$$

such that the rescaled mean-curvature flow

$$\overline{M}_\tau^X=e^{\tau/2}(M_{-e^{-\tau}}-x_0),$$

where $\tau=-\log(t_0-t)$, satisfies

$$\overline{M}_\tau^X\subset\{x_1^2+x_2^2\leqslant\varphi(x_3)^2\}$$

for $\tau\leqslant\mathcal{T}(Z(X))$.

By the corollary, any potential ends must be in direction $x_3\to\pm\infty$.

4. Fine neck analysis

4.1. Setting up the fine neck analysis

Let $\mathcal{M}$ be a non-compact ancient low-entropy flow in $\mathbb{R}^3$, which is not a round shrinking sphere, round shrinking cylinder or a flat plane. Given any point $X_0=(x_0,t_0)\in\mathcal{M}$, we consider the rescaled flow

$$\overline{M}_\tau^{X_0}=e^{\tau/2}\,(M_{-e^{-\tau}}-x_0),$$

where $\tau=-\log(t_0-t)$. By Theorem 3.5 and Colding–Minicozzi [CM2], the rescaled flow converges for $\tau\to-\infty$ to the cylinder $\Sigma=\{x_1^2+x_2^2=2\}$. Moreover, the convergence is uniform in X_0 once we normalize such that $Z(X_0)\leqslant 1$. Hence, we can find universal functions $\sigma(\tau)>0$ and $\rho(\tau)>0$ with

$$\lim_{\tau\to-\infty}\sigma(\tau)=0,\quad \lim_{\tau\to-\infty}\rho(\tau)=\infty,\quad\text{and}\quad -\rho(\tau)\leqslant\rho'(\tau)\leqslant 0, \tag{4.1}$$

such that $\overline{M}_\tau^{X_0}$ is the graph of a function $u(\cdot,\tau)$ over $\Sigma\cap B_{2\rho(\tau)}(0)$, namely

$$\{x+u(x,\tau)\nu_\Sigma(x) : x\in\Sigma\cap B_{2\rho(\tau)}(0)\}\subset\overline{M}_\tau^{X_0},$$

where ν_Σ denotes the outward pointing unit normal to Σ, and

$$\|u(\cdot,\tau)\|_{C^4(\Sigma\cap B_{2\rho(\tau)}(0))}\leqslant\sigma(\tau)\,\rho(\tau)^{-1}. \tag{4.2}$$

We will now set up a fine neck analysis following [ADS1] and [BC].

We denote by $C<\infty$ and $\mathcal{T}>-\infty$ constants that can change from line to line and can depend on various other quantities (such as the function ρ from above, the neck parameter ε from §3.2, etc.), but are independent of the point X_0 with $Z(X_0)\leqslant 1$. We also fix a non-negative smooth cutoff function φ satisfying $\varphi(z)=1$ for $|z|\leqslant\frac{1}{2}$ and $\varphi(z)=0$ for $|z|\geqslant 1$, and set

$$\hat{u}(x,\tau)=u(x,\tau)\varphi\bigg(\frac{x_3}{\rho(\tau)}\bigg).$$

We recall from Angenent–Daskalopoulos–Sesum that there are shrinkers

$$\Sigma_a=\{\text{surface of revolution with profile } r=u_a(z),\ 0\leqslant z\leqslant a\},$$
$$\widetilde{\Sigma}_b=\{\text{surface of revolution with profile } r=\widetilde{U}_b(z),\ 0\leqslant z<\infty\},$$

as illustrated in [ADS1, Figure 1]; see also [KM]. We will refer to these shrinkers as ADS-shrinkers and KM-shrinkers, respectively. The parameter a captures where the curves u_a meet the z-axis, namely $u_a(a)=0$, and the parameter b is the asymptotic slope of the curves u_b, namely $\lim_{z\to\infty}u_b'(z)=b$. A detailed description of these shrinkers can be found in [ADS1, §8]. In particular, the shrinkers can be used for barrier arguments as well as for calibration arguments. To describe the latter, we fix suitable parameters $a_0,b_0,L_0<\infty$ and consider the region $F^+\subset\mathbb{R}^3$ bounded by Σ_{a_0}, Σ_{b_0}, and $\{z=L_0\}$. The region F^+ is foliated by $\{\Sigma_a\}_{a\geqslant a_0}$, Σ, and $\{\widetilde{\Sigma}_b\}_{b\leqslant b_0}$. Denoting by ν_{fol} the outward unit normal of this family, by [ADS1, Lemma 4.10] we have that

$$\operatorname{div}(e^{-x^2/4}\nu_{\text{fol}})=0, \tag{4.3}$$

i.e. the shrinker family forms a calibration for the Gaussian area. Similarly, the region $F^-=\{(x_1,x_2,-x_3):x\in F^+\}$ is calibrated. Let $F=F^-\cup F^+$.

PROPOSITION 4.1. *There exists $\mathcal{T}>-\infty$ independent of X_0 such that*[17]

$$\overline{M}_\tau\cap\{|x_3|\geqslant L_0\}\subset F$$

for all $\tau\leqslant\mathcal{T}$.

[17] Recall that we assume that $Z(X_0)\leqslant 1$.

Proof. By Corollary 3.10 (barrier for the rescaled flow), $\overline{M}_\tau \cap \{x_3 \geqslant L_0\}$ is contained in the region bounded by $\widetilde{\Sigma}_{b_0}$ and $\{x_3 = L_0\}$ for sufficiently negative τ. Moreover, since $\overline{M}_\tau$ is a graph with small norm over a long cylinder, it can not cross $\Sigma_{a_0} \cap \{x_3 \geqslant L_0\}$. This implies the assertion. □

Denote by Δ_τ the region bounded by $\overline{M}_\tau$ and Σ.

PROPOSITION 4.2. (Cf. [BC, Proposition 2.2]) *For all $L \in [L_0, \rho(\tau)]$ and $\tau \leqslant \mathcal{T}$ we have the Gaussian area estimate*

$$\int_{\overline{M}_\tau \cap \{|x_3| \geqslant L\}} e^{-|x|^2/4} - \int_{\Sigma \cap \{|x_3| \geqslant L\}} e^{-|x|^2/4} \geqslant - \int_{\Delta_\tau \cap \{|x_3| = L\}} e^{-|x|^2/4} \, |N \cdot \nu_{fol}|.$$

Proof. Consider the region $\Delta_\tau \cap \{|x_3| \geqslant L\} \cap \{|x| \leqslant R\}$. Integrating (4.3) over this region, and using the divergence theorem and Proposition 4.1, we obtain

$$\begin{aligned} \int_{\overline{M}_\tau \cap \{|x_3| \geqslant L\}} e^{-|x|^2/4} - \int_{\Sigma \cap \{|x_3| \geqslant L\}} e^{-|x|^2/4} \\ \geqslant - \int_{\Delta_\tau \cap \{|x_3| = L\}} e^{-|x|^2/4} \, |N \cdot \nu_{\mathrm{fol}}| - \int_{\Delta_\tau \cap \{|x| = R\}} e^{-|x|^2/4} \, |N \cdot \nu_{\mathrm{fol}}|, \end{aligned}$$

where N denotes the unit normal of the boundary. By the entropy bound, we have that

$$\int_{\Delta_\tau \cap \{|x| = R\}} e^{-|x|^2/4} \, |N \cdot \nu_{\mathrm{fol}}| \leqslant C R^2 e^{-R^2/4},$$

and passing $R \to \infty$ the assertion follows. □

The next proposition shows that closeness to the cylinder in the region $\{|x_3| \leqslant \frac{1}{2} L\}$ implies closeness to the cylinder in the larger region $\{|x_3| \leqslant L\}$.

PROPOSITION 4.3. (Cf. [BC, Proposition 2.3], [ADS1, Lemma 4.7]) *The graph function u satisfies the integral estimates*

$$\int_{\Sigma \cap \{|x_3| \leqslant L\}} e^{-|x|^2/4} |\nabla u(x,\tau)|^2 \leqslant C \int_{\Sigma \cap \{|x_3| \leqslant \frac{L}{2}\}} e^{-|x|^2/4} u(x,\tau)^2$$

and

$$\int_{\Sigma \cap \{L/2 \leqslant |x_3| \leqslant L\}} e^{-|x|^2/4} u(x,\tau)^2 \leqslant C L^{-2} \int_{\Sigma \cap \{|x_3| \leqslant L/2\}} e^{-|x|^2/4} u(x,\tau)^2$$

for all $L \in [L_0, \rho(\tau)]$ and $\tau \leqslant \mathcal{T}$, where $C < \infty$ is a numerical constant.

Proof. By the low-entropy assumption, we have

$$\int_{\overline{M}_\tau} e^{-|x|^2/4} \leqslant \int_\Sigma e^{-|x|^2/4}.$$

Using this and Proposition 4.2, the rest is as in the proof of [BC, Proposition 2.3]. □

Recall that $\overline{M}_\tau$ is expressed as graph of a function $u(x,\tau)$ over $\Sigma\cap B_{2\rho(\tau)}(0)$ satisfying the estimate (4.2). Using that $\overline{M}_\tau$ moves by rescaled mean-curvature flow one obtains the following result.

LEMMA 4.4. (Cf. [BC, Lemma 2.4]) *The function $u(x,\tau)$ satisfies*

$$\partial_\tau u = \mathcal{L}u + E,$$

where $\mathcal{L}$ is the linear operator on Σ defined by

$$\mathcal{L}f = \Delta f - \tfrac{1}{2}\langle x^{\tan}, \nabla f\rangle + f, \tag{4.4}$$

and where the error term can be estimated by

$$|E| \leqslant C\sigma(\tau)\rho^{-1}(\tau)(|u| + |\nabla u|) \tag{4.5}$$

for $\tau \leqslant \mathcal{T}$.

Proof. The proof is similar to that of [BC, Lemma 2.4]. □

Denote by $\mathcal{H}$ the Hilbert space of all functions f on Σ such that

$$\|f\|_{\mathcal{H}}^2 = \int_\Sigma \frac{1}{4\pi} e^{-|x|^2/4} f^2 < \infty. \tag{4.6}$$

LEMMA 4.5. (Cf. [BC, Lemma 2.5]) *The function*

$$\hat{u}(x,\tau) = u(x,\tau)\varphi\Big(\frac{x_3}{\rho(\tau)}\Big)$$

satisfies

$$\partial_\tau \hat{u} = \mathcal{L}\hat{u} + \widehat{E},$$

where

$$\|\widehat{E}\|_{\mathcal{H}} \leqslant C\rho^{-1}\,\|\hat{u}\|_{\mathcal{H}}$$

for $\tau \leqslant \mathcal{T}$.

Proof. As in [BC, p. 8], we compute

$$\begin{aligned}\widehat{E} = E\varphi\Big(\frac{x_3}{\rho(\tau)}\Big) - \frac{2}{\rho(\tau)}\frac{\partial u}{\partial z}\varphi'\Big(\frac{x_3}{\rho(\tau)}\Big) - \frac{1}{\rho(\tau)^2}u\varphi''\Big(\frac{x_3}{\rho(\tau)}\Big)\\ + \frac{x_3}{2\rho(\tau)}u\varphi'\Big(\frac{x_3}{\rho(\tau)}\Big) - \frac{x_3\rho'(\tau)}{\rho(\tau)^2}u\varphi'\Big(\frac{x_3}{\rho(\tau)}\Big).\end{aligned}$$

If $|x_3| \leqslant \frac{1}{2}\rho(\tau)$, then Lemma 4.4 gives

$$|\widehat{E}| = |E| \leqslant C\sigma(\tau)\rho^{-1}(\tau)(|u| + |\nabla u|).$$

If $\frac{1}{2}\rho(\tau) \leqslant |x_3| \leqslant \rho(\tau)$, then using equation (4.1) we obtain

$$|\widehat{E}| \leqslant C|u| + C\rho^{-1}(\tau)|\nabla u|.$$

Using also Proposition 4.3 and equation (4.1), we infer that

$$\begin{aligned}\int_\Sigma e^{-|x|^2/4}|\widehat{E}|^2 &\leqslant \frac{C\sigma^2}{\rho^2}\int_{\Sigma\cap\{|x_3|\leqslant\rho(\tau)/2\}} e^{-|x|^2/4}u^2 \\ &\quad + C\int_{\Sigma\cap\{\rho(\tau)/2\leqslant|x_3|\leqslant\rho(\tau)\}} e^{-|x|^2/4}u^2 \\ &\quad + \frac{C}{\rho^2}\int_{\Sigma\cap\{|x_3|\leqslant\rho(\tau)\}} e^{-|x|^2/4}|\nabla u|^2 \\ &\leqslant \frac{C}{\rho^2}\int_{\Sigma\cap\{|x_3|\leqslant\frac{\rho(\tau)}{2}\}} e^{-|x|^2/4}u^2 \\ &\leqslant \frac{C}{\rho^2}\int_\Sigma e^{-|x|^2/4}\hat{u}^2\end{aligned}$$

for $\tau \leqslant \mathcal{T}$. Thus, we obtain the desired result. □

Let us recall some facts from [BC] about the operator $\mathcal{L}$ defined in (4.4). In cylindrical coordinates this operator takes the form

$$\mathcal{L} = \frac{\partial^2}{\partial z^2}f + \frac{1}{2}\frac{\partial^2}{\partial\theta^2}f - \frac{1}{2}z\frac{\partial}{\partial z}f + f.$$

Analysing the spectrum of $\mathcal{L}$, the Hilbert space $\mathcal{H}$ from (4.6) can be decomposed as

$$\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_0 \oplus \mathcal{H}_-,$$

where $\mathcal{H}_+$ is spanned by the four positive eigenmodes 1, z, $\sin\theta$, $\cos\theta$, and $\mathcal{H}_0$ is spanned by the three zero-modes $z^2 - 2$, $z\cos\theta$, and $z\sin\theta$. We have

$$\begin{cases} \langle \mathcal{L}f, f\rangle_{\mathcal{H}} \geqslant \frac{1}{2}\|f\|^2_{\mathcal{H}} & \text{for } f \in \mathcal{H}_+, \\ \langle \mathcal{L}f, f\rangle_{\mathcal{H}} = 0 & \text{for } f \in \mathcal{H}_0, \\ \langle \mathcal{L}f, f\rangle_{\mathcal{H}} \leqslant -\frac{1}{2}\|f\|^2_{\mathcal{H}} & \text{for } f \in \mathcal{H}_-. \end{cases}$$

Consider the functions

$$\begin{aligned} U_+(\tau) &:= \|P_+\hat{u}(\cdot,\tau)\|^2_{\mathcal{H}}, \\ U_0(\tau) &:= \|P_0\hat{u}(\cdot,\tau)\|^2_{\mathcal{H}}, \\ U_-(\tau) &:= \|P_-\hat{u}(\cdot,\tau)\|^2_{\mathcal{H}}, \end{aligned}$$

where P_+, P_0, and P_- denote the orthogonal projections to $\mathcal{H}_+$, $\mathcal{H}_0$, and $\mathcal{H}_-$, respectively. Using Lemma 4.5, we obtain

$$\begin{aligned}\frac{d}{d\tau}U_+(\tau) &\geqslant U_+(\tau)-C\rho^{-1}(U_+(\tau)+U_0(\tau)+U_-(\tau)),\\ \left|\frac{d}{d\tau}U_0(\tau)\right| &\leqslant C\rho^{-1}(U_+(\tau)+U_0(\tau)+U_-(\tau)),\\ \frac{d}{d\tau}U_-(\tau) &\leqslant -U_-(\tau)+C\rho^{-1}(U_+(\tau)+U_0(\tau)+U_-(\tau)).\end{aligned} \tag{4.7}$$

To proceed, we need the following ODE-lemma.

LEMMA 4.6. (Merle–Zaag [MZ]) *Let $x(\tau)$, $y(\tau)$, $z(\tau)$ be non-negative absolutely continuous functions satisfying $(x+y+z)(\tau)>0$ and*

$$\liminf_{\tau\to-\infty} y(\tau)=0.$$

Moreover, suppose for each $\varepsilon\in\left(0,\frac{1}{100}\right]$, there exists a $\tau_0=\tau_0(\varepsilon)$ such that for $\tau\leqslant\tau_0$ the following holds:

$$\begin{aligned}|x_\tau| &\leqslant \varepsilon(x+y+z),\\ y_\tau &\leqslant -y+\varepsilon(x+z),\\ z_\tau &\geqslant z-\varepsilon(x+y).\end{aligned}$$

Then, we have $y\leqslant 2\varepsilon(x+z)$ for $\tau\leqslant\tau_0$, and either x is dominant in the sense that

$$y+z=o(x)$$

as $\tau\to-\infty$, or z is dominant in the more precise sense that

$$x+y\leqslant 100\varepsilon z$$

for all $\tau\leqslant\tau_0(\varepsilon)$.

Remark 4.7. The statement in [MZ] is slightly different, but a careful inspection of their proof gives the variant of the lemma that we stated; see also [CMa, Appendix B] for a detailed proof.

If $(U_++U_0+U_-)(\hat{\tau})=0$ for some $\hat{\tau}$, then Lemma 4.5 implies that $(U_++U_0+U_-)(\tau)=0$ for all $\tau\leqslant\hat{\tau}$, and by analytic continuation it follows that $\mathcal{M}$ is a round shrinking cylinder; a contradiction. We can thus apply the Merle–Zaag lemma (Lemma 4.6) to conclude that either the neutral mode is dominant, i.e.

$$U_++U_-=o(U_0),$$

or the plus mode is dominant, i.e.

$$U_- + U_0 \leqslant C\rho^{-1} U_+.$$

We will analyze these two cases in turn in the following two sections. Recall also that our analysis above depends on a choice of X_0 and ρ. However, the following proposition shows that either the neutral mode is always dominant or the plus mode is always dominant.

PROPOSITION 4.8. *If a neck centered at some point X_0 satisfies $U_+ + U_- = o(U_0)$ for some admissible choice of ρ_0, then any other neck centered at any point X also satisfy $U_+ + U_- = o(U_0)$ for any admissible choice of ρ.*

Proof. Any two necks converge to the same neck after rescaling as $\tau \to -\infty$. Thus, the above statement is obvious. □

4.2. Fine analysis in the plus mode

In this section, we assume that the plus mode is dominant. We recall from above this means that after fixing a center $X_0 \in \mathcal{M}$ with $Z(X_0) \leqslant 1$, and a graphical scale function ρ, we have that

$$U_- + U_0 \leqslant C\rho^{-1} U_+ \tag{4.8}$$

for all $\tau \leqslant \mathcal{T}$. As before, $C < \infty$ and $\mathcal{T} > -\infty$ denote constants that can change from line to line and are independent of the point $X_0 \in \mathcal{M}$ with $Z(X_0) \leqslant 1$. The main goal of this section is to prove Theorem 4.15.

4.2.1. Graphical radius

Using (4.7) and (4.8), we obtain

$$\frac{d}{d\tau} U_+ \geqslant U_+ - C\rho^{-1} U_+.$$

By integrating this differential inequality, for every $\mu > 0$ we can find a constant

$$\mathcal{T}(\mu) > -\infty$$

such that

$$U_+(\tau) \leqslant Ce^{(1-\mu)\tau}.$$

for all $\tau \leqslant \mathcal{T}(\mu)$. Recalling that $U_+ = \|P_+ \hat{u}\|^2_{\mathcal{H}}$ and using (4.8) we infer that

$$\|\hat{u}\|_{\mathcal{H}} \leqslant Ce^{(1-\mu)\tau/2}.$$

By standard interpolation inequalities, this implies

$$\|u(\cdot,\tau)\|_{C^4([-10L_0,10L_0]\times[0,2\pi])} \leqslant Ce^{\frac{(1-\mu)\tau}{2}} \tag{4.9}$$

for all $\tau \leqslant \mathcal{T}(\mu)$. The estimate (4.9) is not sharp. To improve it to a sharp estimates (i.e. to remove the μ), we start with the following C^0-estimate.

PROPOSITION 4.9. *The rescaled mean-curvature flow $\overline{M}_\tau$ satisfies the estimate*

$$\sup_{\overline{M}_\tau \cap \{|x_3| \leqslant e^{-\tau/10}\}} \left|x_1^2+x_2^2-2\right| \leqslant e^{\tau/10} \tag{4.10}$$

for all $\tau \leqslant \mathcal{T}$.

Proof. Fix μ sufficiently small, e.g. $\mu = \frac{1}{100}$. Recall from §4.1 that there are shrinkers Σ_a and $\widetilde{\Sigma}_b$ with profile functions u_a and $\widetilde{U}_b$.

As preparation for the barrier argument, observe that (4.9) implies that

$$\sup_{\overline{M}_\tau \cap \{|x_3| \leqslant L_0\}} (x_1^2+x_2^2) \leqslant 2+Ce^{(1-\mu)\tau/2}, \tag{4.11}$$

and

$$\inf_{\overline{M}_\tau \cap \{|x_3| \leqslant L_0\}} (x_1^2+x_2^2) \geqslant 2-Ce^{(1-\mu)\tau/2}. \tag{4.12}$$

To prove an upper bound, we will use the shrinkers $\widetilde{\Sigma}_b$ as outer barriers. To this end, we first observe that, for $z \geqslant 2\sqrt{2}$, the profile function $\widetilde{U}_b$ satisfies the estimate

$$2+e^{-1}b^2z^2 \leqslant \widetilde{U}_b(z)^2 \leqslant 2+b^2z^2. \tag{4.13}$$

Indeed, [ADS1, Lemma 4.11] and [ADS1, §8.12] tell us that $\widetilde{U}_b$ satisfies the differential equation

$$\widetilde{U}_b' = \frac{w}{2z\widetilde{U}_b}(\widetilde{U}_b^2-2), \tag{4.14}$$

for $z \geqslant 2\sqrt{2}$, where $w=w(z)$ is a function that satisfies

$$2 \leqslant w(z) \leqslant 2+\frac{16}{z^2}. \tag{4.15}$$

Combining (4.14) and (4.15), we see that

$$0 \leqslant \frac{d}{dz}\log\left(\frac{\widetilde{U}_b(z)^2-2}{z^2}\right) \leqslant \frac{16}{z^3}. \tag{4.16}$$

Integrating (4.16) from z to ∞ and using that the asymptotic slope of $\widetilde{U}_b$ equals b, we infer that

$$0 \leqslant \log b^2 - \log\left(\frac{\widetilde{U}_b(z)^2-2}{z^2}\right) \leqslant \int_{2\sqrt{2}}^{\infty} \frac{16}{x^3}\,dx = 1,$$

which proves (4.13).

Now, fixing $\hat{\tau}\leqslant\mathcal{T}$, we consider the shrinker $\widetilde{\Sigma}_b$ with parameter

$$b=2\sqrt{eC}L_0^{-1}e^{(1-\mu)\hat{\tau}/4}.$$

Using (4.11) and (4.13), we see that $\widetilde{\Sigma}_b\cap\{x_3=L_0\}$ is outside of $\overline{M}_\tau\cap\{x_3=L_0\}$ for all $\tau\leqslant\hat{\tau}$. Moreover, recalling that $\widetilde{U}_b$ has asymptotic slope b and that $\overline{M}_\tau$ has vanishing asymptotic slope (see Corollary 3.10) we also see that $\widetilde{\Sigma}_b\cap\{x_3=h\}$ is outside of

$$\overline{M}_\tau\cap\{x_3=h\}$$

for $h\geqslant h_0(\hat{\tau})$ large enough, and for all $\tau\leqslant\hat{\tau}$. Finally, $\widetilde{\Sigma}_b\cap\{L_0\leqslant x_3\leqslant h\}$ is outside of

$$\overline{M}_\tau\cap\{L_0\leqslant x_3\leqslant h\}$$

for $-\tau$ large enough. Hence, by the avoidance principle, $\widetilde{\Sigma}_{b_1}\cap\{x_3\geqslant L_0\}$ is outside of

$$\overline{M}_\tau\cap\{x_3\geqslant L_0\}$$

for all $\tau\leqslant\hat{\tau}$. Repeating the same argument for x_3 replaced by $-x_3$, and using (4.13), we conclude that

$$\sup_{\overline{M}_\tau\cap\{|x_3|\leqslant e^{-\tau/10}\}}(x_1^2+x_2^2)\leqslant 2+e^{\tau/10} \tag{4.17}$$

for $\tau\leqslant\mathcal{T}$.

Similarly, considering the shrinkers Σ_a as inner barriers as in [BC, pp. 10–11] and using (4.12), we obtain

$$\inf_{\overline{M}_\tau\cap\{|x_3|\leqslant e^{-\tau/10}\}}(x_1^2+x_2^2)\geqslant 2-e^{\tau/10}$$

for $\tau\leqslant\mathcal{T}$. Together with (4.17), this proves the assertion. □

We will now use the C^0-estimate to prove that the rescaled flow $\overline{M}_\tau$ is cylindrical over an exponentially expanding region for $\tau\to-\infty$.

Proposition 4.10. *For $\tau\leqslant\mathcal{T}$ the rescaled mean-curvature flow $\overline{M}_\tau$ can be written as graph of a function $v(\cdot,\tau)$ over the cylinder $\Sigma\cap\{|x_3|\leqslant\frac{1}{2}e^{-\tau/10}\}$ with the estimate*

$$\|v\|_{C^6(\Sigma\cap\{|x_3|\leqslant e^{-\tau/10}/2\})}\leqslant Ce^{\tau/10}.$$

Proof. We first observe that, for $\tau\leqslant\mathcal{T}$, every point $\overline{X}=(\overline{X},\tau)$, where

$$\overline{X}\in\overline{M}_\tau\cap\{|x_3|\leqslant\tfrac{3}{4}e^{-\tau/10}\},$$

has regularity scale comparable to 1, namely

$$CP^{-1} \leqslant R(\overline{X}) \leqslant C \tag{4.18}$$

for some uniform $C<\infty$, where $R(\overline{X})$ denotes the regularity scale of the rescaled flow.

Indeed, if $R(\overline{X})\gg 1$, then we could parameterize $\bar{M}_\tau$ as a graph graph with small norm over a disk of radius 2 centered at $\overline{X}$, contradicting Proposition 4.9. If on the other hand $R(\overline{X})\ll 1$, then by the argument from §3.2 for some $\tau'<\tau$ the rescaled flow would be ε-close to a cylinder or sphere with center $\overline{X}$ and radius $\frac{1}{100}$, contradicting again Proposition 4.9.

To continue, note that the geometric meaning of Proposition 4.9 is that our surface $\overline{M}_\tau$ is trapped between two cylinders of almost equal radii. Combining this with the regularity scale bound (4.18), we see that at every point

$$\overline{X} \in \overline{M}_\tau \cap \{|x_3| \leqslant \tfrac{3}{4} e^{-\tau/10}\}$$

the tangent plane must be almost parallel to the one of the cylinder Σ at $\pi(\overline{X})$, where π denotes the nearest neighbor projection. It follows that

$$\overline{M}_\tau \cap \{|x_3| \leqslant \tfrac{3}{4} e^{-\tau/10}\}$$

can be written as graph of a function $v(\cdot,\tau)$ over the cylinder Σ with small C^1-norm. It is clear by looking at the middle region, that there is one single layer. Moreover, the C^1-norm bound and the regularity scale bound (4.18) imply

$$\|\nabla^2 v\|_{C^2(|x_3|\leqslant 3e^{-\tau/10}/4)} \leqslant C,$$

and thus v is the solution of a linear uniformly parabolic equation[18] with coefficients of uniformly small C^1-norm. Hence, using standard interior estimates, the assertion follows. □

We now repeat the process from §4.1 with improved functions ρ and σ. Namely, by Proposition 4.10, we can now choose

$$\rho(\tau) = e^{-\tau/20} \quad \text{and} \quad \sigma(\tau) = Ce^{\tau/20}.$$

With this new choice of ρ and σ, we can write $\overline{M}_\tau$ as graph of a function $u(\cdot,\tau)$ defined over the exponentially large domain $\Sigma\cap B_{2\rho(\tau)}$ such that it satisfies the estimate (4.2) for $\tau\leqslant\mathcal{T}$.

[18] Namely, $v_t = a_{ij}v_{ij} + b_i v_i + cv$, where $a=a(\nabla v, v)$, $b=b(\nabla v, v)$, $c=c(\nabla v, v)$. cf. (??).

PROPOSITION 4.11. *For $\tau \leqslant \mathcal{T}$ the function*

$$\hat{u}(x,\tau) = u(x,\tau)\varphi\Big(\frac{x_3}{e^{-\tau/20}}\Big)$$

satisfies the estimate

$$\|\hat{u}\|_{\mathcal{H}} \leqslant Ce^{\tau/2}.$$

In particular, we have

$$\sup_{\overline{M}_\tau \cap B_{10L_0}(0)} |x_1^2 + x_2^2 - 2| \leqslant Ce^{\tau/2}.$$

Proof. We define U_+, U_0, and U_- by the formulas from (**??**), where now

$$\hat{u}(x,\tau) = u(x,\tau)\varphi\Big(\frac{x_3}{e^{-\tau/20}}\Big).$$

Using Lemma 4.5 and Proposition 4.8, we then get the inequalities

$$U_0 + U_- \leqslant Ce^{\tau/20}U_+ \tag{4.19}$$

and

$$\frac{d}{d\tau}U_+ \geqslant U_+ - Ce^{\tau/20}U_+.$$

Rewriting the latter inequality as

$$\frac{d}{d\tau}\log(e^{-\tau}U_+) \geqslant -Ce^{\tau/20}$$

and integrating from τ to $\mathcal{T}$ yields the estimate

$$U_+(\tau) \leqslant Ce^{\tau},$$

which together with (4.19) implies that

$$\|\hat{u}\|_{\mathcal{H}} \leqslant Ce^{\tau/2}. \tag{4.20}$$

Using standard interior estimates, we conclude that

$$\sup_{\overline{M}_\tau \cap B_{10L_0}(0)} |x_1^2 + x_2^2 - 2| \leqslant Ce^{\tau/2}.$$

This proves the proposition. □

4.2.2. Constant functions cannot be dominant

To obtain refined information, it is useful to decompose

$$P_+ = P_{1/2} + P_1,$$

where $P_{1/2}$ is the projection to the span of z, $\cos\theta$, and $\sin\theta$, whereas P_1 is the projection to the span of the constant function 1. Accordingly, we can decompose

$$U_+ := \|P_+\hat{u}\|^2_{\mathcal{H}} = \|P_{1/2}\hat{u}\|^2_{\mathcal{H}} + \|P_1\hat{u}\|^2_{\mathcal{H}} =: U_{1/2} + U_1.$$

Using this decomposition, Lemma 4.5, and the assumption (4.8) that the plus mode is dominant,[19] we obtain

$$\left|\frac{d}{d\tau}U_{1/2} - U_{1/2}\right| \leqslant C\rho^{-1}(U_{1/2}+U_1),$$
$$\left|\frac{d}{d\tau}U_1 - 2U_1\right| \leqslant C\rho^{-1}(U_{1/2}+U_1).$$

Applying the Merle–Zaag lemma (Lemma 4.6) with $x = e^{-\tau}U_{1/2}$, $y=0$, and $z = e^{-\tau}U_1$, we infer that either the $U_{1/2}$ is dominant, i.e.

$$U_1 = o(U_{1/2}), \tag{4.21}$$

or the constant function 1 is dominant, i.e.

$$U_{1/2} \leqslant C\rho^{-1}U_1. \tag{4.22}$$

The next proposition shows that U_1 cannot be dominant.

PROPOSITION 4.12. *It must be the case that* $U_1 = o(U_{1/2})$.

Proof. Assume towards a contradiction that the estimate $U_1 = o(U_{1/2})$ does not hold. By the above discussion, we then have

$$U_- + U_0 + U_{1/2} \leqslant Ce^{\tau/20}U_1 \tag{4.23}$$

for all $\tau \leqslant \mathcal{T}$. Using Lemma 4.5, it follows that

$$\left|\frac{d}{d\tau}U_1 - 2U_1\right| \leqslant Ce^{\tau/20}U_1. \tag{4.24}$$

[19] By Proposition 4.8, we can indeed use this inequality with $\rho(\tau) = e^{-\tau/20}$.

These two inequalities imply that

$$\|\hat{u}\|_{\mathcal{H}} \leqslant Ce^{\tau},$$

(see the proof of Proposition 4.11), and thus in particular

$$\|u(\cdot,\tau)\|_{C^4([-10L_0,10L_0]\times[0,2\pi])} \leqslant Ce^{\tau}$$

for all $\tau \leqslant \mathcal{T}$.

Moreover, integrating (4.24) and using (4.23) yields that $e^{-2\tau}U_1$ converges to some positive limit $K_0>0$ for $\tau\to-\infty$ with the estimate

$$|e^{-2\tau}U_1 - K_0^2| \leqslant Ce^{\tau/20}$$

for $\tau \leqslant \mathcal{T}$.

We will now argue similarly as in [ADS1, proof of Lemma 5.11]: By the above estimates, there exists a constant K such that the rescaled flow $\overline{M}_\tau$ satisfies

$$x_1^2+x_2^2 = 2(1+Ke^{\tau})+O(e^{21\tau/20})$$

uniformly on the bounded interval $[-100,100]$, and more precisely

$$K_0 = |K| \int_\Sigma \Big(\frac{e}{8\pi}\Big)^{1/4} \frac{1}{4\pi} e^{-|x|^2/4} = |K| \Big(\frac{\pi}{2e}\Big)^{1/4}.$$

Hence, recalling that $\tau=-\log(-t)$, the original flow satisfies

$$x_1^2+x_2^2 = (-2t)(1+K(-t)^{-1})+O(|t|^{-1/20}) = 2(K-t)+O(|t|^{-1/20})$$

for $|x_3|\leqslant 100\sqrt{-t}$, where we assume for ease of notation that $X_0=(0,0)$. Now, if we instead rescale with respect to the new center $\widetilde{X}_0=(0,K)$, i.e. if we set $\tau=-\log(K-t)$, then the corresponding rescaled flow $\widetilde{M}_\tau$ satisfies

$$x_1^2+x_2^2 = 2+O(e^{21\tau/20}s). \tag{4.25}$$

Write $\widetilde{M}_\tau$ as the graph of a function $\widetilde{U}(\cdot,\tau)$ over

$$\Sigma\cap\{-\rho(\tau)\leqslant x_3\leqslant\rho(\tau)\}$$

with $\rho(\tau)=e^{-\tau/20}$ and consider the spectrum. Arguing as in the proof of Proposition 4.8, we see that $\widetilde{U}_1$ is dominant. Hence,

$$|e^{-2\tau}\widetilde{U}_1 - \widetilde{K}| \leqslant Ce^{\tau/20} \tag{4.26}$$

for some $\tilde{K}>0$. However, using (4.25), we can directly compute

$$\tilde{U}_1 = \Big| \int \varphi\tilde{u}\cdot \Big(\frac{e}{2\pi}\Big)^{1/4} \frac{1}{4\pi} e^{-|x|^2/4} \Big| \leqslant Ce^{21\tau/10},$$

This contradicts (4.26), and finishes the proof of the proposition. □

4.2.3. The fine neck theorem

By Propositions 4.11 and 4.12, we may now assume that

$$U_- + U_0 \leqslant C\rho^{-1}U_+, \tag{4.27}$$

and

$$U_1 = o(U_{1/2}), \tag{4.28}$$

where $\rho(\tau)=e^{-\tau/20}$. Recall in particular that Proposition 4.11 gives

$$\|\hat{u}\|_{\mathcal{H}} \leqslant Ce^{\tau/2}. \tag{4.29}$$

Moreover, using in addition equation (4.28) and the assumption that our solution is not the round shrinking cylinder, we see that

$$\lim_{\tau\to-\infty} e^{-\tau}U_{1/2} > 0. \tag{4.30}$$

Furthermore, by using (4.29), we establish the following coarse estimate in a compact region.

LEMMA 4.13. *For sufficiently large $-\tau$, we have*

$$\|u(\cdot,\tau)\|_{C^2(\Sigma\cap B_{10L_0}(0))} \leqslant Ce^{40\tau/81}. \tag{4.31}$$

Proof. Given a fixed sufficiently large $-\tau_0$, we define parabolic regions Q_r in $\Sigma\times\mathbb{R}$ by

$$Q_r = \{(\theta,z,\tau) : \theta\in(0,2\pi],\ |z|\leqslant 100L_0 r,\ -(100L_0 r)^2 \leqslant \tau_0-\tau\leqslant 0\}.$$

Let $\delta=(\frac{1}{10})^{1/8}$, and consider a smooth non-negative cutoff function $\eta_1(\theta,z,\tau)$ satisfying $\eta_1=0$ on ∂Q_1 and $\eta_1=1$ in Q_δ. Then, the function $\bar{u}_1=u\eta_1$ satisfies

$$\partial_\tau \bar{u}_1 = \mathcal{L}\bar{u}_1 + g_1,$$

where $|g_1|\leqslant C(|u|+|\nabla u|)$, due to Lemma 4.4. Using Proposition 4.3, this implies

$$\int_{Q_1} |g_1|^3 \leqslant C\int_{Q_1} |u|^3+|\nabla u|^3 \leqslant C\|u\|^2_{L^2(Q_1)}\,\|u\|_{C^1(Q_1)} \leqslant Ce^{\tau_0}.$$

Therefore, [WL1, Theorem 3.14] yields

$$\|u\|_{L^\infty(Q_\delta)} \leqslant \|\bar{u}_1\|_{L^\infty(Q_1)} \leqslant Ce^{\tau_0/3}.$$

Hence, the standard interior regularity implies $\|u\|_{C^2(Q_{\delta^2})}\leqslant Ce^{\tau_0/3}$.

Next, we consider a new cutoff η_2 satisfying $\eta_2=0$ on ∂Q_{δ^2} and $\eta_2=1$ in Q_{δ^3}. Then, $\bar{u}_2=u\eta_2$ satisfies $\partial_\tau\bar{u}_2=\mathcal{L}\bar{u}_2+g_2$, where

$$\int_{Q_{\delta^2}} |g_2|^3 \leqslant C\int_{Q_{\delta^2}} (|u|^3+|\nabla u|^3) \leqslant C\|u\|^2_{L^2(Q_{\delta^2})}\,\|u\|_{C^1(Q_{\delta^2})} \leqslant Ce^{4\tau_0/3}.$$

Hence, we have $\|u\|_{C^2(Q_{\delta^4})}\leqslant Ce^{4\tau_0/9}$. We repeat this process twice more, so that we can obtain the desired result. □

We will now express $P_+\hat{u}\in\mathcal{H}_+$ as linear combination of the four eigenfunctions z, $\cos\theta$, $\sin\theta$, and 1. Namely, let

$$\begin{aligned} a^X(\tau) &= \Big(\frac{e}{8\pi}\Big)^{1/4}\int z\hat{u}^X(x,\tau)\frac{1}{4\pi}e^{-|x|^2/4},\\ b^X(\tau) &= \Big(\frac{2e}{\pi}\Big)^{1/4}\int (\cos\theta)\hat{u}^X(x,\tau)\frac{1}{4\pi}e^{-|x|^2/4},\\ c^X(\tau) &= \Big(\frac{2e}{\pi}\Big)^{1/4}\int (\sin\theta)\hat{u}^X(x,\tau)\frac{1}{4\pi}e^{-|x|^2/4},\\ d^X(\tau) &= \Big(\frac{e}{2\pi}\Big)^{1/4}\int \hat{u}^X(x,\tau)\frac{1}{4\pi}e^{-|x|^2/4}, \end{aligned} \tag{4.32}$$

where the superscript X is to remind us that all these coefficients depend (a priori) on the choice of base point X. Then, we have

$$P_+\hat{u}^X = a^X z + b^X\cos\theta + c^X\sin\theta + d^X. \tag{4.33}$$

Moreover, $U_+^X = \|P_+\hat{u}^X\|_{\mathcal{H}}^2$ is given by a sum of coefficients squared:

$$U_+^X = 2^{-1/2}\pi^{1/2}e^{-1/2}(4|a^X|^2+|b^X|^2+|c^X|^2+2|d^X|^2). \tag{4.34}$$

Proposition 4.14. *There coefficients defined in (4.32) satisfy the estimates*

$$|d^X(\tau)| \leqslant Ce^{11\tau/20}, \tag{4.35}$$

and

$$|e^{-\tau/2}a^X(\tau)-\bar{a}^X| + |e^{-\tau/2}b^X(\tau)-\bar{B}^X| + |e^{-\tau/2}c^X(\tau)-\bar{c}^X| \leqslant Ce^{\tau/20} \tag{4.36}$$

for some numbers $\bar{a}^X$, $\bar{B}^X$, and $\bar{c}^X$ that might depend on X.

Proof. Using Lemma 4.5 and $\mathcal{L}1=1$, we compute

$$\begin{aligned} \frac{d}{d\tau}d^X(\tau) &= \Big(\frac{e}{2\pi}\Big)^{1/4}\int(\mathcal{L}\hat{u}+\widehat{E})\frac{1}{4\pi}e^{-|x|^2/4}\\ &= \Big(\frac{e}{2\pi}\Big)^{1/4}\int(\hat{u}+\widehat{E})\frac{1}{4\pi}e^{-|x|^2/4}\\ &= d^X(\tau)+\Big(\frac{e}{2\pi}\Big)^{1/4}\int\frac{\widehat{E}}{4\pi}e^{-|x|^2/4}. \end{aligned}$$

Hence,

$$\Big|\frac{d}{d\tau}(e^{-\tau}d^X(\tau))\Big| \leqslant Ce^{-\tau}\|\widehat{E}\|_{\mathcal{H}} \leqslant Ce^{-\tau}\rho^{-1}\|\hat{u}\|_{\mathcal{H}} \leqslant Ce^{-9\tau/20}.$$

Integrating from τ to $\mathcal{T}$, this implies (4.35).

In a similar manner, we compute

$$\left|\frac{d}{d\tau}(e^{-\tau/2}a^X(\tau))\right|+\left|\frac{d}{d\tau}(e^{-\tau/2}b^X(\tau))\right|+\left|\frac{d}{d\tau}(e^{-\tau/2}c^X(\tau))\right|\leqslant Ce^{-\tau/2}\rho^{-1}\|\hat{u}\|_{\mathcal{H}}\leqslant Ce^{\tau/20}.$$

Integrating from $-\infty$ to τ this implies (4.36). □

We are now ready to prove the main theorem of this section.

THEOREM 4.15. *Let $\mathcal{M}$ be an ancient low-entropy flow which is not the round shrinking sphere, round shrinking cylinder or flat plane. If the plus mode is dominant, then there are constants $\bar{a}=\bar{a}(\mathcal{M})\neq 0$ and $C=C(\mathcal{M})<\infty$, and a decreasing function $\mathcal{T}:\mathbb{R}_+\to\mathbb{R}_-$ (depending on $\mathcal{M}$) with the following significance.*

For every $X\in\mathcal{M}$ the graph function $u^X(\cdot,\tau)$ of the rescaled flow $\overline{M}^X_\tau$ satisfies the estimates

$$\|e^{-\tau/2}\hat{u}^X(x,\tau)-\bar{a}z-\bar{b}^X\cos\theta-\bar{c}^X\sin\theta\|_{\mathcal{H}}\leqslant Ce^{\tau/40}, \tag{4.37}$$

and

$$\sup_{|x_3|\leqslant 10L_0}|e^{-\tau/2}u^X(x,\tau)-\bar{a}z-\bar{b}^X\cos\theta-\bar{c}^X\sin\theta|\leqslant Ce^{\tau/160} \tag{4.38}$$

for $\tau\leqslant\mathcal{T}(Z(X))$. Here, the constant $\bar{a}$ is independent of X, and $\bar{b}^X$ and $\bar{c}^X$ are numbers that may depend on X and satisfy

$$|\bar{b}^X|+|\bar{c}^X|\leqslant C. \tag{4.39}$$

Proof. By scaling, we may pretend without essential loss of generality that we only work with center points X satisfying $Z(X)\leqslant 1$ (this condition is only used to figure out more easily which constants are uniform in X).

Consider the difference

$$D^X:=\hat{u}^X-e^{\tau/2}(\bar{a}^Xz+\overline{B}^X\cos\theta+\bar{c}^X\sin\theta).$$

Using equation (4.33) and Proposition 4.14, we see that

$$|D^X|\leqslant|\hat{u}^X-P_+\hat{u}^X|+C(1+|z|)e^{11\tau/20}.$$

Since by (4.27) and (4.29) we have

$$U_-+U_0\leqslant Ce^{21\tau/20},$$

it follows that

$$\|D^X\|_{\mathcal{H}} \leqslant Ce^{21\tau/40}, \tag{4.40}$$

which proves (4.37) modulo the claim about the coefficients.

Combining the equations (4.29) and (4.34) and Proposition 4.14, we see that

$$|\overline{B}^X| + |\bar{c}^X| \leqslant C, \tag{4.41}$$

which proves (4.39).

We recall that $e^{-\tau/2}u$ corresponds to the original scale. Hence, if instead of

$$X = (x_1, x_2, x_3, t)$$

we consider the new origin

$$X' = (x_1 - \bar{b}^X \cos\theta, x_2 - \bar{c}^X \sin\theta, x_3, t), \tag{4.42}$$

then the estimate (4.40) simplifies to

$$\|\hat{u}^{X'}(x,\tau) - e^{\tau/2}\bar{a}^X z\|_{\mathcal{H}} \leqslant Ce^{21\tau/40}, \tag{4.43}$$

i.e. the estimate (4.37) holds with $\bar{a}^{X'} = \bar{a}^X$, $\bar{b}^{X'} = 0$, and $\bar{c}^{X'} = 0$. If $\bar{a}^X = 0$, then (4.43) implies that $\|\hat{u}^{X'}\|^2_{\mathcal{H}} \leqslant Ce^{21\tau/20}$, contradicting (4.30). Here, we have used Proposition 4.8, as well as Proposition 4.12, to show that, even after re-centering, the $\frac{1}{2}$-mode dominates. Hence, $\bar{a}^X \neq 0$. Since the estimate (4.43) holds for any X, and since $\bar{a}^X$ does not vanish for any X, we see that $\bar{a}^X =: \bar{a}$ is independent of X.

It remains to prove the pointwise estimate (4.38). To this end, we start with

$$\|D^X\|_{L^2(\Sigma \cap \{|z| \leqslant 10L_0\})} \leqslant C\|D^X\|_{\mathcal{H}} \leqslant Ce^{21\tau/40}.$$

Next, using the inequality (4.41) and Lemma 4.13, we estimate

$$\begin{aligned}\|\nabla D^X\|_{L^2(\Sigma \cap \{|z| \leqslant 10L_0\})} + \|\nabla^2 D^X\|_{L^2(\Sigma \cap \{|z| \leqslant 10L_0\})} &\leqslant C\|\hat{u}^X\|_{C^2(\Sigma \cap \{|z| \leqslant 10L_0\})} + Ce^{\tau/2} \\ &\leqslant Ce^{40\tau/81}.\end{aligned}$$

Applying Agmon's inequality, we conclude that

$$\sup_{|x_3| \leqslant 10L_0} |D^X| \leqslant Ce^{81\tau/160}.$$

This finishes the proof of the theorem. □

After a change of coordinates, we may assume without loss of generality that our ancient low-entropy flow $\mathcal{M}$ satisfies $\bar{a}=\bar{a}(\mathcal{M})=1/\sqrt{2}$. Then, after recentering as in (4.42), Theorem 4.15 tells us that the graph $u^X(\cdot,\tau)$ of the rescaled flow $\overline{M}_\tau^X$ satisfies

$$\sup_{|x_3|\leqslant 10L_0}\left|e^{-\tau/2}u^X(x,\tau)-\frac{1}{\sqrt{2}}z\right|\leqslant Ce^{\tau/160} \tag{4.44}$$

for $\tau\leqslant\mathcal{T}(Z(X))$.

COROLLARY 4.16. *If the rescaled flow $\overline{M}_\tau^X$ satisfies (4.44), then we have the estimates*

$$\sup_{\{x_3\leqslant -L_0\}}(x_1^2+x_2^2)\leqslant 2,$$
$$\inf_{\{x_3\geqslant L_0\}}(x_1^2+x_2^2)\geqslant 2,$$

and

$$\inf_{\overline{M}_\tau^X} x_3\leqslant -\mu e^{-\tau/2}$$

for $\tau\leqslant\mathcal{T}(Z(X))$, where $\mu>0$ is a numeric constant. In particular, the unrescaled mean-curvature flow $\mathcal{M}=\{M_t\}$ satisfies

$$\inf_{p\in M_t} x_3(p)>-\infty \quad \textit{and} \quad \sup_{p\in M_t} x_3(p)=\infty.$$

Proof. The first two estimates follow easily by using the KM-shrinkers $\widetilde{\Sigma}_b$ as outer barriers (using also Corollary 3.10) and the ADS-shrinkers Σ_a as inner barriers; since X_0 is arbitrary, the latter also implies that $\sup_{p\in M_t} x_3(p)=\infty$.

The third estimate follows from the improved barrier argument (where the surfaces Σ_a are shifted along the z-axis) from [BC, §3].

Finally, for any $X_0=(x_0,t_0)\in\mathcal{M}$, after recentering in space, the estimates tells us that $\overline{M}_\tau^{X_0}\cap\{x_3\leqslant -L_0\}$ is contained inside the cylinder Σ of radius $\sqrt{2}$ for $\tau\leqslant\mathcal{T}$. Hence, by comparison $M_{t_0}\cap\{x_3\leqslant -L_0\}$ is compact (see also Corollary 4.30 and its proof). Since X_0 is arbitrary, this yields

$$\inf_{p\in M_t} x_3(p)>-\infty. \qquad \square$$

4.3. Fine analysis in the neutral mode

In this section, we assume that the neutral mode is dominant. The goal is to prove Corollary 4.30, which show that the solution is compact. This will follow from Theorem 4.28, which gives a precise inwards quadratic expansion, and a barrier argument.

Given any center $X_0 \in \mathcal{M}$, there exists some functions σ and ρ satisfying (4.1) such that (4.2) holds, and we have

$$U_- + U_+ = o(U_0), \quad |\partial_\tau U_0| \leqslant o(U_0) \tag{4.45}$$

for $\tau \leqslant \mathcal{T}$. In this section, $C < \infty$ and $\mathcal{T} > -\infty$ denote constants that can change from line to line and may also depend on $X_0 \in \mathcal{M}$. To distinguish the initial choice of ρ, we set

$$\rho_0(z) = \rho(z). \tag{4.46}$$

We will later use improved scale functions, but ρ_0 will never change.

4.3.1. Graphical radius

The first goal is to prove a lower bound for the optimal graphical radius. To begin with, we consider the positive function

$$\alpha(\tau) = \left(\int_{|x_3| \leqslant L_0} u^2(x,s) \frac{1}{4\pi} e^{-|x|^2/4} \right)^{1/2}. \tag{4.47}$$

LEMMA 4.17. *For $L \in [L_0, \rho(\tau)]$, we have the estimate*

$$\alpha(\tau)^2 \leqslant \int_{\Sigma \cap \{|x_3| \leqslant L\}} u^2(x,\tau) \frac{1}{4\pi} e^{-|x|^2/4} \leqslant C\alpha(\tau)^2.$$

Proof. By Proposition 4.3, we have

$$\int_{\Sigma \cap \{2^{k-1}L_0 \leqslant |x_3| \leqslant 2^k L_0\}} u^2 e^{-|x|^2/4} \leqslant \frac{C_0}{4^k L_0^2} \int_{\Sigma \cap \{|x_3| \leqslant 2^{k-1} L_0\}} u^2 e^{-|x|^2/4},$$

where C_0 is a constant. Hence,

$$\int_{\Sigma \cap \{|x_3| \leqslant 2L_0\}} u^2 \frac{1}{4\pi} e^{-|x|^2/4} \leqslant \left(1 + \frac{C_0}{4L_0^2}\right) \alpha(\tau)^2 \leqslant \exp\left(\frac{C_0}{4L_0^2}\right) \alpha(\tau)^2,$$

and

$$\begin{aligned} \int_{\Sigma \cap \{|x_3| \leqslant 2^2 L_0\}} u^2 \frac{1}{4\pi} e^{-|x|^2/4} &\leqslant \left(1 + \frac{C_0}{4^2 L_0^2}\right) \exp\left(\tfrac{C_0}{4L_0^2}\right) \alpha(\tau)^2 \\ &\leqslant \exp\left(\left(\frac{1}{4} + \frac{1}{4^2}\right) \frac{C_0}{L_0^2}\right) \alpha(\tau)^2. \end{aligned}$$

Thus, if $2^{I-1} L_0 \leqslant \rho(\tau) \leqslant 2^I L_0$ then we inductively obtain

$$\int_{\Sigma \cap \{|x_3| \leqslant \rho(\tau)\}} u^2 \frac{1}{4\pi} e^{-|x|^2/4} \leqslant \exp\left(\sum_{i=1}^{I} \frac{1}{4^i} \frac{C_0}{L_0^2} \right) \alpha(\tau)^2 \leqslant \exp\left(\frac{2C_0}{L_0^2} \right) \alpha(\tau)^2.$$

This completes the proof. □

Now, define an increasing continuous function by

$$\bar{\alpha}(\tau) = \sup_{\sigma \leqslant \tau} \alpha(\sigma). \tag{4.48}$$

By standard interior estimates, we have

$$|u|(x,\tau) \leqslant C\bar{\alpha}(\tau) \tag{4.49}$$

for $|x_3| \leqslant L_0$ and $\tau \leqslant \mathcal{T}$.

For technical reasons, it will be best to work with a monotone function β, which simultaneously has controlled derivatives. To this end, we define

$$\beta(\tau) = \sup_{\sigma \leqslant \tau} \left(\int_\Sigma u^2(x,\sigma) \varphi^2 \left(\frac{x_3}{\rho_0(\sigma)} \right) \frac{1}{4\pi} e^{-|x|^2/4} \right)^{1/2},$$

where we recall that ρ_0 is defined in (4.46) to be the original graphical scale function, which is the input to this section. Clearly, β is a locally Lipschitz, increasing function. By equation (4.45), we have $\beta' = o(\beta)$ at almost every time,[20] so in particular

$$0 \leqslant \beta'(\tau) \leqslant \tfrac{1}{5}\beta(\tau). \tag{4.50}$$

Moreover, by Lemma 4.17 we have

$$\bar{\alpha}(\tau) \leqslant \beta(\tau) = \left(\int_\Sigma u^2(x,\sigma) \varphi^2 \left(\frac{x_3}{\rho_0(\sigma)} \right) \frac{1}{4\pi} e^{-|x|^2/4} \right)^{1/2} \leqslant C\alpha(\sigma) \leqslant C\bar{\alpha}(\tau),$$

where σ is chosen such that the second equality holds. To recapitulate, we have obtained

$$\bar{\alpha}(\tau) \leqslant \beta(\tau) \leqslant C\bar{\alpha}(\tau). \tag{4.51}$$

We now prove a C^0-estimate in terms of $\beta(\tau)$.

PROPOSITION 4.18. *There are constants $c>0$ and $C<\infty$ such that*

$$|u|(x,\tau) \leqslant C\beta(\tau)^{1/2}$$

holds whenever $|x_3| \leqslant c\beta(\tau)^{-1/4}$ and $\tau \leqslant \mathcal{T}$.

[20] If one prefers thinking about smooth objects, one could further regularize β to be C^1, maintaining similar properties.

Proof. By the estimates (4.49) and (4.51), there is a constant $K<\infty$ such that

$$|u|(x,\tau) \leqslant K\beta(\tau) \tag{4.52}$$

whenever $|x_3|\leqslant L_0$ and $\tau\leqslant\mathcal{T}$.

Fixing $\hat{\tau}\leqslant\mathcal{T}$, we consider the ADS-shrinker Σ_a with parameter

$$a = c(K\beta(\hat{\tau}))^{-1/2}, \tag{4.53}$$

where $c>0$ is a numerical constant depending on L_0. Then, by [ADS1, Lemma 4.4], the profile function u_a satisfies

$$u_a(L_0) \leqslant \sqrt{2} - K\beta(\hat{\tau}). \tag{4.54}$$

Comparing (4.52) and (4.54), we infer that $\overline{M}_\tau \cap \{x_3 = L_0\}$ lies outside of

$$\Sigma_a \cap \{x_3 = L_0\}$$

for $\tau\leqslant\hat{\tau}$.

On the other hand, similarly as in (4.16), we have

$$0 \leqslant \frac{d}{dz}\log\left(\frac{2-u_a(z)^2}{z^2}\right) \leqslant \frac{16}{z^3}.$$

Integration from $\sqrt{a}$ to a yields

$$0 \leqslant \log\left(\frac{2}{a^2}\right) - \log\left(\frac{2-u_a(\sqrt{a})^2}{a}\right) \leqslant 1,$$

and hence

$$u_a(\sqrt{a})^2 \geqslant 2 - \frac{2}{a}. \tag{4.55}$$

Since $\overline{M}_{\hat{\tau}} \cap \{x_3 \geqslant L_0\}$ lies outside of $\Sigma_a \cap \{x_3 \geqslant L_0\}$, comparing the inequalities (4.53) and (4.55) yields

$$u(x,\hat{\tau}) \geqslant -C\beta(\hat{\tau})^{1/2} \tag{4.56}$$

for $\{L_0 \leqslant x_3 \leqslant \sqrt{a}\}$ (and similarly for negative x_3). Hence, (4.52) implies that (4.56) holds for

$$|x_3| \leqslant \sqrt{a} = C_0^{1/2}(K\beta(\hat{\tau}))^{-1/4}.$$

Next, we establish the upper bound. By using (4.16), we have

$$e^{-1}b^2L_0^2 \leqslant \widetilde{U}_b^2(L_0) - 2 \leqslant 4(\widetilde{U}_b(L_0) - \sqrt{2}),$$

for sufficiently small b. Therefore, if we choose

$$b^2 = 4eL_0^{-2}K\beta(\hat{\tau}),$$

then we have

$$K\beta(\hat{\tau}) = \frac{1}{4e}L_0^2 b^2 \leqslant \widetilde{U}_b(L_0) - \sqrt{2}.$$

Hence, (4.52) implies that $\overline{M}_\tau \cap \{x_3 \geqslant L_0\}$ lies inside of $\widetilde{\Sigma}_b \cap \{x_3 \geqslant L_0\}$ for $\tau \leqslant \hat{\tau}$. Then, (4.16) implies that

$$\widetilde{U}_b^2\Big(\frac{1}{\sqrt{b}}\Big) \leqslant 2+b.$$

Thus, we can complete the proof by arguing similarly with the KM-shrinkers $\widetilde{\Sigma}_b$ as outer barriers. □

Similarly as in Proposition 4.10, the C^0-estimate from Proposition 4.18 can be upgraded to a C^4-estimate. Hence, we can now repeat the process from §4.1 with better functions ρ and σ. Namely, we can now choose

$$\rho(\tau) = \beta(\tau)^{-1/5} \quad \text{and} \quad \sigma(\tau) = \beta(\tau)^{1/5}, \tag{4.57}$$

and write $\overline{M}_\tau$ as a graph of a function u over $\Sigma \cap B_{2\rho(\tau)}$ such that

$$\|u(\cdot,\tau)\|_{C^4(\Sigma \cap B_{2\rho(\tau)}(0))} \leqslant \rho(\tau)^{-2} \tag{4.58}$$

for $\tau \leqslant \mathcal{T}$. Note that, by equation (4.50), the derivative ρ' indeed satisfies

$$-\rho(\tau) \leqslant \rho'(\tau) \leqslant 0, \tag{4.59}$$

as required by condition (4.1). From now on, we work with the function

$$\hat{u}(x,\tau) = u(x,\tau)\varphi\Big(\frac{x_3}{\rho(\tau)}\Big),$$

where ρ is the improved graphical radius from (4.57).

The following gives a lower bound for the improved graphical radius.

PROPOSITION 4.19. *There exists constants $\gamma > 0$ and $c > 0$ such that*

$$\rho(\tau) \geqslant c|\tau|^\gamma. \tag{4.60}$$

holds for $\tau \leqslant \mathcal{T}$.

Proof. By Colding–Minicozzi [CM2, Theorem 6.1], there exist $\eta\in\left(\frac{1}{3},1\right)$ and $K<\infty$ such that

$$(F(\Sigma)-F(\overline{M}_\tau))^{1+\eta}\leqslant K(F(\overline{M}_{\tau-1})-F(\overline{M}_{\tau+1}))$$

for $\tau\leqslant\mathcal{T}$. Using the discrete Lojasiewicz lemma [CM2, Lemma 6.9] and also [CM2, formulas (6.19) and (6.20)], this yields

$$(F(\Sigma)-F(\overline{M}_\tau))\leqslant C|\tau|^{-1/\eta},$$

and

$$\sum_{j=J}^{\infty}(F(\overline{M}_{-j-1})-F(\overline{M}_{-j}))^{1/2}\leqslant C(\nu)J^{-\nu} \tag{4.61}$$

for every

$$\nu\in\left(0,\frac{1}{2\eta}-\frac{1}{2}\right).$$

Thus, as in the proof of [CM2, Theorem 0.2], we obtain

$$\int_{-\infty}^{\tau}\int_{\overline{M}_{\tau'}}\left|\mathbf{H}+\frac{x^\perp}{2}\right|e^{-|x|^2/4}\,d\mu_{\tau'}\,d\tau'\leqslant C|\tau|^{-\nu}$$

for $\tau\leqslant\mathcal{T}$. Using also [CM2, Lemma A.48], this yields

$$\int_{\Sigma\cap\{|x_3|\leqslant\rho(\tau)/2\}}|\hat{u}(x,\tau)|e^{-|x|^2/4}\leqslant C|\tau|^{-\nu}.$$

Together with (4.58), this implies

$$\alpha(\tau)\leqslant C|\tau|^{-\nu/2}.$$

Taking also into account (4.51), we infer that

$$\beta(\tau)\leqslant C|\tau|^{-\nu/2}. \tag{4.62}$$

Recalling (4.57), we conclude that (4.60) holds with $\gamma=\frac{1}{10}\nu$. □

4.3.2. Estimates for some error terms

In the following lemma we Taylor expand the rescaled mean-curvature flow to second order.

LEMMA 4.20. *The function*

$$\hat{u}(x,\tau)=u(x,\tau)\varphi\Big(\frac{x_3}{\rho(\tau)}\Big)$$

satisfies

$$\partial_\tau\hat{u}=\mathcal{L}\hat{u}-\frac{1}{2\sqrt{2}}\hat{u}^2-\frac{1}{2\sqrt{2}}\hat{u}_\theta^2-\frac{1}{\sqrt{2}}\hat{u}\hat{u}_{\theta\theta}+E,$$

where the error term can be estimated by

$$\begin{aligned}|E|\leqslant{}&C\varphi(|u|+|\nabla u|)^2(|u|+|\nabla u|+|\nabla^2u|)\\&+C|\varphi'|\rho^{-1}\big(|\nabla u|+|zu|\big)+C|\varphi''|\rho^{-2}|u|\\&+C\varphi(1-\varphi)(|u|^2+|\nabla u|^2+|\nabla^2u|^2).\end{aligned}\tag{4.63}$$

Proof. The rescaled mean-curvature flow for graphs over the cylinder is given by

$$\begin{aligned}\partial_\tau u=\frac{\Big(1+\frac{u_\theta^2}{(\sqrt{2}+u)^2}\Big)u_{zz}-\frac{u_\theta^2u_{\theta\theta}}{(\sqrt{2}+u)^4}-\frac{2u_\theta u_z u_{z\theta}}{(\sqrt{2}+u)^2}-\frac{u_\theta^2}{(\sqrt{2}+u)^3}}{1+u_z^2+u_\theta^2(\sqrt{2}+u)^{-2}}\\+\frac{u_{\theta\theta}}{(\sqrt{2}+u)^2}-\frac{1}{\sqrt{2}+u}+\frac{1}{2}(\sqrt{2}+u-zu_z),\end{aligned}$$

cf. [GKS, Appendix A]. Hence,

$$\partial_\tau u=u_{zz}-\frac{u_\theta^2}{(\sqrt{2}+u)^3}+\frac{u_{\theta\theta}}{(\sqrt{2}+u)^2}-\frac{1}{\sqrt{2}+u}+\frac{1}{2}(\sqrt{2}+u-zu_z)+E_1,$$

where

$$|E_1|\leqslant C|\nabla u|^2(|\nabla u|+|\nabla^2u|).$$

This implies

$$\partial_\tau u=\mathcal{L}u+\mathcal{Q}(u)+E_2,\tag{4.64}$$

where

$$\mathcal{Q}(u)=-\frac{1}{2\sqrt{2}}u^2-\frac{1}{2\sqrt{2}}u_\theta^2-\frac{1}{\sqrt{2}}uu_{\theta\theta},$$

and

$$|E_2|\leqslant C|\nabla u|^2(|u|+|\nabla u|+|\nabla^2u|)+C|u|^2|\nabla^2u|+C|u|^3.$$

We now want to see what this computation translates to in terms of $\hat{u}$. For the linear term, using (4.59) we get

$$|\partial_\tau\hat{u}-\mathcal{L}\hat{u}-\varphi(\partial_\tau u-\mathcal{L}u)|\leqslant\rho^{-2}|\varphi''|\,|u|+2\rho^{-1}|\varphi'|(|\nabla u|+|zu|).\tag{4.65}$$

For the quadratic term, using that ρ is θ-independent, we get

$$|\mathcal{Q}(\hat{u})-\varphi\mathcal{Q}(u)|=|\varphi^2\mathcal{Q}(u)-\varphi\mathcal{Q}(u)|=\varphi(1-\varphi)|\mathcal{Q}(u)|.\tag{4.66}$$

Putting everything together, this implies the assertion. □

For our function

$$\hat{u}(x,\tau) = u(x,\tau)\varphi\Big(\frac{x_3}{\rho(\tau)}\Big),$$

where ρ is the improved graphical scale from (4.57), we let

$$U_+ = \|P_+\hat{u}\|^2_{\mathcal{H}}, \quad U_0 = \|P_0\hat{u}\|^2_{\mathcal{H}}, \quad \text{and} \quad U_- = \|P_-\hat{u}\|^2_{\mathcal{H}}.$$

By assumption (4.45) and Proposition 4.8, we still have

$$U_+ + U_- = o(U_0). \tag{4.67}$$

Therefore, we can now expand

$$\hat{u} = \alpha_1\psi_1 + \alpha_2\psi_2 + \alpha_3\psi_3 + o(|\vec{\alpha}|), \tag{4.68}$$

where

$$\vec{\alpha}(\tau) = (\alpha_1(\tau), \alpha_2(\tau), \alpha_3(\tau))$$

are time-dependent coefficients, and where ψ_1, ψ_2, and ψ_3 are the three zero eigenfunctions satisfying $\mathcal{L}\psi_i = 0$, explicitly

$$\begin{aligned}
\psi_1 &= 2^{-3/2}\Big(\frac{e}{2\pi}\Big)^{1/4}(z^2-2),\\
\psi_2 &= \Big(\frac{e}{2\pi}\Big)^{1/4} z\cos\theta,\\
\psi_3 &= \Big(\frac{e}{2\pi}\Big)^{1/4} z\sin\theta.
\end{aligned}$$

Moreover, Lemma 4.17 and equation (4.67) yield that

$$C^{-1}\alpha(\tau) \leqslant |\vec{\alpha}|(\tau) \leqslant C\alpha(\tau) \tag{4.69}$$

for $\tau \leqslant \mathcal{T}$, where $\alpha(\tau)$ is the function defined in equation (4.47).

PROPOSITION 4.21. *The error term E from Lemma 4.20 satisfies the estimate*

$$|\langle E, \psi_i\rangle| \leqslant C\beta(\tau)^{2+1/5}$$

for $\tau \leqslant \mathcal{T}$.

Proof. Using equation (4.58) and Lemma 4.20, we see that

$$|E| \leqslant C\rho(\tau)^{-2}.$$

Together with $|\psi_i|\leqslant C\rho(\tau)^2$ on the support of E, this yields the coarse estimate

$$|E|\,|\psi_i|\leqslant C.$$

Using this, we compute

$$\begin{aligned}\Big|\int_{\Sigma\cap\{|x_3|\geqslant\rho(\tau)^{1/10}\}} E\psi_i e^{-|x|^2/4}\Big| &\leqslant C\int_{\Sigma\cap\{\rho(\tau)^{1/10}\leqslant|x_3|\leqslant\rho(\tau)\}} e^{-\rho(\tau)^{1/5}/4}\\ &\leqslant C\beta(\tau)^3\end{aligned}$$

for $\tau\leqslant\mathcal{T}$.

On the other hand, if $|x_3|\leqslant\frac{1}{2}\rho(\tau)$, we have

$$\varphi'=\varphi''=1-\varphi=0.$$

Therefore, for $|x_3|\leqslant\rho(\tau)^{1/10}$ we can estimate

$$\begin{aligned}|E|\,|\psi_i| &\leqslant C(z^2+2)(|u|+|\nabla u|)^2(|u|+|\nabla u|+|\nabla^2 u|)\\ &\leqslant C\rho^{3/10}\rho^{-2}(|u|^2+|\nabla u|^2)\\ &\leqslant C\rho^{-1}(|u|^2+|\nabla u|^2).\end{aligned}$$

Thus, using also Proposition 4.3 and Lemma 4.17, we infer that

$$\begin{aligned}\Big|\int_{\Sigma\cap\{|x_3|\leqslant\rho(\tau)^{1/10}\}} E\psi_i e^{-|x|^2/4}\Big| &\leqslant \frac{C}{\rho(\tau)}\int_{\Sigma\cap\{|x_3|\leqslant\rho(\tau)\}}(u^2+|\nabla u|^2)e^{-|x|^2/4}\\ &\leqslant C\rho(\tau)^{-1}\alpha(\tau)^2\\ &\leqslant C\beta(\tau)^{2+1/5}.\end{aligned}$$

This proves the proposition. □

4.3.3. Axis tilt decay

Our next objective is to show that rotations, captured by the coefficients α_2 and α_3 in the expansion (4.68), are decaying rapidly.

Given a point $q\in\mathbb{R}^3$ and a direction $w\in S^2$, we denote by K the *normalized rotation vector field* that corresponds to a rotation by $\frac{1}{2}\pi$ around the axis $W=\{q+wt:t\in\mathbb{R}\}$, namely

$$K(x)=SJS^{-1}(x-q),\quad \text{where } J=\begin{pmatrix}0&-1&0\\1&0&0\\0&0&0\end{pmatrix},$$

and $S\in\mathrm{SO}_3$ is any rotation matrix with $Se_3=w$.[21]

[21] S is only determined up to a 1-parameter choice, but K is well-defined.

Definition 4.22. (Cf. [BC, Definition 4.3]) A point $X \in \mathcal{M}$ with $H(X)>0$ is called *δ-symmetric* if there exists a normalized rotation vector field K such that

$$|\langle K, \nu\rangle H| \leqslant \delta \quad \text{in the parabolic ball } P(X, 10H^{-1}(X)).$$

LEMMA 4.23. *There exists some $\eta_0>0$ such that all points $X \in \mathcal{M}$ that correspond to points in $\overline{M}_\tau \cap \{|x_3| \leqslant \rho(\tau)\}$ are $2^{-\eta_0 \rho(\tau)}$-symmetric for $\tau \leqslant \mathcal{T}$.*

Proof. Fix some time $\tau_0 \leqslant \mathcal{T}$ and set $R:=2\rho(\tau_0)$. Let $\varepsilon_0>0$ and $L<\infty$ be the constant from the neck-improvement theorem [BC, Theorem 4.4].

Let $\overline{X} \in \overline{M}_\tau$ be a point with $|x_3| \leqslant R-L$ and $\tau \leqslant \tau_0$, and denote by $X \in \mathcal{M}$ the corresponding point in the unrescaled flow. Using (4.58) we see that every

$$X' \in P(X, LH^{-1}(X)) \cap \mathcal{M}$$

lies on an ε_0-neck and is ε_0-symmetric. Hence, by the neck improvement theorem, X is $2^{-1}\varepsilon_0$-symmetric.

Similarly, for any $\overline{X} \in \overline{M}_\tau$ with $|x_3| \leqslant R-2L$ and $\tau \leqslant \tau_0$, the arguments above show that the corresponding point $X \in \mathcal{M}$ is $2^{-2}\varepsilon_0$-symmetric. Iterating this k times, until

$$2R-(k+1)L < R, \tag{4.70}$$

we get that for all $\overline{X} \in \overline{M}_\tau$ with $|x_3| \leqslant \frac{1}{2}R$ and $\tau \leqslant \tau_0$, the corresponding point $X \in \mathcal{M}$ is $2^{-k}\varepsilon_0$-symmetric. Since from (4.70), we get $k>R/L-1$, choosing $\eta_0=1/L$ gives the desires result. □

PROPOSITION 4.24. (Axis tilt decay) *There exists a constant $\eta>0$ such that for all $\tau \leqslant \mathcal{T}$ we have*

$$|u_\theta(x,\tau)| \leqslant e^{-\eta|\tau|^\gamma}$$

for all $x \in \Sigma \cap \{|x_3| \leqslant c|\tau|^\gamma\}$.

Proof. For each $\tau \leqslant \mathcal{T}$, consider a point $\overline{X}_\tau \in \overline{M}_\tau \cap \{x_3=0\}$, and let $\omega_\tau \in S^2$ be a direction (close to e_3) in which the corresponding point $X_\tau \in \mathcal{M}$ in the unrescaled flow is $2^{-\eta\rho(\tau)}$-symmetric. Since each $X_{\tau-1}$ is $C2^{-\eta\rho(\tau)}$-symmetric with respect to both ω_τ and $\omega_{\tau-1}$, we get

$$|\omega_\tau - \omega_{\tau-1}| \leqslant C2^{-\eta_0\rho(\tau)}.$$

Hence, using also that $\omega_\tau \to e_3$ as $\tau \to -\infty$ and Proposition 4.19, we obtain

$$|\omega_\tau - e_3| \leqslant C\sum_{m=0}^{\infty} 2^{-\eta_0\rho(\tau-m)} \leqslant C\sum_{m=0}^{\infty} 2^{-c\eta_0|\tau-m|^\gamma}.$$

Note that this can be safely estimated by $Ce^{-\eta|\tau|^\gamma}$, where $\eta:=\frac{1}{10}\eta_0$.

As any $x\in\overline{M}_\tau$ with $|x_3|\leqslant c|\tau|^\gamma$ is also $C2^{-\eta\rho(\tau)}$-symmetric with direction $\omega(x,\tau)$, we get that

$$|\omega(x,\tau)-\omega_\tau|\leqslant C|\tau|^\gamma 2^{-\eta\rho(\tau)},$$

which, again, can be safely estimated by $Ce^{-\eta|\tau|^\gamma}$. Thus, all such points are also $Ce^{-\eta|\tau|^\gamma}$-symmetric with respect to the x_3-axis. Considering the normalized rotation vector field $K:=x_1\partial_{x_2}-x_2\partial_{x_1}$ around the x_3-axis, we conclude that

$$\tfrac{1}{2}|u_\theta|\leqslant H|\langle K,\nu\rangle|\leqslant Ce^{-\eta|\tau|^\gamma}.$$

Slightly decreasing η, this proves the proposition. □

COROLLARY 4.25. *The coefficients α_2 and α_3 from the expansion (4.68) satisfy*

$$|\alpha_2|+|\alpha_3|\leqslant Ce^{-\eta|\tau|^\gamma} \tag{4.71}$$

for $\tau\leqslant\mathcal{T}$.

Proof. Using $\psi_2=\partial_\theta\psi_3$ and Proposition 4.24, we obtain

$$|\alpha_2|=|\langle\hat{u},\partial_\theta\psi_3\rangle|=|\langle\partial_\theta\hat{u},\psi_3\rangle|\leqslant\|\partial_\theta\hat{u}\|_{\mathcal{H}}\leqslant Ce^{-\eta|\tau|^\gamma}.$$

Similarly, we can estimate $|\alpha_3|$ to complete the proof. □

4.3.4. The inwards quadratic neck theorem

The next proposition gives an ODE for the coefficient α_1 from the expansion (4.68).

PROPOSITION 4.26. *The coefficient α_1 from the expansion (4.68) satisfies*

$$\frac{d}{d\tau}\alpha_1=-\Big(\frac{e}{2\pi}\Big)^{1/4}\alpha_1^2+o(\beta^2)+O(e^{-\eta|\tau|^\gamma}). \tag{4.72}$$

Proof. Using Lemma 4.20 and the identities $\mathcal{L}\psi_1=0$ and $\partial_\theta\psi_1=0$, we compute

$$\begin{aligned}\frac{d}{d\tau}\alpha_1&=\langle\partial_\tau\hat{u},\psi_1\rangle\\&=\Big\langle\mathcal{L}\hat{u}-\frac{1}{2\sqrt{2}}\hat{u}^2+\frac{1}{2\sqrt{2}}\hat{u}_\theta^2-\frac{1}{\sqrt{2}}(\hat{u}\hat{u}_\theta)_\theta+E,\psi_1\Big\rangle\\&=\Big\langle-\frac{1}{2\sqrt{2}}\hat{u}^2+\frac{1}{2\sqrt{2}}\hat{u}_\theta^2+E,\psi_1\Big\rangle\\&=-\frac{1}{2\sqrt{2}}\alpha_1^2\langle\psi_1^2,\psi_1\rangle+o(\beta^2)+O(e^{-\eta|\tau|^\gamma}),\end{aligned}$$

where in the last step we used Propositions 4.21 and 4.24 and Corollary 4.25 to estimate the error terms. Together with

$$\int \psi_1^3 \frac{1}{4\pi} e^{-|x|^2/4} = 2^{3/2} \Big(\frac{e}{2\pi}\Big)^{1/4},$$

this implies the assertion. □

Recall that by (4.62) the quantity $\beta(\tau)$ decays at some definite rate as $\tau \to -\infty$. The following lemma shows that $\beta(\tau)$ does not decay rapidly.

LEMMA 4.27. *We have*

$$\limsup_{\tau \to -\infty} |\tau|^{10} \beta(\tau) = \infty.$$

Proof. Suppose towards a contradiction, there is some $C < \infty$ such that

$$\beta(\tau) \leqslant C|\tau|^{-10} \quad \text{for all } \tau \leqslant \mathcal{T}.$$

Then, $\rho(\tau) \geqslant c|\tau|^2$, and equation (4.7), together with the assumption that the neutral mode is dominant, implies

$$\Big| \frac{d}{d\tau} \log U_0 \Big| \leqslant C|\tau|^{-2}.$$

Hence, $|\log U_0| \leqslant C$ for all $\tau \leqslant \mathcal{T}$, which contradicts

$$\lim_{\tau \to -\infty} U_0(\tau) = 0.$$ □

THEOREM 4.28. *The coefficients α_1, α_2, and α_3 from the expansion (4.68) satisfy*

$$\alpha_1(\tau) = \frac{-2^{1/4}\pi^{1/4} e^{-1/4} + o(1)}{|\tau|} \tag{4.73}$$

and

$$|\alpha_2| + |\alpha_3| = o(|\alpha_1|) \tag{4.74}$$

for $\tau \leqslant \mathcal{T}$.

Proof. To keep track of the errors, we consider the monotone quantity

$$\bar{\alpha}_1(\tau) := \sup_{\sigma \leqslant \tau} |\alpha_1(\sigma)|. \tag{4.75}$$

By Proposition 4.26, inequality (4.69), and Corollary 4.25, for every $\varepsilon > 0$ we get

$$\Big| \frac{d}{d\tau} \alpha_1(\tau) + \Big(\frac{e}{2\pi}\Big)^{1/4} \alpha_1^2(\tau) \Big| \leqslant \varepsilon \bar{\alpha}_1^2(\tau) + |\tau|^{-1000} \tag{4.76}$$

for $\tau\leqslant\mathcal{T}=\mathcal{T}(\varepsilon)$. Moreover, taking also into account Lemma 4.27, we have

$$\limsup_{\tau\to-\infty}|\tau|^{10}\bar{\alpha}_1(\tau)=\infty. \tag{4.77}$$

Consider the sets

$$I:=\{\tau\leqslant\mathcal{T}:\bar{\alpha}_1(\tau)=|\alpha_1(\tau)|\},$$
$$J:=\{\tau\leqslant\mathcal{T}:\bar{\alpha}_1(\tau)\geqslant|\tau|^{-10}\}.$$

Due to $\bar{\alpha}_1(\tau)\to0$ and (4.77), the set $I\cap J$ contains a sequence of numbers going to $-\infty$. Clearly, $I\cap J\subseteq(-\infty,\mathcal{T}]$ is closed.

Let $\tau_0\in I\cap J$. Then, by (4.76), we have

$$\frac{d}{d\tau}\alpha_1(\tau)\leqslant-\frac{1}{2}\Big(\frac{e}{2\pi}\Big)^{1/4}\alpha_1^2(\tau)\leqslant-\frac{1}{4}\Big(\frac{e}{2\pi}\Big)^{1/4}\frac{1}{|\tau|^{100}} \tag{4.78}$$

at $\tau=\tau_0$. In particular, remembering (4.75), we see that $\alpha_1(\tau_0)<0$. In fact, we can find a $\delta>0$ such that (4.78) and $\alpha_1(\tau)<0$ hold for $|\tau-\tau_0|<\delta$. Moreover, if there is some $\hat{\tau}\in I$ with $\hat{\tau}\in(-\infty,\tau_0]$ and $\alpha'(\hat{\tau})=0$, then (4.76) yields

$$|\alpha_1^2(\hat{\tau})|\leqslant2|\hat{\tau}|^{-1000}\leqslant2|\hat{\tau}_0|^{-1000}<\tfrac{1}{100}|\alpha_1^2(\tau_0)|.$$

Hence, if

$$\tfrac{1}{10}|\alpha_1(\tau_0)|\leqslant|\alpha_1(\tau)|\leqslant|\alpha_1(\tau_0)|,$$

then $\tau\in I$. Thus, possibly after decreasing δ, we get $(\tau_0-\delta,\tau_0]\subseteq I$. Then, (4.76) implies $(\tau_0-\delta,\tau_0]\subseteq J$.

Summing up, we can find α_1 by solving the ODE

$$\Big|\frac{d}{d\tau}\alpha_1(\tau)+\Big(\frac{e}{2\pi}\Big)^{1/4}\alpha_1^2(\tau)\Big|\leqslant\varepsilon\alpha_1^2(\tau)+|\tau|^{-1000} \tag{4.79}$$

for $\tau\leqslant\mathcal{T}(\varepsilon)$. This yields (4.73). Since α_2 and α_3 are rapidly decaying, we also get (4.74). This concludes the proof of the theorem. □

Corollary 4.29. *For $\tau\leqslant\mathcal{T}$ the rescaled flow $\overline{M}_\tau^{X_0}\cap\{|z|\geqslant L_0\}$ is contained inside the cylinder Σ of radius $\sqrt{2}$.*

Proof. Theorem 4.28 implies that $\overline{M}_\tau^{X_0}$ satisfies

$$u(L_0,\tau)<0$$

for $\tau\leqslant\mathcal{T}$. Hence, given arbitrarily small $b>0$, the KM-barrier $\widetilde{\Sigma}_b$ lies outside of

$$\overline{M}_\tau^{X_0}\cap\{x_3=L_0\}.$$

Using also Corollary 3.10 and the avoidance principle we infer that $\widetilde{\Sigma}_b$ lies outside of $\overline{M}_\tau^{X_0}$ on $\{x_3\geqslant L_0\}$. Namely, $\overline{M}_\tau^{X_0}$ satisfies $x_1^2+x_2^2\leqslant2$ for $x_3\geqslant L_0$. In the same manner, $\overline{M}_\tau^{X_0}$ satisfies $x_1^2+x_2^2\leqslant2$ for $x_3\leqslant L_0$. This proves the corollary. □

COROLLARY 4.30. *For an ancient low-entropy solution, if the neutral mode is dominant, then the solution is compact.*

Proof. Denote the rescaled flow by $\overline{M}_\tau = \overline{M}_\tau^{X_0}$, and the original flow by $\mathcal{M} = \{M_t\}$. Then, by the above corollary, there exists some $t_1 \in (-\infty, t_0)$ and $L < \infty$ such that

$$M_{t_1} \cap \{|z| \geqslant L\}$$

is contained inside the cylinder of radius $\sqrt{2(t_0 - t_1)}$. The lemma below implies that M_t is compact for all $t > t_0$. Since X_0 was arbitrary, this proves the corollary. □

LEMMA 4.31. *Let $M_0 \subset \mathbb{R}^3$ be a complete embedded surface such that*

$$M_0 \setminus B(0, L) = \Sigma \setminus B(0, L)$$

for some $L < \infty$, where Σ denotes the cylinder of radius $\sqrt{2}$. Then the mean-curvature flow $\{M_t\}$ of M_0 is compact for all $t > 1$.

Proof. Let $\mathcal{M}$ be the mean-curvature flow with initial condition M_0. For $z_i \to \pm\infty$ consider the shifted flows $\mathcal{M}^i = \mathcal{M} - (z_i, 0)$. For $i \to \infty$ the flows $\mathcal{M}^i$ converge to a limit $\mathcal{M}^\infty$, whose initial time slice is Σ. By uniqueness, $\mathcal{M}^\infty$ must be a round shrinking cylinder that becomes extinct at time 1. We conclude that M_t is compact (this includes the possibility that M_0 is non-compact, but M_t becomes compact via a contracting cusp singularity at spatial infinity) for all $t > 1$. □

5. Curvature bound and cap size control

Throughout this section, $\mathcal{M}$ will always be an ancient low-entropy flow where the plus mode is dominant. We will frequently use the fine neck theorem (Theorem 4.15) and its corollary (Corollary 4.16). By an affine change of coordinates, we may assume without loss of generality that

$$\bar{a} = \frac{1}{\sqrt{2}},$$

that the axis of the asymptotic cylinder is the x_3 axis, and that

$$\min_{p \in M_0} x_3(p) = 0$$

is attained at the origin.

5.1. Global curvature bound

We will prove a global in space-time curvature bound. To this end, we start with the following proposition.

PROPOSITION 5.1. *For every $\Lambda<\infty$ there exists a $\rho=\rho(\mathcal{M},\Lambda)<\infty$ with the following significance. If $p_0,p\in M_{t_0}$ are points such that $Z(p_0,t_0)\leqslant\Lambda$ and*

$$d:=x_3(p)-x_3(p_0)\geqslant\rho,$$

then

$$Z(p,t_0)>\Lambda.$$

Proof. Let $C=C(\mathcal{M})<\infty$ and $\mathcal{T}(\Lambda)>-\infty$ be the constants from the fine neck theorem (Theorem 4.15) applied at any point with cylindrical scale at most Λ. We will show that if $p_0,p\in M_{t_0}$ are points with $Z(p_0,t_0)\leqslant\Lambda$ and $Z(p,t_0)\leqslant\Lambda$, then $d:=x_3(p)-x_3(p_0)$ is bounded from above.

Let $t_1=t_0-d^2$, and let τ be the corresponding rescaled time, namely

$$\tau=-\log(t_0-t_1)=-2\log d. \tag{5.1}$$

Observe that if d is sufficiently large, at time τ we will see necks around p_0 and p. Those two necks have to align with each other, else looking further back in time they would intersect. Moreover, if d is sufficiently large, then $\tau\leqslant\mathcal{T}(\Lambda)$, as required for applying the fine neck theorem.

Letting r_0 and r be (representitives of) the radii of the fine necks around p_0 and p at time t_1, applying the fine neck theorem (Theorem 4.15) at time τ with $z=0$ twice, once centered at (p_0,t_0) and once centered at (p,t_0), we obtain

$$|r-r_0|\leqslant 2Ce^{\tau/160}. \tag{5.2}$$

On the other hand, applying the fine neck theorem (Theorem 4.15) centered at (p_0,t_0) at time τ with $z=1$, we obtain

$$|r-r_0|\geqslant\frac{1}{\sqrt{2}}-Ce^{\tau/160}. \tag{5.3}$$

Comparing (5.2) and (5.3), we conclude that $-\tau$ is bounded; hence d is bounded above. This finishes the proof of the proposition. □

COROLLARY 5.2. *For every compact interval $I\subseteq(-\infty,T_E(\mathcal{M}))$ there exist constants $\delta>0$, $\Lambda<\infty$ and $C<\infty$ (possibly depending on $\mathcal{M}$ and I) such that for every $p\in M_t$ with $t\in I$ and $x_3(p)\geqslant\Lambda$ either*

$$R(p,t)\geqslant\delta,$$

or

$$(p,t)\text{ is }\varepsilon\text{-spherical at scale }CR(p,t).$$

Proof. Suppose that there exists a sequence of points $p_k \in M_{t_k}$ with $t_k \in I$ such that $x_3(p_k) \to \infty$ and $R(p_k, t_k) \to 0$. Recall that, by §3.2, there are constants $0 \ll C \ll C' < \infty$ such that if (p_k, t_k) is not ε-spherical at scale $CR(p_k, t_k)$, then

$$Z(p_k, t_k) \leqslant C' R(p_k, t_k).$$

For k large, the latter option is impossible by Proposition 5.1, and so the assertion follows. □

LEMMA 5.3. *The surface M_{t_0} has no compact connected components for any*

$$t_0 \in (-\infty, T_E(\mathcal{M})).$$

Proof. Suppose towards a contradiction that there is a compact connected component N_{t_0} at some time t_0. Let $\mathcal{N}$ be the space-time connected component of N_{t_0}, and denote its time-t slice by N_t. By Corollary 4.16, there is some $t_1 < t_0$ such that N_{t_1} is non-compact.

Let δ, Λ, and C be the constants from Corollary 5.2, corresponding to the interval $I = [t_1, t_0]$. Using Corollary 4.16 and spherical barriers, we see that there is a constant $\Lambda' \in (\Lambda, \infty)$ such that

$$N_t \subset B(0, \Lambda') \cup \{x_3 \geqslant \Lambda'\}. \tag{5.4}$$

for all $t \in I$. After possibly increasing the Λ', we may assume that

$$\Lambda' \geqslant \sup_{p \in N_{t_0}} |p| + 10\sqrt{t_1 - t_0} + 10C\delta.$$

Now, if N_t is compact at any given $t \in [t_1, t_0]$, then it must be the case that $R(p, t) \geqslant \delta$ for every $p \in N_t$ with $x_3(p) \geqslant \Lambda'$. Otherwise, Corollary 5.2 would imply that (p, t) is ε-spherical at a scale $\leqslant C\delta$, and N_t could not reach N_{t_0} at time t_0. Hence,

$$-\frac{d}{dt} \sup_{p \in N_t} x_3(p) \leqslant 2\delta^{-1} \tag{5.5}$$

whenever N_t is compact and $\sup_{p \in N_t} x_3(p) \geqslant \Lambda'$. Combining (5.4) and (5.5), we conclude that N_t is compact for all $t \in [t_1, t_0]$; a contradiction. □

COROLLARY 5.4. *There exist a constant $C = C(\mathcal{M}) < \infty$ such that*

$$Z(p, t) \leqslant C R(p, t)$$

for all $(p, t) \in \mathcal{M}$. In particular,

$$\lim_{x_3(p) \to \infty} R(p, t) = \infty.$$

Proof. By Lemma 5.3, the flow cannot be ε-spherical at any scale. Thus, by §3.2, the regularity scale and the cylindrical scale must be comparable. Together with Proposition 5.1, this yields

$$\lim_{x_3(p)\to\infty} R(p,t) = \infty. \qquad \square$$

Lemma 5.3 also quickly implies a local type-I curvature estimate.

PROPOSITION 5.5. *For all $\Lambda<\infty$ there is a $C(\Lambda)<\infty$ with the following significance. If $\mathcal{M}$ reaches a point p_0 at time t_0, then we have the curvature bound*

$$|A(p,t)| \leqslant \frac{C}{\sqrt{t_0-t}}$$

for all $p\in B(p_0, \Lambda\sqrt{t_0-t}\,)$ at all $t<t_0$.

Proof. Suppose towards a contradiction that for some $\Lambda<\infty$ there is a sequence of points $(p_i,t_i)\in\mathcal{M}$ such that the estimate fails for $C_i=i$.

After shifting (p_i,t_i) to $(0,0)$ and rescaling, we get a sequence of flows $\mathcal{M}^i$ which reach the point zero at time zero, such that there are points $q_i\in M^i_{-1}\cap B(0,\Lambda)$ with

$$|A|(q_i,-1) > i.$$

Moreover, the time slices of $\mathcal{M}^i$ have no compact connected components, by Lemma 5.3.

By the entropy assumption, we may pass to a subsequential limit $\mathcal{M}^i\to\mathcal{M}$, which is an ancient low-entropy flow. By Brakke's clearing-out lemma [Bra], the limit flow $\mathcal{M}$ reaches the origin at time zero. Moreover, the limit $\mathcal{M}$ cannot have any compact connected component at any $t<0$.

Thus, $\mathcal{M}$ has no spherical singularities when $t<0$, and so $\mathcal{M}\cap\{t<0\}$ is smooth. In particular, $\mathcal{M}$ satisfies $|A|\leqslant C$ on $B(0,2\Lambda)\times\left[-2,-\frac{1}{2}\right]$ for some $C<\infty$; for i large enough, this contradicts $|A|(q_i,-1)>i$. $\square$

Combining the above results, we now get a global curvature bound:

THEOREM 5.6. (Global curvature bound) *Let $\mathcal{M}$ be an ancient low-entropy flow such that the plus mode is dominant. Then $\mathcal{M}$ is eternal with globally bounded curvature, namely*

$$T_E(\mathcal{M}) = \infty,$$

and there is a constant $C=C(\mathcal{M})<\infty$ such that

$$\sup_{(p,t)\in\mathcal{M}} |A|(p,t) \leqslant C.$$

Proof. By Corollary 5.4 the flow is eternal, i.e. $T_E(\mathcal{M})=\infty$, and by Lemma 5.3 the time slices have no compact connected components. If $(p,t)\in\mathcal{M}$ is any point with $R(p,t)<\frac{1}{10}$, then by Proposition 5.1 and Corollary 5.4 there exists a point (q,t) with $0<x_3(q)-x_3(p)<C\rho$ such that $R(q,t)=1$. But then, by unit regularity there is a point $(q',t+\delta)$ with $|q'-q|\leqslant 1$ and $R(q',t+\delta)\leqslant 2$, where $\delta>0$ is a uniform constant. Using the fine neck theorem (Theorem 4.15), we see that

$$|x_1(p)-x_1(q)|+|x_2(p)-x_2(q)|$$

is bounded. Applying the local type-I curvature estimate (Proposition 5.5) centered at $(q',t+\delta)$ we conclude that $|A(p,t)|\leqslant C$. □

5.2. Cap size control and asymptotics

The goal of this section is to prove Theorem 5.9, which states that our surfaces M_t have a cap of uniformly controlled size and open up like a parabola. To begin with, let us consider the "height of the tip" function

$$\psi(t) := \inf_{x\in M_t} x_3.$$

LEMMA 5.7. *The function ψ is strictly increasing and satisfies*

$$\lim_{t\to\pm\infty} \psi(t) = \pm\infty.$$

Proof. Applying Corollary 4.16 we see that $\psi(t)>-\infty$ for every $t\in\mathbb{R}$, and that the infimum is actually a minimum (using also Corollary 3.10). By the comparison with planes / the strong maximum principle, the function ψ is strictly increasing. Applying Corollary 4.16 again, we infer that

$$\lim_{t\to-\infty} \psi(t) = -\infty.$$

Finally, by comparison with conical expanders (using also Corollary 3.10), we have

$$\lim_{t\to\infty} \psi(t) = \infty.$$ □

PROPOSITION 5.8. (Fast tip points) *There exists a constant $Q=Q(\mathcal{M})<\infty$ such that every $p\in M_t$ with $x_3(p)=\psi(t)$ satisfies $R(p,t)\leqslant Q$.*

Proof. If the assertion fails, we can find a sequence $(p_j,t_j)\in\mathcal{M}$ with $x_3(p_j)=\psi(t_j)$ such that

$$Q_j := R(p_j,t_j)\to\infty.$$

Let $\mathcal{M}^j$ be the flow that is obtained by parabolically rescaling by Q_j^{-1} around

$$X_j=(p_j,t_j).$$

Note that $\mathcal{M}^j$ has expansion parameter $\bar{a}^j=Q_j^{-1}\bar{a}\to 0$.

Up to a subsequence, we can pass to a limit $\mathcal{M}^\infty$. The limit $\mathcal{M}^\infty$ is itself a non-compact ancient low-entropy flow. Since its zero-time slice is contained in a half space, $\mathcal{M}^\infty$ cannot be the cylinder. Hence, by the fine neck theorem (Theorem 4.15) it has an expansion parameter $a^\infty\neq 0$. However, it follows from the fine neck theorem and its proof that $\bar{a}^j\to a^\infty$, contradicting $\bar{a}^j\to 0$. This finishes the proof of the proposition. □

For each time, select a point $p_t\in M_t$ such that $x_3(p_t)=\psi(t)$.

THEOREM 5.9. (Cap size control) *Let $\mathcal{M}$ be an ancient low-entropy flow such that the plus mode is dominant. Then there exists a $C=C(\mathcal{M})<\infty$ such that, for $t\in\mathbb{R}$, every point in $M_t\setminus B_C(p_t)$ lies on a fine neck. In particular, the surface M_t has exactly one end.*

Moreover, $M_t\setminus B_C(p_t)$ is the graph of a function r in cylindrical coordinates around the z-axis satisfying

$$r(t,z,\theta)=\sqrt{2(z-\psi(t))}+o\big(\sqrt{z-\psi(t)}\,\big) \tag{5.6}$$

for $z\geqslant\psi(t)+C$, and the height of the tip function ψ satisfies

$$\psi(t)=t+o(|t|).$$

Proof. Let $p\in M_t$, and let t_* be the time in which $\psi(t_*)=x_3(p)$. By Proposition 5.8 (fast tip points), we have $R(p_{t_*})\leqslant Q$, and by the global curvature bound (Theorem 5.6), we have $t_*-t\geqslant c(x_3(p)-\psi(t))$. Thus, applying the fine neck theorem (Theorem 4.15) and Proposition 4.9 at (p_{t_*},t_*), we see that if $x_3(p)-\psi(t)$ is sufficiently large, then p lies on a fine neck. Together with Corollaries 3.10 and 4.16, this proves that we have a cap of controlled size $C=C(\mathcal{M})<\infty$, and exactly one end.

Fix t. For $z\geqslant\psi(t)+C$, let $r(z)$ be (a representitive of) the radius of the fine neck of M_t at $x_3(x)=z$. By the above and the fine neck theorem (Theorem 4.15), we have

$$r(z+r(z))=r(z)+1+o((z-\psi(t))^{-1/160}). \tag{5.7}$$

This can be iterated to

$$r\big(z+kr(z)+[1+o((z-\psi(t))^{-1/160})]\tfrac{1}{2}k(k-1)\big)=r(z)+k[1+o((z-\psi(t))^{-1/160})].$$

Setting $h=z-\psi(t)$ we get

$$r\big(z+kr(z)+[1+o(h^{-1/160})]\tfrac{1}{2}k(k-1)\big)=r(z)+k[1+o(h^{-1/160})]. \tag{5.8}$$

The argument of r in the left-hand side of (5.8) is

$$s:=z+kr(z)+[1+o(h^{-1/160})]\tfrac{1}{2}k(k-1),$$

and solving for k, we get

$$k=\frac{-\big(r(z)-\frac{1}{2}\big)+\sqrt{[2+o(h^{-1/160})](s-z)}}{1+o(h^{-1/160})}. \tag{5.9}$$

Combining (5.8) and (5.9), and taking h very large, we obtain

$$r(h)=\sqrt{2h}+o(\sqrt{h}\,). \tag{5.10}$$

This proves (5.6).

Finally, in light of (5.10), one can put a a bowl with tip curvature $1+\delta$ inside the domain bounded by M_{t_0}, and a bowl with tip curvature $1-\delta$ outside that domain. By the avoidance principle, this implies

$$(1-2\delta)t\leqslant\psi(t)\leqslant(1+2\delta)t$$

for large enough t, and hence

$$\psi(t)=t+o(t)$$

for positive t. In fact, the above steps are uniform in the original time t_0: For every δ there exist a uniform s such that

$$(1-2\delta)(t-t_0)\leqslant\psi(t)-\psi(t_0)\leqslant(1+2\delta)(t-t_0),$$

whenever $t-t_0\geqslant s$. Thus, for every $t<0$, we have

$$(1-2\delta)|t|\leqslant-\psi(t)\leqslant(1+2\delta)|t|,$$

and so

$$\psi(t)=t+o(|t|)$$

holds for all t. This finishes the proof of the theorem. □

6. Rotational symmetry

6.1. Fine expansion away from the cap

Let $\mathcal{M}$ be as in the previous section. The goal of this section is to prove Theorem 6.6, which shows that the cylindrical end becomes rotationally symmetric at very fast rate, and also controls the position of the cap in the xy-plane uniformly in time.

Given a point $q\in\mathbb{R}^3$ and a direction $w\in S^2$, we denote by K the *normalized rotation vector field* that corresponds to a rotation by $\frac{1}{2}\pi$ around the axis $W=\{q+wt:t\in\mathbb{R}\}$, namely

$$K(x)=SJS^{-1}(x-q), \quad \text{where } J=\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and $S\in\mathrm{SO}_3$ is any rotation matrix with $Se_3=w$.[22]

Definition 6.1. (Cf. [BC, Definition 4.3]) A point $X\in\mathcal{M}$ with $H(X)>0$ is called *δ-symmetric* if there exists a normalized rotation vector field K such that

$$|\langle K,\nu\rangle H|\leqslant\delta \quad \text{in the parabolic ball } P(X,10H^{-1}(X)).$$

The following proposition shows that $\mathcal{M}$ becomes *δ-symmetric* at a very fast rate if one moves away from the cap.

Proposition 6.2. *There exist a constant $C=C(\mathcal{M})<\infty$ with the following significance. If $X=(x,t)\in\mathcal{M}$ is any point with*

$$x_3-\psi(t)\geqslant C,$$

then

$$X \text{ is } (x_3-\psi(t))^{-300}\text{-symmetric.}$$

Proof. We recall from Theorem 5.9 (cap size control) that there exists a constant $C_0<\infty$ such that every point $x\in M_t\setminus B_{C_0}(p_t)$ lies on the center of a fine neck. In particular, every point $X=(x,t)$ with

$$x_3-\psi(t)\geqslant C_0$$

satisfies the a-priori assumptions of the neck-improvement theorem from Brendle–Choi [BC, Theorem 4.4]. Hence, if $X=(x,t)\in\mathcal{M}$ is a point with

$$x_3-\psi(t)\geqslant 2^{j/400}C_0$$

then we can iteratively apply the neck-improvement theorem [BC, Theorem 4.4] to conclude that X is $2^{-j}\varepsilon$-symmetric. This implies the assertion. □

[22] The matrix S is only determined up to a 1-parameter choice, but K is well-defined.

COROLLARY 6.3. (Strong symmetry) *There exist a constant $C=C(\mathcal{M})<\infty$ with the following significance. If $X=(x,t)\in\mathcal{M}$ is any point with $x_3-\psi(t)\geqslant C$, then there exist a direction $w_X\in S^2$ and a point $q_X\in\mathbb{R}^3$ with*

$$|w_X-e_3|\leqslant\tfrac{1}{100}\quad\text{and}\quad\langle q_X,e_3\rangle=x_3,$$

such that the normalized rotation vector field

$$K_X(y)=S_XJS_X^{-1}(y-q_X),$$

where $S_X\in\mathrm{SO}_3$ with $S_Xe_3=w_X$, satisfies the estimate

$$\sup_{P(X,10H^{-1}(X))}|\langle K_X,\nu\rangle H|\leqslant(x_3-\psi(t))^{-300}.$$

Proof. Since fine necks are very close to the asymptotic cylinder, we always have $|w_X-e_3|\leqslant\frac{1}{100}$. In addition, by moving q_X along the axis W, we can always arrange that $\langle q_X,e_3\rangle=x_3$. Hence, the corollary follows from the proposition. □

Definition 6.4. We call any triple (X,w_X,q_X) that satisfies the conclusion of Corollary 6.3 a *strongly symmetric triple.*

The following lemma shows that nearby strongly symmetric triples at the same time align well with each other.

LEMMA 6.5. (Alignment) *There exists a constant $C=C(\mathcal{M})<\infty$ with the following significance. If (X,w_X,q_X) and (Y,w_Y,q_Y) are strongly symmetric triples with $X=(x,t)$, $Y=(y,t)$, and $|x-y|H(X)\leqslant1$, then*

$$|w_X-w_Y|\leqslant C(x_3-\psi(t))^{-300},\tag{6.1}$$

and

$$|\langle q_X-q_Y,e_1\rangle|+|\langle q_X-q_Y,e_2\rangle|\leqslant C(x_3-\psi(t))^{-599/2}.\tag{6.2}$$

Proof. Without loss of generality, after suitable rotations and translations, we may assume that $t=0$, $x_3=0$, $w_X=e_3$, $q_X=0$, and

$$S_Y=\begin{pmatrix}1&0&0\\0&\cos\varphi&-\sin\varphi\\0&\sin\varphi&\cos\varphi\end{pmatrix},$$

where φ is a fixed angle with $0\leqslant\varphi\leqslant\frac{1}{10}$.

We express $\mathcal{M}\cap P(X,10H^{-1}(X))$ in cylindrical coordinates over the z-axis, namely we parameterize as

$$(\theta,z,t)\longmapsto\big(r(\theta,z,t)\cos\theta,r(\theta,z,t)\sin\theta,z\big).$$

In these coordinates, one can directly compute that

$$\nu=\frac{(\cos\theta+r^{-1}r_\theta\sin\theta,\sin\theta-r^{-1}r_\theta\cos\theta,-r_z)}{\sqrt{1+r^{-2}|r_\theta|^2+|r_z|^2}} \tag{6.3}$$

and

$$\big|\langle K_X,\nu\rangle\big|=\frac{|r_\theta|}{\sqrt{1+r^{-2}|r_\theta|^2+|r_z|^2}}.$$

Since X is the center of a (fine) neck, we have

$$r^{-2}|r_\theta|^2+|r_z|^2\leqslant 1+10\varepsilon, \tag{6.4}$$

and

$$\frac{1-\varepsilon}{r}\leqslant H(X)\leqslant\frac{1+\varepsilon}{r}.$$

Combining these equations with Corollary 6.3 (strong symmetry) we infer that

$$\frac{|r_\theta|}{r}\leqslant 2(-\psi(0))^{-300}.$$

Together with Theorem 5.9, this yields the estimate

$$|r_\theta|\leqslant Cr^{-599} \tag{6.5}$$

in the parabolic ball $P(X,10H^{-1}(X))$.

Now, for the normalized rotation vector with center $q_Y=(q_1,q_2,q_3)$ and axis $w_Y=(0,-\sin\varphi,\cos\varphi)$, we compute

$$K_Y(x)=S_YJS_Y^{-1}(x-q_Y)=\begin{pmatrix}(x_2-q_2)\cos\varphi+(x_3-q_3)\sin\varphi\\-(x_1-q_1)\cos\varphi\\-(x_1-q_1)\sin\varphi\end{pmatrix},$$

and

$$\begin{aligned}&\langle K_Y,\nu\rangle\sqrt{1+r^{-2}|r_\theta|^2+|r_z|^2}\\&\quad=r^{-1}(q_1x_2-x_1q_2)\cos\varphi+r^{-1}x_1(x_3-q_3)\sin\varphi+r_z(x_1-q_1)\sin\varphi\\&\qquad+r^{-2}r_\theta[x_2(x_2-q_2)\cos\varphi+x_2(x_3-q_3)\sin\varphi+x_1(x_1-q_1)\cos\varphi].\end{aligned} \tag{6.6}$$

Arguing as above, using Theorem 5.9 and Corollary 6.3, we obtain

$$|\langle K_Y, \nu\rangle| \leqslant Cr^{-599} \tag{6.7}$$

in the parabolic ball $P(X, 8H^{-1}(X))$. In addition, we have the rough estimate

$$|q_1|+|q_2|+|q_3| \leqslant 10r. \tag{6.8}$$

Now, from equation (6.6), using the estimates (6.4), (6.5), (6.7) and (6.8), we infer that

$$\frac{q_1x_2-x_1q_2}{r}\cos\varphi+\frac{x_1}{r}(x_3-q_3)\sin\varphi+r_z(x_1-q_1)\sin\varphi \leqslant Cr^{-599}. \tag{6.9}$$

At time $t=0$, we consider the points with $\theta=0$. Then, $(x_1, x_2)=(r, 0)$ and equation (6.9) yields

$$-q_2\cos\varphi+(x_3-q_3+r_z(r-q_1))\sin\varphi \leqslant Cr^{-599}.$$

In the case $q_2\leqslant 0$, we consider the points with $x_3=20r$. Then, using also $|q_3|\leqslant 10r$ and $\cos\varphi\geqslant\frac{1}{2}$, we obtain

$$\tfrac{1}{2}|q_2|+(10r+r_z(r-q_1))\sin\varphi \leqslant Cr^{-599}.$$

Moreover, since X lies on a (fine) neck, we have $|r_z|\leqslant\varepsilon$. Hence,

$$\tfrac{1}{2}|q_2|+5r\sin\varphi \leqslant Cr^{-599}. \tag{6.10}$$

Since $\sin\varphi\geqslant 0$, we infer that

$$|q_2| \leqslant Cr^{-599} \tag{6.11}$$

and

$$|\sin\varphi| \leqslant Cr^{-600}. \tag{6.12}$$

In the case $q_2\geqslant 0$, we obtain the same estimates by considering points with $x_3=-20r$. Similarly, considering points with $\theta=\frac{1}{2}\pi$, we obtain

$$|q_1| \leqslant Cr^{-599}. \tag{6.13}$$

Since $|w_X-w_Y|\leqslant C|\sin\varphi|$, these inequalities prove the lemma. □

We are now ready to prove the main theorem of this section.

THEOREM 6.6. (Fine asymptotics) *There exist a fixed vector* $q=(q_1,q_2,0)\in\mathbb{R}^3$ *and a large constant* $C<\infty$ *(both depending on* $\mathcal{M}$*) such that, for all* $t\in\mathbb{R}$*, the surface*

$$(M_t-q)\cap\{x_3-\psi(t)\geqslant C\}$$

can be expressed in cylindrical coordinates over the z*-axis with the estimate*

$$|\partial_\theta r|(\theta,z,t)\leqslant r(\theta,z,t)^{-100}. \tag{6.14}$$

Proof. Given any time $t\in\mathbb{R}$, we choose a sequence of points $y_j\in M_t$ satisfying

$$\langle y_j,e_3\rangle=\psi(t)+j.$$

By Corollary 6.3 (strong symmetry), for j large enough (which we will always assume from now on), these points are part of strongly symmetric triples (Y_j,w_j,q_j), where $Y_j=(y_j,t)$. Lemma 6.5 gives the estimates

$$|w_{j+1}-w_j|\leqslant Cj^{-300} \tag{6.15}$$

and

$$|\langle q_{i+1}-q_i,e_1\rangle|+|\langle q_{i+1}-q_i,e_2\rangle|\leqslant Cj^{-299}. \tag{6.16}$$

Since w_j converges to e_3, using (6.15) we get

$$|w_j-e_3|\leqslant C\sum_{k=j}^{\infty}k^{-300}\leqslant Cj^{-299}.$$

Similarly, equation (6.16) implies that

$$\langle q_j,e_1\rangle e_1+\langle q_j,e_2\rangle e_2$$

converges to a limit $q\in\mathbb{R}^2\times\{0\}$ with the estimate

$$|\langle q_j-q,e_1\rangle|+|\langle q_j-q,e_2\rangle|\leqslant C\sum_{k=j}^{\infty}k^{-299}\leqslant Cj^{-298}.$$

By translating $\mathcal{M}$, we may assume $q=0$. Then, we have

$$|w_j-e_3|\leqslant Cj^{-299}, \tag{6.17}$$

$$|\langle q_j,e_1\rangle|+|\langle q_j,e_2\rangle|\leqslant Cj^{-298}. \tag{6.18}$$

By Theorem 5.9 and Corollary 6.3 (strong symmetry), this yields the assertion. □

6.2. Moving plane method

The goal of this section is to prove Theorem 6.14, which says that $\mathcal{M}$ is rotationally symmetric.

Let us start with some preliminaries about the Hopf lemma and the strong maximum principle for graphical mean-curvature flow. Suppose that u and v are graphical solutions of the mean-curvature flow in the parabolic ball $P(0,\delta)$ around the origin of size $\delta>0$, namely

$$u_t = \sqrt{1+|Du|^2}\,\mathrm{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right), \tag{6.19}$$

$$v_t = \sqrt{1+|Dv|^2}\,\mathrm{div}\left(\frac{Dv}{\sqrt{1+|Dv|^2}}\right). \tag{6.20}$$

Then, their difference $w=u-v$ satisfies the linear equation

$$w_t = a_{ij}w_{ij}+b_iw_i, \tag{6.21}$$

where

$$a_{ij}(x,t) = \delta_{ij} - \frac{u_iu_j}{1+|Du|^2}(x,t)$$

and

$$b_i(x,t) = -\frac{v_{ij}(u_j+v_j)}{1+|Du|^2}(x,t) + \frac{v_{kl}v_kv_l(u_i+v_i)}{(1+|Du|^2)(1+|Dv|^2)}(x,t).$$

Hence, standard results for linear parabolic equations (see e.g. [L]) yield the following result.

LEMMA 6.7. (Hopf lemma) *Let u and v be graphical solutions of the mean-curvature flow in $P(0,\delta)\cap\{x_1\leqslant 0\}$. Suppose that $u(0,0)=v(0,0)$, and that $u<v$ in $P(0,\delta)\cap\{x_1<0\}$. Then,*

$$\frac{\partial u}{\partial x_1}(0,0) > \frac{\partial v}{\partial x_1}(0,0).$$

PROPOSITION 6.8. (Strong maximum principle) *Let u and v be graphical solutions of the mean-curvature flow in $P(0,\delta)$. Suppose that $u(0,0)=v(0,0)$, and that $u\leqslant v$ in $P(0,\delta)$. Then, $u=v$ in $P(0,\delta)$.*

Now, as in the previous section, let $\mathcal{M}=\{M_t\}_{t\in\mathbb{R}}$ be an ancient low-entropy flow in $\mathbb{R}^3$, where the plus mode is dominant. By the global curvature bound (Theorem 5.6) and the entropy assumption, there exists some uniform graphical scale $\delta_0>0$ at which the Hopf lemma (Lemma 6.7) and the strong maximum principle (Proposition 6.8) can be applied.

To set up the (parabolic variant of the) moving plane method, given a constant $\mu\geqslant 0$, we consider the surfaces

$$M_t^{\mu-} = M_t \cap \{x_1 < \mu\}, \tag{6.22}$$

$$M_t^{\mu+} = M_t \cap \{x_1 > \mu\}. \tag{6.23}$$

Moreover, we set $M_t^{\mu} = M_t \cap \{x_1 = \mu\}$, and denote by $M_t^{\mu<}$ the surface that is obtained from $M_t^{\mu+}$ by reflection about the plane $\{x_1 = \mu\}$, namely

$$M_t^{\mu<} = \{(2\mu - x_1, x_2, x_3) : x \in M_t^{\mu+}\}.$$

Similarly, denoting by $K_t \subset \mathbb{R}^3$ the closed domain bounded by M_t, we consider the regions

$$K_t^{\mu-} = K_t \cap \{x_1 < \mu\},$$

$$K_t^{\mu+} = K_t \cap \{x_1 > \mu\},$$

and

$$K_t^{\mu<} = \{(2\mu - x_1, x_2, x_3) : x \in K_t^{\mu+}\}.$$

Definition 6.9. We say that *the moving plane can reach* μ if for all $\tilde{\mu}\geqslant\mu$ we have the inclusion $K_t^{\tilde{\mu}<} \subseteq K_t^{\tilde{\mu}-}$ for all $t\in\mathbb{R}$.

The following proposition shows that the reflected domain cannot touch at spatial infinity.

Proposition 6.10. (No contact at infinity) *For every* $\mu>0$, *there exists a constant* $h_\mu<\infty$ *such that*

$$K_t^{\mu<} \cap \{x_3 \geqslant \psi(t) + h_\mu\} \subseteq \operatorname{Int}(K_t^{\mu-})$$

for every $t\in\mathbb{R}$.

Proof. Denote the position vector of $M_t \cap \{x_3 \geqslant \psi(t) + C\}$ by

$$X(\theta, z, t) = (r(\theta, z, t)\cos\theta, r(\theta, z, t)\sin\theta, z).$$

Given (z_0, t_0) with $z_0 - \psi(t_0) \geqslant C$, using Theorem 6.6 (fine asymptotics), we obtain

$$|r(z_0, \theta, t_0) - r(z_0, \theta_0, t_0)| \leqslant \frac{2\pi}{r(z_0, \theta_0, t_0)^{100}},$$

from which we directly infer that

$$K_{t_0}^{\mu<} \cap \{x_3 = z_0\} \cap \left\{x_1 \leqslant \tfrac{1}{2}\mu\right\} \subseteq \operatorname{Int}(K_{t_0}^{\mu-})$$

for sufficiently large $z_0-\psi(t_0)$.

Let $\tilde{\mu}\geqslant\mu$. Using Theorem 6.6 (fine asymptotics), we see that if $\theta\in\left(-\frac{1}{2}\pi,0\right)$ is such that

$$r(\theta,z_0,t_0)\cos\theta\in\left[\tfrac{1}{2}\tilde{\mu},\tfrac{3}{2}\tilde{\mu}\right],$$

then, for sufficiently large $z_0-\psi(t_0)$, we have

$$\langle X_\theta,e_2\rangle=r_\theta\sin\theta+r\cos\theta\geqslant-r^{-100}|\sin\theta|+r\cos\theta>0.$$

Hence, we have a graph with positive slope that can be reflected. Repeating the same argument for $\theta\in\left(0,\frac{1}{2}\pi\right)$, we conclude that

$$K^{\mu<}_{t_0}\cap\{x_3=z_0\}\cap\{x_1\geqslant\tfrac{1}{2}\mu\}\subseteq\operatorname{Int}(K^{\mu-}_{t_0})$$

for sufficiently large $z_0-\psi(t_0)$. This proves the proposition. □

COROLLARY 6.11. (Start plane) *There exists some $\mu<\infty$ such that the moving plane can reach μ.*

Proof. By Theorem 5.9 (cap size control) and Theorem 6.6 (fine asymptotics) there exists a $C=C(\mathcal{M})<\infty$ such that, whenever $x_3-\psi(t)\leqslant C$, we have

$$x_1^2+x_2^2\leqslant 10C, \tag{6.24}$$

namely we uniform control for the position of the cap. Together with Proposition 6.10 (no contact at infinity), this implies the assertion. □

Now, for h_μ as in the proposition and $\delta>0$, we define

$$E^\mu_t=\{x_3\leqslant\psi(t)+h_{\mu/2}\}\quad\text{and}\quad E^{\mu,\delta}_t=\{x\in E^\mu_t:d(x,M^\mu_t)\geqslant\delta\}.$$

LEMMA 6.12. (Distance gap) *Suppose that the moving plane can reach $\mu>0$. Then, there exists a positive increasing function $\alpha:(0,\delta_0)\to\mathbb{R}_+$ such that*

$$d(M^{\mu-}_t,K^{\mu<}_t\cap E^{\mu,\delta}_t)\geqslant\alpha(\delta)>0$$

for all $t\in\mathbb{R}$.

Proof. We will first show that

$$K^{\mu<}_t\subseteq\operatorname{Int}(K^{\mu-}_t) \tag{6.25}$$

for all $t\in\mathbb{R}$. Indeed, by definition, we have $K^{\mu<}_t\subseteq K^{\mu-}_t$. If (6.25) fails, then there must be some $t\in\mathbb{R}$ and some $p\in M^{\mu<}_t\cap M^{\mu-}_t$; this contradicts the strong maximum principle (Proposition 6.8). This proves (6.25).

Now, suppose towards a contradiction that, for some $\delta>0$, we have

$$\inf_{t\in\mathbb{R}} d(M_t^{\mu^-}, K_t^{\mu^<}\cap E_t^{\mu,\delta}) = 0.$$

Choose a sequence of space-time points $(x_i,t_i)\in\mathcal{M}$ such that $x_i\in M_{t_i}^{\mu^<}\cap E_{t_i}^{\mu,\delta}$ and

$$\lim_{i\to\infty} d(x_i, K_{t_i}^{\mu^-}) = 0.$$

By Proposition 6.10 (no contact at infinity), Theorem 6.6 (fine asymptotics), and the uniform cap position control (6.24), the distance between x_i and the point $\psi(t_i)e_3$ is uniformly bounded. Hence, by Theorem 5.6 (global curvature bound) and Theorem 5.9 (cap size control), we can take subsequential limits $\overline{\mathcal{M}}$ and $\overline{X}$ of the flows $\mathcal{M}-(\psi(t_i)e_3,t_i)$ and points $x_i-\psi(t_i)e_3$. Applying the strong maximum principle (Proposition 6.8) for $\overline{\mathcal{M}}$ at the spacetime point $(\overline{X},0)$ gives a contradiction. This proves the lemma. □

LEMMA 6.13. (Angle gap) *Suppose that the moving plane can reach $\mu>0$. Then, there exists a positive constant $\theta_\mu>0$ such that $|\langle\nu(x,t),e_1\rangle|\geqslant\theta_\mu$ holds on $M_t^\mu\cap E_t^\mu$ for all $t\in\mathbb{R}$.*

Proof. First, the Hopf lemma (Lemma 6.7) shows that

$$|\langle\nu(x,t),e_1\rangle|\neq 0.$$

Now, suppose towards a contradiction there is a sequence $(x_i,t_i)\in\mathcal{M}$ such that $x_i\in M_{t_i}^\mu\cap E_{t_i}^\mu$ and

$$\lim_{i\to\infty}|\langle\nu(x_i,t_i),e_1\rangle| = 0.$$

Then, as in the proof of Lemma 6.12, we can take subsequential limits $\overline{\mathcal{M}}$ and $\overline{X}$ of the flows $\mathcal{M}-(\psi(t_i)e_3,t_i)$ and points $x_i-\psi(t_i)e_3$. Applying the Hopf lemma (Lemma 6.7) for $\overline{\mathcal{M}}$ at the spacetime point $(\overline{X},0)$ gives a contradiction. This proves the lemma. □

THEOREM 6.14. *$\mathcal{M}$ is rotationally symmetric.*

Proof. It is enough to show that the moving plane can reach $\mu=0$. Consider the interval

$$I = \{\mu\geqslant 0 : \text{the moving plane can reach } \mu\}$$

Note that $I\neq\varnothing$ by Corollary 6.11 (start plane). Let $\mu:=\inf I$, and observe that $\mu\in I$. Suppose towards a contradiction that $\mu>0$.

First, by Proposition 6.10 (no contact at infinity) we have

$$K_t^{(\mu/2)^<}\cap(E_t^\mu)^c\subseteq K_t^{(\mu/2)^-}$$

for all $t\in\mathbb{R}$.

Next, by Lemma 6.13 (angle gap) there exists a $\delta_1\in\big(0,\min\{\delta_0,\frac{1}{2}\mu\}\big)$ such that, for $\delta\in(0,\delta_1)$, we have

$$K_t^{(\mu-\delta)<}\cap E_t^\mu\cap\{x_1\geqslant\mu-2\delta_1\}\subseteq K_t^{(\mu-\delta)-}$$

for all $t\in\mathbb{R}$.

Finally, combining the above with Lemma 6.12 (distance gap), we conclude that for every $\delta\in(0,\min\{\delta_1,\alpha(\delta_1)\})$ we have

$$K_t^{(\mu-\delta)<}\subseteq K_t^{(\mu-\delta)-}.$$

for all $t\in\mathbb{R}$. Hence, the moving plane can reach $\mu-\delta$; a contradiction. This proves the theorem. □

7. Classification of ancient low-entropy flows

7.1. The non-compact case

Let $\mathcal{M}$ be an ancient non-compact low-entropy flow that is not a round shrinking cylinder or a flat plane. By Corollary 4.30, the plus mode is dominant. As before, we normalize such that

$$a=a(\mathcal{M})=\frac{1}{\sqrt{2}}.$$

THEOREM 7.1. *$\mathcal{M}$ is the bowl soliton.*

Proof. From §5 we know that $\mathcal{M}$ has a cap of controlled size and opens up like a parabola. From §6 we know that $\mathcal{M}$ is rotationally symmetric. Thus, at each time M_t can be described by rotating a curve around the z-axis. The curve $\gamma_t(s)$ $(0\leqslant s<\infty)$ is in the xz-plane and satisfies $x(\gamma(0))=0$ and $\gamma'(0)=(1,0,0)$.

We will first show that the height function $z\colon M_t\to\mathbb{R}$ does not have any local maxima. Assume towards a contradiction that at some time t_0 there is a local maximum of the height function. Denote its value by $h(t_0)$. Since the mean-curvature vector points downward, $h(t)$ is decreasing. On the other hand, the minimum of the height function $\psi(t)$ is an increasing function. In particular, $h-\psi$ is decreasing. Since local maxima of z cannot disappear as we go back in time (cf. [An1]), it follows that the height function has a local maximum for all $t\leqslant t_0$. Using Proposition 5.8, we see that

$$\lim_{t\to-\infty}(h-\psi)(t)=\infty;$$

this contradicts the fact that the cap size is bounded. Thus, the height function z does not have any local maxima, and the z-component of the vector $\gamma_t'(s)$ is non-zero for all $s\neq0$.

We can now describe M_t by a function $r(z,t)$, where $\psi(t)\leqslant z<\infty$. To proceed, we need the following lemma.

LEMMA 7.2. *The radius function $r=r(z,t)$ satisfies*

$$rr_z=1+o(1)$$

uniformly as $z-\psi(t)\to\infty$. In particular, the radius function does not have critical points outside a cap of controlled size.

Proof of Lemma 7.2. We recall from [BC, Lemma 6.5] that on ε-necks one has the estimate

$$|rr_z|+|r^2r_{zz}|\leqslant C,$$

in particular

$$|\partial_z^2r^2|\leqslant\frac{C_0}{r}. \tag{7.1}$$

Now, we consider a point (z_0,t_0) in a neck satisfying $\partial_z(r^2)(z_0,t_0)\neq 2$. Without loss of generality, we assume $\partial_z(r^2)(z_0,t_0)-2=2\delta>0$. Then, we have $\partial_z(r^2)(z,t_0)\geqslant 2+\delta$ for $|z-z_0|\leqslant\tilde{\delta}r_0$, where $r_0=r(z_0,t_0)$ and $\tilde{\delta}=\min\{1,\frac{1}{2}C_0^{-1}\delta\}$. Therefore, we obtain

$$r^2(z_0+\tilde{\delta}r_0,t_0)-r^2(z_0,t_0)\geqslant(2+\delta)\tilde{\delta}r_0.$$

Since $((0,0,z_0),t_0)$ is the center of a fine neck with rescaled time parameter τ satisfying $e^{-\tau/2}=r_0/\sqrt{2}$, using Theorem 5.9 we see that

$$|r(z_0+\tilde{\delta}r_0,t_0)-r(z_0,t_0)-\tilde{\delta}|\leqslant Cr_0^{-1/40}.$$

Combining the above inequalities yields

$$Cr_0^{39/40}+O(1)\geqslant\tilde{\delta}\delta r_0.$$

Therefore, $\tilde{\delta}=\frac{1}{2}C_0^{-1}\delta$ for sufficiently large r_0, and thus $\delta\leqslant Cr_0^{-1/80}$. Namely,

$$\partial_z(r^2)-2=O(r^{-1/80}).$$

This proves the lemma. □

Continuing the proof of the theorem, suppose towards a contradiction that r has a local maximum at some time t_0. The mean-curvature vector at such a maximum points towards the z-axis and has size at least $1/r(z,t)$, hence

$$\frac{d}{dt}r(z_{max}(t),t)\leqslant-\frac{1}{r(z_{max}(t),t)}.$$

Integrating this differential inequality from t_0 back to time $-\infty$ (again, using [An1] to conclude that the maximum point persists), we obtain that $r^2(z_{\max}(t),t)$ grows at least linearly in $-t$. In particular, $z_{\max}(t)-\psi(t)\to\infty$ as $t\to-\infty$, which, as $r_z(z_{\max}(t),t)=0$, contradicts Lemma 7.2.

By the above, the (upwards) unit normal vector ν of our surface can never be horizontal or vertical (except at the tip). Thus,

$$f:=\langle\nu,e_3\rangle>0.$$

By Lemma 7.2, we have the asymptotics

$$f(z,t)=\frac{1+o(1)}{r} \tag{7.2}$$

uniformly as $z-\psi(t)\to\infty$. In particular, there exists some universal $h_0<\infty$ such that

$$f(\psi(t)+h,t)\geqslant\frac{1}{10\sqrt{h}}$$

for all $h\geqslant h_0$. We claim that there exists some universal $\delta>0$, such that $f(z,t)\geqslant\delta$ whenever $\psi(t)\leqslant z\leqslant\psi(t)+h_0$. Otherwise, taking a sequence of (t_i,z_i) with $f(t_i,z_i)\to0$ and passing to a subsequential limit, we obtain a flow (low-entropy, with dominant plus mode, rotationally symmetric, with $a=1/\sqrt{2}$) with $r_z(z_0,0)=0$ for some z_0. By [An1], for all $t<0$, the radius function r would have a local maximum, which was previously seen to be impossible. Hence, we have established the existence of a positive function $g\colon[0,\infty)\to\mathbb{R}_+$ such that

$$f(z,t)\geqslant g(z-\psi(t)). \tag{7.3}$$

Moreover, by Theorem 5.9, we have the asymptotics

$$H(z,t)=\frac{1+o(1)}{r}$$

uniformly as $z-\psi(t)\to\infty$. Combining this with (7.2) and (7.3), we see that H/f converges to 1 at spatial infinity uniformly for all t. In particular, there is some $C<\infty$ such that

$$\left|\frac{H}{f}\right|\leqslant C.$$

Suppose towards a contradiction that

$$c:=\inf\frac{H}{f}<1.$$

Select $X_i=(x_i,t_i)\in\mathcal{M}$ such that

$$\frac{H}{f}(X_i)\to c.$$

Let $\mathcal{M}^i$ be the flow that is obtained from $\mathcal{M}$ by shifting X_i to the space-time origin, and pass to a limit $\mathcal{M}^\infty$. For the limit $\mathcal{M}^\infty$ the function H/f attains a minimum $c<1$ at the space-time origin. Together with $H/f\to 1$ at spatial infinity, this contradicts the strong maximum principle for the evolution equation

$$\partial_t\frac{H}{f}=\Delta\frac{H}{f}+2\left\langle\nabla\log f,\nabla\frac{H}{f}\right\rangle.$$

Hence,

$$\inf\frac{H}{f}\geqslant 1.$$

A similar argument shows that

$$\sup\frac{H}{f}\leqslant 1.$$

Thus, $H=f$. Hence, M_t is a mean-convex and non-collapsed (since the cap is compact) translating soliton, and thus by [Has] must be the bowl soliton. □

7.2. The compact case

In this section, we treat the case where the neutral mode is dominant.

THEOREM 7.3. *If the neutral mode is dominant, then $\mathcal{M}$ is an ancient oval.*

Proof. From §4.3 we know that

$$\mathcal{M}=\{M_t\}_{t\in(-\infty,T_E(\mathcal{M}))}$$

is compact. Hence, the flow becomes extinct in a point at $T_E(\mathcal{M})<\infty$, and each time-slice has the topology of a sphere.

Since the blowdown for $t\to-\infty$ is a cylinder, for $t\leqslant\mathcal{T}$ there is a central neck that divides M_t into two halves $M_t^\pm$. We will show that M_t^+ is mean convex and α-non-collapsed for $t\leqslant\mathcal{T}$.

Consider the "height of the tip" function

$$\psi_+(t)=\max_{p\in M_t}x_3(p).$$

Since the blowdown for $t\to-\infty$ is a cylinder, we have

$$\lim_{t\to-\infty}\frac{\psi_+(t)}{\sqrt{|t|}}=\infty.$$

Let $t_j \to -\infty$ be a sequence such that

$$\psi'_+(t_j)\sqrt{|t_j|} \to \infty.$$

Then, we can find points $p_j \in M^+_{t_j}$ with $\psi_+(t_j) = x_3(p_j)$ and

$$\lim_{j\to\infty} \frac{R(p_j,t_j)}{\sqrt{|t_j|}} = 0.$$

Let $\mathcal{M}^j$ be the sequence of flows that is obtained from $\mathcal{M}$ by shifting (p_j,t_j) to the origin and parabolically rescaling by $R(p_j,t_j)^{-1}$, and pass to a limit $\mathcal{M}^\infty$. The limit $\mathcal{M}^\infty$ is an eternal low-entropy flow, which is non-flat. Hence, by Theorem 7.1 it is a translating bowl soliton.

Therefore, for each j large enough $M^+_{t_j}$ is an embedded disk which has the central neck Z_{t_j} as a collar and also another neck $Z^+_{t_j}$ bounding a convex cap $C^+_{t_j}$ that is α-non-collapsed, say for $\alpha = \frac{1}{100}$. Let $N^+_{t_j}$ be the topological annulus with collars Z_{t_j} and $Z^+_{t_j}$. Let $\{N_t\}_{t\leqslant t_j}$ be the time-dependent topological annulus which is obtained from $N^+_{t_j}$ by following the points backwards in time along the mean curvature evolution. By the preservation of necks backwards in time (see §3.2), the annulus N_t has two necks as collars for all $t \leqslant t_j$. Moreover, for $t \ll t_j$, the entire domain N_t constitutes a single neck. Hence, by the parabolic maximum principle (for H and $|A|/H$) applied with two collar boundaries, the annulus N_t is mean convex and satisfies $|A|/H \leqslant 2$ for all $t \leqslant t_j$.

Now, adding the cap $C^+_{t_j}$, we infer that $M^+_{t_j}$ is mean convex and satisfies $|A|/H \leqslant 100$. Since $t_j \to -\infty$, by applying the parabolic maximum principle again, but this time for the disk M^+_t which has only one collar boundary, the central neck Z_t, we conclude that M^+_t is mean convex and satisfies $|A|/H \leqslant 100$ for $t \leqslant \mathcal{T}$.

The same argument applies to M^-_t. Hence, $\mathcal{M} = \{M_t\}_{t\in(-\infty,T_E(\mathcal{M}))}$ is mean convex and satisfies $|A|/H \leqslant 100$. Together with the entropy assumption, this implies that

$$\mathcal{M} = \{M_t\}_{t\in(-\infty,T_E(\mathcal{M}))}$$

is α-non-collapsed for some $\alpha > 0$. Thus, by [ADS2] it is an ancient oval. □

8. Applications

8.1. Proof of the mean-convex neighborhood conjecture

The purpose of this section is to prove Theorem 1.7 (mean-convex neighborhoods at all times). Since spherical singularities, by compactness, clearly have mean-convex neighborhoods, it suffices to prove.

THEOREM 8.1. (Mean-convex neighborhood theorem) *Let $X_0=(x_0,t_0)$ be a (backwardly) singular point of $\{M_t\}_{t\geqslant 0}$ and suppose that*

$$\lim_{\lambda\to\infty}\lambda(K_{t_0+\lambda^{-2}t}-x_0)=\overline{B}^2(\sqrt{2(t_0-t)}\,)\times\mathbb{R}, \tag{8.1}$$

smoothly with multiplicity 1. *Then there exist an $\varepsilon=\varepsilon(X_0)>0$ such that whenever*

$$t_0-\varepsilon<t_1<t_2<t_0+\varepsilon$$

then

$$K_{t_2}\cap B(x_0,\varepsilon)\subseteq K_{t_1}\setminus M_{t_1}. \tag{8.2}$$

The same result holds with M and K replaced by M' and K', respectively.

We recall that given a closed embedded surface $M\subset\mathbb{R}^3$, we denote by $\{M_t\}_{t\geqslant 0}$ the outer flow. Observe that if $x\in M_t$ is a regular point, then there exists a $\delta>0$ such that $M_t\cap B(x,\delta)$ splits $B(x,\delta)$ into two connected components: one in $\mathrm{Int}(K_t)$ and the other in $\mathbb{R}^3\setminus K_t$.

Theorem 8.1 will be proved after the following two auxiliary results.

PROPOSITION 8.2. *Under the assumptions of Theorem 8.1, there exists a constant $\delta=\delta(X_0)>0$ and a unit-regular, cyclic, integral Brakke flow $\mathcal{M}=\{\mu_t\}_{t\geqslant t_0-\delta}$ whose support is $\{M_t\}_{t\geqslant t_0-\delta}$ such that the following conditions hold.*

(i) *The tangent flow to $\mathcal{M}$ at X_0 is a multiplicity-*1 *cylinder.*

(ii) *The flow $\mathcal{M}$ has only multiplicity-*1 *cylindrical and spherical singularities in*

$$\overline{B}(x_0,2\delta)\times[t_0-\delta,t_0+\delta].$$

(iii) *$\mathcal{M}$ is smooth in $\overline{B}(x_0,2\delta)$ for a.e. $t\in[t_0-\delta,t_0+\delta]$ and is smooth outside of a set of Hausdorff dimension* 1 *for every $t\in[t_0-\delta,t_0+\delta]$.*

(iv) *$H\neq 0$ for every regular point of $\mathcal{M}$ in*

$$\overline{B}(x_0,2\delta)\times[t_0-\delta,t_0+\delta].$$

(v) *There exist $A<\infty$ and $c>0$ such that if $X=(x,t)$ is a point of $\mathcal{M}$ in*

$$\overline{B}(x_0,2\delta)\times[t_0-\delta,t_0+\delta]$$

with $R(X)\leqslant c$, then $\mathcal{M}$ is smooth and connected in $P(X,AR(X))$ and there is a point $X'=(x',t')\in\mathcal{M}\cap P(X,AR(x))$ with $R(X')\geqslant 2R(X)$ and with

$$|x'-x_0|\leqslant\max\{|x-x_0|-cR(X'),\tfrac{1}{2}\delta\}.$$

We remark that condition (v) will be used below to show — via a connectedness argument — that the mean curvature on the regular set does not change sign in a suitable space-time neighborhood of X_0.

Proof. It follows from [HW, Theorem B3] that there exists an outer Brakke flow starting from M, whose support is $\{M_t\}_{t\geqslant 0}$. In particular, this, together with monotonicity, implies that $\mathrm{Ent}[\mathcal{H}^n\lfloor M_t]$ is uniformly bounded. Thus, (8.1) implies that there exists $t_*<t_0$ such that

$$\frac{1}{(4\pi(t_0-t_*))^{n/2}}\int \exp\left(-\frac{|x-x_0|^2}{4(t_0-t_*)}\right) d\mathcal{H}^n\lfloor M_{t_*}<2. \tag{8.3}$$

Therefore, applying [HW, Theorem B3] once more, this time with the initial time t_*, we get a unit-regular, cyclic, integral Brakke flow $\mathcal{M}:=\{\mu_t\}_{t\geqslant t_*}$ whose support is $\{M_t\}_{t\geqslant t_*}$. Together with (8.1) and (8.3), it follows that $\mathcal{M}$ has a multiplicity-1 tangent cylinder at (x_0,t_0). This proves (i).

(ii) now follows from the upper semi-continuity of the density (see §2.2), the proof of Theorem 3.2 (partial regularity), and the gap result from Bernstein–Wang [BW].

Condition (iii) follows from (ii) and standard stratification; see e.g. [W2].

Suppose towards a contradiction that (iv) does not hold. Then, there exists a sequence of smooth point

$$X_i=(x_i,t_i)\to X_0=(x_0,t_0)$$

with $H(X_i)=0$. Denoting by $r_i=R(X_i)$ the regularity scale, since X_0 is singular and $X_i\to X_0$, we have $r_i\to 0$.

Let $\mathcal{M}^i$ be the flow that is obtained from $\mathcal{M}$ by shifting X_i to the origin and parabolically rescaling by $1/r_i$, and pass (using Ilmanen's compactness theorem [I1]) to a Brakke flow limit $\mathcal{M}^\infty$.

By construction as a blowup limit, $\mathcal{M}^\infty$ is an ancient, integral Brakke flow. By the local regularity theorem for the mean-curvature flow and [HW, Theorem B5], $\mathcal{M}^\infty$ is unit regular. By a result of White [W7], the limit $\mathcal{M}^\infty$ is cyclic. Furthermore, we have

$$\mathrm{Ent}[\mathcal{M}^\infty]\leqslant \mathrm{Ent}[S^1\times\mathbb{R}]. \tag{8.4}$$

This entropy bound follows from the same reasoning as in the proof of [He, Theorem 1]. For the sake of completeness, we include it here as well. Since the $\mathcal{M}^i$ have uniformly bounded entropy (by monotonicity), and as $\mathcal{M}^i\to\mathcal{M}^\infty$, it follows from the definition of the entropy of $\mathcal{M}^\infty$ that there exist a sequences of points (x'_k,t'_k) in the original space-time with $(x'_k,t'_k)\to(x_0,t_0)$ and scales $r'_k\to 0$, such that for every $\varepsilon>0$, for k large enough,

$$\frac{1}{4\pi {r'_k}^2}\int \exp\left(-\frac{|x-x'_k|^2}{4(r'_k)^2}\right) d\mu_{t'_k-{r'_k}^2}\geqslant \mathrm{Ent}[\mathcal{M}^\infty]-\varepsilon.$$

On the other hand, since (x_0,t_0) is a cylindrical singularity, there exist $r>0$ such that, for every point in $(p,t)\in B(x_0,r)\times[t_0-r,t_0+r]$, we have

$$\frac{1}{4\pi r^2}\int \exp\left(-\frac{|x-p|^2}{4r^2}\right)d\mu_{t-r^2}\leqslant \mathrm{Ent}[\mathbb{S}^1\times\mathbb{R}]+\varepsilon.$$

By Huisken's monotonicity formula and by the arbitrariness of ε, this implies (8.4). Hence, $\mathcal{M}^\infty$ is an ancient low-entropy flow.

Now, by Theorem 1.2 (classification of ancient low-entropy flows), the limit $\mathcal{M}^\infty$ is either a flat plane, round shrinking sphere, round shrinking cylinder, a bowl soliton, or an ancient oval. If the limit is non-flat, then for i large enough that contradicts $H(X_i)=0$. If the limit is a flat plane, then we obtain a contradiction with the fact X_i has regularity scale $r_i\to 0$, by the local regularity theorem. This proves (iv).

For (v), observe by inspection of the four non-planar ancient low-entropy flows that there exist $A,C<\infty$ such that if $X=(x,t)$ is a point on such a flow $\mathcal{M}^\infty$, then for every unit vector v, there exists a point in

$$X'\in P(X,AR(X))\cap\{y:(y-x)\cdot v\geqslant R(X)\}\cap\mathcal{M}^\infty$$

with

$$4R(X)\leqslant R(X')\leqslant CR(X).$$

Using this, (v) follows by a contradiction argument similarly as in the proof of (iv). This finishes the proof of the proposition. □

By wiggling with the constant δ, we may assume that, in addition to (i)–(v) of Proposition 8.2 we also have:

(vi) $t_0-\delta$ is a smooth time for the flow in $\overline{B}(x_0,2\delta)$, and $M_{t_0-\delta}\cap\overline{B}(x_0,2\delta)$ is ε_0-cylindrical, where $\varepsilon_0>0$ is fixed.

Let $\mathcal{N}$ the unique (space-time) connected component of

$$\mathcal{M}\cap(\overline{B}(x_0,\delta)\times[t_0-\delta,t_0+\delta])$$

which contains the point $X_0=(x_0,t_0)$.[23] In particular, there exist $\varepsilon>0$ such that

$$\mathcal{M}\cap(B(x_0,2\varepsilon)\times[t_0-\varepsilon,t_0+\varepsilon])\subseteq\mathcal{N}. \tag{8.5}$$

We will show that $B(x_0,\varepsilon)\times[t_0-\varepsilon,t_0+\varepsilon]$ is a mean-convex neighborhood.

For any smooth point

$$X=(x,t)\in\mathcal{M}\cap(\overline{B}(x_0,2\delta)\times[t_0-\delta,t_0+\delta]),$$

if $\mathbf{H}(X)$ points into K_t, we say that $H(X)>0$, and if $\mathbf{H}(X)$ points outside of K_t, we say that $H(X)<0$.

[23] In this section $B(x,r)$ always denotes an open ball, and $\overline{B}(x,r)$ a closed ball.

LEMMA 8.3. *We have $H>0$ for all regular points in $\mathcal{N}$.*

Proof. For every time $s\in[t_0-\delta,t_0+\delta]$, let $\mathcal{A}^s$ be the set of (space-time) connected components of $\mathcal{N}\cap(\mathbb{R}^3\times[t_0-\delta,s])$, and let $\mathcal{A}^s_-$ be the set of space-time connected components of $\mathcal{N}\cap(\mathbb{R}^3\times[t_0-\delta,s))$. Set

$$I:=\{s\in[t_0-\delta,t_0+\delta]:\text{For every } \mathcal{F}\in\mathcal{A}^s \text{ and } X_1,X_2\in\mathrm{Reg}\cap\mathcal{F},\\ \mathrm{Sign}(H(X_1))=\mathrm{Sign}(H(X_2))\}.$$

Note that $t_0-\delta\in I$, by property (vi). Also note that, if $s\notin I$, then $s'\notin I$ for every $s'>s$. Thus I is an interval containing $t_0-\delta$. The assertion of the lemma is therefore equivalent to the assertion that $t_0+\delta\in I$. Suppose towards a contradiction that $t_0+\delta\notin I$, and set

$$s_0:=\sup I.$$

CLAIM 8.4. *We have $s_0\notin I$.*

Proof of Claim 8.4. If $s_0=t_0+\delta$ this is clear by assumption. Assume $s_0<t_0+\delta$ were in I. Since $\mathcal{N}\cap\overline{B}(x_0,\delta)$ is compact, each two connected components of $\mathcal{F}_1\neq\mathcal{F}_2$ in $\mathcal{A}^{s_0}$ are a positive distance apart. Moreover, there will be no new connected components entering $\overline{B}(x_0,\delta)$ for some time (again, since supports of Brakke flows are closed). Thus, there exists an $s>s_0$ such that each connected components $\mathcal{F}^s\in A^s$ has a unique connected component $\mathcal{F}^{s_0}\in A^{s_0}$ such that $\mathcal{F}^{s_0}\subseteq\mathcal{F}^s$.

We claim that, if $s-s_0$ is sufficiently small, every smooth point in $\mathcal{F}^s$ can be connected by a path of smooth points in $\overline{B}(x_0,\delta)$ to a point in $\mathcal{F}^{s_0}$. Indeed, by repeated use of Proposition 8.2 (v), any regular point $X=(x,t)\in\mathcal{F}^s$ can be connected via regular points to some $(x',t')=X'\in\mathcal{F}^s$ with $R(X')\geqslant c$ and $|x'-x_0|\leqslant\delta-c^2$. Thus, if $s-s_0$ is sufficiently small, X' can be connected via smooth points to a point in $\mathcal{F}^{s_0}$ (this argument is similar to [KL, Propsition 3.5]).

As $H\neq0$, this implies that H does not change sign on $\mathcal{F}^s$. This proves the claim. □

Thus, if $t_0+\delta\notin I$, then $I^c=[s_0,t_0+\delta]$ for some $s_0\geqslant t_0-\delta$. As $s_0\in I^c$, there exists some connected component $\mathcal{F}\in\mathcal{A}^{s_0}$ such that H does not have a fixed sign over regular points in $\mathcal{F}$. Since for each $s<s_0$, we have $s\in I$, there are three potential ways in which this can happen for the first time at time s_0. We will rule them all out.

Case 1. *Appearance of a new connected component*: $\mathcal{F}\subseteq\partial B(x_0,\delta)\times\{s_0\}$, i.e. a new connected component enters through the boundary. In this case, all points of $\mathcal{F}$ are regular, as the tangent flows are contained in half-spaces. Since $\mathcal{F}$ is connected (by definition), and as $H\neq0$, this implies that the sign of H is constant along $\mathcal{F}$. Thus, this case is excluded.

Case 2. *Merger of two connected components*: There exist two distinct space-time connected components $\mathcal{F}_\pm \in \mathcal{A}^{s_0}_-$ such that $H>0$ (resp. $H<0$) on $\mathcal{F}_+$ (resp. $\mathcal{F}_-$) and such that $\overline{\mathcal{F}}_+ \cap \overline{\mathcal{F}}_- \neq \varnothing$. Since $H \neq 0$ at regular points, the points of $\overline{\mathcal{F}}_+ \cap \overline{\mathcal{F}}_-$ are all singular (in particular, s_0 is a singular time). Moreover, by avoidance,

$$\overline{\mathcal{F}}_+ \cap \overline{\mathcal{F}}_- \subseteq \partial B(x_0, \delta) \times \{s_0\}.$$

Claim 8.5. *There exists a time $t<s_0$ such that $M_t \cap B(x_0, 2\delta)$ is smooth and some* spatial *connected component of $M_t \cap \overline{B}(x_0, 2\delta)$ contains points both from $\mathcal{F}_+$ and $\mathcal{F}_-$.*

Proof of Claim 8.5. Consider the spatial connected components of such $\mathcal{F}_\pm$, $\mathcal{F}_\pm(t)$, and observe how they extend to $\overline{B}(x_0, 2\delta)$. If they never intersect on smooth times, then via a density argument (or, indeed, by the maximum principle), we see that, at singular times as well, if (x', t') is a point reached by both smooth flows, then $x' \in \partial B(x_0, 2\delta)$. We thus obtain two disjoint unit multiplicity Brakke flows in $B\big(x_0, \frac{3}{2}\delta\big)$, which have a point in $\overline{\mathcal{F}}_+ \cap \overline{\mathcal{F}}_-$ as a joint point they reach at time s_0, but this is again impossible since the Huisken density near (x_0, t_0) is less than 2. □

Since $H \neq 0$ on $M_t \cap \overline{B}(x_0, 2\delta)$, this gives a contradiction to the definitions of $\mathcal{F}_\pm$. Indeed, at time t, the signs of H on $\mathcal{F}_\pm$ have to be the same. Thus, this case is excluded as well.

Case 3. *Merger of an existing component with a new boundary component*: In this case, we may assume without loss of generality that $\mathcal{F}$ contains $\mathcal{F}_+ \cup \mathcal{F}_\partial$, where on $\mathcal{F}_+ \in \mathcal{A}^{s_0}_-$ we have $H>0$ on regular points and where

$$\mathcal{F}_\partial \subseteq \mathcal{F} \cap (\partial B(x_0, \delta) \times \{s_0\})$$

is connected, intersects $\overline{\mathcal{F}}_+$, and contains at least one point which is not in

$$\overline{\mathcal{M} \cap (\overline{B}(x_0, \delta) \times [t_0 - \delta, s_0))},$$

and all such points have $H<0$ (note that, as in the first case, such points are always regular). This case is excluded similarly to Case 2 above: Taking such a point, and looking at the connected component of the point reaching it at a slightly earlier smooth time t in a slightly bigger ball, we see it intersects with the continuation of $\mathcal{F}_+(t)$ in that ball, which cannot be.

We have thus excluded the existence of such an $\mathcal{F}$, so $I^c = \varnothing$. This concludes the proof of the lemma. □

Proof of Theorem 8.1. Having established Proposition 8.2 and Lemma 8.3, the final step of the proof of the mean-convex neighborhood theorem is now similar to the final step of the proof of [HW, Theorem 3.5]. For convenience of the reader, we include the argument here as well.

CLAIM 8.6. *If* $t_0-\varepsilon\leqslant t_1<t_2\leqslant t_0+\varepsilon$, *then*

$$K_{t_2}\cap B(x_0,\varepsilon)\subseteq K_{t_1}.$$

Proof of Claim 8.6. Suppose to the contrary that there is point $y\in K_{t_2}\cap B(x_0,\varepsilon)$ that is not in K_{t_1} for some $t_1<t_2$ with $t_1,t_2\in[t_0-\varepsilon,t_0+\varepsilon]$. Let

$$0<\rho<\operatorname{dist}(y,K_{t_1}\cup\partial B(x_0,\varepsilon)).$$

Let $t\in(t_1,t_0)$ be the first time $\geqslant t_1$ such that

$$\operatorname{dist}(y,K_t)=\rho.$$

Let $p\in K_t$ be a point such that $\operatorname{dist}(y,p)=\operatorname{dist}(y,K_t)$. Note that the tangent flow at (p,t) lies in a half-space (more specifically the half-space $\{x:x\cdot(y-p)\leqslant 0\}$). Hence, (p,t) is a regular point of the flow. By the previous lemma, the mean curvature at (p,t) is positive and so, for $s<t$ very close to t, we have $\operatorname{dist}(y,K_s)<\rho$; a contradiction. This proves the claim. □

Continuing the proof of the theorem, it only remains to show that for $t_1<t_2$ with $t_1,t_2\in[t_0-\varepsilon,t_0+\varepsilon]$, if

$$x\in B(x_0,\varepsilon)\cap K_{t_2},$$

then x is in the interior of K_{t_1}. If x is the interior of K_{t_2}, then by Claim 8.6 it is in the interior of K_{t_1}. Thus, we may assume that x is in the boundary of K_{t_2}. For $s\in[t_1,t_2)$ sufficiently close to t_2, the point x is in the interior of K_s. If (x,t_2) is a regular point, this is because the mean curvature is positive. If (x,t_2) is a singular point, this is true by condition (ii) of Proposition 8.2. Since x is in the interior of K_s, and since $K_s\subseteq K_{t_1}$, we conclude that x is in the interior of K_{t_1}. This completes the proof of the mean-convex neighborhood theorem. □

8.2. Proof of the uniqueness conjectures for weak flows

In this final section, we prove the uniqueness results for mean-curvature flow through singularities, namely Theorem 1.9 (non-fattening) and Theorem 1.12 (evolution of 2-spheres).

Proof of Theorem 1.9. Observe that if $T\leqslant T_{\text{disc}}$ then, by definition, the outer flow $\{M_t\}_{t\in[0,T]}$ agrees with the level set flow of M and the inner flow of M. In particular, if (x_0,T) is a multiplicity-1 cylindrical (resp. spherical) singularity of $\{M_t\}$ then *either* $\mathcal{K}_{X,\lambda}$ *or* $\mathcal{K}'_{X,\lambda}$ converges for $\lambda\to 0$ smoothly with multiplicity 1 to a round shrinking solid cylinder or a round shrinking solid ball (see also item (i) of Proposition 8.2 and its proof).

The result now follows from combining the main theorem of Hershkovits–White [HW], which establishes that $T<T_{\text{disc}}$ assuming the existence of mean-convex neighborhoods a priori, and Theorem 1.7, which proves the existence of mean-convex neighborhoods. □

Proof of Theorem 1.12. Let $M\subset\mathbb{R}^3$ be an embedded 2-sphere, and let $\mathcal{M}=\{M_t\}$ be the outer flow starting at M. By a result of White [W1], whenever M_t is smooth, then it has genus zero.

Suppose towards a contradiction that there is some singular point with a tangent flow that is not spherical or cylindrical. Let T be the first time when this happens (more precisely, the infimum of times where this happens). Then $T\leqslant T_{\text{disc}}$ (by Theorem 1.9) and for a.e. $t\in[0,T]$ the flow is smooth. Hence, we can run the argument from Ilmanen [I2], and get, assuming the multiplicity-1 conjecture, that at time T at every singular point there exists a tangent flow that is smooth with multiplicity 1.

Such a tangent flow, by smoothness and unit-regularity, cannot be a non-trivial static or quasistatic cone. Hence, it is a properly embedded shrinker. Suppose towards a contradiction there are two loops in the shrinker N that have non-vanishing intersection number modulo 2. Then, for some smooth time $t'<T$, we would see two loops in $M_{t'}$ with non-vanishing intersection number modulo 2; a contradiction. Hence, by a result of Brendle [Bre2], the shrinker N is a sphere, a cylinder or a plane. Thus, arguing as in the first paragraph of the proof of Theorem 1.9, by the mean-convex neighborhood theorem (Theorem 1.7) and compactness, there is some $\delta>0$ such that the flow has only cylindrical and spherical singularities on $[0,T+\delta)$; this contradicts the definition of T. Hence, the flow $\mathcal{M}$ has only spherical and cylindrical singularities. Thus, by Theorem 1.9 (non-fattening) the solution is unique. □

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