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The linear stability of the Schwarzschild solution to gravitational perturbations

by

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1. Introduction

The Schwarzschild family [69] of spacetimes $(\mathcal{M}, g_M)$, expressed in local coordinates as

$$-\left(1-\frac{2M}{r}\right)dt^2+\left(1-\frac{2M}{r}\right)^{-1}dr^2+r^2(d\theta^2+\sin^2\theta\, d\phi^2), \tag{1}$$

was discovered exactly one hundred years before this writing, and comprises the most basic family of non-trivial solutions to the celebrated Einstein vacuum equations

$$\mathrm{Ric}(g)=0 \tag{2}$$

of general relativity. Though originally geometrically obscured by the coordinate form (1), the family has now long been understood (see the textbook [76]) to yield, for parameter values $M>0$, the simplest examples of spacetimes containing a so-called *black hole*, with the coordinate range $r>2M$ corresponding to the *exterior*.

There is perhaps no question more fundamental to pose concerning Schwarzschild than that of the stability of its exterior:

Fundamental question. Is the Schwarzschild exterior metric (1) stable as a solution to (2)?

The whole tenability of the black hole notion rests on a positive answer to the above. The question is further complicated by the fact that the Schwarzschild family sits as a 1-parameter subfamily of the more elaborate 2-parameter Kerr family $(\mathcal{M}, g_{M,a})$ discovered only much later [43] in 1963.

One can distinguish between three formulations of the above fundamental question, each of increasing difficulty, beginning from the statements initially studied in the physics literature and ending with the definitive formulation of the question as a problem of non-linear stability in the context of the Cauchy problem for (2), in analogy with the non-linear stability of Minkowski space, proven in the monumental work [14] of Christodoulou and Klainerman.

(1) *The formal mode analysis of the linearised equations.* The equations of gravitational perturbations around Schwarzschild (i.e. the linearisation of (2)), "linearised gravity" for short, can be formally decomposed into modes by associating t-derivatives with multiplication by $i\omega$ and angular derivatives with multiplication by $i\ell$. The formal study of fixed modes from the point of view of "metric perturbations" was initiated in a seminal paper of Regge–Wheeler [61]. This study was completed by Vishveshwara [75] and Zerilli [79]. A gauge-invariant formulation of "metric perturbations" was then given by Moncrief [54]. An alternative approach via the Newman-Penrose formalism [58] was conducted by Bardeen–Press [4]. This latter type of analysis was later extended to the Kerr family by Teukolsky [74]. A highlight of this analysis was the discovery that various curvature components in a null frame satisfy a decoupled wave equation, the celebrated *Teukolsky equation*, first discovered in the Schwarzschild case in [4] and generalised to the Kerr case in [74]. The understanding of the problem in the early 1980's is summarised by the magisterial monograph of Chandrasekhar [9], who introduced (see also [8]) an important transformation theory connecting solutions of the two approaches. A highly non-trivial result is the statement of *mode stability* for the Teukolsky equation on Kerr, obtained in a seminal paper of Whiting [77].

(2) *The problem of linear stability of Schwarzschild.* The true problem of linear stability concerns general solutions to the equations of linearised gravity arising from regular initial data, not simply fixed modes. One can in fact distinguish between two linear stability statements:

(2a) the question of whether all solutions to the linearised Einstein equations around Schwarzschild remain bounded for all time by a suitable norm of their initial data and

(2b) the question of asymptotic linear stability–i.e. whether all solutions to the linearised equations asymptotically decay. In view of the existence of the Kerr family and the gauge freedom of the equations, the best result would be that they decay to a linearised Kerr solution in some gauge.

Note that the mode analysis corresponding to formulation 1. described above yields necessary but not sufficient conditions for either statements (2a) and (2b) of true linear stability.(1) In the case of the linear *scalar* wave equation

$$\Box_g \varphi = 0,$$

which can be thought of as a "poor man's" version of linearised gravity, the analogue of (2a) for Schwarzschild was proven by Kay–Wald [41], and the analogue of (2a) and (2b) are shown now for the full subextremal Kerr family in [26], following a host of recent activity [24], [73], [1], [21]. See [25], [23] for a survey. See [6], [2] for generalisations to the Maxwell equations and [3] for a discussion of the extremal case $|a|=M$. Concerning the linearised Einstein equations themselves, work on the wave equation easily generalises to establish physical space decay on certain quantities, for instance those gauge-invariant quantities satisfying the Regge–Wheeler equation on Schwarzschild [32], [7], [29]. *For the full system of linearised gravity however, both problems* (2a) *and* (2b) *have remained open until today.* We note explicitly that even the question of uniform boundedness, let alone decay, for the gauge-invariant quantities satisfying the Teukolsky equation on Schwarzschild has remained open.

(3) *The full non-linear stability of Schwarzschild as a solution to the Cauchy problem for the non-linear Einstein vacuum equations* (2). This is the definitive formulation of the fundamental question. See our previous [18] for a precise statement of the conjecture in the language of the Cauchy problem for (2). In analogy with 2. above, one could distinguish between questions of (3a) orbital stability and (3b) asymptotic stability. Experience from non-linear problems, however, in particular the proof of the non-linear stability of Minkowski space [14] referred to earlier (see also [47], [5]), indicates that (3a) and (3b) are naturally coupled.(2) Since non-linear stability is thus necessarily a question of asymptotic stability, the "Schwarzschild" problem is more correctly re-phrased as *the non-linear asymptotic stability of the Kerr family in a neighbourhood of Schwarzschild.* For even if one restricts to small perturbations of Schwarzschild, it is expected that generically, spacetime dynamically asymptotes to a very slowly rotating Kerr solution with $a\neq 0$. Since in the context of a non-linear stability proof, one effectively must "linearise" around the solution one expects to approach, this suggests that to resolve the full

(1) Thus, the mode analysis can be an effective tool to show instability, but never, on its own, stability. For instability results for related problems proven via the existence of unstable modes, see [70], [28] and references therein. See also discussion in [78].

(2) This coupling arises from the super-criticality of the Einstein vacuum equations (2). Note that under spherical symmetry (where the vacuum equations must be replaced, however, by a suitable Einstein–matter system to restore a dynamical degree of freedom) this super-criticality is broken in the presence of a black hole. Orbital stability can then be proven independently of asymptotic stability, cf. [17] with [19]. Similar symmetric reductions can be studied for the vacuum equations in higher dimensions [11], [35].

non-linear formulation 3. will require a solution of the linear formulation 2. *not just for Schwarzschild, but for very slowly rotating Kerr solutions with* $|a| \ll M$. Though the non-linear stability problem is thus completely open, in [18] we have proven, via a scattering theory construction, the existence of a class of dynamical vacuum spacetimes, without any symmetry assumptions, asymptotically settling down to Kerr in accordance with the expectation of non-linear stability. In view of the fast, exponential rate of approach which we impose in [18], however, the class we construct is expected to be of infinite codimension in the space of all solutions.

The purpose of the present paper is to completely resolve the *linear* stability problem (i.e. formulation (2)) in the Schwarzschild case, in both its aspects (2a) and (2b). A first version of our main result can be stated as follows.

THEOREM. (Linear stability of Schwarzschild) *All solutions to the linearised vacuum Einstein equations* (*in double null gauge*) *around Schwarzschild arising from regular asymptotically flat initial data*

(a) remain uniformly bounded *on the exterior and*

(b) decay inverse polynomially *(through a suitable foliation) to a standard linearised Kerr solution*

after adding a pure gauge solution which can itself be estimated by the size of the data.

See Theorems 3 and 4 of the detailed overview in §2 as well as the more detailed later formulations in the bulk of the paper referred to there.

A word about gauge is already in order. We will express the equations of linearised gravity in a *double null gauge*. This still allows, however, for a residual gauge freedom which in linear theory manifests itself in the existence of "pure gauge solutions" corresponding to one parameter families of deformations of the ambient null foliation of Schwarzschild. To measure geometrically the initial data of a general solution of linearised gravity so as to formulate the boundedness statement (a), one "normalises" the solution on initial data by adding an appropriate pure gauge solution which is computable explicitly from the original solution's initial data. Importantly, this gauge assures that the position of the horizon is fixed and that the "sphere at infinity" is round. We emphasise that at the level of natural energy fluxes, we obtain a boundedness statement controlling the full normalised solution without loss of derivatives. An interesting aspect of our work is that to obtain the decay statement (b), we must add yet another pure gauge solution which effectively re-normalises the gauge on the event horizon. It is fundamental that we can quantitatively control this new pure gauge solution in terms of the geometry of initial data as expressed in the original normalisation, i.e. the new pure gauge solution, though not explicitly given from data, itself satisfies a uniform boundedness statement.

In particular, our theorem is stronger than (and thus includes a fortiori) the statement that the gauge invariant linearised curvature components ($\overset{(1)}{\alpha}$ and $\overset{(1)}{\underline{\alpha}}$ in our notation) satisfying the Teukolsky equation (discussed above in the context of formal mode analysis) themselves remain bounded and in fact decay inverse polynomially. We will in fact prove the boundedness and decay of these components as a preliminary step to proving the full theorem above. More specifically, we will first prove estimates for certain *higher-order* gauge invariant quantities $\overset{(1)}{P}$ and $\overset{(1)}{\underline{P}}$ (at the level of 4 derivatives of the linearised metric and 2 derivatives of $\overset{(1)}{\alpha}$ and $\overset{(1)}{\underline{\alpha}}$), which satisfy the *Regge–Wheeler* equation. (This is the same equation which originally appeared in the "metric perturbations" approach discussed above in the context of formal mode analysis.) The significance of the Regge–Wheeler equation is that it can in fact be understood using the methods developed for the scalar wave equation discussed above. The quantities $\overset{(1)}{P}$ and $\overset{(1)}{\underline{P}}$ will serve as the key to unlocking the whole system, leading first to the control of $\overset{(1)}{\alpha}$ and $\overset{(1)}{\underline{\alpha}}$, and then to the control of the entirety of the system. Let us state explicitly the following corollary of our proof of the main theorem:

COROLLARY. *All the solutions of the Regge–Wheeler and Teukolsky equations on Schwarzschild arising from smooth compactly supported data decay inverse-polynomially with respect to the time function of a suitable foliation of the exterior.*

See Theorems 1 and 2 in §2 and the more detailed later formulations referred to there.

The expression of $\overset{(1)}{P}$ as a second order differential operator applied to $\overset{(1)}{\alpha}$ can be viewed as a physical-space version of the fixed-frequency transformations of Chandrasekhar [9] referred to previously. See also [65]. One of the main points of the present paper is that the physical space understanding of the relation between $\overset{(1)}{\alpha}$ and $\overset{(1)}{P}$ clarifies the fact that the original $\overset{(1)}{\alpha}$ can be recovered from $\overset{(1)}{P}$ and *initial data quantities* by integrating *transport equations* along light cones. Indeed, we succeed in estimating all quantities hierarchically, gauge invariant and gauge dependent, from initial data, by appropriate estimates of such equations, after control of $\overset{(1)}{P}$ and $\overset{(1)}{\underline{P}}$ via Regge–Wheeler.

We collect some additional references relevant for the problem. Recent studies of stability in the physics literature with an eye toward numerical implementation include [51], [64], [50]. For analysis of a model problem related to the axisymmetric reduction of the stability problem, see [40]. We also note [31], [30].

Let us note finally that the inverse polynomial decay bounds shown in our theorem above are in principle sufficiently strong so as to treat quadratic non-linearities of the type present in (2) purely by exploiting the dispersion embodied by our decay results. This allows one already to try to address a restricted non-linear stability conjecture establishing the existence of the full finite-codimension family of solutions which indeed asymptotically settle down to Schwarzschild. (See §2.5 for a precise statement; note that this conjecture would include a fortiori various well-known symmetric reductions of the stability problem which similarly impose a Schwarzschild end-state.) If in future work the main theorem of the present paper can be extended to the Kerr case, at least in the very slowly rotating regime $|a| \ll M$, then it indeed opens the way for study of formulation 3., and thus, for a definitive resolution of the stability question.

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2. Overview

We shall give in this section a complete overview of our paper.

We begin in §2.1 with an introduction to the basic properties of linearised gravity around Schwarzschild in a double null gauge, corresponding to §§3–9 of the body of the paper. We will then state rough versions of the main theorems of the paper in §2.2, corresponding to the precise statements in §10 of the body. We make a brief aside (§2.3) to review the theory of the scalar wave equation which is useful to have in mind before turning to the proofs. We shall then return to outlining the present paper in §2.4, where we shall review the proofs of the main theorems, following closely §§11–14 of the body of the paper. We finally give in §2.5 a restricted version of the non-linear stability conjecture which in principle can be addressed using the results of this paper.

We have included also a guide for reading the paper (§2.6) for the convenience of those who are only interested in a subset of the results proven here.

2.1. Linearised gravity around Schwarzschild in a double null gauge

Our paper will employ a *double null gauge* to express the equations of linearised gravity. This will define an associated *double null frame*. The setup is intimately related with the approach to this problem via the Newman–Penrose formalism [58] studied in the physics literature since [4]. We note that double null gauges have figured prominently in the non-linear analysis of the Einstein vacuum equations [11], [45], [49], [44] and thus provide a promising setting for a future full non-linear analysis of the stability problem. We describe here how the relevant equations are obtained in physical space, as well as their most basic properties, including their initial value formulation and the issue of pure gauge solutions.

The subsections of this section follow closely §§3–9 of the body of the paper: We shall review first in §2.1.1 the form of the Einstein equations in double null gauge (cf. §3). After reviewing the Schwarzschild manifold in §2.1.2 (cf. §4), we shall derive the equations of linearised gravity around Schwarzschild in §2.1.3 (cf. §5). We shall then identify in §2.1.4 two special classes of solutions, pure gauge solutions and a reference linearised Kerr family (cf. §6). The presence of these special solutions motivate looking at a hierarchy of gauge invariant quantities satisfying the Teukolsky and Regge–Wheeler equations; these will be introduced in §2.1.5 (cf. §7). We shall then discuss the characteristic initial value problem for linearised gravity in §2.1.6 (cf. §8), and the issue of gauge normalisation in §2.1.7 (cf. §9).

2.1.1. The Einstein equations in a double null gauge

We first review the general form of the Einstein vacuum equations (2) in a double null gauge, following Christodoulou [10], [11]. This corresponds to §3 of the body of the paper.

A double null gauge is a coordinate system $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{\theta}^1$, $\boldsymbol{\theta}^2$ such that the metric takes the form

$$\boldsymbol{g} = -4\boldsymbol{\Omega}^2 \boldsymbol{du}\boldsymbol{dv} + \boldsymbol{\not g}_{CD}(\boldsymbol{d\theta}^C - \boldsymbol{b}^C\ \boldsymbol{dv})(\boldsymbol{d\theta}^D - \boldsymbol{b}^D\ \boldsymbol{dv}). \tag{3}$$

The hypersurfaces of constant $\boldsymbol{u}$ and $\boldsymbol{v}$ are then manifestly null hypersurfaces. Moreover, the coordinate vector field $\boldsymbol{\partial_u}$ is in the direction of the null generator of the constant-$\boldsymbol{v}$ hypersurfaces.

Associated with a double null gauge is a natural normalised null frame

$$\boldsymbol{e}_3 = \boldsymbol{\Omega}^{-1}\boldsymbol{\partial_u}, \quad \boldsymbol{e}_4 = \boldsymbol{\Omega}^{-1}(\boldsymbol{\partial_v} + \boldsymbol{b}^A\boldsymbol{\partial}_{\boldsymbol{\theta}^A}) \tag{4}$$

which, together with the choice of a local (not necessarily orthonormal) frame $\{\boldsymbol{e}_1, \boldsymbol{e}_2\}$ with $\boldsymbol{g}(\boldsymbol{e}_4, \boldsymbol{e}_1) = 0$, $\boldsymbol{g}(\boldsymbol{e}_3, \boldsymbol{e}_2) = 0$, allows one to decompose components of the second fundamental form and curvature. In our notation this yields *Ricci coefficients*

$$\begin{aligned}
\boldsymbol{\chi}_{AB} &= \boldsymbol{g}(\boldsymbol{\nabla}_A \boldsymbol{e}_4, \boldsymbol{e}_B), & \underline{\boldsymbol{\chi}}_{AB} &= \boldsymbol{g}(\boldsymbol{\nabla}_A \boldsymbol{e}_3, \boldsymbol{e}_B),\\
\boldsymbol{\eta}_A &= -\tfrac{1}{2}\boldsymbol{g}(\boldsymbol{\nabla}_{\boldsymbol{e}_3}\boldsymbol{e}_A, \boldsymbol{e}_4), & \underline{\boldsymbol{\eta}}_A &= -\tfrac{1}{2}\boldsymbol{g}(\boldsymbol{\nabla}_{\boldsymbol{e}_4}\boldsymbol{e}_A, \boldsymbol{e}_3),\\
\widehat{\boldsymbol{\omega}} &= \tfrac{1}{2}\boldsymbol{g}(\boldsymbol{\nabla}_{\boldsymbol{e}_4}\boldsymbol{e}_3, \boldsymbol{e}_4), & \underline{\widehat{\boldsymbol{\omega}}} &= \tfrac{1}{2}\boldsymbol{g}(\boldsymbol{\nabla}_{\boldsymbol{e}_3}\boldsymbol{e}_4, \boldsymbol{e}_3),
\end{aligned} \tag{5}$$

where $\boldsymbol{\nabla}_A = \boldsymbol{\nabla}_{e_A}$ as well as *curvature components*

$$\begin{aligned}
\boldsymbol{\alpha}_{AB} &= \boldsymbol{R}(\boldsymbol{e}_A, \boldsymbol{e}_4, \boldsymbol{e}_B, \boldsymbol{e}_4), & \underline{\boldsymbol{\alpha}}_{AB} &= \boldsymbol{R}(\boldsymbol{e}_A, \boldsymbol{e}_3, \boldsymbol{e}_B, \boldsymbol{e}_3),\\
\boldsymbol{\beta}_A &= \tfrac{1}{2}\boldsymbol{R}(\boldsymbol{e}_A, \boldsymbol{e}_4, \boldsymbol{e}_3, \boldsymbol{e}_4), & \underline{\boldsymbol{\beta}}_A &= \boldsymbol{R}(\boldsymbol{e}_A, \boldsymbol{e}_3, \boldsymbol{e}_3, \boldsymbol{e}_4),\\
\boldsymbol{\varrho} &= \tfrac{1}{4}\boldsymbol{R}(\boldsymbol{e}_4, \boldsymbol{e}_3, \boldsymbol{e}_4, \boldsymbol{e}_3), & \boldsymbol{\sigma} &= \tfrac{1}{4}{}^\star\boldsymbol{R}(\boldsymbol{e}_4, \boldsymbol{e}_3, \boldsymbol{e}_4, \boldsymbol{e}_3).
\end{aligned} \tag{6}$$

The content of the Einstein vacuum equations (2) can be expressed as a system of transport and elliptic equations for the metric (3) and Ricci coefficients (5) coupled with Bianchi identities for the curvature (6), the latter capturing the essential hyperbolicity of the equations (2).

Concerning the metric and Ricci coefficients, examples of transport equations are

$$\underline{\boldsymbol{D}}\boldsymbol{\not{g}} = 2\boldsymbol{\Omega}\underline{\boldsymbol{\chi}} = 2\boldsymbol{\Omega}\widehat{\underline{\boldsymbol{\chi}}} + \boldsymbol{\Omega}\,\mathbf{tr}\,\underline{\boldsymbol{\chi}}\,\boldsymbol{\not{g}}, \quad \boldsymbol{D}\boldsymbol{\not{g}} = 2\boldsymbol{\Omega}\boldsymbol{\chi} = 2\boldsymbol{\Omega}\widehat{\boldsymbol{\chi}} + \boldsymbol{\Omega}\,\mathbf{tr}\,\boldsymbol{\chi}\,\boldsymbol{\not{g}}, \tag{7}$$

$$\boldsymbol{\nabla\!\!\!\!/}_{\mathbf{3}}\widehat{\underline{\boldsymbol{\chi}}} + \mathbf{tr}\,\underline{\boldsymbol{\chi}}\,\widehat{\underline{\boldsymbol{\chi}}} - \widehat{\underline{\boldsymbol{\omega}}}\widehat{\underline{\boldsymbol{\chi}}} = -\underline{\boldsymbol{\alpha}}, \quad \boldsymbol{\nabla\!\!\!\!/}_{\mathbf{4}}\widehat{\boldsymbol{\chi}} + \mathbf{tr}\,\boldsymbol{\chi}\,\widehat{\boldsymbol{\chi}} - \widehat{\boldsymbol{\omega}}\;\widehat{\boldsymbol{\chi}} = \boldsymbol{\alpha}, \tag{8}$$

$$\boldsymbol{\nabla\!\!\!\!/}_{\mathbf{3}}(\mathbf{tr}\,\underline{\boldsymbol{\chi}}) + \tfrac{1}{2}(\mathbf{tr}\,\underline{\boldsymbol{\chi}})^2 - \widehat{\underline{\boldsymbol{\omega}}}\,\mathbf{tr}\,\underline{\boldsymbol{\chi}} = -(\widehat{\underline{\boldsymbol{\chi}}}, \widehat{\underline{\boldsymbol{\chi}}}), \quad \boldsymbol{\nabla\!\!\!\!/}_{\mathbf{4}}(\mathbf{tr}\,\boldsymbol{\chi}) + \tfrac{1}{2}(\mathbf{tr}\,\boldsymbol{\chi})^2 - \widehat{\boldsymbol{\omega}}\,\mathbf{tr}\,\boldsymbol{\chi} = -(\widehat{\boldsymbol{\chi}}, \widehat{\boldsymbol{\chi}}), \tag{9}$$

while an example of an elliptic equation is the Codazzi equation

$$\mathbf{d\!\!\!/iv}\,\widehat{\underline{\boldsymbol{\chi}}} = \frac{1}{2}\widehat{\underline{\boldsymbol{\chi}}}^{\sharp}\cdot(\boldsymbol{\eta} - \underline{\boldsymbol{\eta}}) - \frac{1}{2}\,\mathbf{tr}\,\boldsymbol{\chi}\,\boldsymbol{\eta} + \frac{1}{2\boldsymbol{\Omega}}\boldsymbol{\nabla\!\!\!\!/}(\boldsymbol{\Omega}\,\mathbf{tr}\,\underline{\boldsymbol{\chi}}) + \underline{\boldsymbol{\beta}}. \tag{10}$$

Here $\widehat{\boldsymbol{\chi}}$, known as the *shear*, denotes the trace-free part of $\boldsymbol{\chi}$; the quantity $\mathbf{tr}\,\boldsymbol{\chi}$ is known as the *expansion*. The operators $\underline{\boldsymbol{D}}$ and $\boldsymbol{\nabla\!\!\!\!/}_3$ (respectively, $\boldsymbol{D}$ and $\boldsymbol{\nabla\!\!\!\!/}_4$) are geometric operators associated with $\boldsymbol{g}$ acting on tensors differentiating in the $\boldsymbol{e}_3$ (respectively, $\boldsymbol{e}_4$) directions, while $\mathbf{d\!\!\!/iv}$ is a natural operator on the constant-$(\boldsymbol{u}, \boldsymbol{v})$ spheres. Equations (9) are the celebrated *Raychaudhuri equations*. The full system of "null structure" equations satisfied by the metric and Ricci coefficients is given in §3.3.1.

Concerning the curvature components (6), examples of Bianchi identities are

$$\boldsymbol{\nabla\!\!\!\!/}_{\mathbf{3}}\boldsymbol{\alpha} + \tfrac{1}{2}\,\mathbf{tr}\,\underline{\boldsymbol{\chi}}\,\boldsymbol{\alpha} + 2\widehat{\underline{\boldsymbol{\omega}}}\boldsymbol{\alpha} = -2\boldsymbol{\mathcal{D}\!\!\!\!/}_{\mathbf{2}}^{\star}\boldsymbol{\beta} - 3\widehat{\boldsymbol{\chi}}\boldsymbol{\varrho} - 3\,{}^{\star}\widehat{\boldsymbol{\chi}}\boldsymbol{\sigma} + (4\boldsymbol{\eta} + \boldsymbol{\zeta})\widehat{\otimes}\boldsymbol{\beta}, \tag{11}$$

$$\boldsymbol{\nabla\!\!\!\!/}_{\mathbf{4}}\boldsymbol{\beta} + 2\,\mathbf{tr}\,\boldsymbol{\chi}\,\boldsymbol{\beta} - \widehat{\boldsymbol{\omega}}\boldsymbol{\beta} = \mathbf{d\!\!\!/iv}\,\boldsymbol{\alpha} + (\underline{\boldsymbol{\eta}}^{\sharp} + 2\boldsymbol{\zeta}^{\sharp})\cdot\boldsymbol{\alpha}, \tag{12}$$

where the operator $\boldsymbol{\mathcal{D}\!\!\!\!/}_{\mathbf{2}}^{\star}$ is a suitable adjoint of $\mathbf{d\!\!\!/iv}$ and $2\boldsymbol{\zeta} = \boldsymbol{\eta} - \underline{\boldsymbol{\eta}}$ denotes the torsion. The full system is given in §3.3.2.

2.1.2. The ambient Schwarzschild metric

We note that any Lorentzian metric can *locally* be put in the above form (3). The maximally extended Schwarzschild manifold and–less obviously–the globally hyperbolic region of Kerr (see Pretorius and Israel [59]) can in fact both be *globally* covered by such a coordinate system, where θ^1 and θ^2 are interpreted as coordinates on the sphere $\mathbb{S}^2$, modulo the usual degeneration of spherical coordinates. In the bulk of the paper, fixing an ambient Schwarzschild manifold and double null foliation will be the content of §4.

We summarise briefly here: In the Schwarzschild case with parameter M, one easily derives from expression (1) a double null parametrisation of the exterior region $r > 2M$. Defining first

$$r^* \doteq r + 2M\log(r - 2M) - 2M\log 2M, \tag{13}$$

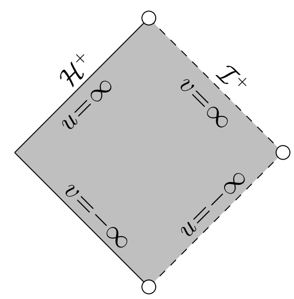

Figure 1.

set

$$u \doteq t - r^* \quad \text{and} \quad v \doteq t + r^*. \tag{14}$$

Then, u and v define null coordinates on the region $r>2M$, parametrising it as $(-\infty,\infty)\times(-\infty,\infty)\times\mathbb{S}^2$. In the notation (3), the Schwarzschild metric takes the form

$$\Omega^2 = 1-\frac{2M}{r}, \quad \not{g} = r^2\gamma, \quad b^A = 0 \tag{15}$$

where γ denotes the standard metric of the unit sphere. Note that $r(u,v)$ is now defined implicitly by (13) and (14). In this context, the coordinates u and v are known as *Eddington–Finkelstein double null coordinates*.

Note that we distinguish Schwarzschild metric quantities in the above differentiable structure by presenting them in regular type, de-bolded compared with quantities associated with a general manifold and metric (3). This notation is used in the remainder of the paper.

Upon rescaling the null coordinate $U=U(u)$ appropriately, one can extend the region on which the metric is defined to include the so-called event horizon $\mathcal{H}^+$, a null hypersurface which then corresponds to $r=2M$. In the body of the paper, it is in fact this U-coordinate which we shall use to *define* the Schwarzschild manifold in §4.1, with u defined by inverting the rescaling, and t and r^* by (14). It is computationally useful however to work with the irregular u coordinate, which formally still parametrises $\mathcal{H}^+$ as $u=\infty$, and to compensate for this by introducing renormalised quantities that are regular on the horizon. See Figure 1. Thus, our basic double null foliation will be that defined by coordinates u and v.

We note that the static Killing field ∂_t whose existence is manifest from (1) is expressed in (u,v) coordinates as

$$T = \tfrac{1}{2}(\partial_u + \partial_v). \tag{16}$$

This vector field extends smoothly to a null vector on the horizon $\mathcal{H}^+$; in (U,v) coordinates, it takes the form $T=\frac{1}{2}\partial_v$ on $\mathcal{H}^+$.

From (15), in the double null foliation defined by the u, v coordinates, we compute that, with the notation of §2.1.1, the non-vanishing Schwarzschild metric coefficients are

$$\Omega=\sqrt{1-\frac{2M}{r}}, \quad \sqrt{\not{g}} \doteq \sqrt{\det \not{g}} = r^2\sqrt{\det\gamma} \doteq r^2\sqrt{\gamma}, \tag{17}$$

and, with respect to the associated null frame,

$$e_3=\Omega^{-1}\partial_u, \quad e_4=\Omega^{-1}\partial_v,$$

the non-vanishing Ricci coefficients are

$$\chi_{AB}=r^{-1}\Omega\not{g}_{AB}, \quad \underline{\chi}_{AB}=-r^{-1}\Omega\not{g}_{AB}, \quad \widehat{\omega}=r^{-2}\Omega^{-1}M, \quad \underline{\widehat{\omega}}=-r^{-2}\Omega^{-1}M, \tag{18}$$

while the only non-vanishing curvature component is

$$\varrho=-\frac{2M}{r^3}. \tag{19}$$

2.1.3. Linearised gravity around Schwarzschild

The equations of interest in this paper (the equations of gravitational perturbations around Schwarzschild, or "linearised gravity" for short) are those that arise from linearising the system in §2.1.1 around their Schwarzschild values (17)–(19). The equations are derived and presented in §5 of the body of the paper. We give a brief outline here.

In deriving the equations of linearised gravity, we will fix the above background differential structure $(\mathcal{M},g)$ with its Schwarzschild metric, and embed (15) in a 1-parameter family of metrics $\boldsymbol{g}$ of the form (3) satisfying the Einstein vacuum equations (2).

The system of linearised gravity concerns linearised quantities associated with the quantites in §2.1.1, namely *linearised metric coefficients*([3])

$$\overset{(1)}{\Omega}, \quad \overset{(1)}{\sqrt{\not{g}}}, \quad \overset{(1)}{\hat{\not{g}}}, \quad \overset{(1)}{b}, \tag{20}$$

linearised Ricci coefficients

$$\overset{(1)}{(\Omega\,\mathrm{tr}\,\chi)}, \quad \overset{(1)}{(\Omega\,\mathrm{tr}\,\underline{\chi})}, \quad \overset{(1)}{\eta}, \quad \overset{(1)}{\underline{\eta}}, \quad \overset{(1)}{\omega}, \quad \overset{(1)}{\underline{\omega}}, \quad \overset{(1)}{\widehat{\chi}}, \quad \overset{(1)}{\underline{\widehat{\chi}}}, \tag{21}$$

([3]) The reader should think of the quantities $\overset{(1)}{\hat{\not{g}}}, \overset{(1)}{\sqrt{\not{g}}}$ as arising from linearising the metric $\not{\boldsymbol{g}}$ and then taking the traceless and trace parts (with respect to the round background metric on the sphere) of this object. The quantities $\overset{(1)}{\omega}, \overset{(1)}{\underline{\omega}}$ arise from linearising the rescaled quantities $\boldsymbol{\omega}=\boldsymbol{\Omega}\widehat{\boldsymbol{\omega}}$ and $\underline{\boldsymbol{\omega}}=\boldsymbol{\Omega}\underline{\widehat{\boldsymbol{\omega}}}$. See §5.1.

as well as *linearised curvature components*

$$\overset{(1)}{\alpha}, \quad \overset{(1)}{\beta}, \quad \overset{(1)}{\varrho}, \quad \overset{(1)}{\sigma}, \quad \overset{(1)}{\underline{\beta}}, \quad \overset{(1)}{\underline{\alpha}}. \tag{22}$$

Our notation is motivated by the formal expansions

$$\boldsymbol{\Omega} = \Omega + \varepsilon \overset{(1)}{\Omega} + O(\varepsilon^2), \tag{23}$$

$$\boldsymbol{\Omega}\, \mathbf{tr}\, \boldsymbol{\chi} = \Omega \operatorname{tr} \chi + \varepsilon (\Omega \overset{(1)}{\operatorname{tr}} \chi) + O(\varepsilon^2), \tag{24}$$

$$\boldsymbol{\alpha} = 0 + \varepsilon \overset{(1)}{\alpha} + O(\varepsilon^2). \tag{25}$$

We recall that the unbolded quantities without any superscripts denote the Schwarzschild values given by (17)–(19), and we have substituted that $\alpha=0$ in (25). See §5.1 for details.

We note that, because our frame (e_3, e_4) is irregular[4] (cf. §2.1.2 and §4.2.3), some quantities require Ω-weights, so as to be regular on the horizon $\mathcal{H}^+$, e.g.

$$\Omega^{-1} \overset{(1)}{\Omega}, \quad \Omega^{-2} (\Omega \overset{(1)}{\operatorname{tr}} \underline{\chi}), \quad \Omega^2 \overset{(1)}{\alpha}.$$

The *linearised Einstein equations* thus take the form of a linear system of equations in the above quantities (20)–(22). They can be derived from plugging in the expansions of type (23)–(25) into the equations of §2.1.1 and collecting the linear terms in ε (see §5.1.2). For instance, the linearised version of the first transport equation (7), decomposed into trace and trace-free part, is

$$\underline{D} \left(\frac{\overset{(1)}{\sqrt{\not g}}}{\sqrt{\not g}} \right) = (\Omega \overset{(1)}{\operatorname{tr}} \underline{\chi}), \quad \sqrt{\not g}\, \underline{D} \left(\frac{\overset{(1)}{\hat{\not g}}_{AB}}{\sqrt{\not g}} \right) = 2\Omega \overset{(1)}{\underline{\hat{\chi}}}_{AB}. \tag{26}$$

Here $\underline{D}$ is a projected Schwarzschild Lie derivative which on scalars reduces simply to ∂_u. Concerning linearised Ricci coefficients, the linearised versions of the equations (8) are

$$\not\nabla_3 (\Omega^{-1} \overset{(1)}{\underline{\hat{\chi}}}) + \Omega^{-1} (\operatorname{tr} \underline{\chi}) \overset{(1)}{\underline{\hat{\chi}}} = -\Omega^{-1} \overset{(1)}{\underline{\alpha}}, \quad \not\nabla_4 (\Omega \overset{(1)}{\hat{\chi}}) + (\operatorname{tr} \chi) \Omega \overset{(1)}{\hat{\chi}} - 2\widehat{\omega} \Omega \overset{(1)}{\hat{\chi}} = -\Omega \overset{(1)}{\alpha}, \tag{27}$$

of (9) is

$$\Omega \not\nabla_4 (\Omega \overset{(1)}{\operatorname{tr}} \underline{\chi}) = \Omega^2 (2 \operatorname{d\!\!\!/iv} \overset{(1)}{\eta} + 2 \overset{(1)}{\varrho} + 4 \varrho \Omega^{-1} \overset{(1)}{\Omega}) - \tfrac{1}{2} (\Omega \operatorname{tr} \chi) ((\Omega \overset{(1)}{\operatorname{tr}} \underline{\chi}) - (\Omega \overset{(1)}{\operatorname{tr}} \chi))$$

and of (10) is

$$\operatorname{d\!\!\!/iv} \overset{(1)}{\underline{\hat{\chi}}} = -\tfrac{1}{2} (\operatorname{tr} \underline{\chi}) \overset{(1)}{\eta} + \overset{(1)}{\underline{\beta}} + \tfrac{1}{2} \Omega^{-1} \not\nabla_A (\Omega \overset{(1)}{\operatorname{tr}} \underline{\chi}). \tag{28}$$

([4]) The linearisation process is frame covariant and thus coincides with the result of linearising with respect to a regular frame. See §5.1.4.

Here $\nabla\!\!\!/_3$, $\nabla\!\!\!/_4$ and $\mathrm{d}\!\!/\mathrm{iv}$ are covariant differentiation operators defined with respect to the Schwarzschild metric. The linearised version of (11)–(12) is

$$\begin{aligned} \nabla\!\!\!/_3 \overset{(1)}{\alpha} + \tfrac{1}{2}(\mathrm{tr}\,\underline{\chi})\overset{(1)}{\alpha} + 2\underline{\widehat{\omega}}\overset{(1)}{\alpha} &= -2\mathcal{D}\!\!\!/_2^\star \overset{(1)}{\beta} - 3\varrho\overset{(1)}{\widehat{\chi}}, \\ \nabla\!\!\!/_4 \overset{(1)}{\beta} + 2(\mathrm{tr}\,\chi)\overset{(1)}{\beta} - \widehat{\omega}\overset{(1)}{\beta} &= \mathrm{d}\!\!/\mathrm{iv}\,\overset{(1)}{\alpha}. \end{aligned} \tag{29}$$

Again, $\mathcal{D}\!\!\!/_2^\star$ is an adjoint of $\mathrm{d}\!\!/\mathrm{iv}$ defined with respect to the Schwarzschild background. The complete system of linearised equations is given in §§5.2.2–5.2.4.

Let us make the following remark: When linearising the equations in §2.1.1 around Minkowski space, the arising linearised Bianchi identities in fact *decouple*, and this allows for them to be studied independently of the full system of linearised gravity. Boundedness and decay results for this decoupled set of equations (the so-called spin 2 equations) were obtained in [13] by Christodoulou and Klainerman using robust vector field methods, and this study played an important role as a preliminary step for their later proof [14] of the non-linear stability of Minkowski space. In contrast, in our setting here, examining (29), we see immediately that the linearised Bianchi identities couple to the linearised null structure equations through the appearance of the term $-3\varrho\overset{(1)}{\widehat{\chi}}$. As we shall see in §2.1.5 below, however, the quantity $\overset{(1)}{\alpha}$ itself satisfies a second order decoupled wave equation.

We shall denote solutions of the above system by

$$\mathscr{S} = \Big(\, \overset{(1)}{\hat{g\!\!/}}, \overset{(1)}{\sqrt{g\!\!/}}, \overset{(1)}{\Omega}, \overset{(1)}{b}, (\Omega \overset{(1)}{\mathrm{tr}} \chi), (\Omega \overset{(1)}{\mathrm{tr}} \underline{\chi}), \overset{(1)}{\widehat{\chi}}, \overset{(1)}{\underline{\widehat{\chi}}}, \overset{(1)}{\eta}, \overset{(1)}{\underline{\eta}}, \overset{(1)}{\omega}, \overset{(1)}{\underline{\omega}}, \overset{(1)}{\alpha}, \overset{(1)}{\beta}, \overset{(1)}{\varrho}, \overset{(1)}{\sigma}, \overset{(1)}{\underline{\beta}}, \overset{(1)}{\underline{\alpha}}, \overset{(1)}{K}\,\Big). \tag{30}$$

For convenience, in the above we have added an additional quantity, the linearised Gauss curvature $\overset{(1)}{K}$ (arising from linearising the Gauss equation (84)), as an unknown. We note that a solution $\mathscr{S}$ is completely determined by its linearised metric coefficients (20), but nonetheless, we prefer to adjoin all quantities as unknowns.

Let us note that one can indeed quantitatively relate *solutions* of this formal linearisation to 1-parameter families of *solutions* of the actual vacuum Einstein equations (2) as expressed by the system in §2.1.1.([5]) In this paper, however, we will develop a self-contained theory of the linearised system without reference to an actual 1-parameter family of solutions of the full non-linear theory.

2.1.4. Special solutions: pure gauge solutions and the linearised Kerr family

Let us discuss immediately two important classes of special solutions of the above system; this corresponds to §6 of the body of the paper.

([5]) See [55] for subtleties that arise for this in the spatially compact case.

For the first class, note that the restriction to coordinates of double null form (3) is not sufficient to uniquely determine them on an abstract Lorentzian manifold. There is residual gauge freedom: Change of coordinates that preserve the form (3), upon linearisation, give rise to a special class of solutions which we shall refer to as *pure gauge solutions*.

An example of a pure gauge solution is one generated by a function $f(v,\theta^A)$:

$$\begin{aligned}\mathscr{G}=\{\overset{(1)}{\Omega}=\Omega^{-1}\partial_v(f\Omega^2),\overset{(1)}{b}_A=-r^2\nabla\!\!\!/_A(\partial_v(r^{-1}f)),\\ \overset{(1)}{\sqrt{g\!\!\!/}}=\sqrt{g\!\!\!/}(r^{-1}\Omega^2 f+r\Delta\!\!\!/ f),\overset{(1)}{\hat{g\!\!\!/}}=-2r\mathcal{D}\!\!\!\!/_2^\star\nabla\!\!\!/ f\},\end{aligned}\tag{31}$$

where we have shown explicitly only the metric perturbation from which all other geometric quantities can be determined. See §6.1 where we shall classify all pure gauge solutions. We will return to discuss pure gauge solutions in §2.1.7 when we discuss gauge-normalised solutions.

Another class of explicit solutions of the linearised system in §2.1.3 arises from linearising the Kerr family itself (in a convenient coordinate representation) around a given Schwarzschild solution.

Linearising the Schwarzschild (i.e. constant $a=0$) sub-family with mass $\widetilde{M}=M+\varepsilon\mathfrak{m}$ (in a particular double null coordinate representation) gives rise to a solution

$$\mathscr{K}=\left\{\overset{(1)}{\Omega}=-\frac{1}{2}\Omega\mathfrak{m},\overset{(1)}{\sqrt{g\!\!\!/}}=-\sqrt{g\!\!\!/}^{-1}\mathfrak{m},\overset{(1)}{\varrho}=-\frac{2M}{r^3}\cdot\mathfrak{m},\overset{(1)}{K}=\frac{\mathfrak{m}}{r^2},\text{rest}=0\right\},$$

while linearising the constant-M mass Kerr subfamily with rotation parameter $\tilde{a}=0+\varepsilon\mathfrak{a}$ gives rise to three linearly independent solutions, each of form:

$$\mathscr{K}=\left\{\overset{(1)}{\Omega}=0,\overset{(1)}{\hat{g\!\!\!/}}=0,\overset{(1)}{\sqrt{g\!\!\!/}}=0,\overset{(1)}{b}_A=\frac{4M\mathfrak{a}}{r}\varepsilon_A{}^C\partial_C Y^1_m\right\},$$

with Y^1_m $(m=-1,0,1)$ being the three linearly independent $\ell=1$ spherical harmonics. See §6.2.1 and §6.2.2. We will call the resulting 4-dimensional subspace of solutions (parameterised by real coefficients $\mathfrak{m}$, s_{-1}, s_0, s_1) *reference linearised Kerr solutions* and denote them by $\mathscr{K}_{\mathfrak{m},s_i}$. We note that these solutions are supported entirely in angular modes $\ell=0$ and $\ell=1$.

In view of the existence of the solutions of this section, the best we can expect of general solutions of our system of linearised gravity is that they decay to a pure gauge solution $\mathscr{G}$ plus a reference linearised Kerr solution $\mathscr{K}$.

2.1.5. Hierarchy of gauge-invariant quantities

In view of the complication provided by the existence of the solutions in §2.1.4 above, it is useful to isolate quantities which vanish for all such solutions. These are the so-called *gauge-invariant quantities*. In the body of the paper, these are discussed in §7.

An example of such a quantity is the linearised curvature component $\overset{(1)}{\alpha}$ from (22): As originally shown by Bardeen and Press [4], the component $\overset{(1)}{\alpha}$ in fact decouples from the full system and satisfies the equation

$$\begin{aligned}\nabla\!\!\!/_4\nabla\!\!\!/_3\overset{(1)}{\alpha}+\left(\tfrac{1}{2}\mathrm{tr}\,\underline{\chi}+2\widehat{\underline{\omega}}\right)\nabla\!\!\!/_4\overset{(1)}{\alpha}+\left(\tfrac{5}{2}\mathrm{tr}\,\chi-\widehat{\omega}\right)\nabla\!\!\!/_3\overset{(1)}{\alpha}-\Delta\!\!\!/\overset{(1)}{\alpha}\\ +\overset{(1)}{\alpha}(5\widehat{\underline{\omega}}\mathrm{tr}\,\chi-\widehat{\omega}\mathrm{tr}\,\underline{\chi}-4\varrho+2K+\mathrm{tr}\,\chi\mathrm{tr}\,\underline{\chi}-4\widehat{\omega}\widehat{\underline{\omega}})=0.\end{aligned}\tag{32}$$

A similar equation (with the roles of the 3 and 4 directions, and underlined and non-underlined quantities, reversed) is satisfied by $\overset{(1)}{\underline{\alpha}}$. These equations are known as the *spin* ± 2 *Teukolsky equations.*(6)

We note already that the vanishing of *both* $\overset{(1)}{\alpha}$ and $\overset{(1)}{\underline{\alpha}}$ identically imply that a solution is a pure gauge solution plus a reference linearised Kerr, provided that it be asymptotically flat (cf. §2.1.6). We shall show this in Appendix B.1.

It turns out that $\overset{(1)}{\alpha}$ and $\overset{(1)}{\underline{\alpha}}$ are best understood in the context of a *hierarchy of gauge-invariant quantities*. We define

$$\overset{(1)}{\psi}\doteq-\tfrac{1}{2}r^{-1}\Omega^{-2}\nabla\!\!\!/_3(r\Omega^2\overset{(1)}{\alpha})=\mathcal{D}\!\!\!/_2^\star\overset{(1)}{\beta}+\tfrac{3}{2}\varrho\overset{(1)}{\widehat{\chi}},\tag{33}$$

$$\overset{(1)}{\underline{\psi}}\doteq\tfrac{1}{2}r^{-1}\Omega^{-2}\nabla\!\!\!/_4(r\Omega^2\overset{(1)}{\underline{\alpha}})=\mathcal{D}\!\!\!/_2^\star\overset{(1)}{\underline{\beta}}-\tfrac{3}{2}\varrho\overset{(1)}{\widehat{\underline{\chi}}}\tag{34}$$

and

$$\overset{(1)}{P}\doteq r^{-3}\Omega^{-1}\nabla\!\!\!/_3(\overset{(1)}{\psi}r^3\Omega)=\mathcal{D}\!\!\!/_2^\star\mathcal{D}\!\!\!/_1^\star(-\overset{(1)}{\varrho},\overset{(1)}{\sigma})+\tfrac{3}{4}\varrho\mathrm{tr}\,\chi(\overset{(1)}{\widehat{\chi}}-\overset{(1)}{\widehat{\underline{\chi}}}),\tag{35}$$

$$\overset{(1)}{\underline{P}}\doteq -r^{-3}\Omega^{-1}\nabla\!\!\!/_4(\overset{(1)}{\underline{\psi}}r^3\Omega)=\mathcal{D}\!\!\!/_2^\star\mathcal{D}\!\!\!/_1^\star(-\overset{(1)}{\varrho},-\overset{(1)}{\sigma})+\tfrac{3}{4}\varrho\mathrm{tr}\,\chi(\overset{(1)}{\widehat{\chi}}-\overset{(1)}{\widehat{\underline{\chi}}}).\tag{36}$$

The second equalities in (33)–(36) above are non-trivial and follow from the linearised Bianchi equations. It follows from equation (32) and the definitions (33)–(36) alone that the quantities $\overset{(1)}{P}$ (and similarly $\overset{(1)}{\underline{P}}$) satisfy the so-called *Regge–Wheeler equation*

$$\begin{aligned}\nabla\!\!\!/_3\nabla\!\!\!/_4\overset{(1)}{P}+\nabla\!\!\!/_4\nabla\!\!\!/_3\overset{(1)}{P}-2\Delta\!\!\!/\overset{(1)}{P}+(5\mathrm{tr}\,\underline{\chi}+\widehat{\underline{\omega}})\cdot\nabla\!\!\!/_4\overset{(1)}{P}+(5\mathrm{tr}\,\chi+\widehat{\omega})\nabla\!\!\!/_3\overset{(1)}{P}\\ +\overset{(1)}{P}(4K-(3\mathrm{tr}\,\chi+\widehat{\omega})2\mathrm{tr}\,\chi-4(\mathrm{tr}\,\chi)^2+2\nabla\!\!\!/_3\mathrm{tr}\,\chi-8\widehat{\omega}\mathrm{tr}\,\chi)=0.\end{aligned}\tag{37}$$

(6) For the Schwarzschild case considered here, these equations are also known as the *Bardeen–Press equations*. It was Teukolsky [74] who showed that the structure allowing decoupled wave equations for gauge invariant quantities survives when linearising the Einstein equations around Kerr. In that case, however, the relevant gauge invariant quantities are defined not with respect to the null frame (4), but with respect to the so-called *algebraically special* frame. In Schwarzschild, the algebraically special frame coincides with the null frame (4) associated with a double null foliation.

The relation defining quantity $\overset{(1)}{P}$ from $\overset{(1)}{\alpha}$ is a physical space interpretation of the fixed-frequency transformation theory of Chandresekhar [9].

In contrast to (32), the equation (37) admits a positive energy and can be understood using the methods developed for studying the scalar wave equation $\Box_g\varphi=0$ (see §2.3 below). This will allow us to view $\overset{(1)}{P}$ and $\overset{(1)}{\underline{P}}$ as the key to unlocking the whole system (§2.4.1).

It is remarkable that the equation (37), which originally appeared as a quantity satisfied by metric perturbations [61], reappears in this context.

We note that there are solutions for which $\overset{(1)}{P}$ and $\overset{(1)}{\underline{P}}$ both vanish identically and are *not* pure gauge, in particular the linearised Robinson–Trautman solutions discussed in Appendix B.2. Nonetheless, as we shall see, we can always estimate $\overset{(1)}{\alpha}$ given control of $\overset{(1)}{P}$ by integrating transport equations (see already §2.4.2 below), picking up also an initial data quantity for $\overset{(1)}{\alpha}$.

In the body of the paper, we will in fact give a self-contained theory of both the Regge–Wheeler equation (37) and the Teukolsky equation (32), defining these in §7.1, proving a well-posedness theorem in §7.2, and deriving (37) from (32) in §7.3. In these sections we naturally drop the superscript $^{(1)}$ from all quantities as we consider a general P satisfying (37) and a general α satisfying (32). Only in §7.4 do we derive that (32) is indeed satisfied by the component $\overset{(1)}{\alpha}$ of a solution to the full system of linearised gravity. The first two main theorems of this paper (Theorem 1 and 2 in §10) will prove boundedness and decay for such general solutions of the Regge–Wheeler and Teukolsky equations. For rough versions of the statements of these theorems, the reader can turn immediately to §2.2.1.

2.1.6. Characteristic initial data, well-posedness and asymptotic flatness

The equations of linearised gravity described in §2.1.3 above admit a *well-posed initial value problem*. Though the structure which makes this possible can be viewed as inherited from the original non-linear system (2), the well-posedness allows us to develop a self-contained theory of solutions for the linear system *without further reference to its origin*. In the body of the paper, this will be discussed in §8. We give a brief treatment here.

As is well known, initial data for the Einstein vacuum equations (2) must satisfy *constraints*. This feature is of course inherited by the linearisation. As we are working in a double null gauge, it is more convenient to discuss characteristic initial data. This has the advantage of reducing the constraints to ordinary differential equations, which can be solved by integrating transport equations after prescribing–freely–suitable seed data.

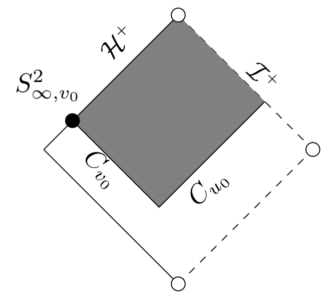

Figure 2.

We will introduce thus first a notion of a *seed initial data set* defined on two null cones C_{u_0} and C_{v_0} of the ambient Schwarzschild metric. Refer to the diagram in Figure 2. This data set will be described by symmetric traceless tensors $\overset{(1)}{\hat{\slashed{g}}}_{\circ,\mathrm{out}}$ and $\overset{(1)}{\hat{\slashed{g}}}_{\circ,\mathrm{in}}$, a 1-form $\overset{(1)}{b}_{\circ,\mathrm{out}}$ and functions $\overset{(1)}{\Omega}_{\circ,\mathrm{out}}$ and $\overset{(1)}{\Omega}_{\circ,\mathrm{in}}$, each defined on C_{u_0} and C_{v_0}, respectively, augmented by certain additional geometric data on the event horizon sphere S^2_{∞,v_0}; see Definition 8.1.

We have the following foundational statement, which we summarise here as follows.

Theorem 0. (Well-posedness of linearised gravity, rough formulation) *A smooth seed initial data set leads to a unique smooth solution $\mathscr{S}$ of the equations of linearised gravity in the region $u_0 \leqslant u \leqslant \infty$, $v_0 \leqslant v < \infty$.*

The precise statement is given in the body of the paper as Theorem 8.1.

The boundedness and decay theorems of our paper will require that data, in addition to smooth, be *asymptotically flat*. (Note that in the full non-linear theory governed by (2), this is necessary even for a local existence theorem with a u-time of existence uniform in v, i.e. up to future null infinity $\mathcal{I}^+$.) We will define asymptotic flatness in terms of seed data in §8.3, and show that it leads to a decay hierarchy for all quantities associated with a solution. These decay rates in fact propagate under evolution by Theorem 0; see Theorem A.1.

2.1.7. Gauge normalisation and final linearised Kerr

Before stating theorems which quantitatively estimate solutions, we must confront the issue of gauge. In addition, we can already identify the final linearised Kerr to which our solution will eventually approach. In the body of the paper, this is the content of §9.

Two "choices of gauge", realised in our linear theory by the addition of two distinct pure gauge solutions $\mathscr{G}$, will play an important role in this work.

(1) To formulate quantitative boundedness (cf. (a) of the main theorem in §1),

we first need a quantitative measure of the initial data. For this, it is necessary to normalise the solution on the initial data hypersurface $C_{u_0} \cup C_{v_0}$, by subtracting a pure gauge solution $\underset{\vee}{\mathscr{G}}$. Given a solution $\mathscr{S}$ asymptotically flat in the sense of §8.3, then Theorem 9.1 (see §9.2) establishes that there indeed exists a $\underset{\vee}{\mathscr{G}}$ normalising the solution on initial data, where addition of $\underset{\vee}{\mathscr{G}}$ ensures in particular that the "location" of the horizon is fixed, and that the sphere at infinity is "round". The resulting solution $\underset{\vee}{\mathscr{S}}$ is known as the *initial-data normalised solution*

$$\underset{\vee}{\mathscr{S}} \doteq \mathscr{S} + \underset{\vee}{\mathscr{G}}. \tag{38}$$

Moreover, it is shown that $\underset{\vee}{\mathscr{G}}$ is itself asymptotically flat.

Eventually, our main *boundedness* result (Theorem 3 in §10) will uniformly bound natural energies (on both spheres and cones) for $\underset{\vee}{\mathscr{S}}$ from an initial energy norm. These integral bounds control all quantities restricted to their $\ell \geqslant 2$ angular frequencies. In Theorem 9.2 (see §9.5), we show that the projection of $\underset{\vee}{\mathscr{S}}$ to its $\ell=0$ and $\ell=1$ modes is precisely a linearised Kerr solution $\mathscr{K}_{\mathfrak{m},s_i}$. Thus, the energies of Theorem 3 are coercive on $\underset{\vee}{\mathscr{S}}' = \underset{\vee}{\mathscr{S}} - \mathscr{K}_{\mathfrak{m},s_i}$. As a simple corollary, we will obtain in particular pointwise uniform bounds on all quantities associated with $\underset{\vee}{\mathscr{S}}$ (see Corollary 10.2) in terms of an initial energy and the Kerr parameters $\mathfrak{m}$, s_i which can be read off explicitly (and thus in particular are bounded) from data. Rough versions of the statement of the theorem and its corollary are given in §2.2.2 below.

(2) To prove decay (cf. (b) of the main theorem in §1) of all quantities, we need to "choose a different gauge", re-normalised at the event horizon $\mathcal{H}^+$. That is to say, we will add yet another pure gauge solution $\overset{\wedge}{\mathscr{G}}$ of the form (31), where $f(v,\theta,\phi)$ is determined from the metric component $\overset{(1)}{\Omega}$ of the initial data-normalised solution $\underset{\vee}{\mathscr{S}}$ by solving the ordinary differential equation (ODE)

$$\partial_v f + \frac{1}{2M} f = -\Omega^{-1} \overset{(1)}{\Omega} \tag{39}$$

along $\mathcal{H}^+$, to obtain the *horizon-renormalised solution* $\overset{\wedge}{\mathscr{S}}$ defined by

$$\overset{\wedge}{\mathscr{S}} \doteq \underset{\vee}{\mathscr{S}} + \overset{\wedge}{\mathscr{G}}. \tag{40}$$

The equation (39) ensures in particular that the (regular) linearised lapse associated with $\overset{\wedge}{\mathscr{S}}$ vanishes on the horizon:

$$\Omega^{-1} \overset{(1)}{\Omega}[\overset{\wedge}{\mathscr{S}}] = 0 \quad \text{on the horizon.} \tag{41}$$

See Proposition 9.3.1 in §9.3. (For convenience, we in fact alter the above definition for the $\ell=0$ mode, so that the linearised Kerr solutions $\mathscr{K}_{\mathfrak{m},s_i}$ are already correctly normalised.)

Eventually, our main *decay* result (Theorem 4 in §10) will give quantitative energy decay estimates for $\hat{\mathscr{S}}$ including the metric components (20) themselves. As with the boundedness theorem, these estimates control the restriction of $\hat{\mathscr{S}}$ to angular frequencies $\ell \geqslant 2$, and thus, as above, they are indeed coercive on $\hat{\mathscr{S}}' = \hat{\mathscr{S}} - \mathscr{K}_{\mathfrak{m}, s_i}$, i.e. they show decay of the solution to a reference linearised Kerr. Moreover, we shall show (using Theorem 3!) that the pure gauge solution $\hat{\mathscr{G}}$ can itself be uniformly bounded (in a weighted sense) by the initial values of $\underset{\vee}{\mathscr{S}}$; thus, the decay result is indeed quantitative. We expect that this global quantitative control of the pure gauge $\hat{\mathscr{G}}$ is potentially of fundamental importance for non-linear applications. A rough version of the statement of Theorem 4 and its pointwise corollary is given in §2.2.3 below.

We note that both the above normalised solutions $\underset{\vee}{\mathscr{S}}$ and $\hat{\mathscr{S}}$ enjoy various additional properties which will be useful later on. In particular, the so called horizon gauge conditions (215) hold globally on the horizon, not just the initial sphere S^2_{∞, v_0}. The roundness of the sphere at infinity and the good properties at the horizon are captured by two quantities $\overset{(1)}{Y}$ and $\overset{(1)}{Z}$, respectively, the former defined by the expression

$$\overset{(1)}{Y} := r(r^2 \not{\mathcal{D}}_2^\star \not{\mathrm{div}}(\Omega^{-1} r \underline{\overset{(1)}{\hat{\chi}}}) - \Omega^{-1} r^3 \underline{\overset{(1)}{\psi}}). \tag{42}$$

It is shown (see §9.4) that $\overset{(1)}{Y}[\underset{\vee}{\mathscr{S}}]$ is uniformly bounded along C_{u_0}. In the course of the proof of Theorem 3, we shall show that this uniform boundedness propagates. In Theorem 4, we shall show that this boundedness holds also for $\overset{(1)}{Y}[\hat{\mathscr{S}}]$. (This will imply, in particular, that the sphere at infinity remains round for the horizon-renormalised solution.)

2.2. The main theorems

We give now our first rough statements of the main theorems of this paper. These will correspond to the more precise statements given in the body of the paper in §10.

2.2.1. Boundedness and decay of gauge invariant quantities: the Regge–Wheeler and Teukolsky equations

The first two main theorems correspond to boundedness and decay statements for the gauge invariant quantities α, ψ and P in §2.1.5, and their corresponding underlined quantities. We will state these as independent statements for general solutions of the Regge–Wheeler and Teukolsky equations.

The first statement concerns Regge–Wheeler. The precise statement is Theorem 1 in §10.1. A rough formulation is as follows.

THEOREM 1. (Rough version) *Let P be a solution of the Regge–Wheeler equation* (37) *arising from regular data described in* §2.1.5, *and define the rescaled quantity* $\Psi = r^5 P$. *Then, the following statements hold*:

(a) *The quantity P remains uniformly bounded with respect to an r-weighted energy norm in terms of its initial flux*

$$\mathbb{F}[\Psi] \lesssim \mathbb{F}_0[\Psi]. \tag{43}$$

(b) *The quantity P decays to zero in the following quantitative senses*: *An r-weighted integrated decay statement*

$$\mathbb{I}[\Psi] \lesssim \mathbb{F}_0[\Psi] \tag{44}$$

holds, as well as polynomial decay of energy fluxes and pointwise polynomial decay.

The flux quantities $\mathbb{F}$ referred to in the above theorem are suprema over integrals on constant u and v null cones (C_u and C_v, respectively). The integral $\mathbb{I}$ is a spacetime intergal over the shaded region of Figure 2. Both will be explained in §10.1.1. The statements (a) and (b) above are in fact just special cases of an r^p hierarchy of flux bounds and integrated decay statements (cf. §2.3.3 below). Although the precise form of the statements obtained in Theorem 1 is new, we again note previous decay-type results for the Regge–Wheeler equation in [32], [7], [29].

Using the above theorem, we can now obtain a result for general solutions of the Teukolsky equation. The precise statement is Theorem 2 in §10.2. A rough formulation is as follows.

THEOREM 2. (Rough version) *Let α be a solution of the spin $+2$ Teukolsky equation* (32) *arising from regular data described in* §2.1.5, *and let ψ and $\Psi = r^5 P$ be the derived quantities defined by* (33) *and* (35). *Then the following statements hold*:

(a) *The triple (Ψ, ψ, α) remains uniformly bounded with respect to an r-weighted energy norm in terms of its initial flux*

$$\mathbb{F}[\Psi, \psi, \alpha] \lesssim \mathbb{F}_0[\Psi, \psi, \alpha].$$

(b) *The triple (Ψ, ψ, α) decays to zero in the following quantitative senses*: *An r-weighted integrated decay statement*

$$\mathbb{I}[\Psi, \psi, \alpha] \lesssim \mathbb{F}_0[\Psi, \psi, \alpha]$$

holds, as well as polynomial decay of suitable fluxes and pointwise polynomial decay.

A similar statement holds for solutions $\underline{\alpha}$ of the spin -2 Teukolsky equation and its derived quantities $\underline{\psi}$ and $\underline{P}$.

Again, the precise versions of the quantities $\mathbb{F}$ and $\mathbb{I}$ referred to in the above theorem will be explained in §10.2.1. As noted above, even a boundedness statement for the Teukolsky equation on Schwarzschild was not previously known.

Applied to the full system of linearised gravity, the above yields the following statement.

COROLLARY. *Let $\mathscr{S}$ be a solution of the full system of linearised gravity arising from regular, asymptotically flat initial data described in* §2.1.6. *Then, Theorem* 2 *applies to yield boundedness and decay for the gauge invariant hierarchy* $(\overset{(1)}{\Psi}, \overset{(1)}{\psi}, \overset{(1)}{\alpha})$ *and* $(\overset{(1)}{\underline{\Psi}}, \overset{(1)}{\underline{\psi}}, \overset{(1)}{\underline{\alpha}})$.

See Corollary 10.1 for the precise statement.

We next turn to the problem of boundedness for *all* quantities (30) associated with $\mathscr{S}$, not just the gauge-invariant ones.

2.2.2. Boundedness of the full system of linearised gravity

The third main theorem is a (quantitative) boundedness statement for the full system of linearised gravity, embodying (a) of the main theorem of the introduction.

The most fundamental statement is again at the level of an energy flux, now augmented by L^2 estimates on spheres, and must be expressed with the help of the initial-data normalised solution $\underset{\vee}{\mathscr{S}}$ discussed already in §2.1.7. The precise statement is formulated as Theorem 3 in §10.3. A rough statement is as follows.

THEOREM 3. (Rough version) *Let $\mathscr{S}$ be a solution of the full system of linearised gravity arising from regular, asymptotically flat initial data described in* §2.1.6, *and let $\underset{\vee}{\mathscr{G}}$ be the pure gauge solution such that*

$$\underset{\vee}{\mathscr{S}} = \mathscr{S} + \underset{\vee}{\mathscr{G}}$$

defined by (38) *is normalised to initial data. Then, the solution $\underset{\vee}{\mathscr{S}}$ remains uniformly bounded with respect to a weighted energy norm $\mathbb{F}$, augmented by the supremum of a weighted L^2-norm on spheres (denoted $\mathbb{D}$), in terms of its initial norm*:

$$\mathbb{D}[\underset{\vee}{\mathscr{S}}] + \mathbb{F}[\underset{\vee}{\mathscr{S}}] \lesssim \mathbb{D}_0[\underset{\vee}{\mathscr{S}}] + \mathbb{F}_0[\underset{\vee}{\mathscr{S}}]. \tag{45}$$

All quantities (30) *of $\underset{\vee}{\mathscr{S}}$ are controlled in L^2 on suitable null cones or spheres by the above norms up to their projections to the $\ell=0$ and $\ell=1$ modes.*

Moreover, there is a unique linearised Kerr solution $\mathscr{K}_{\mathfrak{m},s_i}$, computable explicitly from initial data, such that $\underset{\vee}{\mathscr{S}}' = \underset{\vee}{\mathscr{S}} - \mathscr{K}_{\mathfrak{m},s_i}$ has vanishing $\ell=0$ and $\ell=1$ modes, and thus the above L^2 control is coercive for all quantities (30) *associated with $\underset{\vee}{\mathscr{S}}'$.*

Again, the precise form of the flux $\mathbb{F}$ and the $L^\infty(L^2)$ norm $\mathbb{D}$ is contained in §10.3.1. We emphasise that the bound (45) and resultant control of the restriction of the solution to angular frequencies $\ell \geqslant 2$ is obtained independently of identifying the correct linearised Kerr solution $\mathscr{K}_{\mathfrak{m},s_i}$.

From the above flux bounds we immediately obtain, by standard Sobolev inequalities, pointwise bounds on all quantities (30) associated with $\underset{\vee}{\mathscr{S}}$.

Corollary. *For sufficiently regular, asymptotically flat initial data for* $\mathscr{S}$ *as above,* all *quantities* (30) *associated with* $\underset{\vee}{\mathscr{S}}$ *are uniformly bounded* pointwise *in terms of an initial energy as on the right-hand side of* (45), *and the parameters* $\mathfrak{m}$ *and* s_i *of* $\mathscr{K}_{\mathfrak{m},s_i}$ (*explicitly computable from—and thus also bounded by—initial data*).

See Corollary 10.2 for a precise statement. Note that these pointwise bounds are again r-weighted bounds.

2.2.3. Decay of the full system of linearised gravity in the future-normalised gauge

The final part of our results is the statement of decay for the full system, embodying (b) of the main theorem of the introduction. For this, we have already discussed in §2.1.7 the necessity of adding a pure gauge solution $\overset{\wedge}{\mathscr{G}}$ normalised to the event horizon.

The precise decay theorem is formulated as Theorem 4 in §10.4. A rough statement takes the following form.

Theorem 4. (Rough version) *Let* $\mathscr{S}$ *be a solution of the full system of linearised gravity arising from regular, asymptotically flat initial data described in* §2.1.6, *let* $\underset{\vee}{\mathscr{S}}$ *be as in Theorem* 3, *and let*

$$\overset{\wedge}{\mathscr{S}} = \underset{\vee}{\mathscr{S}} + \overset{\wedge}{\mathscr{G}},$$

defined by (40), *be the solution normalised to the event horizon. Then, the pure gauge solution* $\overset{\wedge}{\mathscr{G}}$, *and thus, in view of* (45), *also* $\overset{\wedge}{\mathscr{S}}$, *satisfy boundedness statements*

$$\mathbb{D}[\overset{\wedge}{\mathscr{G}}] + \mathbb{F}[\overset{\wedge}{\mathscr{G}}] \lesssim \mathbb{D}_0[\underset{\vee}{\mathscr{S}}] + \mathbb{F}_0[\underset{\vee}{\mathscr{S}}] \quad \textit{and} \quad \mathbb{D}[\overset{\wedge}{\mathscr{S}}] + \mathbb{F}[\overset{\wedge}{\mathscr{S}}] \lesssim \mathbb{D}_0[\underset{\vee}{\mathscr{S}}] + \mathbb{F}_0[\underset{\vee}{\mathscr{S}}], \tag{46}$$

from which, as in Theorem 3, *it follows that all quantities* (30) *of* $\overset{\wedge}{\mathscr{G}}$ (*and thus* $\overset{\wedge}{\mathscr{S}}$) *are controlled in* L^2 *on suitable null cones or spheres, while* $\overset{\wedge}{\mathscr{S}}$ *moreover satisfies "integrated local energy decay", schematically*

$$\mathbb{I}\big[\overset{\wedge}{\mathscr{S}}\big] \lesssim \mathbb{D}_0\big[\underset{\vee}{\mathscr{S}}\big] + \mathbb{F}_0\big[\underset{\vee}{\mathscr{S}}\big]. \tag{47}$$

From (47)*, a hierarchy of inverse-polynomial decay estimates follows for* all *quantities* (30) *associated with* $\hat{\hat{\mathscr{S}}}$*. As in Theorem* 3*, these estimates control the quantities* (30) *of* $\hat{\hat{\mathscr{S}}}$ *up to their projections to the* $\ell=0$ *and* $\ell=1$ *modes.*

If $\mathscr{K}_{\mathfrak{m},s_i}$ *is the linearised Kerr solution of Theorem* 3*, then* $\hat{\hat{\mathscr{S}}}'=\hat{\hat{\mathscr{S}}}-\mathscr{K}_{\mathfrak{m},s_i}$ *has vanishing* $\ell=0$ *and* $\ell=1$ *modes and the above control is indeed coercive for all quantities associated with* $\hat{\hat{\mathscr{S}}}'$*.*

Again, for the precise form of the estimates, see the propositions referred to in the full statement of the theorem in §10.4. In analogy with the corollary of Theorem 3, from the above L^2 bounds we immediately obtain pointwise decay estimates by standard Sobolev inequalities.

Corollary. *For sufficiently regular asymptotically flat data for* $\mathscr{S}$ *as above, we have pointwise inverse polynomial decay of all quantities of* $\hat{\hat{\mathscr{S}}}$ *given in* (30) *to those of* $\mathscr{K}_{\mathfrak{m},s_i}$*, in particular quantitative inverse polynomial decay rates for the linearised metric quantities* (20) *themselves.*

See Corollary 10.3 for the precise statement.

2.3. Aside: Review of the case of the scalar wave equation

The proofs of our main theorems build on recent advances in understanding the much simpler problem of the linear scalar wave equation

$$\Box_g \varphi = 0 \tag{48}$$

on a fixed Schwarzschild background $(\mathcal{M}, g)$, discussed already in the introduction. We interrupt the outline of the present paper to review the definitive results for the scalar wave equation (48) on Schwarzschild, following [20], [25] and [22]. The reader very familiar with this material can skip this section altogether. We will resume our outline of the body of the paper in §2.4 below.

Let us note at the outset that the results reviewed in the present section can be viewed firstly as precise scalar wave equation prototypes for the statements in Theorem 1 in §2.2 concerning the Regge–Wheeler equation. In fact, we will see in §2.4.1 below that the proof of Theorem 1 indeed follows closely from the results on the wave equation to be described below. Some of the phenomena, however, that enter will also explicitly appear again in the proofs of the remaining Theorems 2–4, in particular, the red-shift effect (see §2.3.1 below), the notion of an integrated local energy estimate (see §2.3.2 below) and the r^p hierarchy (see §2.3.3 below).

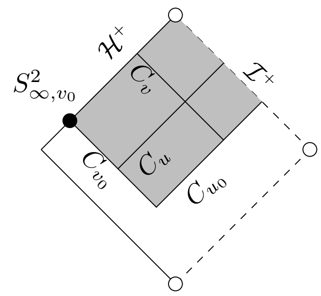

Figure 3.

2.3.1. Boundedness: Conservation laws and the red-shift

The scalar wave equation prototype for statement (43) of Theorem 1 is again a statement that the flux of a non-degenerate, r-weighted energy associated with φ is uniformly bounded from initial energy. As we shall see, this statement naturally arises in stages.

We first consider unweighted, non-degenerate energy boundedness. Explicitly, let us define

$$\begin{aligned} F[\varphi] = \sup_v \int_{C_v} &((\Omega^{-2}\partial_u\varphi)^2 + |\nabla\!\!\!/\varphi|^2) r^2\Omega^2\, du\, d\gamma \\ &+ \sup_u \int_{C_u} ((\Omega^{-1}\partial_v\varphi)^2 + |\Omega^{-1}\nabla\!\!\!/\varphi|^2) r^2\Omega^2\, dv\, d\gamma \end{aligned} \tag{49}$$

and F_0 to be the same quantity where $\sup_v \int_{C_v}$ is replaced by $\int_{C_{v_0}}$, and similarly for u. In regular coordinates, the integrands above represent all tangential derivatives to the cones $C_v = \{u \geqslant u_0\} \times \{v\} \times S^2$ and $C_u = \{u\} \times \{v \geqslant v_0\} \times S^2$, without degeneration ($\nabla\!\!\!/$ denoting the covariant derivative induced by the round metric on the spheres of symmetry), and correspond to the energy flux with respect to the vector field N to be defined below. See Figure 3. The boundedness theorem for the unweighted energy (49) for the scalar wave equation (48) then states as follows.

THEOREM. ([20], [25]) *For solutions φ of the wave equation* (48) *on Schwarzschild, we have*

$$F[\varphi] \lesssim F_0[\varphi]. \tag{50}$$

A similar non-degenerate higher-order statement holds as well. (*This implies uniform pointwise estimates for ψ and for all of its derivatives up to any order.*)

The full statement of (50) can only be proven in conjunction with the integrated decay to be shown in §2.3.2 which follows. We may already now prove, however, a slightly weaker statement, where we remove the Ω^{-1} weight from the $\nabla\!\!\!/\varphi$ term in the second integral of (49). This estimate is still non-degenerate on the cones C_v, and is thus sufficient to obtain pointwise estimates. We sketch the proof of this (slightly weaker) version of (50) below as it is quite elementary and already illustrates two important features: *conservation laws* and the *red-shift*.

Recall the *energy-momentum tensor* associated with φ defined by

$$Q_{\mu\nu}[\varphi] = \partial_\mu\varphi\partial_\nu\varphi - \tfrac{1}{2}g_{\mu\nu}g^{\alpha\beta}\partial_\alpha\varphi\partial_\beta\varphi, \tag{51}$$

which, for solutions of (48), satisfies

$$\nabla^\mu Q_{\mu\nu}[\varphi] = 0. \tag{52}$$

Contracting (52) with the Schwarzschild Killing field T defined in (16), one obtains the conservation law

$$\nabla^\mu J^T_\mu[\varphi] = 0, \tag{53}$$

where we use the notation $J^V_\mu[\varphi]{=}Q_{\mu\nu}[\varphi]V^\nu$ for an arbitrary vector field V. Integrating (53) in a characteristic rectangle bounded by the initial cones C_{u_0} and C_{v_0} and two later cones C_u and C_v, one obtains a conservation law relating flux terms. Using the fundamental positivity property of the tensor (51), namely

$$g(V,V)\leqslant 0, \quad g(V,W)\leqslant 0, \quad g(W,W)\leqslant 0 \quad \Longrightarrow \quad Q_{\mu\nu}[\varphi]V^\mu W^\nu \geqslant 0,$$

it follows that these flux terms arising are non-negative, but degenerate at the horizon $\mathcal{H}^+$, where T becomes null. Thus, this conservation law yields a version of the energy boundedness (50), but where F and F_0 are replaced by fluxes F^T and F^T_0, respectively, that degenerate at the event horizon $\mathcal{H}^+$, i.e. *a flux without the* Ω^{-1} *factor on* both *the* ∂_u *and* $\nabla\!\!\!\!/$ *terms in the definition* (49).

This weaker, degenerate analogue of (50) can be thought of already as a statement of stability, but it does *not* allow one easily to infer uniform pointwise estimates up to the horizon $\mathcal{H}^+$.([7]) It turns out, however, that given the above uniform degenerate energy bound, one can than apply the so-called *red-shift energy identity*, first introduced in [20], satisfied by a well-chosen timelike vector field N, for which the coercive property

$$J^N_\mu[\varphi]N^\mu \lesssim \nabla^\mu J^N_\mu[\varphi] \tag{54}$$

([7]) Here one should mention that the original Kay–Wald [41] approach to boundedness on Schwarzschild obtained pointwise estimates for φ directly from this degenerate energy, applied to an auxilliary solution $\widetilde{\varphi}$ such that $\partial_t\widetilde{\varphi}{=}\varphi$. The method of [41] is fragile, however, and cannot, for instance, obtain boundedness for transversal derivatives. See the discussion in [25].

holds near $\mathcal{H}^+$, and upgrade the degenerate boundedness just obtained to a version of (50), where the Ω^{-1} factor is now indeed obtained in the ∂_u term of the first integral in the definition (49) but must still be removed from the second integral. (To obtain the Ω^{-1} factor in the $\nabla\!\!\!/$ term of the second integral over the C_u cones, we must await for §2.3.2. We note that the flux terms of (49) are precisely the boundary terms that arise from integration of the divergence identity of J^N_μ.) The inequality (54) exploits the celebrated red-shift feature of the horizon (see the discussion in [25]).

To obtain a higher-order analogue of (50), one can of course first commute the wave equation (48) by the Killing vector field T and obtain estimates for $T\varphi$. For a non-degenerate statement, however, we would like to obtain estimates for a *strictly timelike* vector applied to φ. For this, it turns out that one can additionally commute the wave equation (48) by the ingoing null vector field e_3, and observe that the most dangerous new error term in the red-shift identity has a favourable sign (an enhanced red-shift); in fact, for all $k\geqslant 0$,

$$(k+1)J^N_\mu[e_3^k\varphi]N^\mu \lesssim \nabla^\mu J^N_\mu[e_3^k\varphi]-\{\text{controllable terms}\}. \tag{55}$$

From (55) and the fact that $T+e_3$ is timelike, a higher-order analogue of (50) follows, non-degenerate on the C_v cones. Pointwise bounds on φ and all-order derivatives now follow from standard Sobolev-type inequalities.

We note already that inequality (55) in fact "strengthens" the red-shift of (54) by an extra k factor. This strengthening will in fact play an important role in the present paper.

2.3.2. Trapped null geodesics and integrated local energy decay

The scalar wave equation prototype for the statement (44) of Theorem 1 is again a statement of weighted integrated local energy decay. We discuss in this section an *unweighted* version of (44) for equation (48), deferring the question of proper r-weights to §2.3.3 below. It is here already that we shall first encounter one of the fundamental aspects affecting quantitative control of decay: the existence of *trapped null geodesics* associated with the photon sphere at $r=3M$.

Let us define explicitly

$$I[\varphi]=\int_{u_0}^\infty\int_{v_0}^\infty (r^{-3}(r-3M)^2(|T\varphi|^2+|N\varphi|^2+|\nabla\!\!\!/\varphi|^2)+r^{-3}((\partial_u-\partial_v)\varphi)^2+r^{-3}|\varphi|^2) \times r^2\Omega^2\,du\,dv\,d\gamma. \tag{56}$$

The degeneration of the first term in the integrand in (56) at $r=3M$ is related precisely to trapping at the photon sphere. We have the following result.

THEOREM. ([20]) *For solutions φ of the wave equation* (48) *on Schwarzschild, we have the* (*unweighted, degenerate at $r{=}3M$*) *integrated local energy decay*:

$$I[\varphi] \lesssim F_0[\varphi]. \tag{57}$$

To prove integrated local energy decay (57), one applies the energy identity arising from contracting (52) with a well-chosen vector field X orthogonal to the constant-r hypersurfaces: The vector field X is chosen so that the bulk term $\nabla^\mu J^X_\mu$ of the identity has coercivity properties allowing for the control of the integrand of I, but with additional degeneration at the horizon. The trapping constraint implies that to obtain non-negativity of the bulk term, the vector field X must necessarily vanish at $r{=}3M$, hence the degeneration of (56) at $r{=}3M$. The boundary term fluxes of the energy identity, on the other hand, are bounded by the fluxes of J^T. Thus, in view of the trivial energy estimate for the fluxes of J^T, discussed already in §2.3.1, one obtains a version of (57), where one has additional degeneration of the integrand at the horizon and a correspondingly degenerate initial energy F^T_0 on the right-hand side.

Let us note that a general result of Sbierski [66] shows that the estimate (57) could not hold if some degeneration at $r{=}3M$ was not included in the definition (56).

In fact, the existence of the estimate (57) implicitly exploits the fact that trapping itself is "unstable" in the sense that geodesic flow is hyperbolic.([8]) Let us note finally that the actual current applied is in fact more complicated than a pure vector field current J^X, involving in fact order-zero terms. See [20] for details. The analogous construction for P in the proof of Theorem 1 will in fact be simpler.

While the degeneration at $r{=}3M$ is necessary, the degeneration at the event horizon is not. One finally obtains the full (57) by adding the divergence identity associated with the red-shift vector field N of §2.3.1, in view again of (54).([9]) It is at this point that we obtain also the full (50), obtaining also the Ω^{-1} factor on the $\nabla\!\!\!\!/$ term of the second integral in the definition (49).

Let us note that, by commuting the wave equation (48) with T, we may remove the degeneration at $r{=}3M$ in the definition of I in (57) at the cost of requiring loss of differentiability in the estimate, i.e. replacing $F_0[\varphi]$ on the right-hand side of (57) with $F_0[\varphi]{+}F_0[T\varphi]$. In view also of (55), we can further control an analogue of I with all higher derivatives from a suitable higher-order energy of initial data.

([8]) If trapping is stable, then such an estimate cannot hold and the rate of decay is only logarithmic. See [42], [56], [38], [39]. It turns out that the good "unstable" Schwarzschild structure of trapping is preserved in the Kerr case for the full sub-extremal range. The analogous construction cannot, however, be done by a traditional vector field current; see [25], [1], [73] for the $|a|{\ll}M$ case and [26] for the full $|a|{<}M$ case, where in addition to the hyperbolicity of trapping, the fact that trapped frequencies are not superradiant also plays an important role.

([9]) Note, however, that for scattering results, it is useful to have the original degenerate version of the estimate because it is time-reversible; see [27].

2.3.3. The r^p hierarchy, weighted estimates and inverse-polynomial decay bounds

To obtain the scalar wave equation prototype for the full *weighted* boundedness (43) and *weighted* integrated local energy decay (44) of Theorem 1, we must improve the quantities (49) and (56) with growing weights in r. This is similar to—but more elaborate than—the improvement already discussed which arises by adding the red-shift identity associated with the vector field N and exploiting (54).

The fundamental element is the following r^p hierarchy of estimates.

THEOREM. ([22]) *For* $0\leqslant p\leqslant 2$, *the following hierarchy of estimates holds for solutions of the wave equation* (48) *on Schwarzschild*:

$$\begin{aligned}\int_{\{r\geqslant R\}\cap\{u=u_2\}} r^p|\partial_v\varphi|^2+\int_{\{r\geqslant R\}\cap\{u_1\leqslant u\leqslant u_2\}} r^{p-1}(p|\partial_v\varphi|^2+(2-p)|\nabla\!\!\!/\varphi|^2)\\ \lesssim \int_{\{r\geqslant R\}\cap\{u=u_1\}} r^p|\partial_v\varphi|^2+\int_{\{R-1\leqslant r\leqslant R\}\cap\{u_1\leqslant u\leqslant u_2\}} |\partial_u\varphi|^2+|\partial_v\varphi|^2+|\nabla\!\!\!/\varphi|^2.\end{aligned} \tag{58}$$

We note that an analogue of the above identity holds for general asymptotically flat spacetimes. See [57].

In particular, defining now

$$\mathbb{F}[\varphi]=F[\varphi]+\sup_u\int_{C_u\cap\{r\geqslant R\}} r^2|\partial_v\phi|^2 \quad \text{and} \quad \mathbb{I}[\varphi]=I[\varphi]+\int_{r\geqslant R} r|\partial_v\varphi|^2, \tag{59}$$

applying the above theorem in conjunction with the theorems in §2.3.1 and §2.3.2, one infers immediately the following *weighted* boundedness and integrated local decay estimates

COROLLARY. *Let* φ *be a solution of the scalar wave equation* (48). *Then,* φ *satisfies the* r*-weighted uniform boundedness estimate*

$$\mathbb{F}[\varphi]\lesssim\mathbb{F}_0[\varphi] \tag{60}$$

and the r*-weighted integrated local energy estimate*

$$\mathbb{I}[\varphi]\lesssim\mathbb{F}_0[\varphi]. \tag{61}$$

In (60) and (61), we have indeed now obtained the scalar wave-equation prototypes of the estimates (43) and (44) of Theorem 1.

The above estimates only represent the special $p=2$ case of a hierarchy of estimates where the integrands in the second terms of the right-hand side of definitions (59) are replaced by the analogous integrands in (58) arising from a general $0 \leqslant p \leqslant 2$. It turns out that, from repeated use of the above, exploiting the identity also for $p=0,1$ and the pigeonhole principle, one can obtain the following uniform v-decay for the flux of non-degenerate energy

$$\int_{C_u \cap \{v \geqslant \tilde{v}\}} ((\Omega^{-1}\partial_v \varphi)^2 + |\Omega^{-1} \nabla\!\!\!/ \varphi|^2) \lesssim \tilde{v}^{-2}(F_0[\varphi] + F_0[TT\varphi]), \tag{62}$$

as well as similar bounds for fluxes on C_v and for higher-order energies. Pointwise estimates such as

$$|\varphi| \lesssim v^{-1}, \quad |r\varphi| \lesssim u^{-1/2} \quad \text{and} \quad |\sqrt{r}\varphi| \lesssim u^{-1}, \tag{63}$$

the latter two for $r \geqslant r_0 > 2M$, then follow from easy Hardy and Sobolev inequalities, where the implicit constants depend on weighted L^2 norms of initial data. The polynomial decay statements (62)–(63) represent the precise scalar-wave equation prototype for the polynomial decay of Theorem 1.

By commuting the wave equation by the vector field e_4, as well as the generators of spherical symmetry, one can apply an analogue of the above theorem for higher values $p \geqslant 2$. With this, one can further improve (63) to decay rates of the form

$$|\varphi| \lesssim t^{-3/2} \quad \text{and} \quad |\partial_t \varphi| \lesssim t^{-2} \tag{64}$$

for fixed r, where the implicit constants depend on even higher-order weighted norms. See [67] and the recent [57] for a definitive treatment. We will not obtain the analogue of such improvements here, though the methods of [67], [57] easily generalise.

Recall that, in Minkowski space, by the strong Huygens' principle, solutions arising from compactly supported initial data are in fact compactly supported in u and compactly supported in t for fixed r. In the case of Schwarzschild, on the other hand, even for solutions arising from compactly supported initial data, though the decay rate (64) for φ can be indeed improved to t^{-3} [19], [29], [53], it cannot, for generic initial data, be further improved [48]. This is just one of a tower of ℓ-dependent linear obstructions which were first obtained heuristically by Price [60], and there is by now a large literature attached to them. As opposed to estimates of the form (64), which indeed survive for quasilinear problems without symmetry (appearing in particular in the proof of the stability of Minkowski space), the relevance of sharper "Price-law"-type decay estimates for non-linear problems remains unclear.

2.4. Outline of the proofs

Having recalled the theory of the scalar wave equation (48) on Schwarzschild, we now return to the overview of our paper and give an outline of the proof of the main theorems in §2.2, following §§11–14 of the body of the paper. The reader may wish to revisit this section upon reading the actual proofs.

2.4.1. Proof of Theorem 1: Boundedness and decay for Regge–Wheeler

This will be the content of §11 of the body of the paper.

The solution P of the Regge–Wheeler equation can be estimated in direct analogy with our results for the wave equation (48) described in §2.3 above. It is in fact natural to introduce from the beginning rescaled quantities $\Psi = r^5 P$. The equation satisfied by Ψ admits a conserved energy in analogy with the J^T energy discussed in §2.3.1. We immediately obtain in §11.1 the global boundedness of a degenerate flux F^T (Proposition 11.1.1).

In §11.2 we obtain our initial "unweighted" integrated local energy decay, which moreover degenerates at the horizon. This is in close analogy with the estimate for the wave equation (48) via the current associated with a vector field X discussed in §2.3.2. In fact, the construction is easier than in the case of the wave equation, in particular since P, as a symmetric traceless $S^2_{u,v}$ 2-tensor, is necessarily supported only in angular frequencies $\ell \geqslant 2$ (cf. §4.4.3). The initial statement obtained is the integrated local energy decay statement (268), with the degenerate fluxes on the right-hand side.([10]) This is immediately improved in §11.3 by the analogue of the red-shift estimate (55) to obtain non-degenerate boundedness (with an integrated decay which does not degenerate on the horizon), and in §11.4 by (270), which is the analogue of the r^p-weighted estimate for the wave equation following from (58) with $p=2$.

Polynomial decay statements follow by adapting the arguments discussed in §2.3.3 for the wave equation (48). This is accomplished in §11.5. See Proposition 11.5.1.

2.4.2. Proof of Theorem 2: Boundedness and decay for Teukolsky

This will be the content of §12. We outline below.

Let α satisfy the spin $+2$ Teukolsky equation (32), and let ψ and P be defined by (33) and (35), respectively. It follows that P satisfies the Regge–Wheeler equation, and thus, Theorem 1 applies to P. The goal is to ascend the hierarchy, obtaining estimates

([10]) In particular, it is here—and only here—that the structure of trapping at the photon sphere appears directly in this paper.

for ψ and then α from estimates for P by integrating transport equations. (The results for a solution $\underline{\alpha}$ of the spin -2 Teukolsky equation are entirely analogous.)

Multiplying (35) by $r^{2-\varepsilon}\cdot\psi$ we obtain

$$\partial_u(|\psi|^2 r^6\Omega\cdot r^{2-\varepsilon})+|\psi|^2 r^6\Omega^4 r^{1-\varepsilon}\lesssim r^{7+2-\varepsilon}|P|^2\Omega^2. \tag{65}$$

We note that the right-hand side of (65) is indeed bounded when integrated over spacetime by the weighted integrated decay estimate for P discussed above in §2.4.1. Thus, the first term on the left-hand side of (65) gives an estimate for a flux on constant u hypersurfaces, while the second term gives a weighted integrated decay estimate. See (282) and (284) in Proposition 12.1.1. Let us note that both the non-degeneracy at the horizon and the extra r^p weights in our original estimate for P were fundamental for the numerology of (65) to work out. We thus indeed need the full strength of the weights in Theorem 1 to successfully estimate ψ.

Given now estimates for ψ, we similarly multiply (33) by a weighted r-factor times α to obtain

$$\partial_u(r^{4-\varepsilon}\cdot r^2\Omega^4|\alpha|^2)+r^{3-\varepsilon}r^2\Omega^6|\alpha|^2\lesssim r^{-1-\varepsilon}\Omega^4|\psi|^2. \tag{66}$$

Again, the numerology is such that the right-hand side can be estimated by the integrated energy just controlled by Proposition 12.1.1, again, using in an essential way the non-degeneracy at the horizon and the r^p weights. Thus, upon integration over spacetime, (66) yields both an energy flux and integrated energy decay statement for α. See Proposition 12.1.2.

Revisiting the equations, one can now also estimate higher derivatives of ψ, $\underline{\psi}$, α, $\underline{\alpha}$ from control of P and $\underline{P}$ (§12.2). With this, one has all the elements necessary to obtain polynomial decay estimates, following the method of [22] for the scalar wave equation discussed above in §2.3.3. This is achieved in §12.3.

2.4.3. Proof of Theorem 3: boundedness for linearised gravity

This will be the content of §13 of the bulk of the paper. We outline the main points here.

Gauge invariant statements. Let $\check{\mathscr{S}}$ be as in the statement of Theorem 3. By the considerations in §2.1.5, we have that $\overset{(1)}{\alpha}$ and $\overset{(1)}{\underline{\alpha}}$ satisfy the spin ± 2 Teukolsky equations, and thus, Theorem 2 applies, yielding boundedness and decay estimates on the hierarchy of gauge invariant quantities (see §13.1). The goal is to promote this to boundedness estimates on all quantities.

Fluxes on the horizon. The first task (§13.2) is to estimate certain fluxes on the horizon $\mathcal{H}^+$ which contain in their integrands non-gauge invariant quantities. For instance, from the boundedness of the $\overset{(1)}{\psi}$ flux on $\mathcal{H}^+$ and the second identity in (33), we can immediately obtain a flux controlling $\overset{(1)}{\beta}$ and $\overset{(1)}{\widehat{\chi}}$, in particular

$$\int_{\mathcal{H}^+} |\overset{(1)}{\widehat{\chi}}|^2. \tag{67}$$

This is the content of Proposition 13.2.1. Using also the flux of P, we may obtain higher-order fluxes associated with the transversal derivative $\Omega^{-1}\nabla\!\!\!/_3(\Omega\overset{(1)}{\widehat{\chi}})$, namely

$$\int_{\mathcal{H}^+} |\Omega^{-1}\nabla\!\!\!/_3(\Omega\overset{(1)}{\widehat{\chi}})|^2 \tag{68}$$

(see Proposition 13.2.2), as well as the angular derivatives $\mathcal{D}\!\!\!\!/_2^\star\mathcal{D}\!\!\!\!/_1^\star(\overset{(1)}{\varrho},\overset{(1)}{\sigma})$ and $\mathcal{D}\!\!\!\!/_2^*\overset{(1)}{\eta}$. At this point, one can already obtain polynomial v-decay of various fluxes (see Proposition 13.2.4).[11]

We remark that the above estimates control only the projection of the above quantities to angular frequencies $\ell\geqslant 2$. (Note that symmetric traceless $S^2_{u,v}$ 2-tensors like $\overset{(1)}{\widehat{\chi}}$ are necessarily supported only on $\ell\geqslant 2$; see §4.4.3.) This will be a common feature of all estimates obtained in the proof.

Decay estimates for the outgoing shear $\widehat{\chi}$. In analogy to how we estimated ψ from P and α from ψ in §2.4.2, one wishes to integrate the transport equation (27) for $\overset{(1)}{\widehat{\chi}}$. The problem is that the third term on the left-hand side of (27) has a bad (i.e. blue-shift) sign at the horizon. It turns out that, in analogy with the improved k factor in (55) for higher transversal derivatives of the scalar wave equation (48) discussed in §2.3.1, the bad term in (27) can be killed by applying suitable commutation. We see that the quantity

$$\nabla\!\!\!/_3(\Omega^{-1}\nabla\!\!\!/_3(r^2\overset{(1)}{\widehat{\chi}}\Omega)) \tag{69}$$

is in fact "red-shifted". (This is apparent from equation (342) in §13.3.1.) We may now control $\overset{(1)}{\widehat{\chi}}$ in the darker shaded region $r\leqslant r_1$ as indicated in Figure 4.

We couple red-shift estimates for (69) obtained by integrating along outgoing cones C_u in the manner of §2.4.2 to estimates for $\overset{(1)}{\widehat{\chi}}$ and $\Omega^{-1}\nabla\!\!\!/_3(\Omega\overset{(1)}{\widehat{\chi}})$ obtained by integrating (applying the fundamental theorem of calculus, twice) (69) along the cone C_v from the horizon, using our "initial" control of the horizon fluxes (67) and (68). This yields finally

[11] It is worth noting here that, although the integrands of the above fluxes (67) and (68) are gauge-dependent, the *total fluxes* are gauge-invariant, given the horizon gauge conditions (215). The $\overset{(1)}{\widehat{\chi}}$ flux (67) can in fact be directly related to an energy which arises from the Lagrangian structure of the Einstein equations. See Hollands and Wald [34] for a general discussion of such fluxes and the upcoming [37] for the relation to our setting.

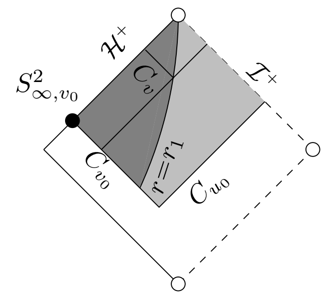

Figure 4.

an integrated decay statement for $\hat{\chi}$ restricted to the darker shaded region (Proposition 13.3.2). Finally, we may extend our estimates of $\overset{(1)}{\hat{\chi}}$ to the lighter shaded region above by integrating directly (27) along C_u in the manner of §2.4.2, restricted to the lighter shaded region, using our "initial" control along $r{=}r_1$ just obtained. This gives a global integrated local energy decay statement for $\overset{(1)}{\hat{\chi}}$ (as well as higher-order derivatives). The estimate is stated as Proposition 13.3.1.

Boundedness estimates on the ingoing shear $\underline{\hat{\chi}}$. Having shown decay for the outgoing shear $\overset{(1)}{\hat{\chi}}$, we now turn to estimate the ingoing shear $\overset{(1)}{\underline{\hat{\chi}}}$. Here, we will only be able to show boundedness, not decay. First, we note that we are not able to estimate the evolution equation (27) for $\overset{(1)}{\underline{\hat{\chi}}}$ directly. Instead, we estimate $\overset{(1)}{\underline{\hat{\chi}}}$ via the quantity $\overset{(1)}{Y}$, defined by (42).

We have already noted that $\overset{(1)}{Y}$ is initially bounded on the outgoing cone C_{u_0}, capturing the roundness of the sphere at infinity which is ensured by our gauge normalisation. It turns out that $\overset{(1)}{Y}$ satisfies an evolution equation in the ingoing direction with gauge-invariant right-hand side which is moreover integrable (see (357)). This allows us to obtain the uniform boundedness of $\overset{(1)}{Y}$ (Proposition 13.4.1). From this, one obtains the boundedness of angular derivatives of $\overset{(1)}{\hat{\chi}}$ (Corollary 13.3), as well as the boundedness of a flux of angular derivatives $\overset{(1)}{\underline{\hat{\chi}}}$ on null cones (Corollary 13.4).

The r-weights above unfortunately do *not* allow us to obtain integrated local energy decay for $\overset{(1)}{\underline{\hat{\chi}}}$. We can obtain, however, at this point such decay for the derivative $\nabla\!\!\!/_4(\Omega^{-1}\overset{(1)}{\underline{\hat{\chi}}})$ (Proposition 13.4.2) and upgrade it to a polynomial decay statement on the event horizon (Proposition 13.4.3).

Boundedness for the remaining quantities. Finally, we briefly discuss the remaining quantities (see §13.5), focussing on the order of the hierarchy and not the precise nature of the estimates.

We first note that the second identity of (35) relating $\overset{(1)}{P}$ with the pair $(\overset{(1)}{\varrho}, \overset{(1)}{\sigma})$ and the shears $\overset{(1)}{\hat{\chi}}$ and $\overset{(1)}{\underline{\hat{\chi}}}$ allows us to obtain estimates for $\overset{(1)}{\varrho}$ and $\overset{(1)}{\sigma}$. Similarly, from the second identities of (34) and (33), we may now obtain estimates for $\overset{(1)}{\underline{\beta}}$ and $\overset{(1)}{\beta}$ (see §13.5.1). The quantities $\overset{(1)}{\eta}$ and $\overset{(1)}{\underline{\eta}}$ can also be controlled, using however propagation equations which we have not introduced in this overview.([12]) We then estimate (see §13.5.4) the quantity $(\Omega \overset{(1)}{\mathrm{tr}} \underline{\chi})$ via the Codazzi equation (28). Finally, we may estimate the metric components (20) themselves by integrating their respective transport equations. For instance, to estimate $\overset{(1)}{\sqrt{\not g}}$, we integrate (26) using the bounds we have just obtained on the right-hand side (see Proposition 13.5.12).

At this point, we have estimated all quantities with the exception of $(\Omega \overset{(1)}{\mathrm{tr}} \chi)$. While this could easily be estimated via the analogue of (28) relating $(\Omega \overset{(1)}{\mathrm{tr}} \chi)$ and $\overset{(1)}{\hat{\chi}}$, that method would not give us the quantitative boundedness as stated in Theorem 3, as our previous estimates on $\overset{(1)}{\hat{\chi}}$ lose regularity. An alternative approach is thus given and it is here where the quantity $\overset{(1)}{Z}$ mentioned already in §2.1.7 is in fact used. We defer further discussion of this to §13.5.5 and §13.5.6.

By the above procedure, we obtain finally L^2 estimates on all quantities (30) associated with $\underset{\vee}{\mathscr{S}}$, either on spheres or cones. As noted in the statement of Theorem 3, these control only the restriction of all quantities to angular frequencies $\ell \geqslant 2$. We can now, however, apply Theorem 9.2 to obtain that the projection of $\underset{\vee}{\mathscr{S}}$ to the $\ell=0$ and $\ell=1$ modes is given precisely by a unique $\mathscr{K}_{\mathfrak{m},s_i}$, and thus we have truly coercive estimates on $\underset{\vee}{\mathscr{S}}' = \underset{\vee}{\mathscr{S}} - \mathscr{K}_{\mathfrak{m},s_i}$. The pointwise estimates of the corollary then follow by standard Sobolev inequalities applied to $\underset{\vee}{\mathscr{S}}'$ (see §13.5.8).

2.4.4. Proof of Theorem 4: Decay for linearised gravity

This will be the content of §14. Again, we outline below.

Boundedness of the pure gauge solution. Let $\hat{\mathscr{S}} = \underset{\vee}{\mathscr{S}} + \hat{\mathscr{G}}$ be as in the statement of Theorem 4. The new normalisation (41) allows us to derive the following evolution equation for the ingoing linearised shear on the horizon:

$$\Omega \not\nabla_4 (\Omega^{-1} \overset{(1)}{\underline{\hat{\chi}}}[\hat{\mathscr{S}}]) + \frac{1}{2M} \Omega^{-1} \overset{(1)}{\underline{\hat{\chi}}}[\hat{\mathscr{S}}] = \frac{\Omega}{2M} \overset{(1)}{\hat{\chi}}[\underset{\vee}{\mathscr{S}}] + 2 \not{\mathcal{D}}_2^{\star} \overset{(1)}{\eta}[\underset{\vee}{\mathscr{S}}].$$

See (408). It follows that the regular quantity $\Omega^{-1} \overset{(1)}{\underline{\hat{\chi}}}$ is "red-shifted", in the sense that the sign of the factor multiplying $\overset{(1)}{\underline{\hat{\chi}}}$ in the second term on the the left-hand side of the

([12]) Specifically, equations (140) and (141). See Proposition 13.5.3.

above identity is positive, and moreover the right-hand side is suitably decaying. This immediately allows for estimating suitably normalised $\overset{(1)}{\hat{\underline{\chi}}}$ and $(\Omega \overset{(1)}{\mathrm{tr}}\, \underline{\chi})$ on the event horizon (see Proposition 14.1.1 and its corollaries). We can then translate this into bounds for derivatives of the gauge function f (see Proposition 14.1.2), and thus, for $\hat{\mathscr{G}}$ itself (see Proposition 14.1.3). This gives the boundedness statement for $\hat{\mathscr{G}}$, and thus, in view of Theorem 3, also for $\hat{\mathscr{S}}$, i.e. the statements schematically represented as (46).

We note that the reader may have expected that one must renormalise the gauge also "at null infinity" in addition to the renormalisation carried out at the horizon. It is indeed remarkable that the horizon-renormalisation is on its own sufficient, at least for the decay to be stated in the present paper. This appears again to be fundamentally connected to the red-shift at the horizon.

Integrated local energy decay. Recall from our discussion in §2.4.3 that it was precisely the quantity $\overset{(1)}{\hat{\underline{\chi}}}$ which provided the first obstruction to showing decay for $\check{\mathscr{S}}$. Having obtained now decay for $\overset{(1)}{\hat{\underline{\chi}}}[\hat{\mathscr{S}}]$ on the horizon, we can now upgrade this to a global integrated decay bound (Proposition 14.2.1), and from this, repeating steps similar to those described in our discussion of the proof of Theorem 3 in §2.4.3, we move up the hierarchy, to obtain now integrated decay at each step for all other quantities (cf. also the proof of Theorem 2). This corresponds to the statement schematically represented as (47). We defer further discussion to §14.2.

Polynomial decay. As in Theorems 1 and 2, polynomial decay can be obtained for all quantities by an adaptation of the method of [22] described in §2.3.3 in the context of the scalar wave equation (48). Refer to §14.3. (We only remark here that for reasons of length, since we have not derived integrated local energy decay for the metric components themselves, we obtain polynomial decay for these by directly integrating transport equations like (26), using the decay already obtained for the Ricci coefficients (see §14.3.2).)

As in the proof of Theorem 3, the above bounds in fact only estimate the restriction of the solution to angular frequencies $\ell \geqslant 2$. In view of Theorem 9.2, however, we again infer that we have indeed true decay to zero for $\hat{\mathscr{S}}' = \hat{\mathscr{S}} - \mathscr{K}_{\mathfrak{m},s_i}$ (without restriction in angular frequency). In other words, all quantities (30) of $\mathscr{S}$ decay (in a suitable L^2 sense) inverse polynomially to their $\mathscr{K}_{\mathfrak{m},s_i}$ values. The pointwise polynomial decay estimates of the corollary then follow immediately by standard Sobolev inequalities.

2.5. A restricted non-linear stability conjecture

For background on the full non-linear problem, see the discussion in our previous paper [18], where we have in particular given a precise formulation of the non-linear stability conjecture for Kerr. We end this overview section by noting that the linear stability around Schwarzschild proven in the present paper is in principle sufficient to try to address the following restricted version of the full non-linear stability conjecture:

Conjecture. (Full finite-codimension non-linear stability of Schwarzschild) Denote by $(\Sigma_M, \bar{g}_M, K_M)$ the induced data on a spacelike asymptotically flat slice of the Schwarzschild solution of mass M crossing the future horizon and bounded by a trapped surface. Then, in the space of all nearby vacuum data $(\Sigma, \bar{g}, K)$, in a suitable norm, there exists a codimension-3 subfamily for which the corresponding maximal vacuum Cauchy development $(\mathcal{M}, g)$ contains a black-hole exterior region (characterized as the past $J^-(\mathcal{I}^+)$ of a complete future null infinity $\mathcal{I}^+$), bounded by a non-empty future affine-complete event horizon $\mathcal{H}^+$, such that in $J^-(\mathcal{I}^+)$ (a) the metric remains close to g_M and moreover (b) asymptotically settles down to a nearby Schwarzschild metric $g_{\widetilde{M}}$ at suitable inverse polynomial rates.

We emphasise that by dimensionality considerations, the above conjecture would construct *all* spacetimes arising from data sufficiently near Schwarzschild whose final state is again Schwarzschild.([13]) In particular, we note that the above class of solutions would a fortiori include the evolution of axisymmetric initial data near Schwarzschild whose total angular momentum vanishes, as these necessarily have a final Schwarzschild endstate (in view of the fact that for the vacuum equations under axisymmetry, angular momentum does not radiate to null infinity).

2.6. Guide to reading the paper

Though the paper has been written to be read linearly, the reader interested only in certain results can skip various sections. We give here a guide to various self-contained, more limited trajectories through the paper.

([13]) The reader should compare this with the result of [18] which gives a scattering construction of a family of solutions asymptoting to *any* given particular subextremal Kerr $|a|<M$. The solutions of [18] are constructed by prescribing exponentially decaying "scattering" data on the event horizon and on null infinity and solving the vacuum Einstein equations (2) *backwards* in time. In view of the strong, exponential rate of approach imposed, however, this family is presumably very exceptional and in particular of *infinite* codimension in the space of all solutions of the Einstein vacuum equations. Thus, when specialised to $a=0$, the result of [18] is far from obtaining the above conjecture. See the comments in [18] for futher discussion.

The reader only interested in results concerning the Regge–Wheeler and Teukolsky equations need only read §4 for notation and basic differential operators defined on Schwarzschild, then §7, up to §7.3, for background on these equations, then §10.1 and §10.2 for the statement of Theorems 1 and 2, and finally §11 and §12 for their proofs.

The boundedness theorem (Theorem 3) is independent of the decay theorem (Theorem 4), and thus the reader interested only in the former can skip §9.3 concerning the horizon-normalised gauge, as well as §10.4 and §14 giving the statement and proof of Theorem 4.

We note that the reader who does not want to concern themselves with some of the intricacies of exploiting pure gauge solutions can skip §9.2, and still understand the proof of Theorem 3.

Finally, the reader interested in the full results but who is willing to take on faith the system of linearised gravity can skip §3 and §5.1, though we note that the pure gauge solutions and reference linearised Kerr solutions in §6 are more easily verified by applying the linearisation in §5.1, than by direct computation.

3. The vacuum Einstein equations in a double null gauge

In this section, we review the form of the vacuum Einstein equations (2) written with respect to a natural null frame attached to a local double null foliation of a Lorentzian manifold. See Christodoulou [11] for a detailed exposition. It is these equations which we shall formally linearise in §5.1 to obtain the equations of linearised gravity. The reader not interested in the derivation of the linearised system can skip immediately to §4.

An outline of the current section is as follows: We begin in §3.1 with preliminaries, defining the notion of double null gauge and associated notation. Ricci coefficients and curvature components are then defined in §3.2. Finally, the vacuum Einstein equations are presented in §3.3.

3.1. Preliminaries

Let $(\boldsymbol{\mathcal{M}},\boldsymbol{g})$ be a (3+1)-dimensional Lorentzian manifold.

3.1.1. Local double null gauge

In a neighbourhood of any point $\mathbf{p}\in\boldsymbol{\mathcal{M}}$, we can introduce local coordinates $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{\theta}^1$ and $\boldsymbol{\theta}^2$ such that the metric is expressed in "canonical double-null form":

$$\boldsymbol{g} = -4\boldsymbol{\Omega}^2\, \boldsymbol{du}\, \boldsymbol{dv} + \boldsymbol{\not{g}}_{CD}(\boldsymbol{d\theta}^C - \boldsymbol{b}^C\, \boldsymbol{dv})(\boldsymbol{d\theta}^D - \boldsymbol{b}^D\, \boldsymbol{dv}) \tag{70}$$

for a spacetime function $\boldsymbol{\Omega}\colon\boldsymbol{\mathcal{M}}\to\mathbb{R}$, an $\boldsymbol{S_{u,v}}$-tangent vector $\boldsymbol{b}^A$ and a symmetric $\boldsymbol{S_{u,v}}$-tangent covariant symmetric 2-tensor $\boldsymbol{\not g}_{AB}$. Here, $\boldsymbol{S_{u,v}}$ denotes the 2-dimensional (Riemannian with metric $\boldsymbol{\not g}_{AB}$) manifold arising as the intersection of the hypersurfaces of constant $\boldsymbol{u}$ and $\boldsymbol{v}$.

3.1.2. Normalised frames

We can define a normalised null frame associated with the above coordinates as follows. We define

$$\boldsymbol{e}_3=\boldsymbol{\Omega}^{-1}\boldsymbol{\partial_u},\quad \boldsymbol{e}_4=\boldsymbol{\Omega}^{-1}(\boldsymbol{\partial_v}+\boldsymbol{b}^A\boldsymbol{\partial}_{\boldsymbol{\theta}^A}),\quad \boldsymbol{e}_A=\boldsymbol{\partial}_{\boldsymbol{\theta}^A} \text{ for } A=1,2, \tag{71}$$

for which we note the relations

$$\boldsymbol{g}(\boldsymbol{e}_3,\quad \boldsymbol{e}_4)=-2,\quad \boldsymbol{g}(\boldsymbol{e}_3,\boldsymbol{e}_A)=0,\quad \boldsymbol{g}(\boldsymbol{e}_4,\boldsymbol{e}_A)=0,\quad \boldsymbol{g}(\boldsymbol{e}_A,\boldsymbol{e}_B)=\boldsymbol{\not g}_{AB}.$$

In particular, $\{\boldsymbol{e}_1,\boldsymbol{e}_2\}$ constitutes a (local) coordinate frame field (not necessarily orthonormal) of the orthogonal complement of the span of $\boldsymbol{e}_3$ and $\boldsymbol{e}_4$ (i.e. in the tangent space of the submanifold $\boldsymbol{S_{u,v}}$). In view of the above relations, we shall refer to the null frame $\boldsymbol{\mathcal{N}}=\{\boldsymbol{e}_3,\boldsymbol{e}_4,\boldsymbol{e}_1,\boldsymbol{e}_2\}$ as being *normalised*.

3.1.3. $\boldsymbol{S_{u,v}}$-tensor algebra

In §3.2, we will express the Ricci coefficients and curvature components of the metric (70) with respect to the null frame (71). These objects will then become $\boldsymbol{S_{u,v}}$-tangent tensors, or $\boldsymbol{S_{u,v}}$-tensors for short (see [11]). Two types of such $\boldsymbol{S_{u,v}}$-tensors will play a particularly important role: 1-forms $\boldsymbol{\xi}$ and symmetric 2-tensors $\boldsymbol{\theta}$, the latter being defined as satisfying $\boldsymbol{\theta}_{AB}=\boldsymbol{\theta}_{BA}$ in any coordinate patch. A *traceless* symmetric $\boldsymbol{S_{u,v}}$ 2-tensor $\boldsymbol{\theta}$ satisfies in addition $\boldsymbol{\not g}^{AB}\boldsymbol{\theta}_{AB}=0$.

Let $\boldsymbol{\xi}$ and $\tilde{\boldsymbol{\xi}}$ be arbitrary $\boldsymbol{S_{u,v}}$ 1-forms, and $\boldsymbol{\theta}$ and $\tilde{\boldsymbol{\theta}}$ be arbitrary symmetric $\boldsymbol{S_{u,v}}$ 2-tensors.

We denote by ${}^\star\boldsymbol{\xi}$ and ${}^\star\boldsymbol{\theta}$ the Hodge-dual (on $(\boldsymbol{S_{u,v}},\boldsymbol{\not g})$) of $\boldsymbol{\xi}$ and $\boldsymbol{\theta}$, respectively, [11], and denote by $\boldsymbol{\theta}^\sharp$ the tensor obtained from $\boldsymbol{\theta}$ by raising an index with $\boldsymbol{\not g}$.

We define the contractions

$$(\boldsymbol{\xi},\tilde{\boldsymbol{\xi}}):=\boldsymbol{\not g}^{AB}\boldsymbol{\xi}_A\tilde{\boldsymbol{\xi}}_B \quad\text{and}\quad (\boldsymbol{\theta},\tilde{\boldsymbol{\theta}}):=\boldsymbol{\not g}^{AB}\boldsymbol{\not g}^{CD}\boldsymbol{\theta}_{AC}\tilde{\boldsymbol{\theta}}_{BD},$$

and $|\boldsymbol{\xi}|^2=(\boldsymbol{\xi},\boldsymbol{\xi})$ and $|\boldsymbol{\theta}|^2=(\boldsymbol{\theta},\boldsymbol{\theta})$. We denote by $\boldsymbol{\theta}^\sharp\cdot\boldsymbol{\xi}$ the 1-form $\boldsymbol{\theta}_A{}^B\boldsymbol{\xi}_B$ arising from the contraction with $\boldsymbol{\not g}$.

We finally define the 2-tensors $\boldsymbol{\theta}\times\tilde{\boldsymbol{\theta}}$ and $\boldsymbol{\xi}\widehat{\otimes}\tilde{\boldsymbol{\xi}}$, and the scalar $\boldsymbol{\theta}\wedge\tilde{\boldsymbol{\theta}}$ via

$$
\begin{aligned}
(\boldsymbol{\theta}\times\tilde{\boldsymbol{\theta}})_{BC} &:= \not{\boldsymbol{g}}^{AD}\boldsymbol{\theta}_{AB}\tilde{\boldsymbol{\theta}}_{DC},\\
(\boldsymbol{\xi}\widehat{\otimes}\tilde{\boldsymbol{\xi}})_{AB} &:= \boldsymbol{\xi}_A\tilde{\boldsymbol{\xi}}_B+\boldsymbol{\xi}_B\tilde{\boldsymbol{\xi}}_A-\not{\boldsymbol{g}}^{CD}\boldsymbol{\xi}_C\tilde{\boldsymbol{\xi}}_D\not{\boldsymbol{g}}_{AB},\\
\boldsymbol{\theta}\wedge\tilde{\boldsymbol{\theta}} &:= \not{\boldsymbol{\epsilon}}^{AB}\not{\boldsymbol{g}}^{CD}\boldsymbol{\theta}_{AC}\tilde{\boldsymbol{\theta}}_{BD},
\end{aligned}
$$

where $\not{\boldsymbol{\epsilon}}_{AB}$ denotes the components of the volume form associated with $\not{\boldsymbol{g}}$ on $\boldsymbol{S}_{\boldsymbol{u},\boldsymbol{v}}$. Note that $\boldsymbol{\xi}\widehat{\otimes}\tilde{\boldsymbol{\xi}}$ is a symmetric traceless 2-tensor.

3.1.4. $\boldsymbol{S}_{\boldsymbol{u},\boldsymbol{v}}$-projected Lie and covariant derivates

We define the derivative operators $\underline{\boldsymbol{D}}$ and $\boldsymbol{D}$ to act on an $\boldsymbol{S}_{\boldsymbol{u},\boldsymbol{v}}$-tensor $\boldsymbol{\phi}$ as the projection onto $\boldsymbol{S}_{\boldsymbol{u},\boldsymbol{v}}$ of the Lie-derivative of $\boldsymbol{\phi}$ in the direction of $\boldsymbol{\Omega}\boldsymbol{e}_3$ and $\boldsymbol{\Omega}\boldsymbol{e}_4$, respectively. We hence have the following relations between the projected Lie-derivatives $\underline{\boldsymbol{D}}$ and $\boldsymbol{D}$, and the $\boldsymbol{S}_{\boldsymbol{u},\boldsymbol{v}}$-projected spacetime covariant derivatives $\not{\nabla}_{\boldsymbol{3}}=\not{\nabla}_{\boldsymbol{e}_3}$ and $\not{\nabla}_{\boldsymbol{4}}=\not{\nabla}_{\boldsymbol{e}_4}$ in the direction $\boldsymbol{e}_3$ and $\boldsymbol{e}_4$, respectively:

$$
\begin{aligned}
&\boldsymbol{D}\boldsymbol{f}=\boldsymbol{\Omega}\not{\nabla}_{\boldsymbol{4}}\boldsymbol{f} && \text{on functions } \boldsymbol{f},\\
&\boldsymbol{D}\boldsymbol{\xi}=\boldsymbol{\Omega}\not{\nabla}_{\boldsymbol{4}}\boldsymbol{\xi}+\boldsymbol{\Omega}\boldsymbol{\chi}^{\sharp}\cdot\boldsymbol{\xi} && \text{on 1-forms } \boldsymbol{\xi},\\
&\boldsymbol{D}\boldsymbol{\theta}=\boldsymbol{\Omega}\not{\nabla}_{\boldsymbol{4}}\boldsymbol{\theta}+\boldsymbol{\Omega}\boldsymbol{\chi}\times\boldsymbol{\theta}+\boldsymbol{\Omega}\boldsymbol{\theta}\times\boldsymbol{\chi} && \text{on symmetric 2-tensors } \boldsymbol{\theta},
\end{aligned}
\tag{72}
$$

and similarly for $\not{\nabla}_{\boldsymbol{3}}$, replacing $\boldsymbol{\chi}$ by $\underline{\boldsymbol{\chi}}$ and $\boldsymbol{D}$ by $\underline{\boldsymbol{D}}$. See [11] for details.

3.1.5. Angular operators on $\boldsymbol{S}_{\boldsymbol{u},\boldsymbol{v}}$

We employ the following notation (adapted from [14]) for operators on the manifolds $\boldsymbol{S}_{\boldsymbol{u},\boldsymbol{v}}$.

Let $\boldsymbol{\xi}$ be an arbitrary 1-form and $\boldsymbol{\theta}$ be an arbitrary symmetric traceless 2-tensor on $\boldsymbol{S}_{\boldsymbol{u},\boldsymbol{v}}$.

- $\not{\boldsymbol{\nabla}}$ denotes the covariant derivative associated with the metric $\not{\boldsymbol{g}}_{AB}$ on $\boldsymbol{S}_{\boldsymbol{u},\boldsymbol{v}}$.
- $\not{\boldsymbol{\mathcal{D}}}_1$ takes $\boldsymbol{\xi}$ into the pair of functions $(\mathbf{d\not{i}v}\,\boldsymbol{\xi},\mathbf{c\not{u}rl}\,\boldsymbol{\xi})$, where $\mathbf{d\not{i}v}\,\boldsymbol{\xi}=\not{\boldsymbol{g}}^{AB}\not{\boldsymbol{\nabla}}_A\boldsymbol{\xi}_B$ and $\mathbf{c\not{u}rl}\,\boldsymbol{\xi}=\not{\boldsymbol{\epsilon}}^{AB}\not{\boldsymbol{\nabla}}_A\boldsymbol{\xi}_B$.
- $\not{\boldsymbol{\mathcal{D}}}_1^{\star}$, the formal([14]) L^2-adjoint of $\not{\boldsymbol{\mathcal{D}}}_1$, takes any pair of scalars $\boldsymbol{\varrho}$ and $\boldsymbol{\sigma}$ into the $\boldsymbol{S}_{\boldsymbol{u},\boldsymbol{v}}$-1-form $-\not{\boldsymbol{\nabla}}_A\boldsymbol{\varrho}+\not{\boldsymbol{\epsilon}}_{AB}\not{\boldsymbol{\nabla}}^B\boldsymbol{\sigma}$.
- $\not{\boldsymbol{\mathcal{D}}}_2$ takes $\boldsymbol{\theta}$ into the $\boldsymbol{S}_{\boldsymbol{u},\boldsymbol{v}}$-1-form $(\mathbf{d\not{i}v}\,\boldsymbol{\theta})_C=\not{\boldsymbol{g}}^{AB}\not{\boldsymbol{\nabla}}_A\boldsymbol{\theta}_{BC}$.
- $\not{\boldsymbol{\mathcal{D}}}_2^{\star}$, the formal L^2 adjoint of $\not{\boldsymbol{\mathcal{D}}}_2$, takes $\boldsymbol{\xi}$ into the symmetric traceless 2-tensor $(\not{\boldsymbol{\mathcal{D}}}_2^{\star}\boldsymbol{\xi})_{AB}=-\frac{1}{2}(\not{\boldsymbol{\nabla}}_B\boldsymbol{\xi}_A+\not{\boldsymbol{\nabla}}_A\boldsymbol{\xi}_B-(\mathbf{d\not{i}v}\,\boldsymbol{\xi})\not{\boldsymbol{g}}_{AB})$.

(14) In our application, the surfaces $\boldsymbol{S}_{\boldsymbol{u},\boldsymbol{v}}$ will be compact topological spheres and this will indeed define an adjoint on appropriate spaces.

3.2. Ricci coefficients and curvature components

We now define the Ricci coefficients and curvature components associated with the metric (70) with respect to the normalised null frame $\mathcal{N}=\{\boldsymbol{e}_3,\boldsymbol{e}_4,\boldsymbol{e}_1,\boldsymbol{e}_2\}$.

For the Ricci coefficients, using the shorthand $\boldsymbol{\nabla}_A=\boldsymbol{\nabla}_{\boldsymbol{e}_A}$ we define

$$\begin{aligned} \boldsymbol{\chi}_{AB}&=\boldsymbol{g}(\boldsymbol{\nabla}_A\boldsymbol{e}_4,\boldsymbol{e}_B), & \underline{\boldsymbol{\chi}}_{AB}&=\boldsymbol{g}(\boldsymbol{\nabla}_A\boldsymbol{e}_3,\boldsymbol{e}_B),\\ \boldsymbol{\eta}_A&=-\tfrac12\boldsymbol{g}(\boldsymbol{\nabla}_{\boldsymbol{e}_3}\boldsymbol{e}_A,\boldsymbol{e}_4), & \underline{\boldsymbol{\eta}}_A&=-\tfrac12\boldsymbol{g}(\boldsymbol{\nabla}_{\boldsymbol{e}_4}\boldsymbol{e}_A,\boldsymbol{e}_3),\\ \widehat{\boldsymbol{\omega}}&=\tfrac12\boldsymbol{g}(\boldsymbol{\nabla}_{\boldsymbol{e}_4}\boldsymbol{e}_3,\boldsymbol{e}_4), & \underline{\widehat{\boldsymbol{\omega}}}&=\tfrac12\boldsymbol{g}(\boldsymbol{\nabla}_{\boldsymbol{e}_3}\boldsymbol{e}_4,\boldsymbol{e}_3),\\ \boldsymbol{\zeta}_A&=\tfrac12\boldsymbol{g}(\boldsymbol{\nabla}_A\boldsymbol{e}_4,\boldsymbol{e}_3). \end{aligned} \tag{73}$$

Note that, in view of $\boldsymbol{\Omega}^{-1}\boldsymbol{e}_3$ and $\boldsymbol{\Omega}^{-1}\boldsymbol{e}_4$ being geodesic vectorfields, all other connection coefficients automatically vanish. It is natural to decompose $\boldsymbol{\chi}$ into its trace-free part $\widehat{\boldsymbol{\chi}}$, a symmetric traceless $\boldsymbol{S}_{\boldsymbol{u},\boldsymbol{v}}$ 2-tensor and its trace $\mathbf{tr}\,\boldsymbol{\chi}$, and similarly $\underline{\boldsymbol{\chi}}$.(15)

With $\boldsymbol{R}$ denoting the Riemann curvature tensor of (70), the null-decomposed curvature components are defined as follows:

$$\begin{aligned} \boldsymbol{\alpha}_{AB}&=\boldsymbol{R}(\boldsymbol{e}_A,\boldsymbol{e}_4,\boldsymbol{e}_B,\boldsymbol{e}_4), & \underline{\boldsymbol{\alpha}}_{AB}&=\boldsymbol{R}(\boldsymbol{e}_A,\boldsymbol{e}_3,\boldsymbol{e}_B,\boldsymbol{e}_3),\\ \boldsymbol{\beta}_A&=\tfrac12\boldsymbol{R}(\boldsymbol{e}_A,\boldsymbol{e}_4,\boldsymbol{e}_3,\boldsymbol{e}_4), & \underline{\boldsymbol{\beta}}_A&=\boldsymbol{R}(\boldsymbol{e}_A,\boldsymbol{e}_3,\boldsymbol{e}_3,\boldsymbol{e}_4),\\ \boldsymbol{\varrho}&=\tfrac14\boldsymbol{R}(\boldsymbol{e}_4,\boldsymbol{e}_3,\boldsymbol{e}_4,\boldsymbol{e}_3), & \boldsymbol{\sigma}&=\tfrac14{}^\star\boldsymbol{R}(\boldsymbol{e}_4,\boldsymbol{e}_3,\boldsymbol{e}_4,\boldsymbol{e}_3), \end{aligned} \tag{74}$$

with ${}^\star\boldsymbol{R}$ denoting the Hodge dual on $(\boldsymbol{\mathcal{M}},\boldsymbol{g})$ of $\boldsymbol{R}$. The above objects are $\boldsymbol{S}_{\boldsymbol{u},\boldsymbol{v}}$-tensors (functions, vectors, symmetric 2-tensors) on $(\boldsymbol{\mathcal{M}},\boldsymbol{g})$; cf. [14]. Note also the relations

$$\underline{\widehat{\boldsymbol{\omega}}}=\frac{\boldsymbol{\nabla\!\!\!/}_3\boldsymbol{\Omega}}{\boldsymbol{\Omega}},\quad \widehat{\boldsymbol{\omega}}=\frac{\boldsymbol{\nabla\!\!\!/}_4\boldsymbol{\Omega}}{\boldsymbol{\Omega}},\quad \boldsymbol{\eta}_A=\boldsymbol{\zeta}_A+\boldsymbol{\nabla\!\!\!/}_A\log\boldsymbol{\Omega},\quad \underline{\boldsymbol{\eta}}_A=-\boldsymbol{\zeta}_A+\boldsymbol{\nabla\!\!\!/}_A\log\boldsymbol{\Omega}.$$

3.3. The Einstein equations

If $(\boldsymbol{\mathcal{M}},\boldsymbol{g})$ satisfies the vacuum Einstein equations

$$\boldsymbol{R}_{\mu\nu}[\boldsymbol{g}]=0, \tag{75}$$

the Ricci coefficients defined in (73) and curvature components (74) satisfy a system of equations, which is presented in this section.

(15) This is of course unrelated to the ˆ in the case of the scalar quantity $\widehat{\boldsymbol{\omega}}$, which distinguishes it from other normalisations; we retain the notation $\widehat{\boldsymbol{\omega}}$ to facilitate comparison with [11].

3.3.1. The null structure equations

First, we have the important first variational formulas:([16])

$$\boldsymbol{D}\boldsymbol{\not g} = 2\boldsymbol{\Omega}\boldsymbol{\chi} = 2\boldsymbol{\Omega}\widehat{\boldsymbol{\chi}} + \boldsymbol{\Omega}\,\mathbf{tr}\,\boldsymbol{\chi}\,\boldsymbol{\not g} \quad \text{and} \quad \underline{\boldsymbol{D}}\boldsymbol{\not g} = 2\boldsymbol{\Omega}\underline{\boldsymbol{\chi}} = 2\boldsymbol{\Omega}\widehat{\underline{\boldsymbol{\chi}}} + \boldsymbol{\Omega}\,\mathbf{tr}\,\underline{\boldsymbol{\chi}}\,\boldsymbol{\not g}. \tag{76}$$

Second,

$$\boldsymbol{\not\nabla}_3\widehat{\underline{\boldsymbol{\chi}}} + \mathbf{tr}\,\underline{\boldsymbol{\chi}}\,\widehat{\underline{\boldsymbol{\chi}}} - \widehat{\underline{\boldsymbol{\omega}}}\;\widehat{\underline{\boldsymbol{\chi}}} = -\underline{\boldsymbol{\alpha}}, \quad \boldsymbol{\not\nabla}_4\widehat{\boldsymbol{\chi}} + \mathbf{tr}\,\boldsymbol{\chi}\,\widehat{\boldsymbol{\chi}} - \widehat{\boldsymbol{\omega}}\widehat{\boldsymbol{\chi}} = \boldsymbol{\alpha}, \tag{77}$$

$$\boldsymbol{\not\nabla}_3(\mathbf{tr}\,\underline{\boldsymbol{\chi}}) + \tfrac{1}{2}(\mathbf{tr}\,\underline{\boldsymbol{\chi}})^2 - \widehat{\underline{\boldsymbol{\omega}}}\,\mathbf{tr}\,\underline{\boldsymbol{\chi}} = -(\widehat{\underline{\boldsymbol{\chi}}}, \widehat{\underline{\boldsymbol{\chi}}}), \quad \boldsymbol{\not\nabla}_4(\mathbf{tr}\,\boldsymbol{\chi}) + \tfrac{1}{2}(\mathbf{tr}\,\boldsymbol{\chi})^2 - \widehat{\boldsymbol{\omega}}\,\mathbf{tr}\,\boldsymbol{\chi} = -(\widehat{\boldsymbol{\chi}}, \widehat{\boldsymbol{\chi}}). \tag{78}$$

Note that the last two equations are the celebrated Raychaudhuri equations. We also have

$$\boldsymbol{\not\nabla}_3\widehat{\boldsymbol{\chi}} + \tfrac{1}{2}\,\mathbf{tr}\,\underline{\boldsymbol{\chi}}\,\widehat{\boldsymbol{\chi}} + \widehat{\underline{\boldsymbol{\omega}}}\widehat{\boldsymbol{\chi}} = -2\boldsymbol{\not{\mathcal{D}}}_2^\star\boldsymbol{\eta} - \tfrac{1}{2}\,\mathbf{tr}\,\boldsymbol{\chi}\,\widehat{\underline{\boldsymbol{\chi}}} + (\boldsymbol{\eta}\widehat{\otimes}\boldsymbol{\eta}), \tag{79}$$

$$\boldsymbol{\not\nabla}_4\widehat{\underline{\boldsymbol{\chi}}} + \tfrac{1}{2}\,\mathbf{tr}\,\boldsymbol{\chi}\,\widehat{\underline{\boldsymbol{\chi}}} + \widehat{\boldsymbol{\omega}}\widehat{\underline{\boldsymbol{\chi}}} = -2\boldsymbol{\not{\mathcal{D}}}_2^\star\underline{\boldsymbol{\eta}} - \tfrac{1}{2}\,\mathbf{tr}\,\underline{\boldsymbol{\chi}}\,\widehat{\boldsymbol{\chi}} + (\underline{\boldsymbol{\eta}}\widehat{\otimes}\underline{\boldsymbol{\eta}}), \tag{80}$$

$$\boldsymbol{\not\nabla}_3(\mathbf{tr}\,\boldsymbol{\chi}) + \tfrac{1}{2}(\mathbf{tr}\,\underline{\boldsymbol{\chi}})(\mathbf{tr}\,\boldsymbol{\chi}) + \widehat{\underline{\boldsymbol{\omega}}}\,\mathbf{tr}\,\boldsymbol{\chi} = -(\widehat{\underline{\boldsymbol{\chi}}}, \widehat{\boldsymbol{\chi}}) + 2(\boldsymbol{\eta}, \boldsymbol{\eta}) + 2\boldsymbol{\varrho} + 2\,\mathbf{d\not{i}v}\,\boldsymbol{\eta}, \tag{81}$$

$$\boldsymbol{\not\nabla}_4(\mathbf{tr}\,\underline{\boldsymbol{\chi}}) + \tfrac{1}{2}(\mathbf{tr}\,\boldsymbol{\chi})(\mathbf{tr}\,\underline{\boldsymbol{\chi}}) + \widehat{\boldsymbol{\omega}}\,\mathbf{tr}\,\underline{\boldsymbol{\chi}} = -(\widehat{\underline{\boldsymbol{\chi}}}, \widehat{\boldsymbol{\chi}}) + 2(\underline{\boldsymbol{\eta}}, \underline{\boldsymbol{\eta}}) + 2\boldsymbol{\varrho} + 2\,\mathbf{d\not{i}v}\,\underline{\boldsymbol{\eta}}, \tag{82}$$

$$\begin{gathered}
\boldsymbol{\not\nabla}_3\underline{\boldsymbol{\eta}} = \underline{\boldsymbol{\chi}}^\sharp\cdot(\boldsymbol{\eta} - \underline{\boldsymbol{\eta}}) + \underline{\boldsymbol{\beta}}, \quad \boldsymbol{\not\nabla}_4\boldsymbol{\eta} = -\boldsymbol{\chi}^\sharp\cdot(\boldsymbol{\eta} - \underline{\boldsymbol{\eta}}) - \boldsymbol{\beta}, \\
\boldsymbol{D}(\boldsymbol{\Omega}\widehat{\underline{\boldsymbol{\omega}}}) = \boldsymbol{\Omega}^2[2(\boldsymbol{\eta}, \underline{\boldsymbol{\eta}}) - |\boldsymbol{\eta}|^2 - \boldsymbol{\varrho}], \quad \underline{\boldsymbol{D}}(\boldsymbol{\Omega}\widehat{\boldsymbol{\omega}}) = \boldsymbol{\Omega}^2[2(\boldsymbol{\eta}, \underline{\boldsymbol{\eta}}) - |\underline{\boldsymbol{\eta}}|^2 - \boldsymbol{\varrho}], \\
\boldsymbol{\Omega}^2\widehat{\underline{\boldsymbol{\omega}}} = \underline{\boldsymbol{D}}\boldsymbol{\Omega}, \quad \boldsymbol{\Omega}^2\widehat{\boldsymbol{\omega}} = \boldsymbol{D}\boldsymbol{\Omega}, \quad \boldsymbol{\eta}_A + \underline{\boldsymbol{\eta}}_A = 2\boldsymbol{\not\nabla}_A\log\boldsymbol{\Omega}, \\
\boldsymbol{\partial_u b}^A = 2\boldsymbol{\Omega}^2(\boldsymbol{\eta}^A - \underline{\boldsymbol{\eta}}^A).
\end{gathered} \tag{83}$$

Finally, we have

$$\mathbf{cu\not{r}l}\,\boldsymbol{\eta} = -\tfrac{1}{2}\boldsymbol{\chi}\wedge\underline{\boldsymbol{\chi}} + \boldsymbol{\sigma} \quad \text{and} \quad \mathbf{cu\not{r}l}\,\underline{\boldsymbol{\eta}} = \tfrac{1}{2}\boldsymbol{\chi}\wedge\underline{\boldsymbol{\chi}} - \boldsymbol{\sigma},$$

the Codazzi equations

$$\begin{aligned}
\mathbf{d\not{i}v}\,\widehat{\boldsymbol{\chi}} &= -\frac{1}{2}\widehat{\boldsymbol{\chi}}^\sharp\cdot(\boldsymbol{\eta} - \underline{\boldsymbol{\eta}}) + \frac{1}{4}\,\mathbf{tr}\,\boldsymbol{\chi}(\boldsymbol{\eta} - \underline{\boldsymbol{\eta}}) + \frac{1}{2}\boldsymbol{\not\nabla}\,\mathbf{tr}\,\boldsymbol{\chi} - \boldsymbol{\beta} \\
&= -\frac{1}{2}\widehat{\boldsymbol{\chi}}^\sharp\cdot(\boldsymbol{\eta} - \underline{\boldsymbol{\eta}}) - \frac{1}{2}\,\mathbf{tr}\,\boldsymbol{\chi}\underline{\boldsymbol{\eta}} + \frac{1}{2\boldsymbol{\Omega}}\boldsymbol{\not\nabla}(\boldsymbol{\Omega}\,\mathbf{tr}\,\boldsymbol{\chi}) - \boldsymbol{\beta},
\end{aligned}$$

$$\begin{aligned}
\mathbf{d\not{i}v}\,\widehat{\underline{\boldsymbol{\chi}}} &= \frac{1}{2}\widehat{\underline{\boldsymbol{\chi}}}^\sharp\cdot(\boldsymbol{\eta} - \underline{\boldsymbol{\eta}}) - \frac{1}{4}\,\mathbf{tr}\,\underline{\boldsymbol{\chi}}(\boldsymbol{\eta} - \underline{\boldsymbol{\eta}}) + \frac{1}{2}\boldsymbol{\not\nabla}\,\mathbf{tr}\,\underline{\boldsymbol{\chi}} + \underline{\boldsymbol{\beta}} \\
&= \frac{1}{2}\widehat{\underline{\boldsymbol{\chi}}}^\sharp\cdot(\boldsymbol{\eta} - \underline{\boldsymbol{\eta}}) - \frac{1}{2}\,\mathbf{tr}\,\underline{\boldsymbol{\chi}}\boldsymbol{\eta} + \frac{1}{2\boldsymbol{\Omega}}\boldsymbol{\not\nabla}(\boldsymbol{\Omega}\,\mathbf{tr}\,\underline{\boldsymbol{\chi}}) + \underline{\boldsymbol{\beta}},
\end{aligned}$$

and the Gauss equation ($\mathbf{K}$ denoting the Gauss curvature of the metric $\boldsymbol{\not g}$)

$$\mathbf{K} = -\tfrac{1}{4}\,\mathbf{tr}\,\boldsymbol{\chi}\,\mathbf{tr}\,\underline{\boldsymbol{\chi}} + \tfrac{1}{2}(\widehat{\boldsymbol{\chi}}, \widehat{\underline{\boldsymbol{\chi}}}) - \boldsymbol{\varrho}. \tag{84}$$

([16]) Note that these formulas are equivalent to the statement that $\boldsymbol{\not\nabla}_3\boldsymbol{\not g} = 0 = \boldsymbol{\not\nabla}_4\boldsymbol{\not g}$.

3.3.2. The Bianchi equations

We finally turn to the equations satisfied by the curvature components of $(\boldsymbol{\mathcal{M}}, \boldsymbol{g})$, which are the well-known Bianchi equations:

$$
\begin{aligned}
\boldsymbol{\nabla\!\!\!/}_3\boldsymbol{\alpha}+\tfrac{1}{2}\,\mathbf{tr}\,\underline{\boldsymbol{\chi}}\,\boldsymbol{\alpha}+2\underline{\widehat{\boldsymbol{\omega}}}\boldsymbol{\alpha} &= -2\boldsymbol{\mathcal{D}\!\!\!/}_{\boldsymbol{2}}^{\star}\boldsymbol{\beta}-3\widehat{\boldsymbol{\chi}}\boldsymbol{\varrho}-3{}^{\star}\widehat{\boldsymbol{\chi}}\boldsymbol{\sigma}+(4\boldsymbol{\eta}+\boldsymbol{\zeta})\widehat{\otimes}\boldsymbol{\beta},\\
\boldsymbol{\nabla\!\!\!/}_4\boldsymbol{\beta}+2\,\mathbf{tr}\,\boldsymbol{\chi}\,\boldsymbol{\beta}-\widehat{\boldsymbol{\omega}}\boldsymbol{\beta} &= \mathbf{d\!\!\!/iv}\,\boldsymbol{\alpha}+(\underline{\boldsymbol{\eta}}^{\sharp}+2\boldsymbol{\zeta}^{\sharp})\cdot\boldsymbol{\alpha},\\
\boldsymbol{\nabla\!\!\!/}_3\boldsymbol{\beta}+\mathbf{tr}\,\underline{\boldsymbol{\chi}}\,\boldsymbol{\beta}+\underline{\widehat{\boldsymbol{\omega}}}\boldsymbol{\beta} &= \boldsymbol{\mathcal{D}\!\!\!/}_1^{\star}(-\boldsymbol{\varrho},\boldsymbol{\sigma})+3\boldsymbol{\eta}\boldsymbol{\varrho}+3{}^{\star}\boldsymbol{\eta}\boldsymbol{\sigma}+2\widehat{\boldsymbol{\chi}}^{\sharp}\cdot\underline{\boldsymbol{\beta}},\\
\boldsymbol{\nabla\!\!\!/}_4\boldsymbol{\varrho}+\tfrac{3}{2}\,\mathbf{tr}\,\boldsymbol{\chi}\,\boldsymbol{\varrho} &= \mathbf{d\!\!\!/iv}\,\boldsymbol{\beta}+(2\underline{\boldsymbol{\eta}}+\boldsymbol{\zeta},\boldsymbol{\beta})-\tfrac{1}{2}(\underline{\widehat{\boldsymbol{\chi}}},\boldsymbol{\alpha}),\\
\boldsymbol{\nabla\!\!\!/}_4\boldsymbol{\sigma}+\tfrac{3}{2}\,\mathbf{tr}\,\boldsymbol{\chi}\,\boldsymbol{\sigma} &= -\,\mathbf{cu\!\!\!/rl}\,\boldsymbol{\beta}-(2\underline{\boldsymbol{\eta}}+\boldsymbol{\zeta})\wedge\boldsymbol{\beta}+\tfrac{1}{2}\underline{\widehat{\boldsymbol{\chi}}}\wedge\boldsymbol{\alpha},
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{\nabla\!\!\!/}_3\boldsymbol{\varrho}+\tfrac{3}{2}\,\mathbf{tr}\,\underline{\boldsymbol{\chi}}\,\boldsymbol{\varrho} &= -\,\mathbf{d\!\!\!/iv}\,\underline{\boldsymbol{\beta}}-(2\boldsymbol{\eta}-\boldsymbol{\zeta},\underline{\boldsymbol{\beta}})-\tfrac{1}{2}(\widehat{\boldsymbol{\chi}},\underline{\boldsymbol{\alpha}}),\\
\boldsymbol{\nabla\!\!\!/}_3\boldsymbol{\sigma}+\tfrac{3}{2}\,\mathbf{tr}\,\underline{\boldsymbol{\chi}}\,\boldsymbol{\sigma} &= -\,\mathbf{cu\!\!\!/rl}\,\underline{\boldsymbol{\beta}}-(2\boldsymbol{\eta}-\boldsymbol{\zeta})\wedge\underline{\boldsymbol{\beta}}-\tfrac{1}{2}\widehat{\boldsymbol{\chi}}\wedge\underline{\boldsymbol{\alpha}},\\
\boldsymbol{\nabla\!\!\!/}_4\underline{\boldsymbol{\beta}}+\mathbf{tr}\,\boldsymbol{\chi}\,\underline{\boldsymbol{\beta}}+\widehat{\boldsymbol{\omega}}\underline{\boldsymbol{\beta}} &= \boldsymbol{\mathcal{D}\!\!\!/}_1^{\star}(\boldsymbol{\varrho},\boldsymbol{\sigma})-3\underline{\boldsymbol{\eta}}\boldsymbol{\varrho}+3{}^{\star}\underline{\boldsymbol{\eta}}\boldsymbol{\sigma}+2\underline{\widehat{\boldsymbol{\chi}}}^{\sharp}\cdot\boldsymbol{\beta},\\
\boldsymbol{\nabla\!\!\!/}_3\underline{\boldsymbol{\beta}}+2\,\mathbf{tr}\,\underline{\boldsymbol{\chi}}\,\underline{\boldsymbol{\beta}}-\underline{\widehat{\boldsymbol{\omega}}}\underline{\boldsymbol{\beta}} &= -\,\mathbf{d\!\!\!/iv}\,\underline{\boldsymbol{\alpha}}-(\boldsymbol{\eta}^{\sharp}-2\boldsymbol{\zeta}^{\sharp})\cdot\underline{\boldsymbol{\alpha}},\\
\boldsymbol{\nabla\!\!\!/}_4\underline{\boldsymbol{\alpha}}+\tfrac{1}{2}\,\mathbf{tr}\,\boldsymbol{\chi}\,\underline{\boldsymbol{\alpha}}+2\widehat{\boldsymbol{\omega}}\underline{\boldsymbol{\alpha}} &= 2\boldsymbol{\mathcal{D}\!\!\!/}_2^{\star}\underline{\boldsymbol{\beta}}-3\underline{\widehat{\boldsymbol{\chi}}}\boldsymbol{\varrho}+3{}^{\star}\underline{\widehat{\boldsymbol{\chi}}}\boldsymbol{\sigma}-(4\underline{\boldsymbol{\eta}}-\boldsymbol{\zeta})\widehat{\otimes}\underline{\boldsymbol{\beta}}.
\end{aligned}
$$

We note that the vacuum equations (75) further imply that the symmetric tensors $\boldsymbol{\alpha}$ and $\underline{\boldsymbol{\alpha}}$ are in addition traceless. The above equations encode the essential hyperbolicity of (75). See [14].

4. The Schwarzschild exterior background

In this section, we shall introduce the Schwarzschild exterior metric as well as relevant background structure which will be useful in the paper.

We first fix in §4.1 an ambient manifold-with-boundary $\mathcal{M}$ on which we define the Schwarzschild exterior metric g with parameter M. We shall then pass to the computationally more convenient Eddington–Finkelstein double null coordinates u and v, and associated null frames in §4.2, computing the Ricci coefficients and curvature components, commenting on issues associated with lack of regularity at the horizon. In §4.3, we shall introduce various natural differential operators associated with Schwarzschild, specialising the definitions in §3, and give some useful commutation formulas. Finally, in §4.4, we recall some elementary properties of the classical $\ell{=}0,1$ spherical harmonics, and derive various important elliptic estimates on spheres.

4.1. Differential structure and metric

We define in this section the underlying differential structure and metric in terms of a Kruskal coordinate system.

4.1.1. An underlying Kruskal coordinate system

Define the manifold with boundary

$$\mathcal{M} := (-\infty, 0] \times (0, \infty) \times S^2 \tag{85}$$

with coordinates $(U, V, \theta^1, \theta^2)$. We will refer to these coordinates as *Kruskal coordinates*. The boundary

$$\mathcal{H}^+ := \{0\} \times (0, \infty) \times S^2$$

will be referred to as the *horizon*. We denote by $S^2_{U,V}$ the 2-sphere $\{U, V\} \times S^2$ in $\mathcal{M}$.

4.1.2. The Schwarzschild metric

We define the Schwarzschild metric on $\mathcal{M}$ as follows.

Fix a parameter $M>0$. Let the function $r: \mathcal{M} \to [2M, \infty)$ be given implicitly as a function of the coordinates U and V by

$$-UV = \frac{1-2M/r}{(2M/r)e^{-r/2M}}, \tag{86}$$

and define also

$$\Omega_K^2(U,V) = \frac{8M^3}{r(U,V)} e^{-r(U,V)/2M}, \quad \gamma = \text{standard metric on } S^2. \tag{87}$$

Then, the Schwarzschild metric g with parameter M is defined to be the metric:

$$g = -4\Omega_K^2(U,V)\, dU\, dV + r^2(U,V)\gamma_{AB}\, d\theta^A\, d\theta^B. \tag{88}$$

Note that the horizon $\mathcal{H}^+ = \partial\mathcal{M}$ is a null hypersurface with respect to g. We will sometimes use the standard spherical coordinates $(\theta^1, \theta^2) = (\theta, \phi)$, in which case the metric γ takes the explicit form

$$\gamma = d\theta^2 + \sin^2\theta\, d\phi^2. \tag{89}$$

The above metric (88) can in fact be extended to define the so-called maximally extended Schwarzschild solution (see [72], [46] and the textbook [76]) on the ambient manifold given by $(-\infty, \infty) \times (\infty, \infty) \times S^2$. In this paper, however, we only need consider the manifold-with-boundary $\mathcal{M}$ as defined in (85).

4.2. Eddington–Finkelstein double null coordinates u and v, and frames

We have defined our manifold and metric as above so that its smoothness is manifest. For computations, however, it is much more convenient to rescale the null coordinates in such a way that quantities are more symmetric. The new coordinates are known as *Eddington–Finkelstein double null coordinates*. Care must be taken at the horizon $\mathcal{H}^+$, however, where these coordinates break down. We explain below.

4.2.1. Eddington–Finkelstein double null coordinates

In this section we will define another double null coordinate system that covers $\mathcal{M}^o$, the interior of $\mathcal{M}$, modulo the degeneration of the angular coordinates. This coordinate system, $(u, v, \theta^1, \theta^2)$, will be referred to as *Eddington–Finkelstein double null coordinates* and are defined via the relations

$$U = -e^{-u/2M} \quad \text{and} \quad V = e^{v/2M}. \tag{90}$$

Using (90), we obtain the Schwarzschild metric on $\mathcal{M}^o$ in $(u, v, \theta^1, \theta^2)$-coordinates:

$$g = -4\Omega^2(u, v)\, du\, dv + r^2(u, v)\gamma_{AB}\, d\theta^A\, d\theta^B, \tag{91}$$

with

$$\Omega^2 := 1 - \frac{2M}{r} \tag{92}$$

and the function $r: (-\infty, \infty) \times (-\infty, \infty) \to (2M, \infty)$ defined implicitly via the relation

$$e^{(v-u)/2M} = \left(\frac{r}{2M} - 1\right) e^{r/2M}. \tag{93}$$

Note that setting $t = u + v$, we may rewrite the above metric in coordinates (t, r, θ, ϕ) in the usual form (1).

In $(u, v, \theta^1, \theta^2)$-coordinates, the horizon $\mathcal{H}^+$ can still be formally parameterised by $(\infty, v, \theta^2, \theta^2)$ with $v \in \mathbb{R}$, $(\theta^1, \theta^2) \in S^2$. This will allow us to use these coordinates at $\mathcal{H}^+$, provided that we appropriately rescale all quantities so as to be regular. We shall see this principle used already in the section below.

Let us also introduce the notation $S^2_{u,v}$ to denote the sphere $S^2_{U,V}$, where U and V are defined by (90). In the spirit of the remark of the previous paragraph, we shall write in addition $S^2_{\infty,v}$ for the spheres of the horizon $\mathcal{H}^+$ (where we are to understand $U = 0$).

Finally, we will often refer informally to the limit $v \to \infty$ as *null infinity* $\mathcal{I}^+$, which can be parameterised as $\mathcal{I}^+ = \{(u, \infty, \theta, \phi)\}$.

4.2.2. Killing fields of the Schwarzschild metric

It is natural at this point to already discuss the Killing fields associated with g.

We define the vectorfield T to be the timelike Killing field ∂_t of the (r,t) coordinates (1), which in Eddington–Finkelstein double null coordinates is given by

$$T = \tfrac{1}{2}(\partial_u + \partial_v). \tag{94}$$

The vector field extends to a smooth Killing field on the horizon $\mathcal{H}^+$, which is moreover null and tangential to the null generator of $\mathcal{H}^+$. See §4.2.3 below.

We can also define a basis of "angular momentum operators" Ω_i, $i=1,2,3$, for instance, fixing standard spherical coordinates (θ,ϕ) on S^2, where γ takes the form (88), by

$$\Omega_1 = \partial_\phi, \quad \Omega_2 = -\sin\phi\,\partial_\theta - \cot\theta\cos\phi\,\partial_\phi, \quad \Omega_3 = \cos\phi\,\partial_\theta - \cot\theta\sin\phi\,\partial_\phi.$$

The Lie algebra of Killing fields of g is then precisely that generated by T and Ω_i, $i=1,2,3$.

4.2.3. The null frames $\mathcal{N}_{EF}$ and $\mathcal{N}_{EF^\star}$

We define in this section two normalised null frames associated with Schwarzschild.

The most important one is the Eddington–Finkelstein frame $\mathcal{N}_{EF}$, which we define first. The vectorfields

$$e_3 = \frac{1}{\Omega}\,\partial_u \quad \text{and} \quad e_4 = \frac{1}{\Omega}\,\partial_v \tag{95}$$

defined with respect to (u,v,θ^1,θ^2)-Eddington–Finkelstein coordinates, together with a local frame field (e_1,e_2) on $S^2_{u,v}$ provide a normalised frame on $\mathcal{M}^o$:

$$\mathcal{N}_{EF} = \text{normalised null frame}\{e_3, e_4, e_1, e_2\}.$$

The above frame does not extend regularly to the horizon $\mathcal{H}^+$ (cf. the comments at the end of §4.2.1). However, it is easy to see that the rescaled null frame

$$\mathcal{N}_{EF^\star} = \text{normalised null frame}\{\Omega^{-1}e_3, \Omega e_4, e_1, e_2\}$$

does extend regularly to a non-vanishing null frame on $\mathcal{H}^+$. Though we shall always compute with respect to $\mathcal{N}_{EF}$, passing to $\mathcal{N}_{EF^\star}$ will be useful to understand which quantities are regular on the horizon.

Note finally that $2T=\Omega e_3+\Omega e_4=\Omega^2(\Omega^{-1}e_3)+\Omega e_4$, from which it follows that, on the horizon, T corresponds up to a factor with the null vector of the EF^* frame $T=\frac{1}{2}\Omega e_4$.

4.2.4. Connection coefficients and curvature components

We compute here the connection coefficients and curvature components with respect to the two null frames of §4.2.3.

With respect to the Eddington–Finkelstein null frame $\mathcal{N}_{EF}=(\Omega^{-1}\partial_u,\Omega^{-1}\partial_v,e_1,e_2)$, we compute the following non-vanishing Ricci coefficients (cf. §3.2) on $(\mathcal{M}^o,g)$:

$$\chi_{AB}:=g(-e_4,\nabla_A e_B)=\frac{\Omega}{r}r^2\gamma_{AB},\qquad \underline{\chi}_{AB}:=g(-e_3,\nabla_A e_B)=-\frac{\Omega}{r}r^2\gamma_{AB} \tag{96}$$

$$\widehat{\omega}:=\frac{1}{2}g(\nabla_{e_4}e_3,e_4)=\frac{M}{r^2\Omega},\qquad \widehat{\underline{\omega}}:=\frac{1}{2}g(\nabla_{e_3}e_4,e_3)=-\frac{M}{r^2\Omega}. \tag{97}$$

Remark 4.1. Recall from the comments in §4.2.3 that the frame $\mathcal{N}_{EF}$ is not regular near the horizon $\mathcal{H}^+$ as is manifest from some of these quantities diverging at the horizon! Converting to the rescaled regular null frame $\mathcal{N}_{EF^\star}$ one easily finds $\chi=(\Omega/r)r^2\gamma$, $\underline{\chi}=-r\gamma$, $\widehat{\omega}=2M/r^2$ and $\widehat{\underline{\omega}}=0$, which makes all components manifestly regular at $\mathcal{H}^+$.

Turning to curvature, the only non-vanishing null curvature component on $(\mathcal{M},g)$ is

$$\varrho:=\frac{1}{4}\,\mathrm{Riem}(e_4,e_3,e_4,e_3)=-\frac{2M}{r^3}, \tag{98}$$

an identity which holds with respect to both null frames $\mathcal{N}_{EF}$ and $\mathcal{N}_{EF^\star}$, as these only differ by a scaling of the two null-directions. We also introduce the notation

$$K=\frac{1}{r^2} \tag{99}$$

for the Gauss curvature of the round $S^2_{u,v}$-spheres.

4.3. Schwarzschild background operators and commutation identities

In this section, we introduce a number of natural differential operators associated with Schwarzschild. In §4.3.1, we specialise the operators discussed in §3.1 to Schwarzschild. We then give in §4.3.2 some important commutation identities and define the additional useful angular operators $\mathcal{A}^{[i]}$.

4.3.1. The $S_{u,v}$-tensor analysis and natural differential operators on Schwarzschild

In this section we simply specialise some of the constructions of §3.1 to Schwarzschild. We explicitly repeat all definitions, however, so that this section can be read independently.

We recall the notion of an $S_{u,v}$ tensor from [11]. Specialised to Schwarzschild, these are simply tensors which when expressed in the frame $\mathcal{N}_{EF}$ only have components in the angular directions e_1 and e_2.

Particularly important for our purpose will be 1-forms ξ and symmetric 2-tensors θ. An $S_{u,v}$-2-tensor θ_{AB} is symmetric if $\theta_{AB}=\theta_{BA}$ and traceless if $\not{g}^{AB}\theta_{AB}=0$.

We quickly repeat the definitions from §3.1.3 for 1-forms ξ and $\tilde{\xi}$ and symmetric 2-tensors θ and $\tilde{\theta}$.

We denote by ${}^\star\xi$ and ${}^\star\theta$ the Hodge duals (with respect to $(S_{u,v},\not{g})$), and by $\theta^\sharp$ the tensor obtained from θ by raising an index with $\not{g}$.

We define the contractions

$$(\xi,\tilde{\xi}):=\not{g}^{AB}\xi_A\tilde{\xi}_B,\quad (\theta,\tilde{\theta}):=\not{g}^{AB}\not{g}^{CD}\theta_{AC}\tilde{\theta}_{BD},\quad \theta^\sharp\cdot\xi=\theta_A{}^B\xi_B.$$

We finally define the 2-tensors $\theta\times\tilde{\theta}$, $\xi\widehat{\otimes}\tilde{\xi}$ and the scalar $\theta\wedge\tilde{\theta}$ via

$$\begin{aligned}(\theta\times\tilde{\theta})_{BC}&:=\not{g}^{AD}\theta_{AB}\tilde{\theta}_{DC},\\ (\xi\widehat{\otimes}\tilde{\xi})_{AB}&:=\xi_A\tilde{\xi}_B+\xi_B\tilde{\xi}_A-\not{g}^{AB}\xi_A\tilde{\xi}_B,\\ \theta\wedge\tilde{\theta}&:=\not{\epsilon}^{AB}\not{g}^{CD}\theta_{AC}\tilde{\theta}_{BD},\end{aligned}$$

where $\not{\epsilon}_{AB}$ denotes the components of the volume form associated with $\not{g}$ on $S_{u,v}$, and where we note again that $\xi\widehat{\otimes}\tilde{\xi}$ is a symmetric traceless $S^2_{u,v}$ 2-tensor.

We now specialise the general definitions of the projected Lie and covariant differential operators in §3.1.4 to the case of the Schwarzschild manifold $(\mathcal{M},g)$, with its Eddington–Finkelstein double null coordinates (u,v,θ^1,θ^2) and normalised null directions $e_3=\Omega^{-1}\partial_u$ and $e_4=\Omega^{-1}\partial_v$.

- The projections to the spheres $S^2_{u,v}$ of the Lie-derivative in the directions ∂_u and ∂_v are denoted by $\underline{D}$ and D, respectively.

Hence, if ξ is an $S^2_{u,v}$ tensor of rank n on $(\mathcal{M},g)$, we have in components(17)

$$(\underline{D}\xi)_{A_1\dots A_n}=\partial_u(\xi_{A_1\dots A_n})\quad\text{and}\quad(D\xi)_{A_1\dots A_n}=\partial_v(\xi_{A_1\dots A_n}).\tag{100}$$

Similarly,

- The projection to the spheres $S^2_{u,v}$ of the covariant derivative in the e_3-direction is denoted by $\not{\nabla}_3$, and that in the e_4-direction by $\not{\nabla}_4$.

The relations (72) now hold true "un-bolding" all quantities. Since χ and $\underline{\chi}$ only have a trace-component in $(\mathcal{M},g)$, one can deduce simplified formulas such as

$$\Omega(\not{\nabla}_3\xi)_A=\partial_u(\xi_A)-\tfrac{1}{2}\Omega\,\mathrm{tr}\,\underline{\chi}\,\xi_A\quad\text{for an } S^2_{u,v}\text{-1-form }\xi\tag{101}$$

(17) Observe that non-trivial terms would appear in the second formula if the background was not spherically symmetric.

and

$$\Omega(\nabla\!\!\!/_4\theta)_{AB} = \partial_v(\theta_{AB}) - \Omega \operatorname{tr}\chi\theta_{AB} \quad \text{for a symmetric traceless } S^2_{u,v}\text{-2-tensor } \theta, \tag{102}$$

which will be used below.

Finally, we specialise the definitions of §3.1.5, i.e. we introduce the following notation:

- $\nabla\!\!\!/$ denotes the covariant derivative associated with the metric $g\!\!\!/$ on $S^2_{u,v}$;
- $\mathcal{D}\!\!\!/_1$ takes any $S_{u,v}$-tangent 1-form ξ into the pair of functions $(\operatorname{d\!\!\!/iv}\xi, \operatorname{cu\!\!\!/rl}\xi)$;
- $\mathcal{D}\!\!\!/_1^\star$, the L^2-adjoint (with respect to $g\!\!\!/$) of $\mathcal{D}\!\!\!/_1$, takes any pair of scalars ϱ and σ into the $S^2_{u,v}$-tangent 1-form $-\nabla\!\!\!/_A\varrho + \epsilon\!\!/_{AB}\nabla\!\!\!/^B\sigma$, with $\epsilon\!\!/_{AB}$ denoting the components of the volume form associated with $g\!\!\!/$ on $S^2_{u,v}$;
- $\mathcal{D}\!\!\!/_2$ takes any 2-covariant symmetric traceless $S^2_{u,v}$-tensor ξ into the $S^2_{u,v}$-tangent 1-form $\operatorname{d\!\!\!/iv}\xi$;
- $\mathcal{D}\!\!\!/_2^\star$, the L^2 adjoint (with respect to $g\!\!\!/$) of $\mathcal{D}\!\!\!/_2$ takes any $S^2_{u,v}$-tangent 1-form ξ into the 2-form $-\frac{1}{2}(\nabla\!\!\!/_B\xi_A + \nabla\!\!\!/_A\xi_B - (\operatorname{d\!\!\!/iv}\xi)g\!\!\!/_{AB})$;
- $\Delta\!\!\!/$ denotes the covariant Laplacian associated with the metric $g\!\!\!/$ on $S^2_{u,v}$.

Recall that the spheres $S^2_{u,v}$ on the Schwarzschild manifold are equipped with the round metric $g\!\!\!/_{AB} = r^2\gamma_{AB}$, with γ being the metric on the unit sphere. We will use the notation $\varepsilon_{AB} = r^{-2}\epsilon\!\!/_{AB}$ for the components of the volume form on the unit sphere.

4.3.2. Commutation formulas and the operators $\mathcal{A}^{[i]}$

We have the following commutation formulas for projected covariant derivatives in the Schwarzschild spacetime. If $\xi = \xi_{A_1 \ldots A_n}$ is an n-covariant $S^2_{u,v}$-tensor on the Schwarzschild manifold $(\mathcal{M}, g)$, then

$$\begin{aligned}
\nabla\!\!\!/_3\nabla\!\!\!/_B\xi_{A_1\ldots A_n} - \nabla\!\!\!/_B\nabla\!\!\!/_3\xi_{A_1\ldots A_n} &= -\tfrac{1}{2}\operatorname{tr}\underline{\chi}\nabla\!\!\!/_B\xi_{A_1\ldots A_n},\\
\nabla\!\!\!/_4\nabla\!\!\!/_B\xi_{A_1\ldots A_n} - \nabla\!\!\!/_B\nabla\!\!\!/_4\xi_{A_1\ldots A_n} &= -\tfrac{1}{2}\operatorname{tr}\chi\nabla\!\!\!/_B\xi_{A_1\ldots A_n},\\
\nabla\!\!\!/_3\nabla\!\!\!/_4\xi_{A_1\ldots A_n} - \nabla\!\!\!/_4\nabla\!\!\!/_3\xi_{A_1\ldots A_n} &= \widehat{\omega}\nabla\!\!\!/_3\xi_{A_1\ldots A_n} - \widehat{\underline{\omega}}\nabla\!\!\!/_4\xi_{A_1\ldots A_n}.
\end{aligned}$$

In particular, we have

$$[\nabla\!\!\!/_4, r\nabla\!\!\!/_A]\xi = 0, \quad [\nabla\!\!\!/_3, r\nabla\!\!\!/_A]\xi = 0, \quad [\Omega\nabla\!\!\!/_3, \Omega\nabla\!\!\!/_4]\xi = 0.$$

Finally, we introduce a shorthand notation for i angular derivatives acting on a symmetric traceless $S^2_{u,v}$-tensor $(i \geqslant 1)$. The crucial feature of these $\mathcal{A}^{[i]}$ is that they commute trivially with $\nabla\!\!\!/_3$ and $\nabla\!\!\!/_4$. We define

$$\mathcal{A}^{[0]} = 1, \text{ and then, inductively, } \quad \mathcal{A}^{[2i+1]} = r\mathcal{D}\!\!\!/_2\mathcal{A}^{[2i]} \text{ and } \mathcal{A}^{[2i+2]} = r^2\mathcal{D}\!\!\!/_2^\star\mathcal{D}\!\!\!/_2\mathcal{A}^{[2i]}. \tag{103}$$

4.4. The $\ell=0,1$ spherical harmonics and elliptic estimates on spheres

We collect in this final subsection some useful properties which require isolating the $\ell=0,1$ angular frequencies of a tensor. More specifically, after defining notation in §4.4.1, we shall recall in §4.4.2 the classical $\ell=0,1$ spherical harmonics and define what it means for $S^2_{u,v}$ tensors of various types to be supported on angular frequencies $\ell\geqslant 2$. This will then allow us to infer in §4.4.3 some useful elliptic estimates on spheres for such tensors.

4.4.1. Norms on spheres

Let (θ,ϕ) denote standard spherical coordinates as in §4.1.2 where the spherical metric takes the form (89).

We define the following pointwise norm for $S^2_{u,v}$-tensors $\xi_{A_1\dots A_n}$ of rank n:

$$|\xi|^2 := \not{g}^{A_1B_1}\dots\not{g}^{A_nB_n}\xi_{A_1\dots A_n}\xi_{B_1\dots B_n}. \tag{104}$$

We also define the $L^2(S^2_{u,v})$-norm

$$\|\xi\|^2_{S^2_{u,v}} := \int_{S^2_{u,v}} r^2(u,v)\sin\theta\, d\theta\, d\phi\, |\xi|^2 \tag{105}$$

and note that([18])

$$\|r^{-1}\cdot\xi\|^2_{S^2_{u,v}} = \int_{S^2_{u,v}} \sin\theta\, d\theta\, d\phi\, |\xi|^2. \tag{106}$$

4.4.2. The $\ell=0,1$ spherical harmonics and tensors supported on $\ell\geqslant 2$

Recall the well-known spherical harmonics Y^ℓ_m (where $\ell\in\mathbb{N}^0$ and $m\in\{-\ell,\dots,\ell\}$ admissible for fixed ℓ) on the unit sphere. The $\ell=0,1$ spherical harmonics are given explicitly by

$$Y^{\ell=0}_{m=0} = \frac{1}{\sqrt{4\pi}} \tag{107}$$

and

$$Y^{\ell=1}_{m=0} = \sqrt{\frac{3}{4\pi}}\cos\theta, \quad Y^{\ell=1}_{m=-1} = \sqrt{\frac{3}{4\pi}}\sin\theta\cos\phi, \quad Y^{\ell=1}_{m=1} = \sqrt{\frac{3}{4\pi}}\sin\theta\sin\phi. \tag{108}$$

We have that the above family is orthogonal with respect to the standard inner product on the sphere, and any arbitrary function $f\in L^2(S^2)$ can be expanded uniquely with respect to such a basis.

([18]) We will often write quantities in this form, as it is easier to read off the decay.

Definition 4.1. We say that a function f on $\mathcal{M}$ is *supported on* $\ell\geqslant 2$ if the projections

$$\int \sin\theta\, d\theta\, d\phi\, f\cdot Y^{\ell}_{m} = 0$$

vanish for (107) and (108). Any function f can be uniquely decomposed orthogonally as

$$f = c(u,v)Y^{\ell=0}_{m=0} + \sum_{i=-1}^{1} c_i(u,v)Y^{\ell=1}_{m=i}(\theta,\phi) + f_{\ell\geqslant 2},$$

where $f_{\ell\geqslant 2}$ is supported on $\ell\geqslant 2$. The functions $c(u,v)$ and $c_i(u,v)$ inherit regularity from f.

Recall that an arbitrary 1-form ξ on S^2 has a unique representation $\xi = r\not{\mathcal{D}}^\star_1(f,g)$ in terms of two unique functions f and g on the unit sphere, both with vanishing mean. We can use this to define an analogous decomposition for $S^2_{u,v}$ 1-forms on $\mathcal{M}$. We then have the following definition.

Definition 4.2. We say that a smooth $S^2_{u,v}$ 1-form ξ on $\mathcal{M}$ is *supported on* $\ell\geqslant 2$ if the functions f and g in the unique representation

$$\xi = r\not{\mathcal{D}}^\star_1(f,g)$$

are supported on $\ell\geqslant 2$. Any smooth $S^2_{u,v}$ 1-form ξ on $\mathcal{M}$ can be uniquely decomposed orthogonally as

$$\xi = \xi_{\ell=1} + \xi_{\ell\geqslant 2},$$

where the two scalar functions

$$(r\,\not{\mathrm{div}}\,\xi_{\ell=1}, r\,\not{\mathrm{curl}}\,\xi_{\ell=1})$$

are in the span of (108) and $\xi_{\ell\geqslant 2}$ is supported on $\ell\geqslant 2$.

For symmetric traceless $S^2_{u,v}$ 2-tensors, we have the following result.

PROPOSITION 4.4.1. *Let ξ be a smooth symmetric traceless $S^2_{u,v}$ 2-tensor. Then, ξ can be uniquely represented as*

$$\xi = r^2\not{\mathcal{D}}^\star_2\not{\mathcal{D}}^\star_1(f,g),$$

where f and g are supported on $\ell\geqslant 2$. In this sense, any symmetric traceless 2-tensor on S^2 is supported on $\ell\geqslant 2$.

Proposition 4.4.1 follows immediately by duality considerations from the following lemma concerning the angular operator

$$\mathcal{T} = r^2 \not{\mathcal{D}}_2^\star \not{\mathcal{D}}_1^\star, \tag{109}$$

which, for fixed u and v can be considered as an operator on the unit sphere([19]) which maps a pair of functions $(f_1(\theta,\phi), f_2(\theta,\phi))$ to a symmetric traceless tensor on S^2. Note that its adjoint, $r^2 \not{\mathcal{D}}_1 \not{\mathcal{D}}_2$, has trivial kernel in L^2.

For the computations in the following lemma we regard $\mathcal{T}$ as an operator defined on pairs of smooth functions, which are dense in $L^2(S^2)$.

Lemma 4.4.1. *The kernel of $\mathcal{T}$ is finite-dimensional. More precisely, if the pair of functions (f_1, f_2) is in the kernel, then*

$$f_1 = cY^{\ell=0}_{m=0} + \sum_{i=-1}^{1} c_i Y^{\ell=1}_{m=i}(\theta,\phi) \quad \textit{and} \quad f_2 = \tilde{c}Y^{\ell=0}_{m=0} + \sum_{i=-1}^{1} \tilde{c}_i Y^{\ell=1}_{m=i}(\theta,\phi)$$

for constants c, c_i, $\tilde{c}$, $\tilde{c}_i$ and $i=-1,0,1$, where Y^ℓ_m are the spherical harmonics defined by (107) and (108).

Proof. If (f_1, f_2) is in the kernel, then clearly

$$\int_{S^2} \sin\theta \, d\theta \, d\phi \, [\not{\mathcal{D}}_2^\star \not{\mathcal{D}}_1^\star (f_1, f_2) \cdot \not{\mathcal{D}}_2^\star \not{\mathcal{D}}_1^\star (f_1, f_2)] = 0.$$

Integrating by parts and using that $\not{\mathcal{D}}_2 \not{\mathcal{D}}_2^\star = -\frac{1}{2}\not{\Delta} - \frac{1}{2}K$ ($K = 1/r^2$ is the Gauss curvature and $\not{\Delta}$ the covariant Laplacian), as well as $\not{\mathcal{D}}_1^\star \not{\mathcal{D}}_1 = -\not{\Delta} + K$ and $\not{\mathcal{D}}_1 \not{\mathcal{D}}_1^\star = -\not{\Delta}$, we find

$$\int_{S^2} \sin\theta \, d\theta \, d\phi \left[\frac{1}{2} f_1 \cdot \not{\Delta}\not{\Delta} f_1 + \frac{1}{r^2} f_1 \not{\Delta} f_1 + \frac{1}{2} f_2 \cdot \not{\Delta}\not{\Delta} f_2 + \frac{1}{r^2} f_2 \not{\Delta} f_2 \right] = 0,$$

and hence

$$\int_{S^2} \sin\theta \, d\theta \, d\phi \left[\frac{1}{2} |\not{\Delta} f_1|^2 - \frac{1}{r^2} |\not{\nabla} f_1|^2 + \frac{1}{2} |\not{\Delta} f_2|^2 - \frac{1}{r^2} |\not{\nabla} f_2|^2 \right] = 0,$$

which can be written

$$\int_{S^2} \sin\theta \, d\theta \, d\phi \left[\frac{1}{2} \left| \not{\Delta} f_1 + \frac{2 f_1}{r^2} \right|^2 + \frac{1}{r^2} |\not{\nabla} f_1|^2 - \frac{2(f_1)^2}{r^4} + \frac{1}{2} \left| \not{\Delta} f_2 + \frac{2 f_2}{r^2} \right|^2 + \frac{1}{r^2} |\not{\nabla} f_2|^2 - 2\frac{(f_2)^2}{r^4} \right] = 0.$$

Clearly, the constant solutions $f_1 = c$ and $f_2 = \tilde{c}$ satisfy this (and are obviously in the kernel). If we assume both f_1 and f_2 to have mean value zero, we see using the Poincaré

([19]) More precisely, it acts on the round spheres $S^2_{u,v}$ which have been rescaled (this is the reason for the factor r^2) to have unit radius.

inequality on the sphere that the only functions satisfying the above condition are the $\ell{=}1$ modes. Finally, one checks directly that the $\ell{=}1$ modes are indeed in the kernel: In components, the equation for f_1,

$$\nabla\!\!\!/_A \nabla\!\!\!/_B Y_m^{\ell=1} + \nabla\!\!\!/_B \nabla\!\!\!/_A Y_m^{\ell=1} = -2 g\!\!\!/_{AB} Y_m^{\ell=1},$$

reads (using ${\Gamma^\theta}_{\phi\phi}{=}-\sin\theta\cos\theta$ and ${\Gamma^\phi}_{\theta\phi}{=}\cos\theta/\sin\theta$ in standard coordinates)

$$\begin{aligned}(2\partial_\theta^2 + 2) Y_m^{\ell=1} &= 0\\ \partial_\theta\partial_\phi Y_m^{\ell=1} - \frac{\cos\theta}{\sin\theta}\partial_\phi Y_m^{\ell=1} &= 0\\ \partial_\phi^2 Y_m^{\ell=1} + \sin\theta\cos\theta\,\partial_\theta Y_m^{\ell=1} + \sin^2\theta Y_m^{\ell=1} &= 0\end{aligned}$$

and these identities are easily verified. The computation for f_2 is similar or can be inferred by duality. □

In particular, we have the following corollary.

Corollary 4.1. *Let ξ be a smooth symmetric traceless $S^2_{u,v}$ 2-tensor on $\mathcal{M}$. Then,*

$$\int_{S^2_{u,v}} \sin\theta\, d\theta\, d\phi\, \mathcal{D}\!\!\!/_1 \mathcal{D}\!\!\!/_2 \xi \cdot (c + c_i Y^{\ell=1}_{m=i}, \tilde{c} + \tilde{c}_i Y^{\ell=1}_{m=i}) = 0.$$

for any choice of constants c, c_i, $\tilde{c}$ and $\tilde{c}_i$.

Note that this, in particular, means that, if ξ is a symmetric traceless $S^2_{u,v}$ 2-tensor, then the scalars $\mathrm{d}\!\!\!/\mathrm{iv}\, \mathrm{d}\!\!\!/\mathrm{iv}\, \xi$ and $\mathrm{c}\!\!\!/\mathrm{url}\, \mathrm{d}\!\!\!/\mathrm{iv}\, \xi$ are supported on $\ell \geqslant 2$.

4.4.3. Elliptic estimates and positivity for angular operators on $S^2_{u,v}$-tensors

We end with a discussion of elliptic estimates giving positivity for various angular operators acting on $S^2_{u,v}$ tensors supported on $\ell \geqslant 2$.

We first give an estimate associated with the operator $\mathcal{T}$ from (109) acting on pairs of scalar functions supported on $\ell \geqslant 2$.

Proposition 4.4.2. *Let (f_1, f_2) be a pair of functions on $S^2_{u,v}$ supported on $\ell \geqslant 2$. Then, we have the elliptic estimate*

$$\sum_{i=0}^{2} \int_{S^2_{u,v}} \sin\theta\, d\theta\, d\phi\, (|r^i \nabla\!\!\!/^i f_1|^2 + |r^i \nabla\!\!\!/^i f_2|^2) \lesssim \int_{S^2_{u,v}} \sin\theta\, d\theta\, d\phi\, |r^2 \mathcal{D}\!\!\!/_2^\star \mathcal{D}\!\!\!/_1^\star (f_1, f_2)|^2.$$

Proof. This follows immediately revisiting the computation of Lemma 4.4.1. □

We next give identities (in the formulas below, recall that $K=1/r^2$) associated with operators acting on symmetric traceless 2-tensors and 1-forms.

Proposition 4.4.3. *Let ξ be a smooth symmetric traceless $S^2_{u,v}$ 2-tensor on $\mathcal{M}$. Then,*

$$\int_{S^2_{u,v}} \sin\theta\, d\theta\, d\phi\, (|\nabla\!\!\!/\,\xi|^2+2K|\xi|^2) = 2\int_{S^2_{u,v}} \sin\theta\, d\theta\, d\phi\, |\mathrm{d}\!\!\!/\mathrm{iv}\,\xi|^2, \tag{110}$$

$$\int_{S^2_{u,v}} \sin\theta\, d\theta\, d\phi\, |\mathcal{D}\!\!\!/_2^\star \,\mathrm{d}\!\!\!/\mathrm{iv}\,\xi|^2 = \int_{S^2_{u,v}} \sin\theta\, d\theta\, d\phi\, \left(\tfrac{1}{4}|\Delta\!\!\!/\,\xi|^2+K|\nabla\!\!\!/\,\xi|^2+K^2|\xi|^2\right). \tag{111}$$

Now, let η be a smooth $S^2_{u,v}$ 1-form on $\mathcal{M}$. Then, we have

$$\|\mathcal{D}\!\!\!/_2^\star\mathcal{D}\!\!\!/_1^\star\mathcal{D}\!\!\!/_1\eta\|^2_{S^2_{u,v}} = \|2\mathcal{D}\!\!\!/_2^\star\,\mathrm{d}\!\!\!/\mathrm{iv}\,\mathcal{D}\!\!\!/_2^\star\eta\|^2_{S^2_{u,v}} + \|2K\mathcal{D}\!\!\!/_2^\star\eta\|^2_{S^2_{u,v}} + 8K\|\,\mathrm{d}\!\!\!/\mathrm{iv}\,\mathcal{D}\!\!\!/_2^\star\eta\|^2_{S^2_{u,v}}. \tag{112}$$

Proof. See [14] for the first and note $\mathcal{D}\!\!\!/_2^\star\,\mathrm{d}\!\!\!/\mathrm{iv}\,\xi = \left(-\frac{1}{2}\Delta\!\!\!/+K\right)\xi$ for the second. For (112) observe that $\mathcal{D}\!\!\!/_2^\star\mathcal{D}\!\!\!/_1^\star\mathcal{D}\!\!\!/_1\eta = \mathcal{D}\!\!\!/_2^\star(-\Delta\!\!\!/+K)\eta = 2\mathcal{D}\!\!\!/_2^\star\,\mathrm{d}\!\!\!/\mathrm{iv}\,\mathcal{D}\!\!\!/_2^\star\eta + 2K\mathcal{D}\!\!\!/_2^\star\eta$ and integrate the cross-term by parts. □

Remark 4.2. The identities (110) and (111) can be paraphrased as saying that the operator $\mathcal{A}^{[n]}$ defined in (103) acting on symmetric traceless $S^2_{u,v}$ 2-tensors is uniformly elliptic and positive definite.

The identity (112), on the other hand, when combined with Proposition 4.4.2 and the identity (111) leads to the following corollary, which can be thought of as an elliptic estimate associated with the operator $\mathcal{D}\!\!\!/_2^\star$ acting on $S^2_{u,v}$ 1-forms η supported on $\ell\geqslant 2$.

Corollary 4.2. *Let η be a smooth $S^2_{u,v}$ 1-form supported on $\ell\geqslant 2$. Then, we have*

$$\sum_{i=0}^{3}\int_{S^2} \sin\theta\, d\theta\, d\phi\, |r^i\nabla\!\!\!/^{\,i}\eta|^2 \lesssim \int_{S^2} \sin\theta\, d\theta\, d\phi\, |\mathcal{A}^{[2]}\mathcal{D}\!\!\!/_2^\star\eta|^2.$$

The statement remains true replacing 3 *by* 1 *in the sum on the left and removing* $\mathcal{A}^{[2]}$ *on the right-hand side.*

We also remark, at this point already, the following result.

Proposition 4.4.4. *Let ξ be a smooth symmetric traceless $S^2_{u,v}$ 2-tensor. Then, we have the estimate*

$$-\int_{S^2} \sin\theta\, d\theta\, d\phi \left(\left(\Delta\!\!\!/-\frac{4}{r^2}\right)\xi\right)_{AB}\xi^{AB} \geqslant \frac{6}{r^2}\int_{S^2} \sin\theta\, d\theta\, d\phi\, |\xi|^2. \tag{113}$$

Proof. We only outline the proof. The desired estimate follows from

$$-\int_{S^2} \sin\theta\, d\theta\, d\phi\, \Delta\!\!\!/\,\xi\cdot\xi = \int_{S^2} \sin\theta\, d\theta\, d\phi\, |\nabla\!\!\!/\,\xi|^2 \geqslant \frac{2}{r^2}\int_{S^2} \sin\theta\, d\theta\, d\phi\, |\xi|^2, \tag{114}$$

which holds for any symmetric traceless $S^2_{u,v}$ 2-tensor ξ. The latter can in turn be shown by representing the tensor ξ as $\xi = \mathcal{D}\!\!\!/^{\star}_2 \mathcal{D}\!\!\!/^{\star}_1(f,g)$ for unique functions f and g supported on $\ell \geqslant 2$ as in Proposition 4.4.1, so in particular

$$-\int_{S^2} \sin\theta\, d\theta\, d\phi\, \Delta\!\!\!/\, f\cdot f \geqslant 6r^{-2}\int_{S^2} \sin\theta\, d\theta\, d\phi |f|^2,$$

and the same estimate for g) and diligently integrating by parts using the properties of spherical harmonics (in particular, their orthogonality). □

5. The equations of linearised gravity around Schwarzschild

In this section, we will present the equations of linearised gravity around Schwarzschild. We begin in §5.1 with a guide to the formal derivation of this system from the equations of §3. The complete system of linearised gravity is then presented in §5.2.

5.1. A guide to the formal derivation from the equations of §3.3

We give in this section a formal derivation of the system from the equations of §3. The reader willing to take the system of linear gravity on faith can skip to §5.2.

5.1.1. Preliminaries

We first identify the general manifold $\boldsymbol{\mathcal{M}}$ and its coordinates $(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{\theta}^1, \boldsymbol{\theta}^2)$ of §3.1 with the interior of the Schwarzschild manifold $\mathcal{M}^\circ$ and its underlying Eddington–Finkelstein double null coordinates $(u, v, \theta^1, \theta^2)$.

On $\mathcal{M}^\circ$, we consider a 1-parameter family of Lorentzian metrics $\boldsymbol{g}(\varepsilon)$

$$\boldsymbol{g}(\varepsilon) \doteq -4\boldsymbol{\Omega}^2(\varepsilon)\, du\, dv + \boldsymbol{g\!\!\!/}_{CD}(\varepsilon)(d\theta^C - \boldsymbol{b}^C(\varepsilon)\, dv)(d\theta^D - \boldsymbol{b}^D(\varepsilon)\, dv), \tag{115}$$

such that $\boldsymbol{g}(0) = g_S$ expressed in the Eddington–Finkelstein double null form, i.e.

$$\boldsymbol{\Omega}^2(0) = \Omega^2 = 1 - \frac{2M}{r}, \quad \boldsymbol{b}(0) = 0, \quad \boldsymbol{g\!\!\!/}_{CD} = r^2\gamma_{CD}.$$

We assume moreover that the family is *smooth in the extended sense*; by this, we mean it defines a smooth family of smooth metrics on the manifold $\mathcal{M}$ of §4.1.1.

This can be characterized explicitly in Eddington–Finkelstein double null coordinates as follows: We require that the function $\boldsymbol{\Omega}^2(\varepsilon)$, the symmetric $S^2_{u,v}$ 2-tensor $\boldsymbol{\not{g}}_{CD}(\varepsilon)$ and the $S^2_{u,v}$ 1-form $\boldsymbol{b}^C(\varepsilon)$ are smooth functions of the double null Eddington–Finkelstein coordinates on the interior $\mathcal{M}^o$, and that for any $n_1, n_2, n_3 \in \mathbb{N}_0$ and in any spherical coordinate chart the functions

$$(e^{u/2M}\partial_u)^{n_1}(\partial_v)^{n_2}(\partial_{\theta_A})^{n_3}(\boldsymbol{\Omega}^2(\varepsilon)e^{u/2M}), \tag{116}$$

$$(e^{u/2M}\partial_u)^{n_1}(\partial_v)^{n_2}(\partial_{\theta_A})^{n_3}\boldsymbol{\not{g}}_{CD}(\varepsilon), \tag{117}$$

$$(e^{u/2M}\partial_u)^{n_1}(\partial_v)^{n_2}(\partial_{\theta_A})^{n_3}\boldsymbol{b}^C(\varepsilon) \tag{118}$$

extend continuously to the boundary $\mathcal{H}^+$ (in particular, the limit $u \to \infty$ of the above quantities exists for any fixed v, θ_1, θ_2).

We similarly say a function $\boldsymbol{f}$ of the Eddington–Finkelstein coordinates $(u, v, \theta^1, \theta^2)$ to be *smooth in the extended sense* if $\boldsymbol{f}$ defines a smooth function on $\mathcal{M}$. Again, we can characterize this directly by requiring that f is a smooth function of its coordinates on $\mathcal{M}^o$, and moreover

$$(e^{u/2M}\partial_u)^{n_1}(\partial_v)^{n_2}(\partial_{\theta_A})^{n_3}\boldsymbol{f}$$

extend continuously to the boundary $\mathcal{H}^+$. Symmetric tensorfields and 1-forms which are smooth in the extended sense are defined completely analogously. We sometimes use the phrase that a function (or $S^2_{u,v}$ 1-form or symmetric traceless $S^2_{u,v}$ tensor) *extends regularly to* $\mathcal{H}^+$ to mean that it is smooth in the extended sense.

In view of the general discussion in §3.1 associated with (115) is a family of null frames

$$\boldsymbol{\mathcal{N}}_{EF} = (\boldsymbol{\Omega}^{-1}(\varepsilon)\partial_u, \boldsymbol{\Omega}^{-1}(\varepsilon)(\partial_v + \boldsymbol{b}^A(\varepsilon)\partial_{\theta^A}), e_1, e_2). \tag{119}$$

We can hence define the Ricci coefficients and curvature components for the family of metrics with respect to these frames as in §3.2, and formally expand them in powers of ε.

Note that the frame (119) does not itself extend smoothly to the event horizon $\mathcal{H}^+$, in the sense that given a smooth (in the extended sense) function $\boldsymbol{f}$ of the Eddington–Finkelstein coordinates, the expression $\boldsymbol{\Omega}^{-1}(\varepsilon)\partial_u\boldsymbol{f}$ does not extend continuously to $\mathcal{H}^+$. It is easily seen on the other hand that the rescaled frame

$$\boldsymbol{\mathcal{N}}_{EF^\star} = (\boldsymbol{\Omega}^{-2}(\varepsilon)\partial_u, \partial_v + \boldsymbol{b}^A(\varepsilon)\partial_{\theta^A}, e_1, e_2) \tag{120}$$

is smooth, in the extended sense in that any element of the frame applied to a smooth (in the extended sense) function produces a smooth (in the extended sense) function.

5.1.2. Outline of the linearisation procedure

We will now linearise the smooth 1-parameter family of metrics (115), i.e. in particular we shall expand the equations of §3.3.1 and §3.3.2 (with the Ricci and curvature components defined with respect to the family of frames (119)) to first order in ε.

We begin by recalling the derivative operators $\underline{\boldsymbol{D}}$ and $\boldsymbol{D}$ associated with (115), which (when acting on functions) read in coordinates:

$$\underline{\boldsymbol{D}} = \partial_u, \boldsymbol{D} = \partial_v + \boldsymbol{b}^A(\varepsilon)\boldsymbol{e}_A, \boldsymbol{e}_A = \frac{\partial}{\partial \theta^A}. \tag{121}$$

To formally linearise the full system of equations of §3.3, we invoke the following general notation: Geometric quantities defined with respect to the full metric (115) are written in bold (e.g. $\boldsymbol{\chi}$). Their Schwarzschild value (i.e. the quantity defined with respect to the Schwarzschild metric) is written without any subscript and their linear perturbation with a superscript "(1)". For instance, we write (recall $\Omega^2 = 1 - 2M/r$)

$$\begin{aligned}
\boldsymbol{\Omega} &\equiv \Omega + \varepsilon \cdot \overset{(1)}{\Omega}, \\
\boldsymbol{\not{g}}_{AB} &\equiv \not{g}_{AB} + \varepsilon \cdot \overset{(1)}{\not{g}}_{AB}, \\
\boldsymbol{\Omega}\, \mathbf{tr}\, \boldsymbol{\chi} &\equiv (\Omega \operatorname{tr} \chi) + \varepsilon \cdot (\Omega \overset{(1)}{\operatorname{tr}} \chi) = \frac{2}{r}\left(1 - \frac{2M}{r}\right) + \varepsilon \cdot (\Omega \overset{(1)}{\operatorname{tr}} \chi), \\
\boldsymbol{\Omega}\, \mathbf{tr}\, \underline{\boldsymbol{\chi}} &\equiv (\Omega \operatorname{tr} \underline{\chi}) + \varepsilon \cdot (\Omega \overset{(1)}{\operatorname{tr}} \underline{\chi}) = -\frac{2}{r}\left(1 - \frac{2M}{r}\right) + \varepsilon \cdot (\Omega \overset{(1)}{\operatorname{tr}} \underline{\chi}), \\
\boldsymbol{\omega} := \boldsymbol{\Omega}\widehat{\boldsymbol{\omega}} &\equiv \Omega \widehat{\omega} + \varepsilon \cdot \overset{(1)}{\omega}, \\
\underline{\boldsymbol{\omega}} := \boldsymbol{\Omega}\underline{\widehat{\boldsymbol{\omega}}} &\equiv \Omega \underline{\widehat{\omega}} + \varepsilon \cdot \overset{(1)}{\underline{\omega}}, \\
\boldsymbol{\varrho} &\equiv -\frac{2M}{r^3} + \varepsilon \cdot \overset{(1)}{\varrho},
\end{aligned} \tag{122}$$

which covers all metric, Ricci and curvature coefficients which have non-trivial Schwarzschild values, cf. (96), and

$$\boldsymbol{b} \equiv 0 + \varepsilon \cdot \overset{(1)}{b}, \quad \widehat{\boldsymbol{\chi}} \equiv 0 + \varepsilon \cdot \overset{(1)}{\widehat{\chi}}, \quad \underline{\widehat{\boldsymbol{\chi}}} \equiv 0 + \varepsilon \cdot \overset{(1)}{\underline{\widehat{\chi}}}, \quad \boldsymbol{\eta} \equiv 0 + \varepsilon \cdot \overset{(1)}{\eta}, \quad \underline{\boldsymbol{\eta}} \equiv 0 + \varepsilon \cdot \overset{(1)}{\underline{\eta}}, \tag{123}$$

as well as

$$\boldsymbol{\alpha} \equiv 0 + \varepsilon \cdot \overset{(1)}{\alpha}, \quad \underline{\boldsymbol{\alpha}} \equiv 0 + \varepsilon \cdot \overset{(1)}{\underline{\alpha}}, \quad \boldsymbol{\beta} \equiv 0 + \varepsilon \cdot \overset{(1)}{\beta}, \quad \underline{\boldsymbol{\beta}} \equiv 0 + \varepsilon \cdot \overset{(1)}{\underline{\beta}}, \quad \boldsymbol{\sigma} \equiv 0 + \varepsilon \cdot \overset{(1)}{\sigma}, \tag{124}$$

for the coefficients which have vanishing Schwarzschild values. In the above, $\equiv$ means to first order in ε.

The linearised equations are now obtained simply by expanding the equations of §3.3 to order ε leading to the equations presented in §5.2.

To give a non-trivial example, we consider the second equation of (76), which, in view of formula for the projected Lie-derivative,

$$(\boldsymbol{D}\boldsymbol{\not{g}})_{AB} = \partial_v(\boldsymbol{\not{g}}_{AB}) + (\boldsymbol{\not{\nabla}}_A \boldsymbol{b}_B + \boldsymbol{\not{\nabla}}_B \boldsymbol{b}_A), \tag{125}$$

with ∂_v acting on the components of $\boldsymbol{\not{g}}$ on the right, can be written as

$$\partial_v(\boldsymbol{\not{g}}_{AB}) = 2\boldsymbol{\Omega}\widehat{\boldsymbol{\chi}}_{AB} + \boldsymbol{\not{g}}_{AB}\boldsymbol{\Omega}\,\mathbf{tr}\,\boldsymbol{\chi} + 2(\boldsymbol{\not{\mathcal{D}}}_2^\star \boldsymbol{b})_{AB} - \boldsymbol{\not{g}}_{AB}\,\mathbf{d\not{i}v}\,\boldsymbol{b}.$$

Note that the operator ∂_v on the left-hand side coincides with the Schwarzschild differential operator D introduced in §4.3.1. Expanding in terms of powers of ε as above, we find

$$\partial_v(\overset{(1)}{\not{g}}_{AB}) = 2\Omega\overset{(1)}{\widehat{\chi}}_{AB} + 2(\not{\mathcal{D}}_2^\star \overset{(1)}{b})_{AB} + \not{g}_{AB}((\Omega\overset{(1)}{\mathrm{tr}}\chi) - \mathrm{d\not{i}v}\,\overset{(1)}{b}) + \Omega\,\mathrm{tr}\,\chi\,\overset{(1)}{\not{g}}_{AB}, \tag{126}$$

where the unbolded operators are the Schwarzschild differential operators of §4.3.1. We now decompose

$$\overset{(1)}{\not{g}}_{AB} = \overset{(1)}{\hat{\not{g}}}_{AB} + \tfrac{1}{2}\not{g}_{AB}\cdot\mathrm{tr}_{\not{g}}\overset{(1)}{\not{g}}, \tag{127}$$

where $\overset{(1)}{\hat{\not{g}}}$ is traceless with respect to the round metric $\not{g}=r^2\gamma$. Note that this implies, to linear order,

$$\mathbf{det}\boldsymbol{\not{g}} \equiv \det\not{g}(1+\varepsilon\cdot\mathrm{tr}_{\not{g}}\overset{(1)}{\not{g}}) \quad \text{and we therefore define } \overset{(1)}{\sqrt{\not{g}}} := \tfrac{1}{2}\sqrt{\not{g}}\cdot\mathrm{tr}_{\not{g}}\overset{(1)}{\not{g}}, \tag{128}$$

using the notation $\sqrt{\not{g}} := \sqrt{\det\not{g}}$. Finally, upon contracting (126) with the inverse $\not{g}^{AB}$, we find

$$\partial_v(\mathrm{tr}_{\not{g}}\overset{(1)}{\not{g}}) = 2(\Omega\overset{(1)}{\mathrm{tr}}\chi) - 2\,\mathrm{d\not{i}v}\,\overset{(1)}{b} \quad \text{and} \quad \sqrt{\not{g}}\partial_v\big(\overset{(1)}{\hat{\not{g}}}_{AB}\big(\sqrt{\not{g}}\big)^{-1}\big) = 2\Omega\overset{(1)}{\widehat{\chi}}_{AB} + 2(\not{\mathcal{D}}_2^\star\overset{(1)}{b})_{AB},$$

leading directly to (131) and (132). The linearisation of the first equation of (76) is completely analogous.

The remaining equations for the metric components and Ricci coefficients are much simpler to linearise, since they are either scalar equations with spherically symmetric background values, or tensorial equations where the background quantity vanishes, in which case one can simply replace all operators and coefficients in the equation by their Schwarzschild ones (see §4.3).

We give two more examples: The non-linear Bianchi equation for $\boldsymbol{\varrho}$ in the 4-direction, which linearises via

$$\partial_v(\varrho+\varepsilon\overset{(1)}{\varrho}) + \varepsilon\overset{(1)}{b}{}^A\frac{\partial}{\partial\theta^A}(\varrho+\varepsilon\overset{(1)}{\varrho}) + \frac{3}{2}((\Omega\,\mathrm{tr}\,\chi) + \varepsilon(\Omega\overset{(1)}{\mathrm{tr}}\chi))(\varrho+\varepsilon\overset{(1)}{\varrho}) = \varepsilon\,\mathrm{d\not{i}v}\,\overset{(1)}{\beta} + \mathcal{O}(\varepsilon^2)$$

to produce the linearised equation (151) in §5.2.4 below, and the Bianchi equation for β in the 3-direction, which using the linearisation formula

$$[\boldsymbol{\mathcal{D}}_1^\star(-\boldsymbol{\varrho},\boldsymbol{\sigma})]_A \equiv -\frac{\partial}{\partial\theta^A}(\varrho+\varepsilon\cdot\overset{(1)}{\varrho})+\not\epsilon_A{}^B\frac{\partial}{\partial\theta^B}(0+\varepsilon\cdot\overset{(1)}{\sigma})\equiv\varepsilon[\mathcal{D}_1^\star(-\overset{(1)}{\varrho},\overset{(1)}{\sigma})]_A,$$

leads to (150).

Remark 5.1. We emphasise that, for the connection coefficients $\boldsymbol{\chi}$ and $\underline{\boldsymbol{\chi}}$, we linearise the equations for their trace-free and (weighted) trace parts $\widehat{\boldsymbol{\chi}}$, $\boldsymbol{\Omega}\,\mathbf{tr}\,\boldsymbol{\chi}$ and $\widehat{\underline{\boldsymbol{\chi}}}$, $\boldsymbol{\Omega}\mathrm{tr}\underline{\boldsymbol{\chi}}$ (taken with respect to the metric $\boldsymbol{g}$); cf. (77)–(82). This is different from linearising the connection coefficients $\underline{\boldsymbol{\chi}}$, and then splitting the resulting tensor into traceless and trace part with respect to the background spherical metric $\not g$, although the two are of course easily related. Consequently, when writing expressions like $(\Omega\,\overset{(1)}{\mathrm{tr}}\,\chi)$, it is understood that this is a weighted linearised trace, not taking an actual trace of a tensor $\overset{(1)}{\chi}$.

5.1.3. Regular quantities at the horizon

While it is indeed convenient to perform the linearisation computation in Eddington–Finkelstein coordinates and with respect to the associated frame (119) as indicated above, one should keep in mind that one will eventually need to consider *regular* quantities. This will require rescaling appropriately some of the linearised quantities near the event horizon.

To understand the correct rescalings, as an example, consider the connection coefficient $\boldsymbol{\chi}$. Since the metric (115) is smooth in the extended sense with respect to our Eddington–Finkelstein differential structure, it is $e^{-u/4M}\boldsymbol{\chi}$ which extends smoothly to the event horizon, and hence it is the linearisation of this quantitity that one should consider. Equivalently, since $\boldsymbol{\Omega}(\varepsilon)e^{u/4M}$ is a function which is smooth in the extended sense, we should consider the linearisation of $\boldsymbol{\Omega\chi}$ near the horizon. Decomposing into the traceless part $\boldsymbol{\Omega}\widehat{\boldsymbol{\chi}}$ and the trace $\boldsymbol{\Omega}\,\mathbf{tr}\,\boldsymbol{\chi}$, we have

$$\boldsymbol{\Omega}\widehat{\boldsymbol{\chi}}\equiv 0+\Omega\overset{(1)}{\widehat{\chi}} \quad\text{and}\quad (\boldsymbol{\Omega}\,\mathbf{tr}\,\boldsymbol{\chi})\equiv\frac{2}{r}\left(1-\frac{2M}{r}\right)+(\Omega\,\overset{(1)}{\mathrm{tr}}\,\chi),$$

and hence the weighted linearised quantities $\Omega\overset{(1)}{\widehat{\chi}}$, as well as $(\Omega\,\overset{(1)}{\mathrm{tr}}\,\chi)$, are the regular linearised quantities that we eventually have to estimate uniformly up to the horizon. Similarly for the metric component $\boldsymbol{\Omega}$, since it is $\boldsymbol{\Omega}e^{u/4M}$, which extends smoothly to $\mathcal{H}^+$, we need to consider the linearisation of

$$\boldsymbol{\Omega}e^{u/4M}\equiv(\Omega+\overset{(1)}{\Omega})e^{u/4M}=(1+\Omega^{-1}\overset{(1)}{\Omega})\Omega e^{u/4M}=\sqrt{\frac{2M}{r}}e^{r/4M}e^{-v/4M}(1+\Omega^{-1}\overset{(1)}{\Omega}).$$

and hence the quantity $\Omega^{-1}\overset{(1)}{\Omega}$. To give a final example, we have

$$\frac{\boldsymbol{\Omega}\,\mathbf{tr}\,\underline{\boldsymbol{\chi}}}{\boldsymbol{\Omega}^2} \equiv \frac{\overset{(1)}{(\Omega\,\mathrm{tr}\,\underline{\chi})}}{\Omega^2} - 2\Omega^{-1}\overset{(1)}{\Omega}\frac{(\Omega\,\mathrm{tr}\,\underline{\chi})}{\Omega^2} = \frac{\overset{(1)}{(\Omega\,\mathrm{tr}\,\underline{\chi})}}{\Omega^2} + \frac{4}{r}\Omega^{-1}\overset{(1)}{\Omega},$$

which extends regularly to $\mathcal{H}^+$, so we need to consider $\Omega^{-2}\overset{(1)}{(\Omega\,\mathrm{tr}\,\underline{\chi})}$ near the horizon.

The full list of rescaled quantities is given by (130) in §5.2.1 below.

5.1.4. Aside: The relation between linearisation in the frames $\boldsymbol{\mathcal{N}}_{\boldsymbol{EF}}$ and $\boldsymbol{\mathcal{N}}_{\boldsymbol{EF^\star}}$

The reader might wonder about the relation between the evolution equations of §5.2 (which we recall arose from linearising metric, Ricci and curvature components in the Eddington–Finkelstein frame $\boldsymbol{\mathcal{N}}_{EF}$ (119), as indicated in the previous section) and the equations one would obtain if one defined these components using the regular frame $\boldsymbol{\mathcal{N}}_{EF^\star}$ (cf. (120)), and then linearised the Einstein equations expressed with respect to that frame.

We collect in this section the formulas allowing oneself to transform the equations from one to another leading to the notion of *linearisation covariance*. To distinguish the components defined with respect to the two frames, we will use the subscripts "EF" and "$EF^\star$" below, which should not be confused with coordinate indices.

For the curvature components we easily see

$$\overset{(1)}{\underline{\alpha}}_{EF^\star} = \Omega^{-2}\,\overset{(1)}{\underline{\alpha}}_{EF}, \qquad \overset{(1)}{\underline{\beta}}_{EF^\star} = \Omega^{-1}\,\overset{(1)}{\underline{\beta}}_{EF}, \qquad \overset{(1)}{\alpha}_{EF^\star} = \Omega^{2}\,\overset{(1)}{\alpha}_{EF}, \qquad \overset{(1)}{\beta}_{EF^\star} = \Omega\overset{(1)}{\beta}_{EF},$$
$$\overset{(1)}{\varrho}_{EF^\star} = \overset{(1)}{\varrho}_{EF}, \qquad \overset{(1)}{\sigma}_{EF^\star} = \overset{(1)}{\sigma}_{EF}.$$

For the Ricci coefficients, we note that

$$\boldsymbol{\chi}_{EF^\star} = \boldsymbol{\Omega}\boldsymbol{\chi}_{EF}, \qquad \widehat{\boldsymbol{\chi}}_{EF^\star} = \boldsymbol{\Omega}\hat{\boldsymbol{\chi}}_{EF}, \qquad \mathrm{tr}\,\boldsymbol{\chi}_{EF^\star} = \boldsymbol{\Omega}\,\mathrm{tr}\,\boldsymbol{\chi}_{EF},$$
$$\underline{\boldsymbol{\chi}}_{EF^\star} = \boldsymbol{\Omega}^{-1}\underline{\boldsymbol{\chi}}_{EF}, \qquad \widehat{\underline{\boldsymbol{\chi}}}_{EF^\star} = \boldsymbol{\Omega}^{-1}\hat{\underline{\boldsymbol{\chi}}}_{EF}, \qquad \mathrm{tr}\,\underline{\boldsymbol{\chi}}_{EF^\star} = \boldsymbol{\Omega}^{-1}\,\mathrm{tr}\,\underline{\boldsymbol{\chi}}_{EF},$$

and hence

$$\overset{(1)}{\widehat{\chi}}_{EF^\star} = \Omega\overset{(1)}{\widehat{\chi}}_{EF}, \qquad \overset{(1)}{\widehat{\underline{\chi}}}_{EF^\star} = \Omega^{-1}\overset{(1)}{\widehat{\underline{\chi}}}_{EF},$$

and

$$\overset{(1)}{(\mathrm{tr}\,\chi)}_{EF^\star} = \overset{(1)}{(\Omega\,\mathrm{tr}\,\chi)}_{EF}, \qquad \overset{(1)}{(\mathrm{tr}\,\underline{\chi})}_{EF^\star} = \Omega^{-2}\overset{(1)}{(\Omega\,\mathrm{tr}\,\underline{\chi})}_{EF} + \frac{4}{r}\Omega^{-1}\overset{(1)}{\Omega}.$$

Also, since $\boldsymbol{\eta}_{EF^\star} = \boldsymbol{\eta}_{EF}$ and $\underline{\boldsymbol{\eta}}_{EF^\star} = \underline{\boldsymbol{\eta}}_{EF}$, we have

$$\overset{(1)}{\eta}_{EF^\star} = \overset{(1)}{\eta}_{EF}, \quad \overset{(1)}{\underline{\eta}}_{EF^\star} = \overset{(1)}{\underline{\eta}}_{EF}, \quad \widehat{\boldsymbol{\omega}}_{EF^\star} = 2\boldsymbol{\omega}_{EF} = 2\boldsymbol{\Omega}\widehat{\boldsymbol{\omega}}_{EF} = \frac{2M}{r^2} + 2\overset{(1)}{\omega}_{EF}, \quad \widehat{\underline{\boldsymbol{\omega}}}_{EF^\star} = 0.$$

Note that the quantity $\overset{(1)}{\underline{\omega}}_{EF^\star}$ is hence automatically zero.

Observe also that the linearised components in the frame $EF^\star$ are automatically regular at the horizon. This is of course consistent with our notion of regular quantities introduced in (130); cf. §5.1.3. Using the formulas above, one may easily reformulate the system of gravitational perturbations of §5.2 as equations for linearised components in the frame $EF^\star$. This yields the same equations as linearising directly the full non-linear equations expressed in the frame $\boldsymbol{\mathcal{N}}_{EF^\star}$ (linearisation covariance).

5.2. The full set of linearised equations

In the following subsections we present the equations arising from the formal linearisation (outlined in §5.1) of the equations of §3.3. These equations are physical space analogues of the equations appearing in Chandrasekhar's [9]. We stress that the system can be studied without reference to the full non-linear Einstein equations. In particular, the discussion below can be read independently of the formal derivation in §5.1.

5.2.1. The complete list of unknowns

The equations will concern a set of quantities

$$\mathscr{S}=(\overset{(1)}{\hat{\not g}},\overset{(1)}{\sqrt{\not g}},\overset{(1)}{\Omega},\overset{(1)}{b},(\Omega\overset{(1)}{\mathrm{tr}}\chi),(\Omega\overset{(1)}{\mathrm{tr}}\underline{\chi}),\overset{(1)}{\widehat{\chi}},\overset{(1)}{\widehat{\underline{\chi}}},\overset{(1)}{\underline{\eta}},\overset{(1)}{\eta},\overset{(1)}{\omega},\overset{(1)}{\underline{\omega}},\overset{(1)}{\alpha},\overset{(1)}{\beta},\overset{(1)}{\varrho},\overset{(1)}{\sigma},\overset{(1)}{\underline{\beta}},\overset{(1)}{\underline{\alpha}},\overset{(1)}{K}) \tag{129}$$

of smooth (to be defined precisely below) functions, $S^2_{u,v}$-vectors and tensors defined on domains of the Schwarzschild manifold $(\mathcal{M},g)$. Specifically, the quantities

- $\overset{(1)}{\hat{\not g}}$, $\overset{(1)}{\widehat{\chi}}$, $\overset{(1)}{\widehat{\underline{\chi}}}$, $\overset{(1)}{\alpha}$ and $\overset{(1)}{\underline{\alpha}}$ are symmetric trace-free $S^2_{u,v}$ 2-tensors;
- $\overset{(1)}{b}$, $\overset{(1)}{\eta}$, $\overset{(1)}{\underline{\eta}}$, $\overset{(1)}{\beta}$ and $\overset{(1)}{\underline{\beta}}$ are $S^2_{u,v}$ 1-forms;
- $\overset{(1)}{\Omega}$, $\overset{(1)}{\sqrt{\not g}}$, $(\Omega\overset{(1)}{\mathrm{tr}}\chi)$, $(\Omega\overset{(1)}{\mathrm{tr}}\underline{\chi})$, $\overset{(1)}{\omega}$, $\overset{(1)}{\underline{\omega}}$, $\overset{(1)}{\varrho}$, $\overset{(1)}{\sigma}$ and $\overset{(1)}{K}$ are scalar functions.([20])

We will sometimes bundle some of these quantities and refer to

- $\overset{(1)}{\hat{\not g}}$, $\overset{(1)}{\sqrt{\not g}}$, $\overset{(1)}{\Omega}$ and $\overset{(1)}{b}$ as the *linearised metric components*;
- $(\Omega\overset{(1)}{\mathrm{tr}}\chi)$, $(\Omega\overset{(1)}{\mathrm{tr}}\underline{\chi})$, $\overset{(1)}{\widehat{\chi}}$, $\overset{(1)}{\widehat{\underline{\chi}}}$, $\overset{(1)}{\eta}$, $\overset{(1)}{\underline{\eta}}$, $\overset{(1)}{\omega}$ and $\overset{(1)}{\underline{\omega}}$ as the *linearised Ricci coefficients*;
- $\overset{(1)}{\alpha}$, $\overset{(1)}{\beta}$, $\overset{(1)}{\varrho}$, $\overset{(1)}{\sigma}$, $\overset{(1)}{\underline{\beta}}$ and $\overset{(1)}{\underline{\alpha}}$ as the *linearised curvature components*.

We also recall from (127) and (128) the notation

$$\overset{(1)}{\not g}_{AB}=\overset{(1)}{\hat{\not g}}_{AB}+\not g_{AB}(\sqrt{\not g})^{-1}\overset{(1)}{\sqrt{\not g}}.$$

([20]) The quantity $\overset{(1)}{\sigma}$ is actually a 2-form on $S^2_{u,v}$ which we will identify with its (pseudo)scalar representative.

When we say that $\mathscr{S}$ is smooth on a set $\mathcal{D}\subset\mathcal{M}$, we mean, following the considerations of §5.1.3 and our previous definition of smooth in the extended sense, that the above quantities (129) are smooth functions of the Eddington–Finkelstein coordinates on $\mathcal{M}^o\cap\mathcal{D}$, and that the following weighted linearised quantities in $\mathscr{S}$ extend regularly to $\mathcal{H}^+\cap\mathcal{D}$:

$$\begin{gathered}\overset{(1)}{\hat{\slashed{g}}},\ \sqrt{\overset{(1)}{\slashed{g}}},\ \overset{(1)}{b},\ \overset{(1)}{\Omega},\ (\Omega\overset{(1)}{\mathrm{tr}}\,\chi),\ \Omega^{-2}(\Omega\overset{(1)}{\mathrm{tr}}\,\underline{\chi}),\ \Omega\overset{(1)}{\widehat{\chi}},\ \Omega^{-1}\overset{(1)}{\widehat{\underline{\chi}}},\ \overset{(1)}{\eta},\ \overset{(1)}{\underline{\eta}},\ \overset{(1)}{\omega},\\ \Omega^{-2}\overset{(1)}{\underline{\omega}},\ \Omega^{2}\overset{(1)}{\alpha},\ \Omega\overset{(1)}{\beta},\ \overset{(1)}{\varrho},\ \overset{(1)}{\sigma},\ \Omega^{-1}\overset{(1)}{\underline{\beta}},\ \Omega^{-2}\overset{(1)}{\underline{\alpha}},\ \overset{(1)}{K}.\end{gathered}\tag{130}$$

We recall that the latter means, for any quantity q from (130) and any $n_1,n_2,n_3\in\mathbb{N}_0$, that, in any spherical coordinate patch,

$$(e^{u/2M}\partial_u)^{n_1}(\partial_v)^{n_2}(\partial_A)^{n_3}q_{\cdot\cdot}$$

extends continuously to $\mathcal{H}^+$. Here $q_{\cdot\cdot}$ stands for the components of the quantity q, so it should be replaced by q_{BC} for the symmetric traceless $S^2_{u,v}$-tensors in (130), by q_B for the $S^2_{u,v}$ 1-forms and by q for the scalars.

We say that a smooth $\mathscr{S}$ defined on $\mathcal{D}\subset\mathcal{M}$ satisfies the equations of *gravitational perturbations around Schwarzschild* (or *linearised gravity*) if (131)–(157) to be given in the subsections below hold on $\mathcal{M}^o\cap\mathcal{D}$.

Note that given a solution of (131)–(157), all quantities of $\mathscr{S}$ can in fact be reconstructed from knowing just the "metric perturbation" $(\overset{(1)}{\Omega},\overset{(1)}{b},\overset{(1)}{\slashed{g}})$. Nonetheless, we shall view all quantities of $\mathcal{S}$ as unknowns.

5.2.2. Equations for the linearised metric components

The equations for the metric components read

$$\underline{D}\left(\frac{\sqrt{\overset{(1)}{\slashed{g}}}}{\sqrt{\slashed{g}}}\right)=(\Omega\overset{(1)}{\mathrm{tr}}\,\underline{\chi}),\quad D\left(\frac{\sqrt{\overset{(1)}{\slashed{g}}}}{\sqrt{\slashed{g}}}\right)=(\Omega\overset{(1)}{\mathrm{tr}}\,\chi)-\slashed{\mathrm{div}}\,\overset{(1)}{b},\tag{131}$$

$$\sqrt{\slashed{g}}\,\underline{D}\left(\frac{\overset{(1)}{\hat{\slashed{g}}}_{AB}}{\sqrt{\slashed{g}}}\right)=2\Omega\overset{(1)}{\widehat{\underline{\chi}}}_{AB},\quad \sqrt{\slashed{g}}\,D\left(\frac{\overset{(1)}{\hat{\slashed{g}}}_{AB}}{\sqrt{\slashed{g}}}\right)=2\Omega\overset{(1)}{\widehat{\chi}}_{AB}+2(\slashed{\mathcal{D}}_2^\star\overset{(1)}{b})_{AB},\tag{132}$$

$$\partial_u\overset{(1)}{b}{}^A=2\Omega^2(\overset{(1)}{\eta}{}^A-\overset{(1)}{\underline{\eta}}{}^A).\tag{133}$$

$$\overset{(1)}{\omega}=D(\Omega^{-1}\overset{(1)}{\Omega}),\quad \overset{(1)}{\underline{\omega}}=\underline{D}(\Omega^{-1}\overset{(1)}{\Omega}),\quad \overset{(1)}{\eta}_A+\overset{(1)}{\underline{\eta}}_A=2\slashed{\nabla}_A(\Omega^{-1}\overset{(1)}{\Omega}).\tag{134}$$

Note that the derivatives D and $\underline{D}$ act on the components of $\overset{(1)}{\hat{\slashed{g}}}$ in (132); cf. formula (100).

5.2.3. Equations for the linearised Ricci coefficients

We start with the equations for the weighted linearised traces of the second fundamental forms:

$$D(\Omega\overset{(1)}{\mathrm{tr}}\,\underline{\chi})=\Omega^2(2\,\mathrm{d}\!\!/\mathrm{iv}\,\overset{(1)}{\eta}+2\overset{(1)}{\varrho}+4\varrho\Omega^{-1}\overset{(1)}{\Omega})-\tfrac12\Omega\,\mathrm{tr}\,\chi((\Omega\overset{(1)}{\mathrm{tr}}\,\underline{\chi})-(\Omega\overset{(1)}{\mathrm{tr}}\,\chi)), \tag{135}$$

$$\underline{D}(\Omega\overset{(1)}{\mathrm{tr}}\,\chi)=\Omega^2(2\,\mathrm{d}\!\!/\mathrm{iv}\,\overset{(1)}{\eta}+2\overset{(1)}{\varrho}+4\varrho\Omega^{-1}\overset{(1)}{\Omega})-\tfrac12\Omega\,\mathrm{tr}\,\chi((\Omega\overset{(1)}{\mathrm{tr}}\,\underline{\chi})-(\Omega\overset{(1)}{\mathrm{tr}}\,\chi)), \tag{136}$$

$$D(\Omega\overset{(1)}{\mathrm{tr}}\,\chi)=-(\Omega\,\mathrm{tr}\,\chi)(\Omega\overset{(1)}{\mathrm{tr}}\,\chi)+2\omega(\Omega\overset{(1)}{\mathrm{tr}}\,\chi)+2(\Omega\,\mathrm{tr}\,\chi)\overset{(1)}{\omega}, \tag{137}$$

$$\underline{D}(\Omega\overset{(1)}{\mathrm{tr}}\,\underline{\chi})=-(\Omega\,\mathrm{tr}\,\underline{\chi})(\Omega\overset{(1)}{\mathrm{tr}}\,\underline{\chi})+2\underline{\omega}(\Omega\overset{(1)}{\mathrm{tr}}\,\underline{\chi})+2(\Omega\,\mathrm{tr}\,\underline{\chi})\overset{(1)}{\underline{\omega}}. \tag{138}$$

For the traceless parts, we have([21])

$$\begin{aligned}\nabla\!\!\!/_3(\Omega^{-1}\overset{(1)}{\hat{\underline{\chi}}})+\Omega^{-1}(\mathrm{tr}\,\underline{\chi})\overset{(1)}{\hat{\underline{\chi}}}&=-\Omega^{-1}\overset{(1)}{\underline{\alpha}},\\ \nabla\!\!\!/_4(\Omega^{-1}\overset{(1)}{\hat{\chi}})+\Omega^{-1}(\mathrm{tr}\,\chi)\overset{(1)}{\hat{\chi}}&=-\Omega^{-1}\overset{(1)}{\alpha},\end{aligned} \tag{139}$$

$$\nabla\!\!\!/_3(\Omega\overset{(1)}{\hat{\chi}})+\tfrac12(\Omega\,\mathrm{tr}\,\underline{\chi})\overset{(1)}{\hat{\chi}}+\tfrac12(\Omega\,\mathrm{tr}\,\chi)\overset{(1)}{\hat{\underline{\chi}}}=-2\Omega\mathcal{D}\!\!\!/_2^{\star}\overset{(1)}{\eta}, \tag{140}$$

$$\nabla\!\!\!/_4(\Omega\overset{(1)}{\hat{\underline{\chi}}})+\tfrac12(\Omega\,\mathrm{tr}\,\chi)\overset{(1)}{\hat{\underline{\chi}}}+\tfrac12(\Omega\,\mathrm{tr}\,\underline{\chi})\overset{(1)}{\hat{\chi}}=-2\Omega\mathcal{D}\!\!\!/_2^{\star}\overset{(1)}{\underline{\eta}}. \tag{141}$$

For $\overset{(1)}{\eta}$ and $\overset{(1)}{\underline{\eta}}$ the equations read

$$\nabla\!\!\!/_3\overset{(1)}{\underline{\eta}}=\tfrac12(\mathrm{tr}\,\underline{\chi})(\overset{(1)}{\eta}-\overset{(1)}{\underline{\eta}})+\overset{(1)}{\underline{\beta}} \quad\text{and}\quad \nabla\!\!\!/_4\overset{(1)}{\eta}=-\tfrac12(\mathrm{tr}\,\chi)(\overset{(1)}{\eta}-\overset{(1)}{\underline{\eta}})-\overset{(1)}{\beta}. \tag{142}$$

The equations for the linearised lapse and its derivatives are given by

$$D\overset{(1)}{\underline{\omega}}=-\Omega^2(\overset{(1)}{\varrho}+2\varrho\Omega^{-1}\overset{(1)}{\Omega}), \tag{143}$$

$$\underline{D}\overset{(1)}{\omega}=-\Omega^2(\overset{(1)}{\varrho}+2\varrho\Omega^{-1}\overset{(1)}{\Omega}), \tag{144}$$

Finally, we have the linearised Codazzi equations

$$\begin{aligned}\mathrm{d}\!\!/\mathrm{iv}\,\overset{(1)}{\hat{\underline{\chi}}}&=-\tfrac12(\mathrm{tr}\,\underline{\chi})\overset{(1)}{\eta}+\overset{(1)}{\underline{\beta}}+\frac{1}{2\Omega}\nabla\!\!\!/(\Omega\overset{(1)}{\mathrm{tr}}\,\underline{\chi}),\\ \mathrm{d}\!\!/\mathrm{iv}\,\overset{(1)}{\hat{\chi}}&=-\tfrac12(\mathrm{tr}\,\chi)\overset{(1)}{\underline{\eta}}-\overset{(1)}{\beta}+\frac{1}{2\Omega}\nabla\!\!\!/(\Omega\overset{(1)}{\mathrm{tr}}\,\chi),\end{aligned} \tag{145}$$

and

$$\mathrm{c}\!\!/\mathrm{url}\,\overset{(1)}{\eta}=\overset{(1)}{\sigma} \quad\text{and}\quad \mathrm{c}\!\!/\mathrm{url}\,\overset{(1)}{\underline{\eta}}=-\overset{(1)}{\sigma}, \tag{146}$$

as well as the linearised Gauss equation

$$\overset{(1)}{K}=-\overset{(1)}{\varrho}-\frac14\frac{\mathrm{tr}\,\chi}{\Omega}((\Omega\overset{(1)}{\mathrm{tr}}\,\underline{\chi})-(\Omega\overset{(1)}{\mathrm{tr}}\,\chi))+\frac12\Omega^{-1}\overset{(1)}{\Omega}(\mathrm{tr}\,\chi\,\mathrm{tr}\,\underline{\chi}). \tag{147}$$

([21]) We use the projected covariant derivatives $\nabla\!\!\!/_3$ and $\nabla\!\!\!/_4$ for equations involving tensorial quantities. This is because later, when we derive estimates, there will be contractions involving $g\!\!\!/$: Since $\nabla\!\!\!/_3 g\!\!\!/=\nabla\!\!\!/_4 g\!\!\!/=0$, such contractions are easier to handle.

5.2.4. Equations for linearised curvature components

We complete the system of linearised gravity with the linearised Bianchi equations:

$$\nabla\!\!\!/_3 \overset{(1)}{\alpha} + \tfrac{1}{2}\,\mathrm{tr}\,\underline{\chi}\,\overset{(1)}{\alpha} + 2\underline{\widehat{\omega}}\,\overset{(1)}{\alpha} = -2\mathcal{D}\!\!\!/_2^\star \overset{(1)}{\beta} - 3\varrho \overset{(1)}{\widehat{\chi}}, \tag{148}$$

$$\nabla\!\!\!/_4 \overset{(1)}{\beta} + 2(\mathrm{tr}\,\chi)\overset{(1)}{\beta} - \widehat{\omega}\,\overset{(1)}{\beta} = \mathrm{d}\!\!\!/\mathrm{iv}\,\overset{(1)}{\alpha}, \tag{149}$$

$$\nabla\!\!\!/_3 \overset{(1)}{\beta} + (\mathrm{tr}\,\underline{\chi})\overset{(1)}{\beta} + \underline{\widehat{\omega}}\,\overset{(1)}{\beta} = \mathcal{D}\!\!\!/_1^\star(-\overset{(1)}{\varrho}, \overset{(1)}{\sigma}) + 3\varrho\overset{(1)}{\eta}, \tag{150}$$

$$\nabla\!\!\!/_4 \overset{(1)}{\varrho} + \frac{3}{2}(\mathrm{tr}\,\chi)\,\overset{(1)}{\varrho} = \mathrm{d}\!\!\!/\mathrm{iv}\,\overset{(1)}{\beta} - \frac{3}{2}\frac{\varrho}{\Omega}(\Omega\,\overset{(1)}{\mathrm{tr}\,\chi}), \tag{151}$$

$$\nabla\!\!\!/_3 \overset{(1)}{\varrho} + \frac{3}{2}(\mathrm{tr}\,\underline{\chi})\,\overset{(1)}{\varrho} = -\,\mathrm{d}\!\!\!/\mathrm{iv}\,\overset{(1)}{\underline{\beta}} - \frac{3}{2}\frac{\varrho}{\Omega}(\Omega\,\overset{(1)}{\mathrm{tr}\,\underline{\chi}}), \tag{152}$$

$$\nabla\!\!\!/_4 \overset{(1)}{\sigma} + \tfrac{3}{2}(\mathrm{tr}\,\chi)\overset{(1)}{\sigma} = -\,\mathrm{c}\!\!\!/\mathrm{url}\,\overset{(1)}{\beta}, \tag{153}$$

$$\nabla\!\!\!/_3 \overset{(1)}{\sigma} + \tfrac{3}{2}(\mathrm{tr}\,\underline{\chi})\overset{(1)}{\sigma} = -\,\mathrm{c}\!\!\!/\mathrm{url}\,\overset{(1)}{\underline{\beta}}, \tag{154}$$

$$\nabla\!\!\!/_4 \overset{(1)}{\underline{\beta}} + (\mathrm{tr}\,\chi)\overset{(1)}{\underline{\beta}} + \widehat{\omega}\,\overset{(1)}{\underline{\beta}} = \mathcal{D}\!\!\!/_1^\star(\overset{(1)}{\varrho}, \overset{(1)}{\sigma}) - 3\varrho\overset{(1)}{\underline{\eta}}, \tag{155}$$

$$\nabla\!\!\!/_3 \overset{(1)}{\underline{\beta}} + 2(\mathrm{tr}\,\underline{\chi})\overset{(1)}{\underline{\beta}} - \underline{\widehat{\omega}}\,\overset{(1)}{\underline{\beta}} = -\,\mathrm{d}\!\!\!/\mathrm{iv}\,\overset{(1)}{\underline{\alpha}}, \tag{156}$$

$$\nabla\!\!\!/_4 \overset{(1)}{\underline{\alpha}} + \tfrac{1}{2}(\mathrm{tr}\,\chi)\overset{(1)}{\underline{\alpha}} + 2\widehat{\omega}\,\overset{(1)}{\underline{\alpha}} = 2\mathcal{D}\!\!\!/_2^\star \overset{(1)}{\underline{\beta}} - 3\varrho\overset{(1)}{\underline{\widehat{\chi}}}. \tag{157}$$

Note the coupling of the linearised Bianchi system with the linearised Ricci coefficients $\overset{(1)}{\widehat{\chi}}$, $\overset{(1)}{\eta}$, $\overset{(1)}{\underline{\eta}}$ and $\overset{(1)}{\underline{\widehat{\chi}}}$, remarked already in §2.1.3. (For comparison, recall that these terms do not arise in the Minkowski case since the background curvature components vanish, in particular ϱ=0.)

6. Special solutions: pure gauge and linearised Kerr

In this section, we shall look at two classes of special solutions of the system of linearised gravity given in §5.2. We begin in §6.1 with a discussion of *pure gauge solutions* followed by a presentation of a 4-dimensional family of *reference linearised Kerr solutions* in §6.2.

6.1. Pure gauge solutions $\mathscr{G}$

As described already in §2.1.4, pure gauge solutions are those derived from linearising the families of metrics that arise from applying to Schwarzschild smooth 1-parameter families of coordinate transformations which preserve the double null form (70) of the metric. We will classify such solutions here, deriving them from the formal linearisation of §5.1 in order to best illustrate their geometric significance. (The reader can alternatively simply directly verify that these solutions satisfy the system of linearised gravity of §5.2.)

6.1.1. General computations and formal developments

Let us fix functions $\Omega^2(u,v)\cdot f_1(u,\theta,\phi)$, $f_2(v,\theta,\phi)$, $j_3(v,\theta,\phi)$ and $j_4(v,\theta,\phi)$ on $\mathcal{M}^\circ$ extending smoothly to $\mathcal{M}$.

Consider a smooth 1-parameter family of coordinates on $\mathcal{M}^\circ$ defined by

$$\begin{aligned}\tilde{\boldsymbol{u}}&=\tilde{\boldsymbol{u}}_\varepsilon:=u+\varepsilon f_1(u,\theta,\phi),\\ \tilde{\boldsymbol{v}}&=\tilde{\boldsymbol{v}}_\varepsilon:=v+\varepsilon f_2(v,\theta,\phi),\\ \tilde{\boldsymbol{\theta}}&=\tilde{\boldsymbol{\theta}}_\varepsilon:=\theta+\varepsilon f_3(u,v,\theta,\phi)=\theta+\varepsilon\frac{2}{r(u,v)}(f_2)_\theta(v,\theta,\phi)+\varepsilon j_3(v,\theta,\phi),\\ \tilde{\boldsymbol{\phi}}&=\tilde{\boldsymbol{\phi}}_\varepsilon:=\phi+\varepsilon f_4(u,v,\theta,\phi)=\phi+\varepsilon\frac{2}{r(u,v)}\frac{1}{\sin^2\theta}(f_2)_\phi(v,\theta,\phi)+\varepsilon j_4(v,\theta,\phi).\end{aligned}\tag{158}$$

If we express the Schwarzschild metric in the form (91)

$$\boldsymbol{g}_S=-4\Omega^2(\tilde{\boldsymbol{u}},\tilde{\boldsymbol{v}})d\tilde{\boldsymbol{u}}d\tilde{\boldsymbol{v}}+r^2(\tilde{\boldsymbol{u}},\tilde{\boldsymbol{v}})(d\tilde{\boldsymbol{\theta}}^2+\sin^2\tilde{\boldsymbol{\theta}}\,d\tilde{\boldsymbol{\phi}}^2),\tag{159}$$

where Ω^2, r^2 and γ are defined by the expressions (92), (93) and (89), where however u, v, θ and ϕ are replaced by the new coordinates $\tilde{\boldsymbol{u}}$, $\tilde{\boldsymbol{v}}$, $\tilde{\boldsymbol{\theta}}$ and $\tilde{\boldsymbol{\phi}}$, then, in view of (158), this defines with respect to the original fixed coordinates u, v, θ and ϕ of §4.2.1 a 1-parameter family of metrics, whose first-order ε-dependence can be expressed as

$$\begin{aligned}\boldsymbol{g}_S(\varepsilon)\equiv{}&du\,dv\left[-4\Omega^2+\varepsilon\left(-\frac{8M}{r^2}\Omega^2[f_2-f_1]-4\Omega^2(f_2)_v-4\Omega^2(f_1)_u\right)\right]\\ &+dv\,d\theta\,(-4\Omega^2\varepsilon(f_1)_\theta+2r^2\varepsilon(f_3)_v)+dv\,d\phi\,(-4\Omega^2\varepsilon(f_1)_\phi+2r^2\sin^2\theta\varepsilon(f_4)_v)\\ &+d\theta\,d\theta\,(r^2+2r\Omega^2\varepsilon[f_2-f_1]+2r^2\varepsilon(f_3)_\theta)+d\theta\,d\phi\,(2\varepsilon r^2(f_3)_\phi+2\varepsilon r^2\sin^2\theta(f_4)_\theta)\\ &+\sin^2\theta\,d\phi\,d\phi\left[r^2+2r\Omega^2\varepsilon[f_2-f_1]+2r^2\varepsilon(f_4)_\phi+2r^2\frac{\cos\theta}{\sin\theta}\varepsilon f_3\right].\end{aligned}\tag{160}$$

Here, as before, "$\equiv$" indicates that we are ignoring terms of order ε^2 or higher and the subscripts v, θ, etc. indicate a partial derivative with respect to this variable. The smoothness assumptions on f_1, f_2, j_3 and j_4 ensure that this defines in fact a smooth 1-parameter family of metrics on $\mathcal{M}$, i.e. including the boundary $\mathcal{H}^+$.

Note that the right-hand side of (160) is of the double-null form (115) with $\boldsymbol{\Omega}^2(0)=\Omega^2$, $\boldsymbol{b}(0)=0$ and $\boldsymbol{\not{g}}(0)=\not{g}$. Since the right-hand side of (160) thus defines a family diffeomorphic to Schwarzschild to first order in ε, it in particular satisfies the Einstein equations (75) to first order, and thus, still gives rise to a solution of linearised gravity, which can be read off as in §5.1.2 from (160) by collecting the terms at $\mathcal{O}(\varepsilon)$.

Specifically, from (160) we can read off

$$\overset{(1)}{b}{}^{\theta}=\frac{1}{r^2}(2\Omega^2(f_1)_\theta-r^2(f_3)_v)\overset{(1)}{b}{}^{\phi}=\frac{1}{r^2}\left(2\Omega^2\frac{(f_1)_\phi}{\sin^2\theta}-r^2(f_4)_v\right), \tag{161}$$

$$2\Omega^{-1}\overset{(1)}{\Omega}=(f_2)_v+\frac{2M}{r^2}f_2+(f_1)_u-\frac{2M}{r^2}f_1, \tag{162}$$

$$\frac{\overset{(1)}{\sqrt{\not g}}}{\sqrt{\not g}}=\frac{2\Omega^2}{r}(f_2-f_1)+\frac{1}{\sin\theta}\partial_\theta(\sin\theta f_3)+(f_4)_\phi, \tag{163}$$

$$\begin{aligned}
\overset{(1)}{\hat{\not g}}_{\theta\theta}&=2r^2(f_3)_\theta-\frac{r^2}{\sin\theta}\partial_\theta(\sin\theta f_3)-r^2(f_4)_\phi,\\
\overset{(1)}{\hat{\not g}}_{\theta\phi}&=r^2(f_3)_\phi+r^2\sin^2\theta(f_4)_\theta,\\
\sin^{-2}\theta\,\overset{(1)}{\hat{\not g}}_{\phi\phi}&=2r^2(f_4)_\phi+2r^2\frac{\cos\theta}{\sin\theta}f_3-\frac{r^2}{\sin\theta}\partial_\theta(\sin\theta f_3)-r^2(f_4)_\phi.
\end{aligned} \tag{164}$$

Note that, since $\Omega^2(u,v)\cdot f_1(u,\theta,\phi)$, $f_2(v,\theta,\phi)$, $j_3(v,\theta,\phi)$, $j_4(v,\theta,\phi)$ are functions smooth in the extended sense on $\mathcal{M}$, the perturbation $(\Omega^{-1}\overset{(1)}{\Omega},\overset{(1)}{b},\overset{(1)}{\not g})$ is smooth on $\mathcal{M}$.

All geometric quantities can now be computed from the above using the system of gravitational perturbations. For future reference we collect here the formulas for

$$\begin{aligned}
(\Omega\overset{(1)}{\operatorname{tr}}\chi)&=\partial_v\left(\frac{\overset{(1)}{\sqrt{\not g}}}{\sqrt{\not g}}\right)+\not{\mathrm{div}}\,\overset{(1)}{b}=\partial_v\left(\frac{2\Omega^2}{r}(f_2-f_1)\right)+\frac{2\Omega^2}{r^2}\Delta_{S^2}f_1\\
&=\frac{2\Omega^2}{r}\partial_v f_2+\frac{\Omega^2}{r^2}[(2-4\Omega^2)(f_2-f_1)+2\Delta_{S^2}f_1],
\end{aligned} \tag{165}$$

where $\Delta_{S^2}=r^2\not\Delta$ denotes the Laplacian on the unit sphere and

$$\begin{aligned}
(\Omega\overset{(1)}{\operatorname{tr}}\underline{\chi})&=\partial_u\left(\frac{\overset{(1)}{\sqrt{\not g}}}{\sqrt{\not g}}\right)=\partial_u\left(\frac{2\Omega^2}{r}(f_2-f_1)\right)+\frac{2\Omega^2}{r^2}\Delta_{S^2}f_2\\
&=\frac{2\Omega^2}{r}(-\partial_u f_1)-\frac{\Omega^2}{r^2}[(2-4\Omega^2)(f_2-f_1)-2\Delta_{S^2}f_2],
\end{aligned} \tag{166}$$

which are easily determined from (131). We conclude the following.

Proposition 6.1.1. *Let $\Omega^2\cdot f_1(u,\theta,\phi)$, $f_2(v,\theta,\phi)$, $j_3(v,\theta,\phi)$, $j_4(v,\theta,\phi)$ be smooth functions on $\mathcal{M}$ and $f_3(u,v,\theta,\phi)$, $f_4(u,v,\theta,\phi)$ be defined through (158). Then, the metric perturbation with $\Omega^{-1}\overset{(1)}{\Omega}$ defined as in (162), $\overset{(1)}{b}{}^A$ defined as in (161) and $\overset{(1)}{\not g}$ defined as in (163) and (164) determines a smooth solution of the system of gravitational perturbations on $\mathcal{M}$.*

We call such a solution a pure gauge solution *and denote it by $\mathscr{G}$ or, to indicate how it is generated, by $\mathscr{G}(f_1,f_2,j_3,j_4)$.*

Note that non-trivial (f_1, f_2, j_3, j_4) can generate the trivial solution $\mathscr{G}=0$. For instance, choosing $f_1=1$, $f_2=1$ and $j_3=j_4=0$ generate the zero solution.

The validity of the above proposition is clear by the above formal computations. One can also check explicitly that all equations of the system of gravitational perturbations (131)–(157) are satisfied.

In the next three subsections we will look at the basic building blocks of pure gauge solutions arising from Proposition 6.1.1. Specifically, we compute explicitly all Ricci coefficients and curvature quantities of three special pure gauge solutions produced by Proposition 6.1.1:

- setting $(f_1=0, f_2=f(v,\theta,\phi), j_3=0, j_4=0)$: Lemma 6.1.1;
- setting $(f_1=f(u,\theta,\phi), f_2=0, j_3=0, j_4=0)$: Lemma 6.1.2;
- setting $(f_1=0, f_2=0, j_3(v,\theta,\phi), j_4(v,\theta,\phi))$: Lemma 6.1.3.

In view of linearity, the general pure gauge solution can be obtained from summing the three special ones.

We will also include the computation that all equations of (131)–(157) are indeed satisfied for Lemma 6.1.1, since we have omitted the lengthy but straightforward proof of Proposition 6.1.1.

6.1.2. Pure gauge solutions of the form $(f_1=0, f_2=f(v,\theta,\phi), j_3=0, j_4=0)$

The following solution is the explicit form of the pure gauge solution $(f_1=0, f_2=f(v,\theta,\phi), j_3=0, j_4=0)$ arising from Proposition 6.1.1.

Lemma 6.1.1. *For any smooth function $f=f(v,\theta,\phi)$, the following is a pure gauge solution of the system of gravitational perturbations:*

$$2\Omega^{-1}\overset{(1)}{\Omega}=\frac{1}{\Omega^2}\partial_v(f\Omega^2),\quad \overset{(1)}{\hat{\not{g}}}=\frac{4}{r}r^2\not{\mathcal{D}}_2^\star\not{\mathcal{D}}_1^\star(f,0),\quad \frac{\overset{(1)}{\sqrt{\not{g}}}}{\sqrt{\not{g}}}=\frac{2\Omega^2 f}{r}+\frac{2}{r}r^2\not{\Delta}f,$$

$$\overset{(1)}{b}=2r^2\not{\mathcal{D}}_1^\star\left(\partial_v\left(\frac{f}{r}\right),0\right),\quad \overset{(1)}{\eta}=-\frac{\Omega^2}{r^2}r\not{\mathcal{D}}_1^\star(f,0),\quad \overset{(1)}{\underline{\eta}}=-\frac{1}{\Omega^2}r\not{\mathcal{D}}_1^\star\left(\partial_v\left(\frac{\Omega^2}{r}f\right),0\right)$$

$$\overset{(1)}{\underline{\widehat{\chi}}}=2\frac{\Omega}{r^2}r^2\not{\mathcal{D}}_2^\star\not{\mathcal{D}}_1^\star(f,0),\quad \overset{(1)}{(\Omega\operatorname{tr}\chi)}=2\partial_v\left(\frac{f\Omega^2}{r}\right),\quad \overset{(1)}{(\Omega\operatorname{tr}\underline{\chi})}=2\frac{\Omega^2}{r^2}[\Delta_{S^2}f-f(1-2\Omega^2)],$$

$$\overset{(1)}{\varrho}=\frac{6M\Omega^2}{r^4}f,\quad \overset{(1)}{\underline{\beta}}=-\frac{6M\Omega}{r^4}r\not{\mathcal{D}}_1^\star(f,0),\quad \overset{(1)}{K}=-\frac{\Omega^2}{r^3}(\Delta_{S^2}f+2f),$$

$\overset{(1)}{\omega}$ *and* $\overset{(1)}{\underline{\omega}}$ *determined by* (134) *and*

$$\overset{(1)}{\widehat{\chi}}=\overset{(1)}{\alpha}=\overset{(1)}{\underline{\alpha}}=0,\quad \overset{(1)}{\beta}=0,\quad \overset{(1)}{\sigma}=0.$$

We will call f a gauge function.

Remark 6.1. Note that $\overset{(1)}{\eta}$, $\overset{(1)}{\varrho}$ and $(\Omega \overset{(1)}{\mathrm{tr}}\, \chi)$ vanish on the horizon $\mathcal{H}^+$.

Remark 6.2. In §8.3, we will introduce the notion of asymptotically flat seed data (Definition 8.2). In this language, note that the above pure gauge solution induces asymptotically flat seed data with weight s to order zero on $C_{u_0} \cap C_{v_0}$, provided the gauge function f satisfies $|(r\nabla\!\!\!/)^n f| \lesssim v$ for $n \leqslant 2$, $|(r\nabla\!\!\!/)^n r^2 \partial_v (f/r)| \lesssim 1$ for $n \leqslant 1$, and

$$\left| r^{1+s} \partial_v \left(r^2 \partial_v \left(\frac{f}{r} \right) \right) \right| \lesssim 1.$$

In particular, f does not have to be bounded. It is also easy to see what higher-order assumptions on f need to be imposed to guarantee that the seed data are asymptotically flat to higher order.

Proof of Lemma 6.1.1. We verify some of the null stucture and all of the Bianchi equations below leaving the remaining equations to the reader. The left-hand side of the (renormalised) (135) is

$$D[r(\Omega \overset{(1)}{\mathrm{tr}}\, \underline{\chi})] = 2\Delta_{S^2} \left(\partial_v \left(\frac{\Omega^2}{r} f \right) \right) - 2\partial_v \left(\frac{\Omega^2 (1-2\Omega^2)}{r} f \right),$$

while the right-hand side is

$$2\Omega^2 r \left(\frac{1}{\Omega^2 r} \Delta_{S^2} \left(\partial_v \left(\frac{\Omega^2}{r} f \right) \right) + \frac{6M}{r^4} \Omega^2 f - \frac{2M}{\Omega^2 r^3} \partial_v (f\Omega^2) \right) + r \frac{\Omega^2}{r} 2\partial_v \left(\frac{f\Omega^2}{r} \right).$$

The term involving the angular Laplacian cancels and since

$$\begin{aligned} -2\partial_v \left(\frac{\Omega^2 (1-2\Omega^2)}{r} f \right) &= 2\partial_v \left(\frac{\Omega^2}{r} f \right) - \partial_v \left(\frac{8M}{r^2} \Omega^2 f \right) \\ &= 2\Omega^2 \partial_v \left(\frac{\Omega^2}{r} f \right) + \frac{4M}{r} \partial_v \left(\frac{\Omega^2}{r} f \right) - \partial_v \left(\frac{8M}{r^2} \Omega^2 f \right) \\ &= 2\Omega^2 \partial_v \left(\frac{\Omega^2}{r} f \right) - \frac{4M}{r^2} \partial_v (f\Omega^2) + \frac{12M}{r^3} \Omega^4 f, \end{aligned}$$

we have established that (135) holds. Let us check the two Codazzi equations. The one involving β can be read off directly, while the one involving $\underline{\beta}$ reads

$$2\Omega \mathcal{D}\!\!\!\!/_2 \mathcal{D}\!\!\!\!/_2^\star \mathcal{D}\!\!\!\!/_1^\star (f,0) = + \frac{\Omega}{r} \frac{\Omega^2}{r^2} r \nabla\!\!\!\!/ f + \frac{6M\Omega}{r^4} r \nabla\!\!\!\!/ f + \frac{1}{2\Omega} 2 \frac{\Omega^2}{r^2} \nabla\!\!\!\!/ [\Delta_{S^2} f - f(1-2\Omega^2)]. \tag{167}$$

Now, use the fact that

$$\mathcal{D}\!\!\!\!/_2 \mathcal{D}\!\!\!\!/_2^\star \mathcal{D}\!\!\!\!/_1^\star (f,0) = \left(-\frac{1}{2} \Delta\!\!\!\!/ - \frac{1}{2} K \right) \mathcal{D}\!\!\!\!/_1^\star (f,0) = \frac{1}{2} \mathcal{D}\!\!\!\!/_1^\star \mathcal{D}\!\!\!\!/_1 \mathcal{D}\!\!\!\!/_1^\star (f,0) - K \mathcal{D}\!\!\!\!/_1^\star (f,0) = \frac{1}{2} \nabla\!\!\!\!/ \Delta\!\!\!\!/ f + \frac{1}{r^2} \nabla\!\!\!\!/ f \tag{168}$$

to validate (167). To validate (138), note that

$$\partial_u[(\Omega \overset{(1)}{\operatorname{tr}} \underline{\chi})r^2\Omega^{-2}] = -\frac{8M}{r^2}f\Omega^2 = -4r\frac{1}{2}f\partial_u\left(\frac{2M}{r^2}\right) = -4r\overset{(1)}{\underline{\omega}}.$$

The Bianchi equations (148) and (149) are trivially satisfied, (150) is easily checked by inspection. For (151) note that

$$\partial_v(r^3\overset{(1)}{\varrho}) = \partial_v\left(r^3\frac{6M}{r^4}\Omega^2 f\right) = \frac{3M}{\Omega}2\partial_v\left(\frac{f\Omega^2}{r}\right) = \frac{3M}{\Omega r^3}r^3(\Omega \overset{(1)}{\operatorname{tr}} \chi).$$

For (152) note similarly that

$$\partial_u(r^3\overset{(1)}{\varrho}) = f\partial_u\left(r^3\frac{6M}{r^4}\Omega^2\right) = -6M\Omega(\Delta_{S^2}f) + \frac{3M}{\Omega}2\frac{\Omega^2}{r^2}[\Delta_{S^2}f - f(1-2\Omega^2)],$$

which is readily verified through the identity $\partial_u(6M\Omega^2/r) = 6M\Omega^2(\Omega^2/r^2 - 2M/r^3)$. The Bianchi equations (153) and (154) hold trivially and so does (157). The equation (156) is also easily verified using that f does not depend on u. It remains to verify (155). We renormalise it to

$$\nabla\!\!\!/_4(r^2\Omega\overset{(1)}{\underline{\beta}})_A = -r^2\Omega\nabla\!\!\!/_A\overset{(1)}{\varrho} + \frac{6M}{r}\Omega\overset{(1)}{\underline{\eta}}_A.$$

The left-hand side is

$$\nabla\!\!\!/_4(r^2\Omega\overset{(1)}{\underline{\beta}})_A = \nabla\!\!\!/_4\left(\frac{6M\Omega}{r^2}r\nabla\!\!\!/_A f\right) = \frac{6Mr}{\Omega}\nabla\!\!\!/_A\partial_v\left(\frac{f\Omega}{r^2}\right),$$

while the right-hand side is

$$-r^2\Omega\nabla\!\!\!/_A\overset{(1)}{\varrho} + \frac{6M}{r}\Omega\overset{(1)}{\underline{\eta}}_A = -\frac{6M}{r^2}\Omega\left(1-\frac{2M}{r}\right)\nabla\!\!\!/_A f + \frac{6M}{r}\frac{1}{\Omega^2}r\nabla\!\!\!/_A\left(\partial_v\left(\frac{\Omega^2}{r}f\right)\right).$$

Equation (155) is now immediate. □

6.1.3. Pure gauge solutions of the form $(f_1 = f(u,\theta,\phi), f_2 = 0, j_3 = 0, j_4 = 0)$

Completely analogously, one proves the following lemma, which provides the explicit form of the pure gauge solution $(f_1 = f(u,\theta,\phi), f_2 = 0, j_3 = 0, j_4 = 0)$ arising from Proposition 6.1.1.

Lemma 6.1.2. *For any function $f = f(u,\theta,\phi)$ such that $\Omega^2 \cdot f(u,\theta,\phi)$ is smooth in the extended sense on $\mathcal{M}$, the following is a (pure gauge) solution of the system of* *gravitational perturbations*:

$$2\Omega^{-1}\overset{(1)}{\Omega}=\frac{1}{\Omega^2}\partial_u(f\Omega^2),\quad \frac{\overset{(1)}{\sqrt{\not g}}}{\sqrt{\not g}}=-\frac{2\Omega^2 f}{r},\quad \overset{(1)}{b}=2\Omega^2\not{\mathcal{D}}_1^\star(-f,0),$$

$$\overset{(1)}{\widehat{\chi}}=2\frac{\Omega}{r^2}r^2\not{\mathcal{D}}_2^\star\not{\mathcal{D}}_1^\star(f,0),\quad \overset{(1)}{\eta}=-\frac{1}{\Omega^2}r\not{\mathcal{D}}_1^\star(\partial_u(\frac{\Omega^2}{r}f),0),\quad \overset{(1)}{\underline{\eta}}=\frac{\Omega^2}{r^2}r\not{\mathcal{D}}_1^\star(f,0),$$

$$(\Omega\overset{(1)}{\mathrm{tr}}\,\underline{\chi})=-2\partial_u(\frac{f\Omega^2}{r}),\quad (\Omega\overset{(1)}{\mathrm{tr}}\,\chi)=2\frac{\Omega^2}{r^2}[\Delta_{S^2}f-f(1-2\Omega^2)],$$

$$\overset{(1)}{\beta}=\frac{6M\Omega}{r^4}r\not{\mathcal{D}}_1^\star(f,0),\quad \overset{(1)}{\varrho}=-\frac{6M\Omega^2}{r^4}f,\quad \overset{(1)}{K}=\frac{\Omega^2}{r^3}(\Delta_{S^2}f+2f),$$

$\overset{(1)}{\omega}$ *and* $\overset{(1)}{\underline{\omega}}$ *determined by* (134) *and*

$$0=\overset{(1)}{\hat{\not g}}=\overset{(1)}{\widehat{\underline{\chi}}}=\overset{(1)}{\alpha}=\overset{(1)}{\underline{\alpha}}=0,\quad \overset{(1)}{\underline{\beta}}=0,\quad \overset{(1)}{\sigma}=0.$$

We will call f a gauge function*.*

Remark 6.3. Recall that, as long as $\Omega^2\cdot f(u,\theta,\phi)$ is smooth in the extended sense and uniformly bounded, the corresponding pure gauge solution is smooth in the extended sense all the way to the horizon. In addition, the linearised curvature components and Ricci coefficients of the above gauge solution satisfy the decay rates (439) towards null infinity; cf. Remark 6.2.

6.1.4. Pure gauge solutions of the form $(f_1=0, f_2=0, j_3, j_4)$

We finally consider pure gauge solutions of the form $(f_1=0, f_2=0, j_3, j_4)$. As we will see, they will only generate non-trivial values for the metric components $\overset{(1)}{\sqrt{\not g}}$, $\overset{(1)}{\hat{\not g}}$ and $\overset{(1)}{b}$, while all other quantities of the solution vanish. To discuss these solutions, it will be desirable to bring the formulas (161)–(164) into a more geometric form. For this, it is useful to think about the associated underlying coordinate transformation (158) on the sphere as

$$\tilde{\boldsymbol{\theta}}^A=\theta^A+\varepsilon j^A(v,\theta,\phi),\quad \text{with } \tilde{\boldsymbol{\theta}}^1=\tilde{\boldsymbol{\theta}},\ \tilde{\boldsymbol{\theta}}^2=\tilde{\boldsymbol{\phi}},\ \theta^1=\theta \text{ and } \theta^2=\phi,$$

for a v-dependent vectorfield $j^A(v,\theta,\phi)$ on $S^2_{u,v}$ with components j_3 and j_4. We can then solve for each u and v the equation

$$j=r^2\not{\mathcal{D}}_1^\star(-q_1,-q_2)\quad\text{or, in components,}\quad j^A=\gamma^{AB}\partial_B q_1-\varepsilon^{AC}\partial_C q_2,$$

for a pair of functions $(q_1(v,\theta,\phi),q_2(v,\theta,\phi))$. Note that q_1 and q_2 do not depend on u, as there is no u-dependence, if the equation is written with indices upstairs. Note that q_1 and q_2 are unique, up to their spherical mean. The following lemma parametrises the pure gauge solutions of the title in terms of smooth functions $q_1(v,\theta,\phi)$ and $q_2(v,\theta,\phi)$, which, by the above considerations, exploit the full freedom given by j_3 and j_4.

LEMMA 6.1.3. *For any smooth functions $q_1(v,\theta,\phi)$ and $q_2(v,\theta,\phi)$, the following is a pure gauge solution of the system of gravitational perturbations*:

$$\overset{(1)}{\hat{\slashed{g}}} = 2r^2 \slashed{\mathcal{D}}_2^\star \slashed{\mathcal{D}}_1^\star (q_1, q_2), \quad \frac{\overset{(1)}{\sqrt{\slashed{g}}}}{\sqrt{\slashed{g}}} = r^2 \slashed{\Delta} q_1, \quad \overset{(1)}{b} = r^2 \slashed{\mathcal{D}}_1^\star (\partial_v q_1, \partial_v q_2),$$

while the linearised metric coefficient $\overset{(1)}{\Omega}$, as well as all linearised connection coefficients and curvature components vanish.

Proof. Note that, since $\overset{(1)}{b}$ with indices upstairs does not depend on the variable u, (133) indeed holds. The equations (131) and (132) are also readily checked, and the remaining equations hold trivially. □

6.2. A 4-dimensional reference linearised Kerr family $\mathscr{K}$

The other class of interesting special solutions which we shall identify corresponds to the 4-dimensional family that arises by linearising 1-parameter representations of Kerr (which of course solves the non-linear equations (75)) around Schwarzschild in an appropriate coordinate system. We will present such a family here, giving first in §6.2.1 a 1-dimensional linearised Schwarzschild family, and then in §6.2.2 a 3-dimensional family corresponding to Kerr with fixed mass M.

6.2.1. Linearised Schwarzschild solutions

We begin by reminding the reader that, in view of the pure gauge solutions identified in §6.1, there is no unique way of identifying a 1-parameter family of linearised Schwarzschild solutions. This *uniqueness up to pure gauge solutions* is reflected in the choice of double null coordinates in which one linearises the 1-parameter Schwarzschild family. A particularly simple such choice is given by writing the 1-parameter Schwarzschild family in rescaled null coordinates

$$g_M = 4M^2 \left(-4\left(1 - \frac{1}{x}\right) d\hat{u}\, d\hat{v} + x^2 d\sigma_2 \right), \tag{169}$$

with x defined via the relation $(x-1)e^x = e^{\hat{v}-\hat{u}}$. Note that, setting $r=2Mx$, $u=2M\hat{u}$ and $v=2M\hat{v}$ produces the metric in standard Eddington– Finkelstein coordinates u and v. Since the x in (169) does not depend on M at all, the linearisation of (169) in the parameter M is immediate.

One obtains thus a proof of the following proposition (which can alternatively be proven by directly verifying that the system of linearised gravity is satisfied).

Proposition 6.2.1. *For every* $\mathfrak{m}\in\mathbb{R}$, *the following is a (spherically symmetric) solution of the system of gravitational perturbations* (131)–(157) *in* $\mathcal{M}$:

$$\overset{(1)}{\hat{\not g}} = \overset{(1)}{\widehat{\chi}} = \overset{(1)}{\widehat{\underline{\chi}}} = \overset{(1)}{\alpha} = \overset{(1)}{\underline{\alpha}} = 0,$$
$$\overset{(1)}{b} = \overset{(1)}{\eta} = \overset{(1)}{\underline{\eta}} = \overset{(1)}{\beta} = \overset{(1)}{\underline{\beta}} = 0,$$
$$(\Omega \overset{(1)}{\mathrm{tr}}\,\chi) = \Omega^{-2}(\Omega \overset{(1)}{\mathrm{tr}}\,\underline{\chi}) = \overset{(1)}{\omega} = \overset{(1)}{\underline{\omega}} = \sigma = 0,$$

and

$$2\Omega^{-1}\overset{(1)}{\Omega} = -\mathfrak{m}, \quad \mathrm{tr}_{\not g}\,\overset{(1)}{\not g} = -2\mathfrak{m}, \quad \overset{(1)}{\varrho} = -\frac{2M}{r^3}\cdot\mathfrak{m}, \quad \overset{(1)}{K} = \frac{\mathfrak{m}}{r^2}.$$

We refer to the above 1*-parameter family as the* reference linearised Schwarzschild solutions.

As mentioned, the above proposition exhibits the family of linearised Schwarzschild solutions in a particular gauge.

Remark 6.4. With respect to standard Eddington–Finkelstein coordinates (u,v), the Schwarzschild family is given by

$$g_M = -4\left(1-\frac{2M}{r_M}\right) du\, dv + (r_M)^2\, d\sigma_2, \tag{170}$$

with r_M defined via $(r_M/2M-1)e^{r_M/2M}=e^{(v-u)/2M}$. If one linearises (170) with respect to the parameter M fixing the (u,v)-differential structure, one obtains the sum of the family of Lemma 6.2.1 and the pure-gauge transformation generated by $f_1=u/2M$ and $f_2=v/2M$ (and $f_3=f_4=j_3=j_4=0$) in (158), the reason being that the coordinate transformation relating (u,v) and $(\hat{u},\hat{v})$ mentioned above depends on M itself.

6.2.2. Linearised Kerr solutions leaving the mass unchanged

Recall from the discussion in §2.1.2 that the Kerr family can globally be brought into the double null form (115) in its exterior. This was achieved in [59] (see also [18]). One can linearise a 1-parameter representation of the metric in this form with respect to the angular momentum parameter $a=\varepsilon\mathfrak{a}$, to obtain what we shall call the (reference) linearised Kerr solution below. Alternatively one can take a shortcut and start from the Kerr metric expressed in standard Boyer–Lindquist coordinates ignoring all terms quadratic or higher in a:

$$g_{\mathrm{Kerr}} = -\left(1-\frac{2M}{r}\right)dt^2 + \frac{dr^2}{1-2M/r} + r^2(d\theta^2+\sin^2\theta\, d\phi^2) - \frac{4Ma}{r}\sin^2\theta\, d\phi\, dt + \mathcal{O}(a^2).$$

One can now introduce the standard Eddington–Finkelstein coordinates (u,v) for the Schwarzschild part and do a coordinate transformation $\phi \mapsto \tilde{\phi}+f(v-u)$ for an appropriate function f to bring the metric into the form (115) to first order in ε and still read off the metric perturbation. Either of these procedures leads to the ($m{=}0$ case of the) following proposition.

Proposition 6.2.2. *Let $Y_m^{\ell=1}$, for $m=-1,0,1$, denote the spherical harmonics (108). For any $\mathfrak{a}\in\mathbb{R}$, the following is a smooth solution of the system of gravitational perturbations (131)–(157) on $\mathcal{M}$. The non-vanishing metric coefficients are*

$$\overset{(1)}{b}{}^A = (b^{\mathrm{Kerr},m})^A = \frac{4M\mathfrak{a}}{r}\not{\epsilon}^{AB}\partial_B Y_m^{\ell=1}. \tag{171}$$

The non-vanishing Ricci coefficients are

$$\overset{(1)}{\eta}{}^A = (\eta^{\mathrm{Kerr},m})^A = \frac{3M\mathfrak{a}}{r^2}\not{\epsilon}^{AB}\partial_B Y_m^{\ell=1} \quad \textit{and} \quad \overset{(1)}{\underline{\eta}} = \underline{\eta}^{\mathrm{Kerr},m} = -\eta^{\mathrm{Kerr},m}. \tag{172}$$

The non-vanishing curvature components are

$$\overset{(1)}{\beta} = \beta^{\mathrm{Kerr},m} = \frac{\Omega}{r}\eta^{\mathrm{Kerr},m}, \quad \overset{(1)}{\underline{\beta}} = \underline{\beta}^{\mathrm{Kerr},m} = -\beta^{\mathrm{Kerr},m}, \quad \overset{(1)}{\sigma} = \sigma^{\mathrm{Kerr},m} = \frac{6}{r^4}\mathfrak{a}M\cdot Y_m^{\ell=1}.$$

We will refer to this 3-parameter family spanned by the above solutions ($m=-1,0,1$) as the reference $\ell{=}1$ linearised Kerr solutions.

Note that the above family may be parameterised by the $\ell{=}1$-modes of the curvature component $\overset{(1)}{\sigma}$.([22])

Proof. To ease notation, we suppress the superscript m for the proof. We first note that $\not{\mathrm{div}}\, b^{\mathrm{Kerr}}=0$ and $\not{\mathcal{D}}_2^\star b^{\mathrm{Kerr}}=0$, as well as

$$\partial_u (b^{\mathrm{Kerr}})^A = \partial_u\left(\frac{4Ma}{r^3}\gamma^{AC}\varepsilon_C{}^B\partial_B Y_m^{\ell=1}\right) = \frac{12Ma\Omega^2}{r^2}\not{\epsilon}^{AB}\partial_B Y^{\ell=1} = 2\Omega^2(\eta^{\mathrm{Kerr}}-\underline{\eta}^{\mathrm{Kerr}})^A,$$

where we recall that $\not{g}=r^2\gamma$ and that $\varepsilon_A{}^B=\not{\epsilon}_A{}^B$ does not depend on r. Hence, (131)–(133) all hold.

Since also $\not{\mathrm{div}}\,\eta^{\mathrm{Kerr}}=\not{\mathrm{div}}\,\underline{\eta}^{\mathrm{Kerr}}=0$ and $\not{\mathcal{D}}_2^\star\eta^{\mathrm{Kerr}}=\not{\mathcal{D}}_2^\star\underline{\eta}^{\mathrm{Kerr}}=0$, all null structure equations (135)–(147) hold trivially except (142) and (146). Note that the Codazzi equations (145) hold by definition of $\overset{(1)}{\beta}$ and $\overset{(1)}{\underline{\beta}}$ in terms of $\overset{(1)}{\eta}$ and $\overset{(1)}{\underline{\eta}}$. To verify (146), we compute

$$\begin{aligned}\not{\mathrm{curl}}\,\eta^{\mathrm{Kerr}} &= \not{\epsilon}^{BA}\partial_B\eta_A^{\mathrm{Kerr}} = \frac{3Ma}{r^2}\not{\epsilon}^{BA}\partial_B\not{\epsilon}_A{}^C\partial_C Y_m^{\ell=1}\\ &= -\frac{3Ma}{r^4}\Delta_{S^2}Y_m^{\ell=1} = \frac{6}{r^4}aM\cdot Y_m^{\ell=1} = \sigma^{\mathrm{Kerr}},\end{aligned}$$

([22]) The scalar $\overset{(1)}{\sigma}$ has no $\ell{=}0$ mode, as it satisfies the equation $\overset{(1)}{\sigma}=curl\overset{(1)}{\eta}$.

and similarly for $\underline{\eta}^{\mathrm{Kerr}}$. To verify (142), we compute

$$(\Omega \nabla\!\!\!\!/_4 r^2 \eta^{\mathrm{Kerr}})_A = \partial_v (r^2 \eta^{\mathrm{Kerr}}_A) - \frac{\Omega^2}{r}(r^2 \eta^{\mathrm{Kerr}}_A) = -\Omega \beta^{\mathrm{Kerr}}_A,$$
$$(\Omega \nabla\!\!\!\!/_3 r^2 \underline{\eta}^{\mathrm{Kerr}})_A = \partial_u (r^2 \underline{\eta}^{\mathrm{Kerr}}_A) + \frac{\Omega^2}{r}(r^2 \underline{\eta}^{\mathrm{Kerr}}_A) = \Omega \underline{\beta}^{\mathrm{Kerr}}_A.$$

We finally turn to verifying the Bianchi equations (148)–(157). We first note that the ones for $\overset{(1)}{\alpha}$ and $\overset{(1)}{\underline{\alpha}}$, as well as those for $\overset{(1)}{\rho}$, are trivially satisfied. Also,

$$(\Omega \nabla\!\!\!\!/_4 (r^4 \Omega^{-1} \beta^{\mathrm{Kerr}}))_A = (\Omega \nabla\!\!\!\!/_4 (r^3 \eta^{\mathrm{Kerr}}))_A = 0,$$
$$(\Omega \nabla\!\!\!\!/_3 (r^4 \Omega^{-1} \underline{\beta}^{\mathrm{Kerr}}))_A = (\Omega \nabla\!\!\!\!/_3 (r^3 \underline{\eta}^{\mathrm{Kerr}}))_A = 0,$$

verifying (149) and (156). It remains to check that (150) and (153)–(155) are satisfied. For the $\overset{(1)}{\sigma}$ equations, we note

$$\Omega \nabla\!\!\!\!/_4 (r^3 \sigma^{\mathrm{Kerr}}) = -\frac{6}{r^2} a M \Omega^2 Y^{\ell=1}_m = -r^2 \sigma^{\mathrm{Kerr}} \Omega^2 = -\Omega r^3\, \mathrm{c}\!\!\!/\mathrm{url}\, \beta^{\mathrm{Kerr}},$$

and similarly for equation (154). We finally verify (150), noting that (155) is verified analogously:

$$\begin{aligned}
\Omega \nabla\!\!\!\!/_3 (r^2 \Omega \underline{\beta}^{\mathrm{Kerr}})_A &= \Omega \nabla\!\!\!\!/_3 (r \Omega^2 \underline{\eta}^{\mathrm{Kerr}})_A \\
&= \partial_u \left(\frac{3Ma}{r} \Omega^2 \varepsilon_A{}^B \partial_B Y^{\ell=1}_m \right) + \frac{\Omega^2}{r} \left(\frac{3Ma}{r} \Omega^2 \varepsilon_A{}^B \partial_B Y^{\ell=1}_m \right) \\
&= \frac{6Ma}{r^2} \Omega^2 \left(1 - \frac{2M}{r} \right) \varepsilon_A{}^B \partial_B Y^{\ell=1}_m - \frac{3Ma}{r^3} 2M \Omega^2 \varepsilon_A{}^B \partial_B Y^{\ell=1}_m \\
&= r^2 \Omega^2 \varepsilon_A{}^B \partial_B \sigma^{\mathrm{Kerr}} - \frac{6M}{r} \Omega^2 \eta^{\mathrm{Kerr}}_A. \qquad \square
\end{aligned}$$

The reader might wonder why the family in Proposition 6.2.2 is a 3-parameter family of solutions, while the full Kerr metric with fixed mass is a 1-parameter family. This can be explained as follows. When writing down the Kerr metric one fixes an axis of symmetry. Rotations of this axis in space correspond to the same Kerr metric expressed in different coordinates. At the linear level, if we linearised the metric at a *non-trivial* ($a \neq 0$) member of the Kerr family, this would manifest itself in the existence of non-trivial pure gauge solutions corresponding to a rotation of the axis. In contrast, here we are linearising with respect to the *spherically symmetric* Schwarzschild metric. The associated pure gauge solutions of rotating the axis are then trivial in view of the isometry group of the round sphere. Hence, we must see three "basis" Kerr metrics which cannot be connected by a pure gauge transformation. Note the aforementioned trivial pure gauge solutions are seen as $(q_1 = 0, q_2 = Y^1_m)$ generating the trivial solution in Lemma 6.1.3.

Finally, let us combine the 1-dimensional space of reference linearised Schwarzschild solutions and the 3-dimensional space of reference $\ell = 1$ linearised Kerrs in the following definition.

Definition 6.1. Let $\mathfrak{m}$, s_{-1}, s_0 and s_1 be four real parameters. We call the sum of the solution of Proposition 6.2.1 with parameter $\mathfrak{m}$ and the solution of Proposition 6.2.2 satisfying $\sigma^{\mathrm{Kerr}}=\sum_m s_m Y_m^{\ell=1}$ the *reference linearised Kerr solution* with parameters $(\mathfrak{m}, s_{-1}, s_0, s_1)$, and denote it by $\mathscr{K}_{\mathfrak{m},s_i}$, or simply $\mathscr{K}$.

7. The Teukolsky and Regge–Wheeler equations and the gauge invariant hierarchy

In this section, we will introduce the celebrated spin ± 2 Teukolsky equations and the Regge–Wheeler equation and explain the connection between the two and their relation to the full system of linearised gravity.

We begin in §7.1 by defining the above three equations, considering them as second order hyperbolic partial differential equations (PDEs) for independent unknowns α, $\underline{\alpha}$ and P, independently that is of the system of gravitational perturbations. In §7.2, we shall state a well-posedness theorem for the characteristic initial value problem for these equations. We then introduce in §7.3 a fundamental transformation mapping solutions α, $\underline{\alpha}$ of the spin ± 2 Teukolsky equations to solutions P, $\underline{P}$ of the Regge–Wheeler equation. This transformation will play a key role in understanding solutions to the Teukolsky equation itself. As was remarked in §2.1.5, this transformation is a physical space version of transformations appearing in Chandrasekhar [9] for fixed frequencies. Note that §§7.1–7.3 are independent of §5 and §6.

The relation of the above PDEs with the full system of linearised gravity is finally explained in §7.4, where we will see that, given a smooth solution of the system of gravitational perturbations, the curvature components $\overset{(1)}{\alpha}$ and $\overset{(1)}{\underline{\alpha}}$ satisfy the Teukolsky equation. By the above transformations, this gives rise to $\overset{(1)}{P}$ and $\overset{(1)}{\underline{P}}$ satisfying the Regge–Wheeler equation. We shall see also (Proposition 7.4.1) how the latter can be re-expressed in various ways using the Bianchi identities.

7.1. The spin ± 2 Teukolsky equations and the Regge–Wheeler equation

The spin ± 2 Teukolsky equations concern symmetric traceless $S_{u,v}$ 2-tensors, which we shall denote by α and $\underline{\alpha}$, in anticipation of §7.4. For now, let these be defined on a subset $\mathcal{D}\subset\mathcal{M}$. Note that, with our normalisations, it is natural to assume that the rescaled quantities $\Omega^2\alpha$ and $\Omega^{-2}\underline{\alpha}$ are smooth on $\mathcal{D}$, up to and including the horizon $\mathcal{D}\cap\mathcal{H}^+$.

Definition 7.1. Let α be a symmetric traceless $S^2_{u,v}$ 2-tensor defined on a subset $\mathcal{D}\subset\mathcal{M}$ such that $\Omega^2\alpha$ is smooth on $\mathcal{D}$. We say that α satisfies the (tensorial form of the)

Teukolsky equation of spin $+2$, if α satisfies the following PDE:

$$\begin{aligned}\nabla\!\!\!/_4\nabla\!\!\!/_3\alpha+\left(\tfrac{1}{2}\operatorname{tr}\underline{\chi}+2\underline{\widehat{\omega}}\right)\nabla\!\!\!/_4\alpha+\left(\tfrac{5}{2}\operatorname{tr}\chi-\widehat{\omega}\right)\nabla\!\!\!/_3\alpha-\Delta\!\!\!/\alpha\\+\alpha(5\underline{\widehat{\omega}}\operatorname{tr}\chi-\widehat{\omega}\operatorname{tr}\underline{\chi}-4\varrho+2K+\operatorname{tr}\chi\operatorname{tr}\underline{\chi}-4\widehat{\omega}\underline{\widehat{\omega}})=0.\end{aligned}\tag{173}$$

Let $\underline{\alpha}$ be a symmetric traceless $S^2_{u,v}$-tensor on $\mathcal{D}$ such that $\Omega^{-2}\underline{\alpha}$ is smooth on $\mathcal{D}$. We say that $\underline{\alpha}$ satisfies the (tensorial form of the) *Teukolsky equation of spin* -2, if $\underline{\alpha}$ satisfies the following PDE:

$$\begin{aligned}\nabla\!\!\!/_3\nabla\!\!\!/_4\underline{\alpha}+\left(\tfrac{1}{2}\operatorname{tr}\chi+2\widehat{\omega}\right)\nabla\!\!\!/_3\underline{\alpha}+\left(\tfrac{5}{2}\operatorname{tr}\underline{\chi}-\underline{\widehat{\omega}}\right)\nabla\!\!\!/_4\underline{\alpha}-\Delta\!\!\!/\underline{\alpha}\\+\underline{\alpha}(5\widehat{\omega}\operatorname{tr}\underline{\chi}-\underline{\widehat{\omega}}\operatorname{tr}\chi-4\varrho+2K+\operatorname{tr}\chi\operatorname{tr}\underline{\chi}-4\widehat{\omega}\underline{\widehat{\omega}})=0.\end{aligned}\tag{174}$$

We note that the Teukolsky equation of spin -2 is obtained from that of spin $+2$ by interchanging $\nabla\!\!\!/_3$ with $\nabla\!\!\!/_4$ and underlined Schwarzschild quantities with non-underlined ones.

Let us repeat the explicit characterization of smoothness up to the horizon from §5 in terms of Eddington–Finkelstein double null coordinates u and v: An $S^2_{u,v}$-tensor Θ extends smoothly to the horizon if, in the spherical coordinate chart, the components Θ_{CD} are smooth functions of the double null Eddington–Finkelstein coordinates on the interior $\mathcal{M}^o$, and if, for any $n_1,n_2,n_3\in\mathbb{N}_0$ and $A\in\{1,2\}$, the functions

$$(e^{u/2M}\partial_u)^{n_1}(\partial_v)^{n_2}(\partial_{\theta_A})^{n_3}\Theta_{CD}$$

extend continuously to the boundary $\mathcal{H}^+$.

According to Proposition 7.4.1, it will follow that, given a solution $\mathscr{S}$ of linearised gravity, then the quantities $\overset{(1)}{\alpha}$ and $\overset{(1)}{\underline{\alpha}}$ satisfy the spin ±2 Teukolsky equations, respectively. For now, however, we will study the Teukolsky equation in its own right, independently of the full system.

The other equation to be defined in this section is the Regge–Wheeler equation, to be satisfied again by a symmetric traceless $S^2_{u,v}$-tensor P.

Definition 7.2. Let P be a smooth, symmetric traceless $S^2_{u,v}$-tensor on $\mathcal{D}$. We say that P satisfies the (tensorial form of the) *Regge–Wheeler equation*, if P satisfies the following PDE:

$$\begin{aligned}\nabla\!\!\!/_3\nabla\!\!\!/_4P+\nabla\!\!\!/_4\nabla\!\!\!/_3P-2\Delta\!\!\!/P+(5\operatorname{tr}\underline{\chi}+\underline{\widehat{\omega}})\cdot\nabla\!\!\!/_4P+(5\operatorname{tr}\chi+\widehat{\omega})\nabla\!\!\!/_3P\\+P(4K-(3\operatorname{tr}\chi+\widehat{\omega})2\operatorname{tr}\chi-4(\operatorname{tr}\chi)^2+2\nabla\!\!\!/_3\operatorname{tr}\chi-8\widehat{\omega}\operatorname{tr}\chi)=0.\end{aligned}\tag{175}$$

In §7.3, we shall show that, given solutions α and $\underline{\alpha}$ of the spin ±2 Teukolsky equations, respectively, we can derive two solutions P and $\underline{P}$, respectively, of the Regge–Wheeler equation. In view of the above remarks, it follows that we can associate such solutions to a solution $\mathscr{S}$ of the full system of linearised gravity. As with (173) and (174), however, for now we shall consider the equation (175) in its own right. We again note that the Regge–Wheeler equation was first derived in [61] in the context of the theory of "metric perturbations".

To bring (175) in a more familiar form, we define the weighted symmetric traceless $S^2_{u,v}$-tensors

$$\Psi = r^5 P \quad \text{and} \quad \underline{\Psi} = r^5 \underline{P}, \tag{176}$$

and conclude

Corollary 7.1. *If P satisfies the Regge–Wheeler equation, then the weighted symmetric traceless $S^2_{u,v}$-tensor $\Psi = r^5 P$ satisfies the equation*

$$\Omega \nabla\!\!\!/_3(\Omega \nabla\!\!\!/_4 \Psi) - \left(1 - \frac{2M}{r}\right) \Delta\!\!\!/ \Psi + V \Psi = 0 \quad \textit{with } V = \left(\frac{4}{r^2} - \frac{6M}{r^3}\right)\left(1 - \frac{2M}{r}\right). \tag{177}$$

Proof. Direct computation using the Schwarzschild background values of §4.2.4. □

In the context of the proof of Theorem 1, we will do estimates directly at the level of the tensorial equation (177).

Remark 7.1. In the literature, the Regge–Wheeler equation is typically stated for a scalar function f as

$$\frac{1}{1 - 2M/r} \partial_u \partial_v f - \Delta\!\!\!/ f - \frac{6M}{r^3} f = 0.$$

It is not hard to see that, if two smooth functions f and g satisfy the scalar Regge–Wheeler equation, then the symmetric traceless tensor $\phi = r^2 \mathcal{D}\!\!\!/^{\star}_2 \mathcal{D}\!\!\!/^{\star}_1 (f, g)$ satisfies (177), the additional factor of $4/r^2$ appearing from the commutation of the angular operators with $\Delta\!\!\!/$. Note also that one can reconstruct f and g uniquely from ϕ, up to the $\ell = 0, 1$ modes.

7.2. The characteristic initial value problem

For completeness, we state here a standard well-posedness theorem for both the Teukolsky and the Regge–Wheeler equation. In view of future applications, we formulate it in the context of a characteristic problem: We fix a sphere $S^2_{u_0,v_0}$ in $\mathcal{M}$ and consider the outgoing Schwarzschild light cone $C_{u_0} = \{u_0\} \times \{v \geqslant v_0\} \times S^2$ and the ingoing light cone $C_{v_0} = \{u \geqslant u_0\} \times \{v_0\} \times S^2$ on which the data are being prescribed. In our convention, C_{v_0} includes the horizon sphere S^2_{∞,v_0}.

We begin with the spin ± 2 Teukolsky equations.

THEOREM 7.1. (Well-posedness for Teukolsky of spin +2) *Given a sphere* $S^2_{u_0,v_0}$ *with corresponding null cones* C_{u_0} *and* C_{v_0}*, prescribe*

- *along* C_{v_0} *a symmetric traceless* $S^2_{u,v}$*-tensor* $\alpha_{\circ,\mathrm{in}}$*, such that* $\Omega^2\alpha_{\circ,\mathrm{in}}$ *is smooth*;
- *along* C_{u_0} *a smooth symmetric traceless* $S^2_{u,v}$*-tensor* $\alpha_{\circ,\mathrm{out}}$ *satisfying* $\alpha_{\circ,\mathrm{out}}=\alpha_{\circ,\mathrm{in}}$ *on* $S^2_{u_0,v_0}$.

Then, there exists a unique smooth symmetric traceless $S^2_{u,v}$ *2-tensor* $\Omega^2\alpha$ *defined on* $\mathcal{M}\cap\{u\geqslant u_0\}\cap\{v\geqslant v_0\}$ *such that*

- α *satisfies the Teukolsky equation of spin* +2 (173) *in* $\mathcal{M}\cap\{u\geqslant u_0\}\cap\{v\geqslant v_0\}$;
- $\Omega^2\alpha\big|_{C_{u_0}}=\Omega^2\alpha_{\circ,\mathrm{in}}$ *and* $\alpha\big|_{C_{v_0}}=\alpha_{\circ,\mathrm{out}}$.

We emphasise that, in our convention, the set $\mathcal{M}\cap\{u\geqslant u_0\}\cap\{v\geqslant v_0\}$ includes the intersection $\mathcal{H}^+\cap\{v\geqslant v_0\}$.

THEOREM 7.2. (Well-posedness for Teukolsky of spin −2) *Theorem* 7.1 *holds replacing all* α *by* $\underline{\alpha}$*, all* Ω^2 *by* Ω^{-2} *and* (173) *by* (174) *in the above statement.*

The well-posedness statement for the Regge–Wheeler equation (175) is entirely analogous.

THEOREM 7.3. (Well-posedness for Regge–Wheeler) *Given a sphere* $S^2_{u_0,v_0}$ *with corresponding null cones* C_{u_0} *and* C_{v_0}*, prescribe*

- *along* C_{v_0} *a smooth symmetric traceless* $S^2_{u,v}$*-tensor* $P_{\circ,\mathrm{in}}$;
- *along* C_{u_0} *a smooth symmetric traceless* $S^2_{u,v}$*-tensor* $P_{\circ,\mathrm{out}}$ *satisfying* $P_{\circ,\mathrm{out}}=P_{\circ,\mathrm{in}}$ *on* $S^2_{u_0,v_0}$.

Then, there exists a unique smooth symmetric traceless $S_{u,v}$ *2-tensor* P *defined on* $\mathcal{M}\cap\{u\geqslant u_0\}\cap\{v\geqslant v_0\}$ *such that*

- P *satisfies the Regge–Wheeler equation* (175) *in* $\mathcal{M}\cap\{u\geqslant u_0\}\cap\{v\geqslant v_0\}$;
- $P\big|_{C_{u_0}}=P_{\circ,\mathrm{in}}$ *and* $P\big|_{C_{v_0}}=P_{\circ,\mathrm{out}}$.

The proof of all theorems above are easily obtained following [12] and [33]. See also [62].

7.3. The transformation theory: Definitions of ψ, $\underline{\psi}$, P and $\underline{P}$

We now describe a transformation theory relating solutions of the Teukolsky equations to solutions of the Regge–Wheeler equation.

Given a solution α of the Teukolsky equation of spin +2, we can define the following *derived* quantities:

$$\psi := -\frac{1}{2r\Omega^2} \nabla\!\!\!\!/_3 (r\Omega^2\alpha), \tag{178}$$

$$P := \frac{1}{r^3\Omega} \nabla\!\!\!\!/_3 (\psi r^3 \Omega). \tag{179}$$

Similarly, given a solution $\underline{\alpha}$ of the Teukolsky equation of spin -2, we can define the following *derived* quantities:

$$\underline{\psi} := \frac{1}{2r\Omega^2} \nabla\!\!\!\!/_4 (r\Omega^2\underline{\alpha}), \tag{180}$$

$$\underline{P} := -\frac{1}{r^3\Omega} \nabla\!\!\!\!/_4 (\underline{\psi} r^3 \Omega). \tag{181}$$

These quantities are again symmetric traceless $S^2_{u,v}$ 2-tensors.

The following proposition can be proven by a straightforward computation.

PROPOSITION 7.3.1. *Let α be a solution of the Teukolsky equation of spin $+2$ on $\mathcal{M}\cap\{u\geqslant u_0\}\cap\{v\geqslant v_0\}$, as arising from Theorem 7.1. Then, the symmetric traceless tensor P as defined through (178) and (179) satisfies the Regge–Wheeler equation on the intersection $\mathcal{M}\cap\{u\geqslant u_0\}\cap\{v\geqslant v_0\}$.*

Now, let $\underline{\alpha}$ be a solution of the Teukolsky equation of spin -2 on the intersection $\mathcal{M}\cap\{u\geqslant u_0\}\cap\{v\geqslant v_0\}$ as arising from Theorem 7.2. Then, the symmetric traceless tensor $\underline{P}$ as defined through (180) and (181) satisfies the Regge–Wheeler equation on the intersection $\mathcal{M}\cap\{u\geqslant u_0\}\cap\{v\geqslant v_0\}$.

We note that Fourier space analogues of the above transformations were first discovered by Chandrasekhar [9], who also discussed differential transformations mapping solutions of Regge–Wheeler to solutions of Teukolsky. In this paper, however, it is the physical space structure of the above transformations which we shall exploit, in particular, the fact that they can be interpreted as transport equations, which allow α (respectively $\underline{\alpha}$) to be recovered from P (respectively $\underline{P}$) *and initial data*.

7.4. The connection with the system of gravitational perturbations

We will now finally relate the equations presented above to the full system of linearised gravity.

Let $\mathscr{S}$ be a smooth solution of the system of gravitational perturbations, and recall the quantities $\overset{(1)}{\alpha}$ and $\overset{(1)}{\underline{\alpha}}$ of §5.2.1. Note that both these symmetric traceless $S^2_{u,v}$ 2-tensors are *gauge invariant* in the sense that any of the pure gauge solutions discussed in §6.1 satisfies $\overset{(1)}{\alpha}=\overset{(1)}{\underline{\alpha}}=0$. The latter fact can be checked directly from (161)–(164). We note also that $\overset{(1)}{\alpha}$ and $\overset{(1)}{\underline{\alpha}}$ vanish for the 4-dimensional reference linearised Kerr family of §6.2. We will in fact show in Appendix B.1 that, provided that $\mathscr{S}$ is asymptotically flat (see §8.3), then the vanishing identically of $\overset{(1)}{\alpha}$ and $\overset{(1)}{\underline{\alpha}}$ implies that $\mathscr{S}$ is a sum of a pure gauge solution and a reference linearised Kerr.

Remarkably, as was first shown by Bardeen–Press [4], $\overset{(1)}{\alpha}$ and $\overset{(1)}{\underline{\alpha}}$ satisfy the Teukolsky equation of spin $+2$ and spin -2 respectively. Combining this fact with results of the previous section yields the following.

Proposition 7.4.1. *Let $\mathscr{S}$ be a smooth solution of the system of gravitational perturbations on a domain $\mathcal{D}\subset\mathcal{M}$, and consider the curvature components $\overset{(1)}{\alpha}$ and $\overset{(1)}{\underline{\alpha}}$ which are part of the solution $\mathscr{S}$. Then, $\overset{(1)}{\alpha}$ satisfies the Teukolsky equation of spin 2, and $\overset{(1)}{\underline{\alpha}}$ satisfies the Teukolsky equation of spin -2. Moreover, the derived quantities (178)–(181) that are defined for any solution of the Teukolsky equation, now denoted $\overset{(1)}{\psi}$, $\overset{(1)}{P}$ and $\overset{(1)}{\underline{\psi}}$, $\overset{(1)}{\underline{P}}$, can also be re-expressed in terms of derivatives of curvature components and Ricci coefficients, using the Bianchi and null structure equations. We have*

$$\overset{(1)}{\psi}=\not{\mathcal{D}}_2^\star\overset{(1)}{\beta}+\tfrac{3}{2}\varrho\overset{(1)}{\widehat{\chi}}\quad\textit{and}\quad\overset{(1)}{\underline{\psi}}=\not{\mathcal{D}}_2^\star\overset{(1)}{\underline{\beta}}-\tfrac{3}{2}\varrho\overset{(1)}{\underline{\widehat{\chi}}},\tag{182}$$

as well as

$$\overset{(1)}{P}=\not{\mathcal{D}}_2^\star\not{\mathcal{D}}_1^\star(-\overset{(1)}{\varrho},\overset{(1)}{\sigma})+\tfrac{3}{4}\varrho\,\mathrm{tr}\,\chi(\overset{(1)}{\widehat{\chi}}-\overset{(1)}{\underline{\widehat{\chi}}}),\tag{183}$$

$$\overset{(1)}{\underline{P}}=\not{\mathcal{D}}_2^\star\not{\mathcal{D}}_1^\star(-\overset{(1)}{\varrho},-\overset{(1)}{\sigma})+\tfrac{3}{4}\varrho\,\mathrm{tr}\,\chi(\overset{(1)}{\widehat{\chi}}-\overset{(1)}{\underline{\widehat{\chi}}}).\tag{184}$$

Proof. The equation for $\overset{(1)}{\alpha}$ is easily derived from taking a 4-derivative of (148) and using (149), (139) and (103). The equation for $\overset{(1)}{\underline{\alpha}}$ follows from taking a 3-derivative of (157) and using (156). The identities for $\overset{(1)}{\psi}$ and $\overset{(1)}{\underline{\psi}}$ are immediate from (148) and (157). To see the identity for $\overset{(1)}{\underline{P}}$, one uses the Bianchi equation (155) and the null structure equation (141), as well as the formulas of §4.3.2, to obtain

$$\begin{aligned}\not\nabla_4\overset{(1)}{\underline{\psi}}&=\not{\mathcal{D}}_2^\star(\not{\mathcal{D}}_1^\star(\overset{(1)}{\varrho},\overset{(1)}{\sigma})-3\overset{(1)}{\underline{\eta}}\varrho-\mathrm{tr}\,\chi\,\overset{(1)}{\underline{\beta}}-\widehat{\omega}\overset{(1)}{\underline{\beta}})-\tfrac{1}{2}\,\mathrm{tr}\,\chi\,\not{\mathcal{D}}_2^\star\overset{(1)}{\underline{\beta}}\\&\quad+\tfrac{9}{4}\varrho\,\mathrm{tr}\,\chi\,\overset{(1)}{\underline{\widehat{\chi}}}-\tfrac{3}{2}\varrho(-\widehat{\omega}\overset{(1)}{\underline{\widehat{\chi}}}-\tfrac{1}{2}\,\mathrm{tr}\,\chi\,\overset{(1)}{\underline{\widehat{\chi}}}-\tfrac{1}{2}\,\mathrm{tr}\,\underline{\chi}\,\overset{(1)}{\widehat{\chi}}-2\not{\mathcal{D}}_2^\star\overset{(1)}{\underline{\eta}})\\&=\not{\mathcal{D}}_2^\star\not{\mathcal{D}}_1^\star(\overset{(1)}{\varrho},\overset{(1)}{\sigma})-\tfrac{3}{4}\varrho\,\mathrm{tr}\,\chi\,(\overset{(1)}{\widehat{\chi}}-\overset{(1)}{\underline{\widehat{\chi}}})-\tfrac{3}{2}\,\mathrm{tr}\,\chi\,(\not{\mathcal{D}}_2^\star\overset{(1)}{\underline{\beta}}-\tfrac{3}{2}\varrho\overset{(1)}{\underline{\widehat{\chi}}})-\widehat{\omega}(\not{\mathcal{D}}_2^\star\overset{(1)}{\underline{\beta}}-\tfrac{3}{2}\varrho\overset{(1)}{\underline{\widehat{\chi}}}).\end{aligned}\tag{185}$$

The computation for $\overset{(1)}{P}$ is completely analogous. □

Remark 7.2. In view of the gauge invariance of $\overset{(1)}{\alpha}$ and $\overset{(1)}{\underline{\alpha}}$ in the sense above, it follows from the definitions (178)–(181) that the quantities $\overset{(1)}{P}$, $\overset{(1)}{\underline{P}}$, $\overset{(1)}{\psi}$ and $\overset{(1)}{\underline{\psi}}$ are also manifestly gauge invariant. We note however (see Appendix B.2) that there exist asymptotically flat solutions $\mathscr{S}$ which are *not* pure gauge such that $\overset{(1)}{P}$ and $\overset{(1)}{\underline{P}}$ identically vanish.

The fact that $\overset{(1)}{P}$ and $\overset{(1)}{\underline{P}}$ above satisfy the Regge–Wheeler equation (175), together with the relations (178)–(181), but also (182)–(184), will be the key to estimating the Teukolsky equations and unlocking the system of linearised gravity.

8. Initial data and well-posedness of linearised gravity

We turn in this section to the well-posedness of the system of linearised gravity of §5.2.

We first describe how to prescribe initial data in §8.1 below. Then, we shall formulate the well-posedness theorem in §8.2. Finally, in §8.3, we define what it means for data to be asymptotically flat.

8.1. Seed data on an initial double null cone

In this section, we describe how to prescribe initial data for the system of gravitational perturbations derived in §5.

The setting will be that of a characteristic initial value problem: We fix a sphere $S^2_{u_0,v_0}$ in $\mathcal{M}$, and consider the outgoing Schwarzschild light cone $C_{u_0}=\{u_0\}\times\{v\geqslant v_0\}\times S_2$ and the ingoing light cone $C_{v_0}=\{u\geqslant u_0\}\times\{v_0\}\times S^2$ on which the data are being prescribed. Initial data are determined by so-called "seed data" that can be prescribed freely. Recall that, in our convention, C_{v_0} includes the horizon sphere S^2_{∞,v_0}. The definition of seed data is given below. We remark that this is essentially a linearised version of the prescription given in [11].

Definition 8.1. Given a sphere $S^2_{u_0,v_0}$ with corresponding null cones C_{u_0} and C_{v_0}, a *smooth seed initial data set* consists of prescribing

- along C_{v_0} a smooth symmetric traceless $S^2_{u,v}$ 2-tensor $\overset{(1)}{\hat{\not{g}}}_{\circ,\mathrm{in}}(u,\theta,\phi)$;
- along C_{u_0} a smooth symmetric traceless $S^2_{u,v}$ 2-tensor $\overset{(1)}{\hat{\not{g}}}_{\circ,\mathrm{out}}(v,\theta,\phi)$;
- along C_{v_0} a smooth function $\Omega^{-1}\overset{(1)}{\Omega}_{\circ,\mathrm{in}}(u,\theta,\phi)$;
- along C_{u_0} a smooth function $\overset{(1)}{\Omega}_{\circ,\mathrm{out}}(v,\theta,\phi)$;
- along C_{u_0} a smooth $S^2_{u,v}$-1-form $\overset{(1)}{b}_{\circ}(v,\theta,\phi)$;
- on the sphere S^2_{∞,v_0} a smooth function $\Omega^{-2}(\Omega\overset{(1)}{\mathrm{tr}}\,\underline{\chi})_{\circ}(\theta,\phi)$;
- on the sphere S^2_{∞,v_0} a smooth function $\mathrm{tr}_{\not{g}}\,\overset{(1)}{\not{g}}_{\circ}(\theta,\phi)$;
- on the sphere S^2_{∞,v_0} a smooth function $(\Omega\overset{(1)}{\mathrm{tr}}\,\chi)_{\circ}(\theta,\phi)$;
- on the sphere S^2_{∞,v_0} a smooth 1-form $\overset{(1)}{\eta}_{\circ}(\theta,\phi)$.

We emphasise that since by our convention that C_{v_0} includes S^2_{∞,v_0}, smoothness is to be understood above geometrically, up to and including the horizon.

8.2. The well-posedness theorem

We can now state the fundamental well-posedness theorem for linearised gravity on Schwarzschild:

THEOREM 8.1. (Well-posedness) *Fix a sphere $S^2_{u_0,v_0}$ and consider a smooth seed initial data set as in Definition 8.1. Then, there exists a unique smooth solution*

$$\mathscr{S} = (\overset{(1)}{\hat{\not g}}, \overset{(1)}{\sqrt{\not g}}, \overset{(1)}{\Omega}, \overset{(1)}{b}, (\Omega \overset{(1)}{\mathrm{tr}} \chi), (\Omega \overset{(1)}{\mathrm{tr}} \underline{\chi}), \overset{(1)}{\widehat{\chi}}, \overset{(1)}{\widehat{\underline{\chi}}}, \overset{(1)}{\underline{\eta}}, \overset{(1)}{\eta}, \overset{(1)}{\omega}, \overset{(1)}{\underline{\omega}}, \overset{(1)}{\alpha}, \overset{(1)}{\beta}, \overset{(1)}{\varrho}, \overset{(1)}{\sigma}, \overset{(1)}{\underline{\beta}}, \overset{(1)}{\underline{\alpha}}, \overset{(1)}{K})$$

of linearised gravity defined in $\mathcal{M}\cap\{u\geqslant u_0\}\cap\{v\geqslant v_0\}$ which agrees with the seed data on C_{u_0} and C_{v_0}.

Remark 8.1. Recall that from §5.2.1 that our notion of smoothness includes the statement that the weighted quantities (130) of $\mathscr{S}$ extend smoothly to $\mathcal{H}^+\cap\{v\geqslant v_0\}$. Note that our initial smoothness assumptions on the weighted quantities $\Omega^{-1}\overset{(1)}{\Omega}_{\circ,\mathrm{in}}$, etc., is indeed consistent with this.

Proof. We give a brief outline, exploiting Theorem 7.1, leaving the details to the reader.

First, we show that the equations uniquely determine from seed data all dynamical quantities on $C_{u_0}\cup C_{v_0}$ such that all tangential equations are satisfied. (This is in fact implicitly carried out in Appendix A.)

Now, since by Proposition 7.4.1, given a solution $\mathscr{S}$, the quantities $\overset{(1)}{\alpha}$ and $\overset{(1)}{\underline{\alpha}}$ satisfy the spin ± 2 Teukolsky equations, we can determine these globally from their initial values on $C_{u_0}\cup C_{v_0}$ by applying Theorem 7.1. Once these are determined we may order a subset of the remaining equations hierarchically so all remaining quantities are determined by the previous by integrating transport equations or by taking derivatives. For instance, given $\overset{(1)}{\alpha}$ and $\overset{(1)}{\underline{\alpha}}$, then (139) can be integrated as a linear o.d.e. to determine $\overset{(1)}{\widehat{\chi}}$ and $\overset{(1)}{\widehat{\underline{\chi}}}$ from seed data. (Note that the projection to the $\ell=0,1$ modes behaves differently to the rest; cf. §9.5.) Finally, the fact that the tangential equations hold initially on $C_{u_0}\cup C_{v_0}$ can be used to show that the complete system of equations, i.e. not just those used to construct the solution, is satisfied ("propagation of constraints"). $\square$

8.3. Pointwise asymptotic flatness

As we shall see, our main boundedness theorem (Theorem 3) is most naturally formulated in terms of a solution $\underset{\vee}{\mathscr{S}}$ arising from Theorem 8.1, where the seed data satisfy certain "gauge normalisation conditions" (see §9.1) and such that certain weighted energy quantities are bounded (see the norms in §10.1.1 and §10.3.1). A sufficient condition which ensures that given a general solution $\mathscr{S}$ arising from non-normalised seed data as in Theorem 8.1, a solution $\underset{\vee}{\mathscr{S}}$ can be associated with $\mathscr{S}$ (by addition of a pure gauge solution) satisfying the assumptions of Theorem 3 is to assume pointwise asymptotic flatness on the seed data. (See the statement of Theorem 9.1.) We give the relevant definition of this notion in this section.

To keep the notation concise, we first define the following derived quantities along C_{u_0} from a smooth seed initial data set as in Definition 8.1:

$$2\overset{(1)}{\hat{\chi}}_{\circ,\mathrm{out}} = \nabla\!\!\!/_4 \overset{(1)}{\hat{\not{g}}}_{\circ,\mathrm{out}} - 2\Omega^{-1}\mathcal{D}\!\!\!/_2^{\star}\overset{(1)}{b}_{\circ},$$

which is a symmetric traceless $S^2_{u,v}$ 2-tensor,

$$2\overset{(1)}{\alpha}_{\circ,\mathrm{out}} = r^{-2}\Omega\nabla\!\!\!/_4(r^2\Omega^{-1}\nabla\!\!\!/_4\overset{(1)}{\hat{\not{g}}}_{\circ,\mathrm{out}} - 2\Omega^{-2}r^2\mathcal{D}\!\!\!/_2^{\star}\overset{(1)}{b}_{\circ}),$$

which is a symmetric traceless $S^2_{u,v}$ 2-tensor,

$$\overset{(1)}{\omega}_{\circ} = \partial_v(\Omega^{-1}\overset{(1)}{\Omega}_{\circ,\mathrm{out}}),$$

which is a scalar function. Note that these quantities are uniquely determined in terms of the seed data.

Let us also agree on a shorthand notation to handle higher derivatives. For an $S^2_{u,v}$-tensor ξ of rank n on $\mathcal{M}$, we define, for any $n_1 \geqslant 0$ and $n_2 \geqslant 0$,

$$\mathfrak{D}_{n_1,n_2}\xi = (r\nabla\!\!\!/)^{n_1}(r\Omega\nabla\!\!\!/_4)^{n_2}\xi,$$

producing an $S^2_{u,v}$-tensor of rank $n+n_1$ on $\mathcal{M}$.

We may now state our pointwise notion of asymptotic flatness.

Definition 8.2. We call a seed initial data set *asymptotically flat* with weight s to order n provided the seed data satisfies the following estimates along C_{u_0} for some $0<s\leqslant 1$ and any $n_1 \geqslant 0$ and $n_2 \geqslant 0$ with $n_1+n_2 \leqslant n$:

$$|\Omega^{-1}\overset{(1)}{\Omega}_{\circ,\mathrm{out}}| + |\mathfrak{D}_{n_1,n_2}(r^{2+s}\overset{(1)}{\omega}_{\circ})| \leqslant C_{\circ,n_1,n_2} \tag{186}$$

$$|\mathfrak{D}_{n_1,n_2}(r\overset{(1)}{b}_{\circ})| \leqslant C_{\circ,n_1,n_2}, \tag{187}$$

$$|\overset{(1)}{\hat{\not{g}}}_{\circ,\mathrm{out}}| + |\mathfrak{D}_{n_1,n_2}(r^2\overset{(1)}{\hat{\chi}}_{\circ,\mathrm{out}})| + |\mathfrak{D}_{n_1,n_2}(r^{3+s}\overset{(1)}{\alpha}_{\circ,\mathrm{out}})| \leqslant C_{\circ,n_1,n_2}, \tag{188}$$

for some constant $C_{\circ,n_1,n_2}$ depending on n_1 and n_2. We say that the seed data are asymptotically flat to all orders if the above bounds hold for any n.

Observe that a trivial choice to construct (physically interesting) asymptotically flat seed data is to choose $\overset{(1)}{\hat{\slashed{g}}}_{\circ,\mathrm{out}}$ of compact support on C_{u_0}, and

$$\Omega^{-1}\overset{(1)}{\Omega}_{\circ,\mathrm{out}} \equiv 0 \quad \text{and} \quad \overset{(1)}{b}_{\circ} \equiv 0$$

along C_{u_0}.

We will show in Theorem A.1 of Appendix A that asymptotically flat seed data lead in particular to a hierarchy of decay for all quantities that moreover propagates under evolution by Theorem 8.1.

9. Gauge-normalised solutions and identification of the Kerr parameters

In this section, we define the two gauge-normalisations which will play a fundamental role in this paper and we identify the correct linearised Kerr parameters of a general asymptotically flat solution $\mathscr{S}$ to our system.

We first define in §9.1 what it means for a solution $\mathscr{S}$ to be initial-data normalised. (This condition can be read explicitly from the seed data.) Our main boundedness theorem (Theorem 3 of §10.3) will then concern such normalised solutions.

We then show, in §9.2, that given a solution $\mathscr{S}$ arising from asymptotically flat seed data in the sense of Definition 8.2 above, we can indeed associate with it an initial data-normalised solution $\underset{\vee}{\mathscr{S}}$, which is realised by adding to $\mathscr{S}$ a pure gauge solution $\underset{\vee}{\mathscr{G}}$. Importantly, $\underset{\vee}{\mathscr{G}}$ can be explicitly determined by the seed data of $\mathscr{S}$ and is itself asymptotically flat in the sense of Definition 8.2. This result is stated as Theorem 9.1.

We next define in §9.3 a renormalised solution $\overset{\wedge}{\mathscr{S}}$ realised by the addition of an additional pure gauge solution $\overset{\wedge}{\mathscr{G}}$ determined by the behaviour of $\underset{\vee}{\mathscr{S}}$ along the event horizon. We will call $\overset{\wedge}{\mathscr{S}}$ the horizon-renormalised solution, and it will be the object of our main decay theorem (Theorem 4 of §10.4). As opposed to the pure gauge solution defining $\underset{\vee}{\mathscr{S}}$, the pure gauge solution defining $\overset{\wedge}{\mathscr{S}}$ is not explicitly computable from the seed data of $\mathscr{S}$. Only in the final proof of the decay theorem will we show that the pure gauge solution defining $\overset{\wedge}{\mathscr{S}}$ is itself bounded (with the appropriate weights near null infinity) by the seed data of $\underset{\vee}{\mathscr{S}}$.

In §9.4, we prove several global properties satisfied by the solutions $\underset{\vee}{\mathscr{S}}$ and $\overset{\wedge}{\mathscr{S}}$ that will be exploited later.

Finally, in §9.5, we show that the projection of $\underset{\vee}{\mathscr{S}}$ to the $\ell=0,1$ modes defines a unique linearised Kerr solution $\mathscr{K}_{\mathfrak{m},s_i}$. This is the statement of Theorem 9.2. In particular, the final Kerr parameters can indeed be read off from initial data.

9.1. Initial-data normalised solutions $\underset{\vee}{\mathscr{S}}$

In this section we define the notion of an initial-data normalised solution. As we will show in Theorem 9.1 below, given a seed initial data set and its associated solution $\mathscr{S}$, we can find a pure gauge solution $\mathscr{G}$ such that the initial data for the sum $\mathscr{S}+\mathscr{G}$ satisfies all of these conditions.

Definition 9.1. Consider a seed data set as in Definition 8.1 and let $\mathscr{S}$ be the resulting solution given by Theorem 8.1. We say that the initial data satisfies

- the *lapse and shift condition* if

$$\begin{aligned} \partial_u(\Omega^{-1}\overset{(1)}{\Omega}) = \overset{(1)}{\underline{\omega}} = 0 \quad & \text{along the null hypersurface } C_{v_0}, \\ \partial_v(\Omega^{-1}\overset{(1)}{\Omega}) = \overset{(1)}{\omega} = 0 \quad & \text{along the null hypersurface } C_{u_0}, \end{aligned} \tag{189}$$

$$\overset{(1)}{b}{}^A = \tfrac{2}{3} r^3 \not{g}^{AB} \partial_B \overset{(1)}{\sigma}_{\ell=1} \quad \text{along the null hypersurface } C_{u_0}. \tag{190}$$

- the *round sphere condition at infinity* provided

$$\lim_{v\to\infty} r^2 \overset{(1)}{K}_{\ell\geqslant 2}(u_0, v, \theta, \phi) = 0 \quad \text{along the null hypersurface } C_{u_0}, \tag{191}$$

$$\lim_{v\to\infty} r^2 \not{\mathcal{D}}_2^\star \not{\mathcal{D}}_2 \overset{(1)}{\hat{\not{g}}}(u_0, v, \theta, \phi) = 0 \quad \text{along the null hypersurface } C_{u_0}. \tag{192}$$

- the *horizon gauge conditions* if the following hold on $S_{\mathcal{H}} := S^2_{\infty, v_0}$:

$$(\overset{(1)}{\Omega \operatorname{tr} \chi}) = 0, \tag{193}$$

$$\overset{(1)}{\varrho} - \overset{(1)}{\varrho}_{\ell=0} + \not{\mathrm{d}}\mathrm{iv}\, \overset{(1)}{\eta} = 0, \tag{194}$$

- the *auxiliary gauge conditions* if the following holds on $S_{\mathcal{H}} := S^2_{\infty, v_0}$:

$$2\Omega^{-1}\overset{(1)}{\Omega}|_{\ell=0} = 4M^2 \overset{(1)}{\varrho}_{\ell=0}, \tag{195}$$

$$2\Omega^{-1}\overset{(1)}{\Omega}|_{\ell=1} = 0, \tag{196}$$

$$\frac{1}{\Omega^2}(\overset{(1)}{\Omega \operatorname{tr} \underline{\chi}})_{\ell=0,1} = 0, \tag{197}$$

$$\overset{(1)}{\sqrt{\not{g}}}_{\ell=1} = 0. \tag{198}$$

Finally, we call the solution $\mathscr{S}$ *initial-data normalised* if it satisfies all gauge conditions (189)–(198) above. We typically will denote such solutions by $\underset{\vee}{\mathscr{S}}$.

We note that the above conditions can all be written explicitly in terms of the seed data. The auxilliary gauge conditions are related to chosing a centre of mass frame.

We also immediately note, by straightforward computation, the following result.

PROPOSITION 9.1.1. *The reference Kerr solutions $\mathscr{K}$ of Definition 6.1 are initial data normalised.*

9.2. Achieving the initial-data normalisation for a general $\mathscr{S}$

In this section, we prove the existence of a pure gauge solution $\underset{\vee}{\mathscr{G}}$ such that, upon adding these to a given solution $\mathscr{S}$ arising from regular asymptotically flat seed data, the resulting solution $\underset{\vee}{\mathscr{S}}$ is generated by data satisfying all conditions of §9.1. This will define the initial-data normalised solution.

THEOREM 9.1. *Consider a seed data set as in Definition 8.1 and let $\mathscr{S}$ be the resulting solution given by Theorem 8.1. Assume that the seed data are asymptotically flat with weight s to order $n\geqslant 10$, as in Definition 8.2.*

Then there exists a pure gauge solution $\underset{\vee}{\mathscr{G}}$, explicitly computable from the seed data of $\mathscr{S}$, such that the sum

$$\underset{\vee}{\mathscr{S}} \doteq \mathscr{S} + \underset{\vee}{\mathscr{G}}$$

is initial-data normalised, i.e. all gauge conditions (189)–(198) of Definition 9.1 hold for $\underset{\vee}{\mathscr{S}}$. The pure gauge solution $\underset{\vee}{\mathscr{G}}$ is unique and arises itself from seed data which are asymptotically flat to order $n-2$.

9.2.1. Overview of the proof of Theorem 9.1

The proof of Theorem 9.1 requires a few preparatory propositions, collected in the four §§9.2.2–9.2.5. The proof proper will then be carried out in §9.2.6. For the propositions, we make frequent use of Lemmas 6.1.1–6.1.3. Let us describe briefly what is achieved in each individual section:

(1) In §9.2.2 we prove that the families of Lemmas 6.1.1 and 6.1.2 can generate pure gauge solutions with arbitrary prescribed linearised lapse $\Omega^{-1}\overset{(1)}{\Omega}$ along a double null-cone $C_{u_0}\cup C_{v_0}$ emanating from a fixed sphere $S^2_{u_0,v_0}$. Such a solution will clearly be useful to achieve (189), (195) and (196).

(2) The question of uniqueness of such pure gauge solutions is then addressed in §9.2.3. In view of the linearity of the theory, this is equivalent to understanding all gauge solutions generated by Lemmas 6.1.1 and 6.1.2, which do not change $\Omega^{-1}\overset{(1)}{\Omega}$ on $C_{u_0}\cup C_{v_0}$. It turns out that uniqueness holds within the class of pure gauge solutions of Lemmas 6.1.1 and 6.1.2 *up to specifying three free functions on a fixed sphere*. The reason for this freedom essentially arises from integration "constants" when imposing the vanishing of (162) along $C_{u_0}\cup C_{v_0}$. These free functions can essentially (up to $\ell=0,1$ modes) be used to prescribe the horizon gauge conditions (193) and (194), and the round sphere condition (191). We remark that the $\ell=0$ modes require a special treatment, as one needs to address the existence of the reference linearised Schwarzschild solutions, which requires special care in achieving (195) and (197).

(3) Having fully exploited the special gauge solutions of Lemmas 6.1.1 and 6.1.2 in the first two steps above, we turn to Lemma 6.1.3. Note that such pure gauge solutions only generate non-trivial values for $\overset{(1)}{\hat{\not{g}}}$, $\overset{(1)}{\sqrt{\not{g}}}$ and $\overset{(1)}{b}$, hence they do not interfere with the gauge conditions in the first two steps above. In §9.2.4 we construct a pure gauge solution which will allow us to achieve (190), the gauge solution being unique up to a pure gauge solution changing only $\overset{(1)}{\hat{\not{g}}}$ and $\overset{(1)}{\sqrt{\not{g}}}$.

(4) In §9.2.5 we finally exploit the "residual freedom" mentioned at the end of (3) to construct a pure gauge solutions allowing us to achieve (198) and (192).

9.2.2. Pure gauge solutions with prescribed initial lapse

We show that we can use Lemmas 6.1.1 and 6.1.2 to obtain a pure gauge solution $\mathscr{G}$ with prescribed linearised lapse $\Omega^{-1}\overset{(1)}{\Omega}$ on $C_{u_0} \cup C_{v_0}$.

Proposition 9.2.1. *Fix a sphere $S^2_{u_0,v_0}$ in $\mathcal{M}$ with corresponding outgoing cone C_{u_0} and ingoing cone C_{v_0}. Let $\Omega_{\text{out}}(v,\theta,\phi)$ be a bounded smooth function on C_{u_0} and $\Omega_{\text{in}}(u,\theta,\phi)$ be a bounded function, smooth in the extended sense on C_{v_0}, such that $\Omega_{\text{in}}(u_0,\theta,\phi)=\Omega_{\text{out}}(v_0,\theta,\phi)$ holds on the sphere $S^2_{u_0,v_0}$. Then, there exists a pure gauge solution $\mathscr{G}$ of the system of gravitational perturbations such that*

$$2\Omega^{-1}\overset{(1)}{\Omega}(u_0,v,\theta,\phi)=\Omega_{\text{out}}(v,\theta,\phi) \quad \textit{and} \quad 2\Omega^{-1}\overset{(1)}{\Omega}(u,v_0,\theta,\phi)=\Omega_{\text{in}}(u,\theta,\phi).$$

Proof. Let f_{out} be a function along C_{u_0} determined as the solution to the ODE (for each θ and ϕ)

$$\partial_v f_{\text{out}}+\frac{2M}{r^2(u_0,v)}f_{\text{out}}=\Omega_{\text{out}}-\frac{1}{2}\Omega_{\text{in}}(u_0,\theta,\phi)\ \ f_{\text{out}}(v_0,\theta,\phi)=0. \tag{199}$$

Let f_{in} be a function along C_{v_0} determined as the solution to the ODE

$$\partial_u f_{\text{in}}-\frac{2M}{r^2(u,v_0)}f_{\text{in}}=\Omega_{\text{in}}-\frac{1}{2}\Omega_{\text{out}}(v_0,\theta,\phi)\ \ f_{\text{in}}(u_0,\theta,\phi)=0.$$

We claim that the pure gauge solution generated by applying Lemma 6.1.1 with f_{out} added to the pure gauge solution generated by applying Lemma 6.1.2 with f_{in}, yields the desired solution. To see this, we compute

$$\begin{aligned}2\Omega^{-1}\overset{(1)}{\Omega}&=\partial_v f_{\text{out}}+\frac{2M}{r^2(u,v)}f_{\text{out}}+\partial_u f_{\text{in}}-\frac{2M}{r^2(u,v)}f_{\text{in}}\\&=-\frac{2M}{r^2(u_0,v)}f_{\text{out}}+\Omega_{\text{out}}-\frac{1}{2}\Omega_{\text{in}}(u_0,\theta,\phi)+\frac{2M}{r^2(u,v)}f_{\text{out}}\\&\quad+\frac{2M}{r^2(u,v_0)}f_{\text{in}}+\Omega_{\text{in}}-\frac{1}{2}\Omega_{\text{out}}(v_0,\theta,\phi)-\frac{2M}{r^2(u,v)}f_{\text{in}},\end{aligned}$$

and hence

$$2\Omega^{-1}\overset{(1)}{\Omega}(u_0,v,\theta,\phi)=\Omega_{\mathrm{out}}(v,\theta,\phi) \quad \text{and} \quad 2\Omega^{-1}\overset{(1)}{\Omega}(u,v_0,\theta,\phi)=\Omega_{\mathrm{in}}(v_0,\theta,\phi). \qquad \square$$

Writing out the explicit solution of the ordinary differential equation (199), one also deduces from Lemma 6.1.1 the following corollary (cf. Remark 6.2).

Corollary 9.1. *If the function Ω_{out} on the outgoing cone C_{u_0} arises as the quantity $\Omega^{-1}\overset{(1)}{\Omega}_{\circ,\mathrm{out}}$ of a seed data set which is asymptotically flat to order $n\geqslant 2$, i.e. in particular (186) holds, then the pure gauge solution $\mathscr{G}$ constructed in Proposition 9.2.1 arises itself from seed data which are asymptotically flat to order $n-2$.*

Proof. For f_{out} one easily checks the required estimates (186)–(188) on C_{u_0} from Lemma 6.1.1, in particular that $|f_{\mathrm{out}}/r|$ and $|r^2\partial_v f_{\mathrm{out}}/r|$ are uniformly bounded. For f_{in} one uses Lemma 6.1.2 and the fact that f_{in} is uniformly bounded with $f_{\mathrm{in}}=0$ for $u=u_0$. $\square$

For later purposes, we also note the following special case of Proposition 9.2.1.

Corollary 9.2. *Let $\mathfrak{m}\in\mathbb{R}$. The functions*

$$\begin{aligned} f_{\mathrm{out}}(v) &= \tfrac{1}{2}\mathfrak{m}\Omega^{-2}(u_0,v)[r(u_0,v)-r(u_0,v_0)],\\ f_{\mathrm{in}}(u) &= \tfrac{1}{2}\mathfrak{m}\Omega^{-2}(u,v_0)[-r(u,v_0)+r(u_0,v_0)] \end{aligned}$$

generate a (spherically symmetric) pure gauge solution satisfying $2\Omega^{-1}\overset{(1)}{\Omega}=\mathfrak{m}$ along both C_{u_0} and C_{v_0}. It furthermore satisfies

$$\begin{aligned} \overset{(1)}{\varrho}(\infty,v_0,\theta,\phi) &= -\frac{3}{(2M)^3}\frac{\mathfrak{m}}{2}(r(u_0,v_0)-2M),\\ (\Omega\overset{(1)}{\mathrm{tr}}\,\chi)(\infty,v_0,\theta,\phi) &= -\frac{1}{4M^2}\mathfrak{m}(r(u_0,v_0)-2M),\\ \Omega^{-2}(\Omega\overset{(1)}{\mathrm{tr}}\,\underline{\chi})(\infty,v_0,\theta,\phi) &= -\frac{1}{2M}\mathfrak{m}\cdot\frac{r(u_0,v_0)}{2M}. \end{aligned} \tag{200}$$

Proof. Direct computation using Lemmas 6.1.1 and 6.1.2. For (200) recall that $f_{\mathrm{out}}(v_0)=0$. $\square$

9.2.3. Pure gauge solutions with vanishing initial lapse

In this section we explicitly parametrise the space of (the sum of) all special gauge transformations arising from Lemmas 6.1.1 and 6.1.2, which satisfy the condition

$$\Omega^{-1}\overset{(1)}{\Omega}=0 \quad \text{along both } C_{u_0} \text{ and } C_{v_0}. \tag{201}$$

LEMMA 9.2.1. *Let h_1, h_2 and h_3 be smooth functions on the unit sphere and let $R:=r(u_0,v_0)$. Then,*

$$f_1(u,\theta,\phi)=\frac{1}{\Omega^2(u,v_0)}\left(h_2(\theta,\phi)r(u,v_0)+h_3(\theta,\phi)-\frac{2M}{r(u,v_0)}h_1(\theta,\phi)\right),$$

$$\begin{aligned}f_2(v,\theta,\phi)=\frac{1}{\Omega^2(u_0,v)\Omega^2(u_0,v_0)}&\left[h_2(\theta,\phi)\left(r(u_0,v)\left(1-\frac{4M}{R}\right)-\frac{2M}{r(u_0,v)}R-R+6M\right)\right.\\&+h_1(\theta,\phi)\left(\frac{2M}{R^2}r(u_0,v)+\frac{4M^2}{Rr(u_0,v)}+1-\frac{6M}{R}\right)\\&\left.+h_3(\theta,\phi)\left(-\frac{2M}{R^2}r(u_0,v)-\frac{2M}{r(u_0,v)}+\frac{4M}{R}\right)\right],\end{aligned}$$

and $j_3\equiv j_4\equiv 0$ generates a pure gauge solution satisfying (201). Moreover, any pure gauge solution in Proposition 6.1.1 satisfying $j_3=j_4=0$ as well as (201) is of the form above.

Before we embark on the proof, note that the function $f_1(u,\theta,\phi)\cdot\Omega^2(u,v)$ is indeed smooth in the extended sense on $\mathcal{M}$, since

$$\Omega^2(u,v)\cdot\Omega^{-2}(u,v_0)=\frac{r(u,v_0)}{r(u,v)}e^{(-r(u,v)+r(u,v_0))/2M}e^{(v-v_0)/2M}$$

is smooth in the extended sense.

Proof. We note that $f_2(v_0,\theta,\phi)=h_1(\theta,\phi)$, $\partial_v(f_2)(v_0,\theta,\phi)=h_2(\theta,\phi)$ and

$$\frac{1}{\Omega^2(u,v)}(f_1\Omega^2(u,v))_u\bigg|_{v=v_0}=-\left(h_2(\theta,\phi)+h_1(\theta,\phi)\frac{2M}{r^2(u,v_0)}\right),$$

which verifies (162)=0 along C_{v_0}. Along C_{u_0}, we compute

$$\begin{aligned}&\frac{1}{\Omega^2(u,v)}(f_2\Omega^2(u,v))_v\bigg|_{u=u_0}\\&\quad=\Omega^2(u_0,v_0)\left[h_2(\theta,\phi)\left(1-\frac{4M}{R}+\frac{2M}{r^2(u_0,v)}R\right)\right.\\&\qquad\left.+h_1(\theta,\phi)\left(\frac{2M}{R^2}-\frac{4M^2}{Rr^2(u_0,v)}\right)+h_3(\theta,\phi)\left(-\frac{2M}{R^2}+\frac{2M}{r^2(u_0,v)}\right)\right],\end{aligned}$$

which verifies (162)=0 along C_{u_0}, after observing that

$$\begin{aligned}&(f_1)_u(u_0,\theta,\phi)-\frac{2M}{r^2(u_0,v)}f_1(u_0,\theta,\phi)\\&\quad=\frac{1}{\Omega^2(u_0,v_0)}\left[\left(-\frac{2M}{r^2(u_0,v)}+\frac{2M}{R^2}\right)\left(h_2(\theta,\phi)R+h_3(\theta,\phi)-\frac{2M}{R}h_1(\theta,\phi)\right)\right.\\&\qquad\left.-\left(h_2(\theta,\phi)+\frac{2M}{R^2}h_1(\theta,\phi)\right)\left(1-\frac{2M}{R}\right)\right].\end{aligned}$$

For the uniqueness assertion note that an arbitrary pure gauge solution f_1, f_2, $j_3{=}0$, $j_4{=}0$ satisfying (201) is uniquely determined by specifying $f_2(v_0,\theta,\phi)$, $\partial_v f_2(v_0,\theta,\phi)$ and $f_1(u_0,\theta,\phi)$ via elementary ODE theory applied to (162)=0 along C_{u_0} and C_{v_0}. On the other hand, the aforementioned values are seen to be in one-to-one correspondence with the functions h_1, h_2 and h_3. □

COROLLARY 9.3. *The pure gauge solution of Lemma 9.2.1 induces seed initial data on $C_{u_0}\cup C_{v_0}$ which is asymptotically flat to any order.*

Proof. This follows from carefully going through Lemmas 6.1.1 and 6.1.2. The key is to note that f_1 is uniformly bounded and that f_2 satisfies the estimates $|f_2/r|\lesssim 1$, $|r^2\partial_v(f_2/r)|\lesssim 1$ and $|r^2\partial_v(r^2\partial_v(f_2/r))|\lesssim 1$, the latter allowing one to obtain the estimates (186). The statements about higher derivatives in (186)–(188) are then straightforward. □

The gauge transformations of Lemma 9.2.1 can be used to prescribe additional geometric quantities on the horizon sphere S^2_{∞,v_0}. We first state the three fundamental propositions before proving them.

PROPOSITION 9.2.2. *Let X^1 and X^2 be smooth functions on S^2_{∞,v_0} with vanishing projection to $\ell{=}0$ and $\ell{=}1$. There exists a pure gauge solution $\mathscr{G}(f_1,f_2,j_3{=}0,j_4{=}0)$ satisfying* (201), *the round sphere condition* (191) *and*

$$\begin{aligned}(\Omega \overset{(1)}{\mathrm{tr}}\,\chi)(\infty,v_0,\theta,\phi)&=X^1,\\ \operatorname{d\!\!/iv}\overset{(1)}{\eta}+\overset{(1)}{\varrho}(\infty,v_0,\theta,\phi)&=X^2.\end{aligned}\tag{202}$$

Moreover, the functions f_1 and f_2 of the pure gauge solution $(f_1,f_2,j_3{=}0,j_4{=}0)$ are uniquely determined with their projection to $\ell{=}0$ and $\ell{=}1$ vanishing. Finally, the pure gauge solution induces asymptotically flat (*to any order*) *seed data on $C_{u_0}\cup C_{v_0}$.*

For the $\ell{=}1$ modes we have an additional degree of freedom which stems from the fact the round sphere condition is always satisfied as the pure gauge solutions below cannot alter the linearised Gaussian curvature $\overset{(1)}{K}$.

PROPOSITION 9.2.3. *Let X^1, X^2 and X^3 be smooth functions on S^2_{∞,v_0} all supported on $\ell{=}1$ only. There exists a pure gauge solution $\mathscr{G}(f_1,f_2,j_3{=}0,j_4{=}0)$ satisfying* (201), *which in addition satisfies*

$$\begin{aligned}(\Omega \overset{(1)}{\mathrm{tr}}\,\chi)(\infty,v_0,\theta,\phi)&=X^1,\\ \operatorname{d\!\!/iv}\overset{(1)}{\eta}+\overset{(1)}{\varrho}(\infty,v_0,\theta,\phi)&=X^2,\\ \Omega^{-2}(\Omega \overset{(1)}{\mathrm{tr}}\,\underline{\chi})(\infty,v_0,\theta,\phi)&=X^3.\end{aligned}\tag{203}$$

Moreover, the functions f_1 and f_2 of the pure gauge solution $(f_1, f_2, j_3{=}0, j_4{=}0)$ are uniquely determined and supported on $\ell{=}1$ only. Finally, the pure gauge solution induces asymptotically flat (to any order) seed data on $C_{u_0}\cup C_{v_0}$.

For the $\ell{=}0$ modes we cannot fix all three geometric quantities on the horizon independently because of a spherically symmetric degree of freedom that is *not* pure gauge and corresponding to a linearised Schwarzschild solution, discussed in §6.2.1. However, we can combine Lemma 9.2.1 with Corollary 9.2 to prove the following.

PROPOSITION 9.2.4. *Let $\mathfrak{m}$, X^1 and X^3 be constants on S^2_{∞,v_0}. There exists a pure gauge solution $\mathscr{G}$ generated by $(f_1, f_2, j_3{=}0, j_4{=}0)$, with f_1 and f_2 being spherically symmetric, which satisfies (189) and in addition*

$$2\Omega^{-1}\overset{(1)}{\Omega}(\infty, v_0, \theta, \phi) = \mathfrak{m}, \tag{204}$$

$$(\Omega \overset{(1)}{\operatorname{tr}} \chi)(\infty, v_0, \theta, \phi) = X^1, \tag{205}$$

$$\Omega^{-2}(\Omega \overset{(1)}{\operatorname{tr}} \underline{\chi})(\infty, v_0, \theta, \phi) = X^3. \tag{206}$$

Moreover, f_1 and f_2 are unique up to a constant $f_1{=}f_2{=}\lambda$, and hence the pure gauge solution is unique (as $(\lambda, \lambda, 0, 0)$ generates the zero solution). The solution also necessarily satisfies

$$\overset{(1)}{\varrho}(\infty, v_0, \theta, \phi) = \frac{3}{4M} X^1.$$

Finally, the pure gauge solution induces asymptotically flat (to any order) seed data on $C_{u_0}\cup C_{v_0}$.

Proof. We will prove Propositions 9.2.2–9.2.4 all at the same time. We first compute from the general solution of Lemma 9.2.1 the following geometric quantities on the horizon:

$$\begin{aligned}
(\Omega \overset{(1)}{\operatorname{tr}} \chi)(\infty, v_0, \theta, \phi) &= \frac{1}{2M^2}[(\Delta_{S^2}-1)(2M\cdot h_2(\theta,\phi)+h_3(\theta,\phi)-h_1(\theta,\phi))],\\
(\operatorname{d\!/iv} \overset{(1)}{\eta} + \overset{(1)}{\varrho})(\infty, v_0, \theta, \phi) &= \frac{1}{(2M)^3}[\Delta_{S^2}(h_3(\theta,\phi)-2h_1(\theta,\phi))\\
&\qquad -3(2M\cdot h_2(\theta,\phi)+h_3(\theta,\phi)-h_1(\theta,\phi))],\\
\Omega^{-2}(\Omega \overset{(1)}{\operatorname{tr}} \underline{\chi})(\infty, v_0, \theta, \phi) &= \frac{1}{2M^2}(\Delta_{S^2} h_1(\theta,\phi)+h_1(\theta,\phi)-h_3(\theta,\phi)).
\end{aligned}$$

To prove Proposition 9.2.2, we choose

$$h_3 - h_1 = h_2 \frac{R^2}{2M}\left(1-\frac{4M}{R}\right). \tag{207}$$

One checks that, with this choice, both f_1 and f_2 (as well as angular derivatives thereof) are uniformly bounded, hence $r^3 \cdot \overset{(1)}{K}$ is uniformly bounded and in particular the round sphere condition (191) holds for any h_1, h_2 and h_3 satisfying (207). Plugging this relation into the first equation above and recalling that $R>2M$, we see that the first equation uniquely determines h_2 to satisfy the condition in the proposition. Plugging (207) into the second equation to isolate h_1, we see that we can uniquely solve for h_1 to determine the second condition in the proposition. Of course h_3 is determined by (207).

To prove Proposition 9.2.3, we can restrict to $\ell=1$. The three equations above then turn into a simple algebraic system with non-zero determinant which admits a unique solution for any X_1, X_2, X_3 prescribed.

To prove Proposition 9.2.4, we project to $\ell=0$ to see the resulting algebraic system has 1-dimensional kernel $h_2=0$, $h_1=h_3$. It is easy to see that such gauge solutions are trivial. We now add the pure gauge solution from Corollary 9.2 to the aforementioned projection to obtain (setting $h_4=h_1-h_3$)

$$\begin{aligned}
(\Omega \overset{(1)}{\mathrm{tr}}\,\chi)(\infty, v_0, \theta, \phi) &= -\frac{1}{2M^2}(2M\cdot h_2 - h_4) - \frac{1}{4M^2}\mathfrak{m}(R-2M),\\
\Omega^{-2}(\Omega \overset{(1)}{\mathrm{tr}}\,\underline{\chi})(\infty, v_0, \theta, \phi) &= \frac{1}{2M^2}h_4 - \frac{1}{2M}\mathfrak{m}\frac{R}{2M},\\
\overset{(1)}{\varrho}(\infty, v_0, \theta, \phi) &= -\frac{3}{(2M)^3}(2M\cdot h_2 - h_4) - \frac{3}{(2M)^3}\frac{\mathfrak{m}}{2}(R-2M).
\end{aligned}$$

Note that the right-hand side of the third is a multiple of the first. It is immediate that we can solve the first two equations uniquely for h_2 and h_4, given any left-hand side and given any $\mathfrak{m}$. This provides the statement in the proposition recalling that h_2 and h_4 do not alter $\Omega^{-1}\overset{(1)}{\Omega}$ on $C_{u_0}\cup C_{v_0}$. The uniqueness follows since (189) and (204) at the horizon fix $\Omega^{-1}\overset{(1)}{\Omega}$ on $C_{u_0}\cup C_{v_0}$, so the remaining gauge solution must be of the type of Lemma 9.2.1 projected to $\ell=0$. This solution is trivial if

$$(\Omega \overset{(1)}{\mathrm{tr}}\,\chi)(\infty, v_0, \theta, \phi) = \Omega^{-2}(\Omega \overset{(1)}{\mathrm{tr}}\,\underline{\chi})(\infty, v_0, \theta, \phi) = 0.$$

Finally, the assertion about asymptotic flatness is a consequence of Corollary 9.3. □

9.2.4. Pure gauge solutions with prescribed shift

In this section we exploit pure gauge solutions arising from Lemma 6.1.3, which we recall only generate non-trivial metric components $\overset{(1)}{\hat{\slashed{g}}}$, $\overset{(1)}{\sqrt{\slashed{g}}}$ and $\overset{(1)}{b}$, while all Ricci and curvature coefficients vanish. In particular, any such pure gauge solution will automatically satisfy all gauge conditions of Definition 9.1, except (190) and (198).

PROPOSITION 9.2.5. *Let $\tilde{b}$ be a smooth $S^2_{u,v}$-valued 1-form prescribed along C_{u_0}. There exists a pure gauge solution of the type of Lemma 6.1.3, which satisfies*

$$\overset{(1)}{b} = \tilde{b} \quad \text{along } C_{u_0}.$$

Moreover, except for $\overset{(1)}{\hat{\slashed{g}}}$ and $\overset{(1)}{\sqrt{\slashed{g}}}/\sqrt{\slashed{g}}$ which are potentially non-vanishing, all linearised Ricci and curvature components of this pure gauge solution vanish. The solution is unique up to a pure gauge solution generated by functions q_1 and q_2 depending only on the angular variables (Proposition 9.2.6).

Proof. By Lemma 6.1.3 we need to determine q_1 and q_2 solving

$$\partial_v(\Delta_{S^2} q_1) = -\,\slashed{\mathrm{div}}\, \tilde{b} \ \text{ and } \ \partial_v(\Delta_{S^2} q_2) = -\,\slashed{\mathrm{curl}}\, \tilde{b} \quad \text{along } C_{u_0}, \tag{208}$$

where $\Delta_{S^2} = r^2 \slashed{\Delta}$ is defined with respect to the round *unit* sphere. We can solve these ODEs uniquely prescribing $\Delta_{S^2} q_1$ and $\Delta_{S^2} q_2$ freely (as functions with vanishing mean) initially at $v = v_0$ accounting for the non-uniqueness asserted in the proposition. The conclusions now follow from Lemma 6.1.3. □

COROLLARY 9.4. *If $\tilde{b}$ in Proposition 9.2.5 satisfies $|(r\slashed{\nabla})^{n_1}(r\slashed{\nabla}_4)^{n_2}\tilde{b}| \lesssim v^{-1}$ along C_{u_0} for $n_1 + n_2 \leqslant n$ (as is the case when $\tilde{b}$ arises as the quantity $\overset{(1)}{b}_{\circ}$ of a seed data set which is asymptotically flat to order n; cf. (187)), then the pure gauge solution induces data on $C_{u_0} \cup C_{v_0}$ which are asymptotically flat to order n.*

9.2.5. Residual pure gauge solutions

We finally give an explicit parametrisation of the kernel in Proposition 9.2.5, which we recall is generated by q_1 and q_2 being smooth functions of the unit sphere the proof of which is immediate from Lemma 6.1.3.

PROPOSITION 9.2.6. *Let q_1 and q_2 be smooth functions on the unit sphere. Then, there exists a pure gauge solution satisfying*

$$\overset{(1)}{\hat{\slashed{g}}}(u,v,\theta,\phi) = 2r^2 \slashed{\mathcal{D}}_2^\star \slashed{\mathcal{D}}_1^\star(q_1,q_2) \quad \text{and} \quad \frac{\overset{(1)}{\sqrt{\slashed{g}}}}{\sqrt{\slashed{g}}} = r^2 \slashed{\Delta} q_1, \tag{209}$$

and with all other metric and Ricci coefficients and curvature components globally vanishing. In particular, the pure gauge solution is asymptotically flat to any order.

Note the above solutions in particular do not change the linearised Gaussian curvature $\overset{(1)}{K}$. We will use them below to bring the metric into "standard form" on the sphere at infinity, i.e. to achieve (192), once we have (191).

9.2.6. Proof of Theorem 9.1

We can now prove Theorem 9.1. Let $\mathscr{S}$ be as in the proposition.

Applying Proposition 9.2.1 with $\Omega_{\text{out}}=-\Omega^{-1}\overset{(1)}{\Omega}_{\circ,\text{out}}$ and $\Omega_{\text{in}}=-\Omega^{-1}\overset{(1)}{\Omega}_{\circ,\text{in}}$, we achieve that the sum of the original solution $\mathscr{S}$ and the pure gauge solution generated by Proposition 9.2.1 satisfies $\Omega^{-1}\overset{(1)}{\Omega}=0$ along $C_{u_0}\cup C_{v_0}$. We denote this solution by $\mathscr{S}_1$.

The weighted geometric quantity $r^2\overset{(1)}{K}_{\ell\geqslant 2}$ of the solution $\mathscr{S}_1$ converges pointwise with at least $n-4$ angular derivatives $r\nabla\!\!\!\!/$ to a smooth function $X_4(\theta,\phi)$ on the unit sphere along the cone C_{u_0} as $v\to\infty$. This follows from (the first part of) Theorem A.1 in the appendix and the fact that $r^2\overset{(1)}{K}$ has such a limit for the pure gauge solution applied in the previous step. Let $\bar{f}$ be the unique solution of the equation $\Delta_{S^2}\bar{f}+2\bar{f}=X_4$ on the unit sphere which has vanishing projection to $\ell=1$.

We now apply Proposition 9.2.1 again, this time with $\Omega_{\text{out}}=\bar{f}$ and $\Omega_{\text{in}}=\bar{f}$ and add the resulting pure gauge solution to $\mathscr{S}_1$. The solution thus obtained will be denoted $\mathscr{S}_2$. The solution $\mathscr{S}_2$ clearly satisfies (189) and also (191), in fact it satisfies

$$\lim_{v\to\infty}((r\nabla\!\!\!\!/)^k r^2\overset{(1)}{K}_{\ell\geqslant 2}[\mathscr{S}_2](u_0,v,\theta,\phi))=0 \tag{210}$$

for $k\leqslant n-4$. To see the last claim, note that the f_{out} associated with Proposition 9.2.1 precisely cancels the weighted Gaussian curvature $r^2\overset{(1)}{K}$ at infinity of the solution $\mathscr{S}_1$, as can be seen directly from Lemma 6.1.1. On the other hand, the contribution from f_{in} through Lemma 6.1.2 does not affect $r^2\overset{(1)}{K}$ at infinity since f_{in} is uniformly bounded.

Let now

$$(\Omega\overset{(1)}{\operatorname{tr}}\chi)|_{S^2_{\infty,v_0}}=X_1,\ (\operatorname{d\!\!\!/iv}\overset{(1)}{\eta}+\overset{(1)}{\varrho})|_{S^2_{\infty,v_0}}=X_2,\ \Omega^{-2}(\Omega\overset{(1)}{\operatorname{tr}}\underline{\chi})|_{S^2_{\infty,v_0}}=X_3,$$

for X_1, X_2 and X_3 smooth functions on the sphere S^2_{∞,v_0}, denote the geometric quantities on the horizon for $\mathscr{S}_2$. We apply Proposition 9.2.2 to generate a pure gauge solution $\mathscr{G}_1$ satisfying (189) and (191), and in addition $(\Omega\overset{(1)}{\operatorname{tr}}\chi)|_{S^2_{\infty,v_0}}=X_1-(X_1)_{\ell=0,1}$ and

$$(\operatorname{d\!\!\!/iv}\overset{(1)}{\eta}+\overset{(1)}{\varrho})|_{S^2_{\infty,v_0}}=X_2-(X_2)_{\ell=0,1},$$

where the notation indicates that the projection to $\ell=0$ and $\ell=1$ has been removed. We apply Proposition 9.2.3 to generate a pure gauge solution $\mathscr{G}_2$ satisfying (189) and (191) and in addition $(\Omega\overset{(1)}{\operatorname{tr}}\chi)|_{S^2_{\infty,v_0}}=(X_1)_{\ell=1}$, $(\operatorname{d\!\!\!/iv}\overset{(1)}{\eta}+\overset{(1)}{\varrho})|_{S^2_{\infty,v_0}}=(X_2)_{\ell=1}$ and

$$\Omega^{-2}(\Omega\overset{(1)}{\operatorname{tr}}\underline{\chi})|_{S^2_{\infty,v_0}}=(X_3)_{\ell=1}.$$

We apply Proposition 9.2.4 to generate a pure gauge solution $\mathscr{G}_3$ which satisfies

$$(\Omega \overset{(1)}{\mathrm{tr}} \chi)|_{S^2_{\infty,v_0}} = (X_1)_{\ell=0}, \quad \Omega^{-2}(\Omega \overset{(1)}{\mathrm{tr}} \underline{\chi})|_{S^2_{\infty,v_0}} = (X_3)_{\ell=0},$$
$$2\Omega^{-1}\overset{(1)}{\Omega}|_{S^2_{\infty,v_0}} = 4M^2\left(-(X_2)_{\ell=0} + \frac{3}{4M}(X_1)_{\ell=0}\right),$$

the last holding on all of $C_{u_0} \cup C_{v_0}$ by (189). By Proposition 9.2.4, the solution necessarily satisfies $\overset{(1)}{\varrho}|_{S^2_{\infty,v_0}} = (3/4M)(X_1)_{\ell=0}$.

If we now define the solution $\mathscr{S}_3 := \mathscr{S}_2 - \mathscr{G}_1 - \mathscr{G}_2 - \mathscr{G}_3$, then this solution satisfies (189), (191) and also (195), (196) as well as the horizon gauge conditions and the auxiliary condition (197). One also checks directly that any pure gauge solution of the form $(f_1, f_2, j_3 = 0, j_4 = 0)$ with these properties is necessarily trivial.

The solution $\mathscr{S}_3$ satisfies $\overset{(1)}{b} = \tilde{b}$ along C_{u_0} for some smooth v-valued $S^2_{u_0,v}$-1-form $\tilde{b}$ along C_{u_0}. We apply Proposition 9.2.5 for $\tilde{b}^A - \frac{2}{3}r^3 \not{g}^{AB}\partial_B \overset{(1)}{\sigma}_{\ell=1}[\mathscr{S}_3]$ and subtract (a representative of) the pure gauge solution generated by it from $\mathscr{S}_3$. We denote the resulting solution by $\mathscr{S}_4$.

The solution $\mathscr{S}_4$ satisfies all of the desired gauge conditions except (198) and (192) and any such solution is determined up to a pure gauge solution of Proposition 9.2.6. The geometric quantity $\overset{(1)}{\hat{\not{g}}}$ has a smooth limit along the cone C_{u_0} and converges to a symmetric traceless $S^2_{u,v}$-tensor $\overset{(1)}{\hat{\not{g}}}_\infty(\theta, \phi)$ on the unit sphere. This follows from the assumptions in Proposition A.1 in conjunction with the constraint equations along C_{u_0} and the fact that this is true for all pure gauge transformations applied so far. We solve the elliptic equation

$$2r^2 \not{\mathcal{D}}_2^\star \not{\mathcal{D}}_1^\star (q_1, q_2) = \overset{(1)}{\hat{\not{g}}}_\infty(\theta, \phi)$$

for q_1 and q_2 on the unit sphere, which can be done uniquely up to $\ell=0$ and $\ell=1$ modes of q_1 and q_2. Noting that $\ell=0$ modes and the $\ell=1$ mode of q_2 generate trivial pure gauge solutions, we have determined q_1 and q_2 up to trivial pure gauge solutions and the three $\ell=1$ modes for q_1. Let X_5 be the value of $\overset{(1)}{\sqrt{\not{g}}}_{\ell=1}$ on the horizon sphere S^2_{∞,v_0} of the solution $\mathscr{S}_4$ we determine q_1 uniquely by solving $\Delta_{S^2} q_1 = X_4$. We now subtract the gauge solution generated by Proposition 9.2.6 for q_1 and q_2 as above from $\mathscr{S}_4$. We denote the resulting solution by $\mathscr{S}_5$. It is easy to see that all of the desired gauge conditions are now satisfied. The assertion about $\underset{\vee}{\mathscr{G}}$ (and hence $\underset{\vee}{\mathscr{S}}$) being asymptotically flat follows from Corollaries 9.1, 9.3 and 9.4.

9.3. The horizon-renormalised solution $\hat{\mathscr{S}}$

As discussed already in §2.1.7, our main decay theorem, Theorem 4, will require passing to a new gauge normalised from the event horizon values of $\underset{\vee}{\mathscr{S}}$. The following proposition defines and proves the existence of the horizon-renormalised solution $\hat{\mathscr{S}}$.

PROPOSITION 9.3.1. *Let $\underset{\vee}{\mathscr{S}}$ be an initial-data normalised solution as in Definition 8.1. Then, there exists a unique pure gauge solution $\hat{\mathscr{G}}$ of the type of Lemma 6.1.1, computable from the trace of $\underset{\vee}{\mathscr{S}}$ on the event horizon $\mathcal{H}^+$, such that the sum*

$$\hat{\mathscr{S}} \doteq \underset{\vee}{\mathscr{S}} + \hat{\mathscr{G}} \tag{211}$$

has the following properties:

(1) *The projection to $\ell \geqslant 2$ of the linearised lapse vanishes along the event horizon for $\hat{\mathscr{S}}$, i.e.*

$$\Omega^{-1} \overset{(1)}{\Omega}_{\ell\geqslant 2} = 0 \text{ holds along the event horizon } \mathcal{H}^+. \tag{212}$$

(2) *The pure gauge solution $\hat{\mathscr{G}}$ satisfies*

$$\Omega^{-2}(\Omega \overset{(1)}{\mathrm{tr}}\, \underline{\chi})|_{S^2_{\infty,v_0}} = 0 \text{ and } \Omega^{-1}\overset{(1)}{\hat{\underline{\chi}}}|_{S^2_{\infty,v_0}} = 0 \quad \text{on the horizon sphere } S^2_{\infty,v_0}. \tag{213}$$

(3) *The function f generating $\hat{\mathscr{G}}$ has vanishing projection to $\ell = 0, 1$.*

We call $\hat{\mathscr{S}}$ the horizon-renormalised solution.

Proof. Let f be determined as the unique solution to the ODE

$$\partial_v f + \frac{1}{2M} f = -2\Omega^{-1}\overset{(1)}{\Omega}_{\ell\geqslant 2}[\underset{\vee}{\mathscr{S}}](\infty, v, \theta, \phi) \quad \text{with } f(v_0, \theta, \phi) = 0, \tag{214}$$

where the right-hand side of the ODE denotes the projection to $\ell \geqslant 2$ of the trace of the linearised lapse of $\underset{\vee}{\mathscr{S}}$ along the event horizon. Given f as above, we define $\hat{\mathscr{G}}$ to be the pure gauge solution associated by Lemma 6.1.1 and $\hat{\mathscr{S}} = \underset{\vee}{\mathscr{S}} + \hat{\mathscr{G}}$. It is easy to check that $\hat{\mathscr{S}}$ satisfies (212) and $\hat{\mathscr{G}}$ satisfies (213). The uniqueness statement follows since satisfying the ODE is a necessary condition for (212) to hold and $f(v_0, \theta, \phi) = 0$ is required by the expression for $\Omega^{-2}(\Omega \overset{(1)}{\mathrm{tr}}\, \underline{\chi})|_{S^2_{\infty,v_0}}$ in Lemma 6.1.1. □

Remark 9.1. Since f is supported for $\ell \geqslant 2$ only one easily sees from Lemma 6.1.1 and Theorem 9.2 below that in fact one also has $\Omega^{-1}\overset{(1)}{\Omega}_{\ell=1} = 0$ on $\mathcal{H}^+$ as this holds both for a reference linearised Kerr solution and for the solution $\hat{\mathscr{G}}$.

Note that $\hat{\mathscr{S}}$ still satisfies the horizon gauge conditions (193), (194). Note also that if we apply the proposition for a reference linearised Kerr solution, i.e. with $\underset{\vee}{\mathscr{S}}=\mathscr{K}$, then $\hat{\mathscr{S}}=\hat{\mathscr{K}}=\mathscr{K}$, so the reference Kerr is both in the initial data and in the horizon normalised gauge. Another way to say this is that the pure gauge solution $\hat{\mathscr{G}}$ is not supported on $\ell=0,1$, the terminology being introduced in Definition 9.2 below.

In contrast to Theorem 9.1 concerning the initial-data normalised solution $\underset{\vee}{\mathscr{S}}$ which states asymptotic flatness for $\underset{\vee}{\mathscr{G}}$, at this point, we do not know that $\hat{\mathscr{G}}$ enjoys this property. Thus, a priori the $\hat{\mathscr{S}}$ defined by Proposition 9.3.1 may have data which are not asymptotically flat, even if the data corresponding to $\underset{\vee}{\mathscr{S}}$ are asymptotically flat.

While showing that $\hat{\mathscr{S}}$ is asymptotically flat in this case in the sense of Definition 8.2 would require an improvement of our polynomial decay bounds for gauge invariant quantities and decay estimates at all orders of derivatives, we will prove, in the context of the proof of Theorem 4, *weighted boundedness* estimates for $\hat{\mathscr{G}}$. See Remarks 10.4 and 10.8.

9.4. Global properties of the gauge-normalised solutions

In this section, we collect some global properties of the system of linearised gravity that follow for the initial data-normalised and horizon-renormalised solutions $\underset{\vee}{\mathscr{S}}$ and $\hat{\mathscr{S}}$.

9.4.1. Propagation along the event horizon

We first deduce two conservation laws along the event horizon $\mathcal{H}^+$.

PROPOSITION 9.4.1. *Consider a seed data set as in Definition 8.1, let $\mathscr{S}$ be the resulting solution given by Theorem 8.1. If $\mathscr{S}$ satisfies the horizon gauge conditions (193) and (194), then we have*

$$(\Omega \overset{(1)}{\mathrm{tr}}\,\chi)=0 \quad \textit{and} \quad \overset{(1)}{\varrho}-\overset{(1)}{\varrho}_{\ell=0}+\slashed{\mathrm{div}}\,\overset{(1)}{\eta}=0 \quad \textit{pointwise along } \mathcal{H}^+. \tag{215}$$

The assumption, and hence the conclusion, holds in particular for the solutions $\underset{\vee}{\mathscr{S}}$ and $\hat{\mathscr{S}}$ defined above.

Proof. We write the linearised Raychaudhuri equation (138) as

$$\partial_v\big(e^{-v/2M}r^2(\Omega \overset{(1)}{\mathrm{tr}}\,\chi)\big)=0 \quad \text{along } \mathcal{H}^+, \tag{216}$$

and use the assumption that the quantity in brackets is zero on the initial sphere. To show the second bound, we note that, using the first, we have the following propagation equation along $\mathcal{H}^+$:

$$\slashed{\nabla}_4\big(r^3\overset{(1)}{\varrho}-r^3\overset{(1)}{\varrho}_{\ell=0}+r^3\,\slashed{\mathrm{div}}\,\overset{(1)}{\eta}\big)=0, \tag{217}$$

and from (194) we conclude $r^3\overset{(1)}{\varrho}-r^3\overset{(1)}{\varrho}_{\ell=0}+r^3\,\mathrm{d}\!\!/\mathrm{iv}\,\overset{(1)}{\eta}=0$ pointwise on $\mathcal{H}^+$. The claim about $\underset{\vee}{\mathscr{S}}$ is immediate as it satisfies (193) and (194) by definition. The claim about $\hat{\mathscr{S}}$ follows since $\hat{\mathscr{G}}$ satisfies $\overset{(1)}{\eta}=0$ and $(\Omega\overset{(1)}{\mathrm{tr}}\chi)=\overset{(1)}{\varrho}=0$ on the horizon $\mathcal{H}^+$. $\square$

9.4.2. The geometric quantities $\overset{(1)}{Y}$ and $\overset{(1)}{Z}$

In this section we will define two auxiliary quantities which will play a key role later in the analysis. Specifically, we will later assume uniform boundedness of these quantities on the initial data and show that this is propagated in evolution.

Besides the definition, we also prove two propositions, which show that the initial uniform boundedness of these quantities can in fact be *deduced* for the solution $\underset{\vee}{\mathscr{S}}$ arising from Theorem 9.1.

The quantities are defined as follows:

$$\overset{(1)}{Y}:=r[r^2\mathcal{D}\!\!\!/_2^\star\,\mathrm{d}\!\!/\mathrm{iv}(\Omega^{-1}r\underline{\overset{(1)}{\hat{\chi}}})-\Omega^{-1}r^3\underline{\overset{(1)}{\psi}}]=r^3\mathcal{D}\!\!\!/_2^\star\overset{(1)}{\eta}-3Mr\underline{\overset{(1)}{\hat{\chi}}}\Omega^{-1}+\frac{1}{2}\frac{r^4}{\Omega^2}\mathcal{D}\!\!\!/_2^\star\nabla\!\!\!\!/\,(\Omega\overset{(1)}{\mathrm{tr}}\,\underline{\chi}),\tag{218}$$

$$\overset{(1)}{Z}_A:=\frac{r^3}{\Omega^2}\nabla\!\!\!\!/_A(\Omega\overset{(1)}{\mathrm{tr}}\,\chi)-2r^2(\overset{(1)}{\eta}_A+\underline{\overset{(1)}{\eta}}_A)=\frac{r^3}{\Omega^2}\nabla\!\!\!\!/_A\left((\Omega\overset{(1)}{\mathrm{tr}}\,\chi)-\frac{4}{r}\Omega^2\Omega^{-1}\overset{(1)}{\Omega}\right),\tag{219}$$

where the non-defining equalities hold for solutions for the system of gravitational perturbations.

We start with a proposition on $\overset{(1)}{Y}$.

Proposition 9.4.2. *Consider a seed data set as in Definition 8.1, which is asymptotically flat to order $n\geqslant 12$. Let $\mathscr{S}$ be the resulting solution given by Theorem 8.1 and let $\underset{\vee}{\mathscr{S}}$ be as in Theorem 9.1. Then, the geometric quantity $\overset{(1)}{Y}$ associated with $\underset{\vee}{\mathscr{S}}$ is uniformly bounded along C_{u_0}.*

Before we prove the proposition, let us remark that we will eventually also prove that $\overset{(1)}{Y}$ is bounded for $\hat{\mathscr{S}}$, but this will require global boundedness estimates on the pure gauge solution $\hat{\mathscr{G}}$. See Theorem 4.

Proof. One first derives a propagation equation for the Gaussian curvature $\overset{(1)}{K}$ along C_{u_0} which follows by taking a ∂_v derivative of (147). This reads schematically

$$\partial_v((r\nabla\!\!\!\!/)^k r^2\overset{(1)}{K})=\mathcal{Q},$$

where $\mathcal{Q}$ satisfies $|\mathcal{Q}|\leqslant Cr^{-2}$ for $k\leqslant n-5$ from the seed data being asymptotically flat to order $n\geqslant 12$; cf. Theorem A.1. Using the round sphere condition at infinity, (191), we obtain that $r^3\overset{(1)}{K}$ is uniformly bounded along C_{u_0}. Commuting with angular derivatives and using that the proof of Theorem 9.1 actually gave (210), one obtains that $r^2\mathcal{D}_2^\star \nabla_A r^3 \overset{(1)}{K}$ is similarly uniformly bounded along C_{u_0}; cf. (210). One finally looks at the commuted linearised Gauss equation (147),

$$r^2\mathcal{D}_2^\star \nabla_A r^3 \overset{(1)}{K} = r^5 \mathcal{D}_2^\star \nabla_A \overset{(1)}{\varrho} - \Omega^2 \overset{(1)}{Y} - 3Mr\Omega \overset{(1)}{\hat{\underline{\chi}}} + \tfrac{1}{2} r^4 r^2 \mathcal{D}_2^\star \nabla_A (\Omega \overset{(1)}{\operatorname{tr}} \chi) - \Omega^2 r^3 \mathcal{D}_2^\star \overset{(1)}{\underline{\eta}},$$

which when combined with the decay rates (439) yields boundedness of $\overset{(1)}{Y}$. □

Remarkably, as we will see, the global uniform boundedness of $\overset{(1)}{Y}$ is actually propagated by the equations; cf. Proposition 13.4.1.

We now turn to the quantity $\overset{(1)}{Z}$ above.

PROPOSITION 9.4.3. *Consider a seed data set as in Definition 8.1, let $\mathscr{S}$ be the resulting solution given by Theorem 8.1 and let $\underset{\vee}{\mathscr{S}}$ be as in Theorem 9.1. Then, the geometric quantity $\overset{(1)}{Z}\Omega^{-2}$ associated with $\underset{\vee}{\mathscr{S}}$ is uniformly bounded along C_{v_0}. Moreover, since $\overset{(1)}{Z}_A = -2r\Omega^2 \nabla_A f$ near the horizon for pure gauge solutions of Lemma 6.1.1, the boundedness statement holds equivalently for $\overset{(1)}{Z}\Omega^{-2}$ associated with $\hat{\mathscr{S}}$.*

Proof. Note that, along C_{v_0}, we have

$$\partial_u \left((\Omega \overset{(1)}{\operatorname{tr}} \chi) - \frac{4}{r} \Omega^2 \Omega^{-1} \overset{(1)}{\Omega} \right) = -\frac{1}{M^2} \Omega^{-1} \overset{(1)}{\Omega} + \frac{1}{M^2} \Omega^{-1} \overset{(1)}{\Omega} + \mathcal{O}(\Omega^4) = \mathcal{O}(\Omega^4),$$

where we have used the horizon gauge conditions and the lapse gauge condition. The same statement holds for arbitrary angular commutations. Since the quantity in brackets vanishes initially on the horizon $\mathcal{H}^+$, it actually vanishes to order Ω^4 by the estimate. The fact that $\overset{(1)}{Z}_A = -2r\Omega^2 \nabla_A f$ for pure gauge solution of Lemma 6.1.1 is read off directly from this lemma, so that the last statement follows from recalling that $\hat{\mathscr{G}} = \hat{\mathscr{S}} - \underset{\vee}{\mathscr{S}}$ arises from Lemma 6.1.1. □

Remarkably, as we will see, the uniform boundedness of $\overset{(1)}{Z}\Omega^{-2}$ near the horizon is again actually propagated by the equations; cf. Proposition 13.5.6.

9.5. The projection to the $\ell=0,1$ modes and the Kerr parameters

The initial-data normalisation, as we have defined it, will allow to completely understand the projection of solutions $\mathscr{S}$ to their $\ell=0,1$ modes. The main result of this section is the following.

THEOREM 9.2. *Let $\mathscr{S}$ and $\underset{\vee}{\mathscr{S}}$ be as in Theorem 9.1. Then the projection of $\underset{\vee}{\mathscr{S}}$ to its $\ell=0,1$ modes (see Definition 9.3) is a reference linearised Kerr solution $\mathscr{K}_{\mathfrak{m},s_i}$, where the parameters $\mathfrak{m}$ and s_i are given by*

$$\mathfrak{m} = -4M^2 \overset{(1)}{\varrho}_{\ell=0}|_{S^2_{\infty,v_0}}, \quad s_{-1} = \overset{(1)}{\sigma}_{\ell=1,m=-1}|_{S^2_{\infty,v_0}},$$
$$s_0 = \overset{(1)}{\sigma}_{\ell=1,m=0}|_{S^2_{\infty,v_0}}, \quad s_1 = \overset{(1)}{\sigma}_{\ell=1,m=1}|_{S^2_{\infty,v_0}}.$$

Here, $\overset{(1)}{\sigma}_{\ell=L,m=S}$ denotes the projection of $\overset{(1)}{\sigma}$ to the spherical harmonic Y^L_S. (Thus, in particular, solutions $\mathscr{S}$ supported only on $\ell=0,1$ are a linearised Kerr plus a pure gauge solution.)

9.5.1. The projection to $\ell=0,1$

We begin with the following definition.

Definition 9.2. We say that a solution $\mathscr{S}$ of the system of gravitational perturbations is *supported only on* $\ell=0,1$ if

- all scalars s in $\mathscr{S}$ are supported on $\ell=0,1$ only (cf. Definition 4.1);
- all 1-forms ξ in $\mathscr{S}$ are supported on $\ell=1$ only (cf. Definition 4.2);
- all symmetric traceless tensors θ in $\mathscr{S}$ vanish (cf. Proposition 4.4.1).

Conversely, we define a solution $\mathscr{S}$ to have *support outside* $\ell=0,1$ if

- all scalars s in $\mathscr{S}$ are supported on $\ell\geqslant 2$ only (cf. Definition 4.1);
- all 1-forms ξ in $\mathscr{S}$ are supported on $\ell\geqslant 2$ only (cf. Proposition 4.4.1).

Observe that the reference linearised Kerr solutions $\mathscr{K}$ are supported only on $\ell=0,1$. Note also that, by Lemma 4.4.1, a solution that is supported only on $\ell=0,1$ satisfies

$$r^2 \mathcal{D}_2^\star \nabla_A s = 0 \text{ for all scalars } s \text{ in } \mathscr{S} \quad \text{and} \quad r\mathcal{D}_2^\star \xi = 0 \text{ for all 1-forms } \xi \text{ in } \mathscr{S}.$$

In general, it is easy to see that one has the following result.

LEMMA 9.5.1. *Let $\mathscr{S}$ be a smooth solution of the system of gravitational perturbations on $\mathcal{M}\cap\{u\geqslant u_0\}\cap\{v\geqslant v_0\}$. We have the unique decomposition*

$$\mathscr{S} = \mathscr{S}_{\ell=0,1} + \mathscr{S}_{\ell\geqslant 2},$$

where $\mathscr{S}_{\ell=0,1}$ and $\mathscr{S}_{\ell\geqslant 2}$ are both solutions to the system of gravitational perturbations with $\mathscr{S}_{\ell=0,1}$ supported only on $\ell=0,1$ and $\mathscr{S}_{\ell\geqslant 2}$ supported outside $\ell=0,1$.

Definition 9.3. We call the map $\mathscr{S}\mapsto\mathscr{S}_{\ell=0,1}$ in Lemma 9.5.1 *the projection of $\mathscr{S}$ to its $\ell=0,1$ modes.*

9.5.2. Proof of Theorem 9.2

Let $\underset{\vee}{\mathscr{S}}$ be as in Theorem 9.1. The solution $\underset{\vee}{\mathscr{S}}$ satisfies

$$\overset{(1)}{\varrho}_{\ell=0}[\underset{\vee}{\mathscr{S}}]=-\frac{1}{4M^2}\mathfrak{m} \quad \text{and} \quad \overset{(1)}{\sigma}_{\ell=1}[\underset{\vee}{\mathscr{S}}]=\sum_{i=-1}^{1} s_i Y_i^{\ell=1}$$

on the sphere S^2_{∞,v_0} for some real numbers $\mathfrak{m}$, s_{-1}, s_0 and s_1. We can hence subtract from $\underset{\vee}{\mathscr{S}}$ a reference linearised Kerr solution $\mathscr{K}_{\mathfrak{m},s_i}$ such that the projection to $\ell=0,1$ of the solution $\underset{\vee}{\mathscr{S}}-\mathscr{K}_{R,S_i}$ satisfies in particular the following conditions:

- $\overset{(1)}{\varrho}_{\ell=0}=0$ and $\overset{(1)}{\sigma}_{\ell=1}=0$ on S^2_{∞,v_0};
- $\Omega^{-1}\overset{(1)}{\Omega}_{\ell=0,1}=0$ on C_{u_0} and C_{v_0};
- $(\operatorname{d\!\!/iv}\overset{(1)}{b})_{\ell=1}=0$ and $(\operatorname{cu\!\!/rl}\overset{(1)}{b})_{\ell=1}=0$ along C_{u_0};
- $(\Omega\overset{(1)}{\operatorname{tr}}\chi)_{\ell=0,1}=0$ and $(\overset{(1)}{\varrho}+\operatorname{d\!\!/iv}\overset{(1)}{\eta})_{\ell=1}=0$ on S^2_{∞,v_0};
- $\Omega^{-2}(\Omega\overset{(1)}{\operatorname{tr}}\underline{\chi})_{\ell=0,1}=0$ and $\overset{(1)}{\sqrt{\not g}}_{\ell=1}=0$ on S^2_{∞,v_0};

where we recall Proposition 9.1.1. Note that the parameters $\mathfrak{m}$ and s_i are precisely the ones claimed in the theorem. We will now show that this implies the following identities for $\underset{\vee}{\mathscr{S}}-\mathscr{K}_{\mathfrak{m},s_i}$ globally on $\mathcal{M}\cap\{u\geqslant u_0\}\cap\{v\geqslant v_0\}$:

$$\begin{aligned}
&\Omega^{-1}\overset{(1)}{\Omega}|_{\ell=0,1}=\overset{(1)}{\sqrt{\not g}}_{\ell=0,1}=\overset{(1)}{\sigma}_{\ell=0,1}=(\Omega\overset{(1)}{\operatorname{tr}}\chi)_{\ell=0,1}=\Omega^{-2}(\Omega\overset{(1)}{\operatorname{tr}}\underline{\chi})_{\ell=0,1}=\overset{(1)}{\varrho}_{\ell=0,1}=0,\\
&(\operatorname{d\!\!/iv}\overset{(1)}{b})_{\ell=0,1}=\Omega(\operatorname{d\!\!/iv}\overset{(1)}{\eta})_{\ell=0,1}=\Omega^{-1}(\operatorname{d\!\!/iv}\overset{(1)}{\underline{\eta}})_{\ell=0,1}=\Omega(\operatorname{d\!\!/iv}\overset{(1)}{\beta})_{\ell=0,1}=\Omega^{-1}(\operatorname{d\!\!/iv}\overset{(1)}{\underline{\beta}})_{\ell=0,1}=0,\\
&(\operatorname{cu\!\!/rl}\overset{(1)}{b})_{\ell=0,1}=\Omega(\operatorname{cu\!\!/rl}\overset{(1)}{\eta})_{\ell=0,1}=\Omega^{-1}(\operatorname{cu\!\!/rl}\overset{(1)}{\underline{\eta}})_{\ell=0,1}=\Omega(\operatorname{cu\!\!/rl}\overset{(1)}{\underline{\beta}})_{\ell=0,1}=\Omega^{-1}(\operatorname{cu\!\!/rl}\overset{(1)}{\beta})_{\ell=0,1}=0.
\end{aligned}$$

which in turn implies that $\underset{\vee}{\mathscr{S}}-\mathscr{K}_{\mathfrak{m},s_i}$ is supported outside $\ell=0,1$ providing the conclusion of the theorem. It is also easy to see that $\mathscr{K}_{\mathfrak{m},s_i}$ is unique, as any other choice of parameters would make the solution non-trivial on the horizon sphere.

We first obtain the above identities on the initial null hypersurfaces, and then show how the identities can be propagated globally.

The $\ell=0,1$ modes vanish on C_{u_0} and C_{v_0}

(1) We first obtain additional identities on the sphere S_{∞,v_0} via elliptic equations. The linearised Gauss equation simplifies on the horizon $\mathcal{H}^+$ to

$$\overset{(1)}{K}=-\overset{(1)}{\varrho}. \tag{220}$$

Computing the linearised Gauss curvature in terms of $\overset{(1)}{\not g}$, we find

$$2\overset{(1)}{K}=-\not\Delta(\operatorname{tr}_{\not g}\overset{(1)}{\not g})+\operatorname{d\!\!/iv}\operatorname{d\!\!/iv}\overset{(1)}{\not g}-\frac{1}{r^2}\operatorname{tr}_{\not g}\overset{(1)}{\not g}=-\frac{1}{2}\not\Delta(\operatorname{tr}_{\not g}\overset{(1)}{\not g})+\operatorname{d\!\!/iv}\operatorname{d\!\!/iv}\overset{(1)}{\hat{\not g}}-\frac{1}{r^2}\operatorname{tr}_{\not g}\overset{(1)}{\not g}. \tag{221}$$

Projecting on the $\ell=1$ modes, we see that $\overset{(1)}{K}_{\ell=1}=0$; cf. Corollary 4.1. By (220), also $\overset{(1)}{\varrho}_{\ell=1}=0$. From $(\overset{(1)}{\varrho}+\mathrm{d}\!\!/\mathrm{iv}\,\overset{(1)}{\eta})_{\ell=1}=0$ and the fact that $\Omega^{-1}\overset{(1)}{\Omega}_{\ell=1}=0$ implies $\mathrm{d}\!\!/\mathrm{iv}(\overset{(1)}{\eta}+\overset{(1)}{\underline{\eta}})_{\ell=1}=0$, we conclude $(\mathrm{d}\!\!/\mathrm{iv}\,\eta)_{\ell=1}=(\mathrm{d}\!\!/\mathrm{iv}\,\overset{(1)}{\eta})_{\ell=1}=0$. Also, $\overset{(1)}{\sigma}_{\ell=1}=0$ implies

$$(\mathrm{c}\!\!/\mathrm{url}\,\overset{(1)}{\eta})_{\ell=1}=-(\mathrm{c}\!\!/\mathrm{url}\,\overset{(1)}{\underline{\eta}})_{\ell=1}=0.$$

Taking a divergence (and curl respectively) of the Codazzi equations (145), we find $(\Omega\,\mathrm{d}\!\!/\mathrm{iv}\,\overset{(1)}{\beta})_{\ell=1}=(\Omega^{-1}\,\mathrm{d}\!\!/\mathrm{iv}\,\overset{(1)}{\underline{\beta}})_{\ell=1}=0$, as well as $(\Omega\,\mathrm{c}\!\!/\mathrm{url}\,\overset{(1)}{\beta})_{\ell=1}=(\Omega^{-1}\,\mathrm{c}\!\!/\mathrm{url}\,\overset{(1)}{\underline{\beta}})_{\ell=1}=0$ on $\mathcal{S}_{\mathcal{H}}$. Recall also that $\overset{(1)}{\sqrt{\not g}}_{\ell=0,1}=0$ on S_{∞,v_0} by (198), and the fact that $\overset{(1)}{\varrho}_{\ell=0}$ implies $\overset{(1)}{\sqrt{\not g}}_{\ell=0}=0$ on $S_{\mathcal{H}}$ by combining (220) and (221).

(2) Commuting the Bianchi equation (156) with $\mathrm{d}\!\!/\mathrm{iv}$ (and $\mathrm{c}\!\!/\mathrm{url}$), we conclude that $(\mathrm{d}\!\!/\mathrm{iv}\,\overset{(1)}{\underline{\beta}})_{\ell=1}=0$ and $(\mathrm{c}\!\!/\mathrm{url}\,\overset{(1)}{\underline{\beta}})_{\ell=1}=0$ hold along $v=v_0$. We then conclude that $(\mathrm{d}\!\!/\mathrm{iv}\,\overset{(1)}{\underline{\eta}})_{\ell=1}=0$ and $(\mathrm{c}\!\!/\mathrm{url}\,\overset{(1)}{\underline{\eta}})_{\ell=1}=-\overset{(1)}{\sigma}_{\ell=1}=0$ hold along $v=v_0$ from commuting (142).

(3) Propagating from the horizon outwards in the 3-direction, we see from (138) using $\Omega^{-1}\overset{(1)}{\Omega}_{\ell=0,1}=0$ that

$$\partial_u[(\Omega\,\overset{(1)}{\mathrm{tr}\,\underline{\chi}})_{\ell=0,1}\Omega^{-2}r^2]=0, \tag{222}$$

and hence, taking into account the projection of (193),

$$\Omega^{-2}(\Omega\,\overset{(1)}{\mathrm{tr}\,\underline{\chi}})_{\ell=0,1}=0 \quad \text{along } C_{v_0}. \tag{223}$$

(4) From (152) and (142) we derive, using that $\mathrm{d}\!\!/\mathrm{iv}(\overset{(1)}{\eta}+\overset{(1)}{\underline{\eta}})_{\ell=1}=\mathrm{c}\!\!/\mathrm{url}(\overset{(1)}{\eta}+\overset{(1)}{\underline{\eta}})_{\ell=1}=0$ along the cone,

$$\frac{1}{\Omega}\nabla\!\!\!/_3(r^3\overset{(1)}{\varrho}+r\,\mathrm{d}\!\!/\mathrm{iv}(r^2\overset{(1)}{\underline{\eta}}))|_{\ell=0,1}=0, \tag{224}$$

where the right-hand side vanishes along C_{v_0}, by the previous step. Moreover, the quantity in brackets on the left is also zero initially on $S_{\mathcal{H}}$ when projected to $\ell=0$ and $\ell=1$. We conclude that

$$\overset{(1)}{\varrho}_{\ell=0}=0 \quad\text{and}\quad \overset{(1)}{\varrho}_{\ell=1}+(\mathrm{d}\!\!/\mathrm{iv}\,\overset{(1)}{\underline{\eta}})_{\ell=1}=0 \quad \text{along } C_{v_0}. \tag{225}$$

By item (2) above, this means that $\overset{(1)}{\varrho}_{\ell=1}=(\mathrm{d}\!\!/\mathrm{iv}\,\overset{(1)}{\underline{\eta}})_{\ell=1}=(\mathrm{d}\!\!/\mathrm{iv}\,\overset{(1)}{\eta})_{\ell=1}=0$ individually on C_{v_0}. We can also conclude that $\overset{(1)}{\sqrt{\not g}}_{\ell=0}=0$ along C_{v_0} from the projection to $\ell=0$ of (131) along C_{v_0}.

(5) The previous step allows us to conclude that the condition $(\Omega\,\overset{(1)}{\mathrm{tr}\,\chi})_{\ell=0,1}=0$ is propagated along $v=v_0$. This follows by writing (136) as

$$\frac{1}{\Omega^2}\partial_u(r(\Omega\,\overset{(1)}{\mathrm{tr}\,\chi}))=2r\,\mathrm{d}\!\!/\mathrm{iv}\,\overset{(1)}{\eta}+2r\,\overset{(1)}{\varrho}-\frac{1}{2}\frac{r}{\Omega^2}\Omega\,\mathrm{tr}\,\chi(\Omega\,\overset{(1)}{\mathrm{tr}\,\underline{\chi}})$$

and noting that the right-hand side vanishes on C_{v_0} when projected on $\ell=0,1$.

(6) The commuted (with $\not{d}\mathrm{iv}$ and $\not{c}\mathrm{url}$) Codazzi equation now gives $(\not{d}\mathrm{iv}\,\overset{(1)}{\beta})_{\ell=1}=0$ and $(\not{c}\mathrm{url}\,\overset{(1)}{\beta})_{\ell=1}=0$ along $v=v_0$. With this, all required identities have been established on C_{v_0}.

(7) We finally need to propagate the identities along the outgoing hypersurface C_{u_0} from the sphere of intersection $S^2_{u_0,v_0}$, where all the desired identities have already been established. This follows analogously to what we have done before and will only be sketched. For $(\Omega\,\overset{(1)}{\mathrm{tr}}\,\chi)_{\ell=0,1}$, this follows from the Raychaudhuri equation (137). For $(\not{d}\mathrm{iv}\,\overset{(1)}{\beta})_{\ell=1}$ and $(\not{c}\mathrm{url}\,\overset{(1)}{\beta})_{\ell=1}$, this follows from commuting (with $\not{d}\mathrm{iv}$ and $\not{c}\mathrm{url}$) the Bianchi equation (149). Equation (142) shows that $(\not{d}\mathrm{iv}\,\overset{(1)}{\eta})_{\ell=1}=0$ and $(\not{c}\mathrm{url}\,\overset{(1)}{\eta})_{\ell=1}=0$ (and, by the lapse gauge condition, the $\overset{(1)}{\underline{\eta}}$-analogues). Finally, $\overset{(1)}{\varrho}_{\ell=1}=0$ and $\overset{(1)}{\sigma}_{\ell=1}=0$ from their Bianchi equations in the 4-direction and $(\Omega\,\overset{(1)}{\mathrm{tr}}\,\underline{\chi})_{\ell=0,1}=0$ from integrating the projection of (135). Note that, once we have that $\overset{(1)}{\sigma}_{\ell=1}$ vanishes along C_{u_0}, we can conclude $\overset{(1)}{b}_{\ell=1}=0$.

The $\ell=0,1$ modes vanish globally

To obtain the identities of Theorem 9.2 globally we first observe that from the $\ell=1$-projection of the Bianchi equation (149) and (156) one concludes that

$$\Omega(\not{d}\mathrm{iv}\,\overset{(1)}{\beta})_{\ell=0,1}=\Omega^{-1}(\not{d}\mathrm{iv}\,\overset{(1)}{\underline{\beta}})_{\ell=0,1}=0 \quad\text{and}\quad \Omega(\not{c}\mathrm{url}\,\overset{(1)}{\beta})_{\ell=0,1}=\Omega^{-1}(\not{c}\mathrm{url}\,\overset{(1)}{\underline{\beta}})_{\ell=0,1}=0.$$

Because these identities hold globally, the commuted Bianchi equations (150) and (155) yield the equations

$$\ell(\ell+1)\,\overset{(1)}{\varrho}_{\ell=1}+3\varrho(\not{d}\mathrm{iv}\,\overset{(1)}{\eta})_{\ell=1}=0 \quad\text{and}\quad -\ell(\ell+1)\,\overset{(1)}{\varrho}_{\ell=1}-3\varrho(\not{d}\mathrm{iv}\,\overset{(1)}{\underline{\eta}})_{\ell=1}=0,$$

from which we conclude that $(\not{d}\mathrm{iv}\,\overset{(1)}{\eta})_{\ell=1}=(\not{d}\mathrm{iv}\,\overset{(1)}{\underline{\eta}})_{\ell=1}$ globally. Note also that the Bianchi equations for $\overset{(1)}{\sigma}_{\ell=1}$ ensure that $\overset{(1)}{\sigma}_{\ell=1}=0$, and hence $(\not{c}\mathrm{url}\,\overset{(1)}{\eta})_{\ell=1}=(\not{c}\mathrm{url}\,\overset{(1)}{\underline{\eta}})_{\ell=1}$ globally. We obtain $(\not{d}\mathrm{iv}\,\overset{(1)}{\eta})_{\ell=1}=0=(\not{d}\mathrm{iv}\,\overset{(1)}{\underline{\eta}})_{\ell=1}$ individually (and similarly for the $\not{c}\mathrm{url}$) from the commuted (142). We conclude from (134) that $\Omega^{-1}\overset{(1)}{\Omega}|_{\ell=1}=0$. This allows to use Raychaudhuri (137), (138) to conclude that $(\Omega\,\overset{(1)}{\mathrm{tr}}\,\chi)_{\ell=1}=0=\Omega^{-2}(\Omega\,\overset{(1)}{\mathrm{tr}}\,\underline{\chi})_{\ell=1}$ globally.

It remains to show that the $\ell=0$ modes vanish globally. For this, note first that the

$\ell=0$ projected linearised Raychaudhuri equations([23]) can be written as

$$D\left(\frac{(\Omega\,\mathrm{tr}\,\overset{(1)}{\chi})_{\ell=0}}{\Omega^2}r-4\Omega^{-1}\overset{(1)}{\Omega}|_{\ell=0}+\frac{\overset{(1)}{\sqrt{\not g}}_{\ell=0}}{\sqrt{\not g}}\right)=0$$

$$\underline{D}\left(\frac{(\Omega\,\mathrm{tr}\,\overset{(1)}{\underline{\chi}})_{\ell=0}}{\Omega^2}r+4\Omega^{-1}\overset{(1)}{\Omega}|_{\ell=0}-\frac{\overset{(1)}{\sqrt{\not g}}_{\ell=0}}{\sqrt{\not g}}\right)=0.$$

Hence, the quantities in brackets vanish identically, and in particular

$$(\Omega\,\mathrm{tr}\,\overset{(1)}{\chi})_{\ell=0}+(\Omega\,\mathrm{tr}\,\overset{(1)}{\underline{\chi}})_{\ell=0}=0$$

globally. With this, we can combine (135) and (143), as well as (136) and (144), as

$$D(r^2(\Omega\,\mathrm{tr}\,\overset{(1)}{\underline{\chi}})_{\ell=0}-2\overset{(1)}{\underline{\omega}}_{\ell=0})=0, \underline{D}(r^2(\Omega\,\mathrm{tr}\,\overset{(1)}{\chi})_{\ell=0}-2\overset{(1)}{\omega}_{\ell=0})=0.$$

The quantities in brackets vanish, so in particular

$$(\underline{D}+\underline{D})\Omega^{-1}\overset{(1)}{\Omega}|_{\ell=0}=0$$

and, since $\Omega^{-1}\overset{(1)}{\Omega}|_{\ell=0}$ is zero on both C_{u_0} and C_{v_0}, we can conclude global vanishing of $\Omega^{-1}\overset{(1)}{\Omega}|_{\ell=0}$, and hence of $\overset{(1)}{\underline{\omega}}_{\ell=0}$ and $\overset{(1)}{\omega}_{\ell=0}$, hence of $(\Omega\,\mathrm{tr}\,\overset{(1)}{\chi})_{\ell=0}$ and $(\Omega\,\mathrm{tr}\,\overset{(1)}{\underline{\chi}})_{\ell=0}$ individually. Global vanishing of $\overset{(1)}{\sqrt{\not g}}_{\ell=0}$ follows from (131).

10. Precise statements of the main theorems

In this section, we present the precise statements of the main theorems of this paper. These will correspond to the rough statements already given in the overview §2.2.

§10.1 will concern boundedness and decay statements for general solutions P of the Regge–Wheeler equation. The main result is Theorem 1.

§10.2 will concern boundedness and decay statements for general solutions α and $\underline{\alpha}$ of the spin ± 2 Teukolsky equations. The main result is Theorem 2, while in Corollary 10.1 we will apply this to linearised gravity, and infer boundedness and decay for the gauge invariant quantities $(\overset{(1)}{\alpha},\overset{(1)}{\psi},\overset{(1)}{\Psi})$ and $(\overset{(1)}{\underline{\alpha}},\overset{(1)}{\underline{\psi}},\overset{(1)}{\underline{\Psi}})$ characterizing a solution $\mathscr{S}$ of the full system.

§10.3 will concern the boundedness of all quantities (130) associated with an initial-data normalised solution $\overset{\vee}{\mathscr{S}}$ of linearised gravity, not just the gauge invariant quantities. The main result is Theorem 3 and its pointwise Corollary 10.2.

Finally, §10.4 will concern the decay theorem for the future-renormalised solution $\hat{\mathscr{S}}$. The main result is Theorem 4 and its pointwise Corollary 10.3.

The remainder of the paper will then concern the proofs of these theorems.

([23]) We remark that the second holds without projection on $\ell=0$, while the first has a $\not{\mathrm{d}}\mathrm{iv}\, b$-term in general.

10.1. Theorem 1: Boundedness and decay for solutions to Regge–Wheeler

Our first theorem (Theorem 1) is concerned purely with solutions of the Regge–Wheeler equation. We will state the theorem in §10.1.2 below, after first defining in §10.1.1 the norms and energies appearing in its formulation.

10.1.1. Energies and norms

We begin with the definition of various norms which will appear in Theorem 1. Let P below denote a solution of the Regge–Wheeler equation, as arising from Theorem 7.3.

We define the following energies for the rescaled solution $\Psi = r^5 P$ defined in (176). The energy fluxes

$$F_u[\Psi](v_1,v_2) = \int_{v_1}^{v_2} d\bar{v}\, (\|r^{-1}\Omega \nabla\!\!\!/_4 \Psi\|^2_{S^2_{u,v}} + \|r^{-1} \nabla\!\!\!/ \Psi\|^2_{S^2_{u,v}} + r^{-2}\|r^{-1}\Psi\|^2_{S^2_{u,v}}), \tag{226}$$

$$F_v[\Psi](u_1,u_2) = \int_{u_1}^{u_2} d\bar{u}\, \Omega^2 (\|r^{-1}\ \Omega^{-1} \nabla\!\!\!/_3 \Psi\|^2_{S^2_{u,v}} + \|r^{-1} \nabla\!\!\!/ \Psi\|^2_{S^2_{u,v}} + r^{-2}\|r^{-1}\Psi\|^2_{S^2_{u,v}}), \tag{227}$$

as well as the weighted (near infinity) fluxes

$$F^{\mathcal{I}}_u[\Psi](v_1,v_2) = \int_{v_1}^{v_2} d\bar{v}\, (r^2\|r^{-1}\Omega \nabla\!\!\!/_4 \Psi\|^2_{S^2_{u,v}} + \|r^{-1} \nabla\!\!\!/ \Psi\|^2_{S^2_{u,v}} + r^{-2}\|r^{-1}\Psi\|^2_{S^2_{u,v}}), \tag{228}$$

$$F^{\mathcal{I}}_v[\Psi](u_1,u_2) = \int_{u_1}^{u_2} d\bar{u}\, \Omega^2 (\|r^{-1} \Omega^{-1} \nabla\!\!\!/_3 \Psi\|^2_{S^2_{u,v}} + r^2\|r^{-1}\ \nabla\!\!\!/ \Psi\|^2_{S^2_{u,v}} + \|r^{-1}\Psi\|^2_{S^2_{u,v}}), \tag{229}$$

where we recall the norms on the spheres $S^2_{u,v}$ defined in (105). From these, we define

$$\mathbb{F}[\Psi] = \sup_u F^{\mathcal{I}}_u[\Psi](v_0,\infty) + \sup_v F^{\mathcal{I}}_v[\Psi](u_0,\infty), \tag{230}$$

with corresponding initial energies

$$\mathbb{F}_0[\Psi] = F^{\mathcal{I}}_{u_0}[\Psi](v_0,\infty) + F^{\mathcal{I}}_{v_0}[\Psi](u_0,\infty). \tag{231}$$

To estimate higher-order energies, we also introduce the following notation, tailored to the fact that the Regge–Wheeler equation (255) commutes with T and the angular momentum operators Ω_i (cf. §4.2.2):

$$\mathbb{F}^{n,T}[\Psi] := \sum_{i=0}^{n} \sup_u F^{\mathcal{I}}_u[T^i\Psi](v_0,\infty) + \sup_v \sum_{i=0}^{n} F^{\mathcal{I}}_v[T^i\Psi](u_0,\infty), \tag{232}$$

$$\mathbb{F}^{n,T,\nabla\!\!\!/}[\Psi] := \sum_{i+j\leqslant n} \sup_u F^{\mathcal{I}}_u[T^i(r\nabla\!\!\!/_A)^j\Psi](v_0,\infty) + \sup_v \sum_{i+j\leqslant n} F^{\mathcal{I}}_v[T^i(r\nabla\!\!\!/_A)^j\Psi](u_0,\infty), \tag{233}$$

which initially become

$$\mathbb{F}_0^{n,T}[\Psi] := \sum_{i=0}^{n} F_{u_0}^{\mathcal{I}}[T^i\Psi](v_0,\infty) + \sum_{i=0}^{n} F_{v_0}^{\mathcal{I}}[T^i\Psi](u_0,\infty), \tag{234}$$

$$\mathbb{F}_0^{n,T,\nabla\!\!\!/}[\Psi] := \sum_{i+j\leqslant n} F_{u_0}^{\mathcal{I}}[T^i(r\nabla\!\!\!/_A)^j\Psi](v_0,\infty) + \sum_{i=0}^{n} F_{v_0}^{\mathcal{I}}[T^i(r\nabla\!\!\!/_A)^j\Psi](u_0,\infty). \tag{235}$$

We also define spacetime energies, which will be used in the integrated local energy decay estimate. These will be denoted by the letter $\mathbb{I}$. We define (denoting $d\mathrm{vol}_{S^2} = \sin\theta\, d\theta\, d\phi$)

$$\begin{aligned}\mathbb{I}_{\mathrm{deg}}[\Psi] := \int_{u_0}^{\infty}\int_{v_0}^{\infty}\int_{S^2_{\bar u,\bar v}} d\bar u\, d\bar v\, d\mathrm{vol}_{S^2}\, \Omega^2 \bigg[& \frac{1}{r^2}|\Omega\nabla\!\!\!/_4\Psi - \Omega\nabla\!\!\!/_3\Psi|^2 + \frac{1}{r^3}|\Psi|^2 \\ & + \frac{(r-3M)^2}{r^2}\left(\frac{1}{r}|\nabla\!\!\!/\Psi|^2 + \frac{1}{r^2}|\Omega\nabla\!\!\!/_4\Psi|^2 + \frac{1}{r^2}|\Omega^{-1}\nabla\!\!\!/_3\Psi|^2\right)\bigg],\end{aligned} \tag{236}$$

which degenerates near the trapped set $r=3M$ and a weighted energy localised to $r\geqslant 4M$

$$\begin{aligned}\mathbb{I}_{\mathcal{I},\varepsilon}[\Psi] := \int_{u_0}^{\infty}\int_{v_0}^{\infty}\int_{S^2_{\bar u,\bar v}} du\, dv\, d\mathrm{vol}_{S^2}\, \iota_{r\geqslant 4M} [& r|\Omega\nabla\!\!\!/_4\Psi|^2 + r^{-1-\varepsilon}|\Omega\nabla\!\!\!/_3\Psi|^2 \\ & + r^{1-\varepsilon}|\nabla\!\!\!/\Psi|^2 + r^{-1-\varepsilon}|\Psi|^2],\end{aligned}$$

for some $0<\varepsilon<\frac{1}{8}$ now fixed once and for all, and $\iota_{r\geqslant R}$ being the indicator function which equals 1 for $r\geqslant R$ and is zero otherwise. The higher-order analogues are defined in the obvious way:

$$\mathbb{I}_{\mathcal{I},\varepsilon}^{n,T,\nabla\!\!\!/}[\Psi] := \sum_{i+j\leqslant n}^{n} \mathbb{I}_{\mathcal{I},\varepsilon}[T^i(r\nabla\!\!\!/_A)^j\Psi], \tag{237}$$

and similarly for $\mathbb{I}_{\mathrm{deg}}^{n,T,\nabla\!\!\!/}[\Psi]$.

10.1.2. Statement of the theorem

We are now ready to state the boundedness and decay theorem for solutions P of the Regge–Wheeler equation.

THEOREM 1. *Let P be a solution of the Regge–Wheeler equation as arising from Theorem 7.3. Then, the weighted symmetric traceless $S^2_{u,v}$-tensor $\Psi = r^5 P$ satisfies equation (177) and the following estimates hold, provided the initial energies on the right-hand sides are finite*:

(1) *the basic boundedness and integrated decay estimates of Proposition 11.3.1, as well as the weighted boundedness estimate*

$$\mathbb{F}[\Psi] \lesssim \mathbb{F}_0[\Psi]; \tag{238}$$

(2) *the higher-order estimates* (*for any integer* $n\geqslant 0$)

$$\mathbb{F}^{n,T,\nabla\!\!\!/}[\Psi]\lesssim\mathbb{F}_0^{n,T,\nabla\!\!\!/}[\Psi], \tag{239}$$

(3) *the weighted integrated decay estimate* (*for any integer* $n\geqslant 0$)

$$\mathbb{I}_{\mathcal{I},\varepsilon}^{n,T,\nabla\!\!\!/}[\Psi]+\mathbb{I}_{\mathrm{deg}}^{n,T,\nabla\!\!\!/}[\Psi]\lesssim\mathbb{F}_0^{n,T,\nabla\!\!\!/}[\Psi]. \tag{240}$$

(4) *Finally, the polynomial decay estimates of Proposition* 11.5.1 *hold.*

The proof of the above theorem will be the content of §11.

10.2. Theorem 2: Boundedness and decay for solutions to Teukolsky

Our second theorem is concerned purely with solutions to the spin ± 2 Teukolsky equations. We define relevant energies and norms in §10.2.1 below. We state the theorem in §10.2.2. We shall then infer an immediate application of the result to the full system of linearised gravity in §10.2.3.

10.2.1. Energies and norms

Let α be a solution to the Teukolsky equation of spin $+2$ as arising from Theorem 7.1 and $\underline{\alpha}$ be a smooth solution to the Teukolsky equation of spin -2 as arising from Theorem 7.2.

Recall that associated with a solution to the Teukolsky equation of spin $+2$ are the derived quantities ψ and P defined in (178) and (179) of §7.3, and associated with a solution to the Teukolsky equation of spin -2 are the quantities $\underline{\psi}$ and $\underline{P}$ defined in (180) and (181).

We define the following energies for the solution α and its derived quantities ψ and P, and the solution $\underline{\alpha}$ and its derived quantities $\underline{\psi}$ and $\underline{P}$:

$$\begin{aligned}\mathbb{F}[\Psi,\psi]=\mathbb{F}[\Psi]+\sup_u\int_{v_0}^{\infty}dv\,\|r^{-1}\cdot\psi\|^2_{S^2_{u,v}}r^{8-\varepsilon}\Omega^2,\ \mathbb{F}[\underline{\Psi},\underline{\psi}]\\ =\mathbb{F}[\,\underline{\Psi}]+\sup_v\int_{u_0}^{\infty}du\,\|r^{-1}\cdot\underline{\psi}\|^2_{S^2_{u,v}}r^{6},\end{aligned}$$

with the obvious definition for $\mathbb{F}_0[\Psi,\psi]$ and $\mathbb{F}_0[\underline{\Psi},\underline{\psi}]$. Also,

$$\begin{aligned}\mathbb{F}[\Psi,\psi,\alpha]&=\mathbb{F}[\Psi,\psi]+\sup_u\int_{v_0}^{\infty}dv\,\|r^{-1}\alpha\|^2_{S^2_{u,v}}r^{6-\varepsilon}\Omega^4,\\ \mathbb{F}[\,\underline{\Psi},\underline{\psi},\underline{\alpha}\,]&=\mathbb{F}[\,\underline{\Psi},\underline{\psi}]+\sup_v\int_{u_0}^{\infty}du\,\|r^{-1}\underline{\alpha}\|^2_{S^2_{u,v}}\Omega^{-2},\end{aligned}$$

again with the obvious definition for $\mathbb{F}_0[\Psi,\psi,\alpha]$ and $\mathbb{F}_0[\,\underline{\Psi},\underline{\psi},\underline{\alpha}\,]$. Finally, the higher-order norms

$$\begin{aligned}
\mathbb{F}[\Psi,\mathfrak{D}\psi,\alpha] &= \mathbb{F}[\Psi,\psi,\alpha]+\sup_u \int_{v_0}^{\infty} dv\,\|r^{-1}\cdot\mathfrak{D}(\psi\Omega)\|^2_{S^2_{u,v}} r^{8-\varepsilon},\\
\mathbb{F}[\underline{\Psi},\mathfrak{D}\underline{\psi},\underline{\alpha}\,] &= \mathbb{F}[\,\underline{\Psi},\underline{\psi},\underline{\alpha}\,]+\sup_v \int_{u_0}^{\infty} du\,\|r^{-1}\cdot\mathfrak{D}(\underline{\psi}\Omega^{-1})\|^2_{S^2_{u,v}} \Omega^2 r^6,\\
\mathbb{F}[\Psi,\mathfrak{D}\psi,\mathfrak{D}\alpha] &= \mathbb{F}[\Psi,\mathfrak{D}\psi,\alpha]+\sup_u \int_{v_0}^{\infty} dv\,\|r^{-1}\cdot\mathfrak{D}(\alpha\Omega^2)\|^2_{S^2_{u,v}} r^{6-\varepsilon},\\
\mathbb{F}[\,\underline{\Psi},\mathfrak{D}\underline{\psi},\mathfrak{D}\underline{\alpha}\,] &= \mathbb{F}[\,\underline{\Psi},\mathfrak{D}\underline{\psi},\underline{\alpha}\,]+\sup_v \int_{u_0}^{\infty} du\,\|r^{-1}\cdot\mathfrak{D}(\underline{\alpha}\Omega^{-2})\|^2_{S^2_{u,v}} \Omega^2,
\end{aligned}$$

where we have employed the shorthand notation

$$\|r^{-1}\cdot\mathfrak{D}\xi\|^2_{S^2_{u,v}} := \|r^{-1}\cdot r\nabla\!\!\!\!/_A\xi\|^2_{S^2_{u,v}} + \|r^{-1}\cdot\Omega^{-1}\nabla\!\!\!\!/_3\xi\|^2_{S^2_{u,v}} + \|r^{-1}\cdot r\Omega\nabla\!\!\!\!/_4\xi\|^2_{S^2_{u,v}}$$

for an $S^2_{u,v}$-tensor ξ.([24])

We also define a basic spacetime energy measuring some form of integrated decay:

$$\begin{aligned}
\mathbb{I}_{\text{master}}[\Psi,\mathfrak{D}\psi,\mathfrak{D}\alpha] &= \mathbb{I}_{\deg}[\Psi]+\mathbb{I}^{\mathcal{I}}_{\varepsilon}[\Psi]+\int_{u_0}^{\infty}\int_{v_0}^{\infty} d\bar u\,d\bar v\,\Omega^2[r^{7-\varepsilon}\|r^{-1}\cdot\mathfrak{D}(\Omega\psi)\|^2_{S^2_{\bar u,\bar v}}\\
&\qquad + r^{5-\varepsilon}\|r^{-1}\cdot\mathfrak{D}(\Omega^2\alpha)\|^2_{S^2_{\bar u,\bar v}}],\\
\mathbb{I}_{\text{master}}[\underline{\Psi},\mathfrak{D}\underline{\psi},\mathfrak{D}\underline{\alpha}\,] &= \mathbb{I}_{\deg}[\underline{\Psi}]+\mathbb{I}^{\mathcal{I}}_{\varepsilon}[\underline{\Psi}]+\int_{u_0}^{\infty}\int_{v_0}^{\infty} d\bar u\,d\bar v\,\Omega^2[r^{5-\varepsilon}\|r^{-1}\cdot\mathfrak{D}(\Omega^{-1}\underline{\psi})\|^2_{S^2_{\bar u,\bar v}}\\
&\qquad + r^{1-\varepsilon}\|r^{-1}\cdot\mathfrak{D}(\Omega^{-2}\underline{\alpha})\|^2_{S^2_{\bar u,\bar v}}].
\end{aligned}$$

The following higher-order energies are then defined in the obvious way

$$\mathbb{F}^{n,T,\nabla\!\!\!\!/}[\,\underline{\Psi},\mathfrak{D}\underline{\psi},\mathfrak{D}\underline{\alpha}\,],\quad \mathbb{F}_0^{n,T,\nabla\!\!\!\!/}[\,\underline{\Psi},\mathfrak{D}\underline{\psi},\mathfrak{D}\underline{\alpha}\,]\quad\text{and}\quad \mathbb{I}^{n,T,\nabla\!\!\!\!/}_{\text{master}}[\,\underline{\Psi},\mathfrak{D}\underline{\psi},\mathfrak{D}\underline{\alpha}\,],$$

as are their non-underlined counterparts.

10.2.2. Statement of the theorem

We are now ready to state the boundedness and decay theorem for solutions of the spin ± 2 Teukolsky equations.

([24]) Note that, whenever the argument of $\mathbb{F}$ involves *underlined* quantities, it is generally a different flux compared to the non-underlined quantities. The only exception is $\mathbb{F}[\underline{\Psi}]$, which is obtained by inserting $\underline{\Psi}$ instead of Ψ in $\mathbb{F}[\Psi]$.

THEOREM 2. *Let α be a solution of the spin +2 Teukolsky equation as arising from Theorem 7.1. Then, the derived quantity $\Psi{=}r^5 P$, with P defined through (178) and (179), satisfies the conclusions of Theorem 1. Moreover, provided the initial energies on the right-hand side of (241)–(242) are finite, we have the following estimates:*

(1) *the weighted boundedness estimate*

$$\mathbb{F}[\Psi, \psi, \alpha] \lesssim \mathbb{F}_0[\Psi, \psi, \alpha]; \tag{241}$$

(2) *the higher-order statements (for any integer $n \geqslant 0$)*

$$\mathbb{F}^{n,T,\nabla\!\!\!/}[\Psi, \mathfrak{D}\psi, \mathfrak{D}\alpha] \lesssim \mathbb{F}_0^{n,T,\nabla\!\!\!/}[\Psi, \mathfrak{D}\psi, \mathfrak{D}\alpha]; \tag{242}$$

(3) *the weighted integrated decay estimate (for any integer $n \geqslant 0$)*

$$\mathbb{I}^{n,T,\nabla\!\!\!/}_{\mathrm{master}}[\Psi, \mathfrak{D}\psi, \mathfrak{D}\alpha] \lesssim \mathbb{F}_0^{n,T,\nabla\!\!\!/}[\Psi, \mathfrak{D}\psi, \mathfrak{D}\alpha]. \tag{243}$$

(4) *Finally, the polynomial decay estimates of Propositions 12.3.4– 12.3.7 and the L^1-estimate of Corollary 12.6 hold.*

Now, let $\underline{\alpha}$ be a smooth solution of the spin −2 Teukolsky equation as arising from Theorem 7.2. Then, $\underline{\Psi}{=}r^5\underline{P}$, with $\underline{P}$ defined through (178), (179), satisfies the conclusions of Theorem 1. Moreover, the estimates (1)–(4) above hold replacing the quantities α, ψ and Ψ by $\underline{\alpha}$, $\underline{\psi}$ and $\underline{\Psi}$, respectively, provided the energies on the right-hand side are finite.

Note that the second sentence of Theorem 2, that Ψ satisfies the conclusions of Theorem 1, is already immediate from Proposition 7.3.1. The same proposition immediately yields the analogous statement for $\underline{\Psi}$ claimed in the second part of the theorem.

The proof of Theorem 2 will be carried out in §12. Key to the proof is to exploit the transformation formulas of §7.3.

10.2.3. Application to the full system of linearised gravity: Boundedness and decay for the gauge invariant hierarchy

In view of Proposition 7.4.1, we infer the following application to the full system of linearised gravity.

COROLLARY 10.1. *Let $\mathscr{S}$ be a smooth solution of the system of gravitational perturbations arising from a smooth seed initial data set on $C_{u_0} \cup C_{v_0}$ through Theorem 8.1. Then, the following statements hold:*

- *the gauge invariant curvature component $\overset{(1)}{\alpha}$ of the solution $\mathscr{S}$ satisfies the Teukolsky equation of spin +2, and hence the first part of Theorem 2 applies yielding boundedness and decay for $(\overset{(1)}{\Psi}, \overset{(1)}{\psi}, \overset{(1)}{\alpha})$.*

- *the gauge invariant curvature component* $\underline{\overset{(1)}{\alpha}}$ *of the solution* $\mathscr{S}$ *satisfies the Teukolsky equation of spin* -2, *and hence the second part of Theorem 2 applies yielding boundedness and decay for* $(\underline{\overset{(1)}{\Psi}}, \underline{\overset{(1)}{\psi}}, \underline{\overset{(1)}{\alpha}})$.

Let us note that we know more information about $\overset{(1)}{\alpha}$ and $\underline{\overset{(1)}{\alpha}}$ than the statement that they satisfy the spin ± 2 Teukolsky equations. The solutions $\overset{(1)}{\alpha}$ and $\underline{\overset{(1)}{\alpha}}$ are in fact non-trivially related to each other through the system of linearised Bianchi and null structure equations. (For fixed frequency solutions, these relations are well known; see [71].) We stress that the estimates inferred in Corollary 10.1 for $\mathscr{S}$ are derived without exploiting this relation. The above corollary will be the starting point (see §13.1) for the proof of Theorem 3 to which we now turn.

10.3. Theorem 3: Boundedness for solutions to the full system

We now consider the full system of linearised gravity. Our next theorem (Theorem 3) asserts boundedness of initial-data normalised solutions $\overset{\vee}{\mathscr{S}}$ as in Definition 9.1. In view of Theorem 9.1, we will be able to apply Theorem 3 to solutions $\mathscr{S}$ arising from general, smooth asymptotically flat seed data. We first define some additional energies and norms in §10.3.1 before stating the precise formulation of the theorem in §10.3.2.

10.3.1. Energies and norms

Let $\mathscr{S}$ be a solution of the system of gravitational perturbations as arising from Theorem 8.1.

Recall that, by Proposition 7.4.1, the components $\overset{(1)}{\alpha}$ and $\underline{\overset{(1)}{\alpha}}$ of the solution satisfy the spin ± 2 Teukolsky equations and thus, by Proposition 7.3.1, the quantities $\overset{(1)}{P}$ and $\underline{\overset{(1)}{P}}$ derived from $\overset{(1)}{\alpha}$ and $\underline{\overset{(1)}{\alpha}}$, respectively, satisfy the Regge–Wheeler equation. Thus, we may use the notation of §10.1.1 and §10.2.1 to denote energies associated with these gauge-invariant quantities. We will augment these with the following combined notation

$$\mathbb{F}[\overset{(1)}{\Psi}, \underline{\overset{(1)}{\Psi}}, \overset{(1)}{\psi}, \underline{\overset{(1)}{\psi}}, \overset{(1)}{\alpha}, \underline{\overset{(1)}{\alpha}}] := \mathbb{F}[\overset{(1)}{\Psi}, \overset{(1)}{\psi}, \overset{(1)}{\alpha}] + \mathbb{F}[\underline{\overset{(1)}{\Psi}}, \underline{\overset{(1)}{\psi}}, \underline{\overset{(1)}{\alpha}}].$$

We proceed to define additional (gauge-dependent) energies.

We define the flux

$$\|(\nabla\!\!\!/_3)^2 \overset{(1)}{\hat{\chi}}\|^2_{L^\infty_v L^2(C_v)} = \sup_{v \geqslant v_0} \int_{u_0}^{\infty} d\bar{u} \int_{S^2_{\bar{u},v}} \sin\theta \, d\theta \, d\phi \, \Omega^2 \left[\left| \frac{1}{\Omega} \nabla\!\!\!/_3 \left(\frac{1}{\Omega} \nabla\!\!\!/_3 (r^2 \overset{(1)}{\hat{\chi}} \Omega) \right) \right|^2 + \left| \frac{1}{\Omega} \nabla\!\!\!/_3 (r^2 \overset{(1)}{\hat{\chi}} \Omega) \right|^2 + \frac{1}{r^\varepsilon} |r^2 \overset{(1)}{\hat{\chi}} \Omega|^2 \right].$$

Recall now the two auxiliary quantities $\overset{(1)}{Y}$ and $\overset{(1)}{Z}$ defined in (218) and (219). We define the following energy for the Ricci coefficients on spheres (the superscript (5) stands for the fact that this energy is at the level of 5 derivatives of the Ricci coefficients):

$$\begin{aligned}\mathbb{D}^{[5]}[\overset{(1)}{Y},\overset{(1)}{Z}] = & \sup_{u,v}\|r^{-1}\cdot r^3 \not{\mathcal{D}}_2^\star \,\mathrm{d}\not{\mathrm{i}}\mathrm{v}\, \not{\mathcal{D}}_2^\star \overset{(1)}{Y}\|^2_{S^2_{u,v}} + \sup_{u,v}\|r^{-1}\cdot r^2\,\mathrm{d}\not{\mathrm{i}}\mathrm{v}\,\not{\mathcal{D}}_2^\star r\Omega\not{\nabla}_4\overset{(1)}{Y}\|^2_{S^2_{u,v}} \\ & + \sup_{u,v} r^{2+\varepsilon}\|r^{-1}\cdot\Omega^{-1}r^4\,\mathrm{d}\not{\mathrm{i}}\mathrm{v}\,\not{\mathcal{D}}_2^\star\,\mathrm{d}\not{\mathrm{i}}\mathrm{v}\,\not{\mathcal{D}}_2^\star\overset{(1)}{Z}\|^2_{S^2_{u,v}} \\ & + \sup_{u,v}\|r^{-1}\cdot r\not{\nabla}_3(r^4\not{\mathcal{D}}_2^\star\,\mathrm{d}\not{\mathrm{i}}\mathrm{v}\,\not{\mathcal{D}}_2^\star\not{\nabla}(r\Omega^{-2}(\Omega\,\overset{(1)}{\mathrm{tr}}\,\chi)))\|^2_{S^2_{u,v}},\end{aligned} \tag{244}$$

which, at the level of data, is

$$\begin{aligned}\mathbb{D}_0^{[5]}[\overset{(1)}{Y},\overset{(1)}{Z}] = & \sup_{v}\|r^{-1}\cdot r^3 \not{\mathcal{D}}_2^\star \,\mathrm{d}\not{\mathrm{i}}\mathrm{v}\, \not{\mathcal{D}}_2^\star \overset{(1)}{Y}\|^2_{S^2_{u_0,v}} + \sup_{v}\|r^{-1}\cdot r^2\,\mathrm{d}\not{\mathrm{i}}\mathrm{v}\,\not{\mathcal{D}}_2^\star r\Omega\not{\nabla}_4\overset{(1)}{Y}\|^2_{S^2_{u_0,v}} \\ & + \sup_{u} r^{2+\varepsilon}\|r^{-1}\cdot\Omega^{-1}r^4\,\mathrm{d}\not{\mathrm{i}}\mathrm{v}\,\not{\mathcal{D}}_2^\star\,\mathrm{d}\not{\mathrm{i}}\mathrm{v}\,\not{\mathcal{D}}_2^\star\overset{(1)}{Z}\|^2_{S^2_{u,v_0}} \\ & + \sup_{u}\|\not{\nabla}_3(r^4\not{\mathcal{D}}_2^\star\,\mathrm{d}\not{\mathrm{i}}\mathrm{v}\,\not{\mathcal{D}}_2^\star\not{\nabla}(r\Omega^{-2}(\Omega\,\overset{(1)}{\mathrm{tr}}\,\chi)))\|^2_{S^2_{u,v_0}}.\end{aligned} \tag{245}$$

Recall Propositions 9.4.3 and 9.4.2, which guarantee that the norm $\mathbb{D}_0^{[5]}[\overset{(1)}{Y},\overset{(1)}{Z}]$ is indeed finite for the initial data of the solution $\underset{\vee}{\mathscr{S}}$ defined in Theorem 9.1.

Remark 10.1. One should think of the last term in (244) as the $\not{\nabla}_3$ derivative of $\overset{(1)}{Z}$, but without the $(\overset{(1)}{\eta}+\overset{(1)}{\underline{\eta}})$-part. There is a small technical advantage in that the quantity in the energy satisfies a "more decoupled" equation. Note also that, for both $\overset{(1)}{Z}$ and the last term, we do not put the optimal weight near the horizon (which would allow another factor of Ω^{-1} in both terms of the second line; cf. Proposition 9.4.3).

Let us note finally that, if $\mathscr{S}$ is supported on $\ell=0,1$ only, then all the above energies manifestly vanish. In particular, the above energies vanish for the reference linearised Kerr solutions $\mathscr{K}_{\mathfrak{m},s_i}$.

10.3.2. Statement of the theorem

We are now ready to state our boundedness theorem for the initial-data normalised solution $\underset{\vee}{\mathscr{S}}$ of the full system of linearised gravity.

THEOREM 3. *Let $\underset{\vee}{\mathscr{S}}$ be a smooth solution of the system of gravitational perturbations arising from a smooth seed initial data set on $C_{u_0}\cup C_{v_0}$ through Theorem 8.1, which is moreover initial-data normalised according to Definition 9.1.*

(*In particular, given a general smooth seed initial data set which is asymptotically flat with weight s to order* $n \geqslant 10$ *according to Definition* 8.2, *then defining*

$$\underset{\vee}{\mathscr{S}} = \mathscr{S} + \underset{\vee}{\mathscr{G}}$$

by applying Theorem 9.1, *it follows that* $\underset{\vee}{\mathscr{S}}$ *satisfies the above assumption.*)

Then, the curvature quantities α *and* $\underline{\alpha}$ *associated with* $\underset{\vee}{\mathscr{S}}$ *satisfy the conclusions of Theorem* 2.

We assume finiteness of the following initial energy, which is at the level of five derivatives of curvature and five derivatives of the Ricci coefficients

$$\begin{aligned} \mathbb{E}_0 := &\|\nabla\!\!\!/_3^2 (r^3 \, \mathrm{d}\!\!/\mathrm{iv}\, \mathcal{D}\!\!\!/_2^\star \, \mathrm{d}\!\!/\mathrm{iv} \, \overset{(1)}{\hat{\chi}})\|^2_{L^2(C_{v_0})} + \mathbb{D}_0^{[5]}[\overset{(1)}{Z}, \overset{(1)}{Y}] \\ &+ \mathbb{F}_0^{2,T,\nabla\!\!\!/}[\overset{(1)}{\underline{\Psi}}, \mathfrak{D}\overset{(1)}{\underline{\psi}}, \mathfrak{D}\overset{(1)}{\underline{\alpha}}] + \mathbb{F}_0^{2,T,\nabla\!\!\!/}[\overset{(1)}{\Psi}, \mathfrak{D}\overset{(1)}{\psi}, \mathfrak{D}\overset{(1)}{\alpha}] < \infty. \end{aligned} \tag{246}$$

Then, we have the estimates

$$\mathbb{F}_0^{2,T,\nabla\!\!\!/}\big[\overset{(1)}{\underline{\Psi}}, \mathfrak{D}\overset{(1)}{\underline{\psi}}, \mathfrak{D}\overset{(1)}{\underline{\alpha}}\big] + \mathbb{F}_0^{2,T,\nabla\!\!\!/}\big[\overset{(1)}{\Psi}, \mathfrak{D}\overset{(1)}{\psi}, \mathfrak{D}\overset{(1)}{\alpha}\big] \lesssim \mathbb{E}_0 \tag{247}$$

and

$$\|\nabla\!\!\!/_3^2 (r^3 \, \mathrm{d}\!\!/\mathrm{iv}\, \mathcal{D}\!\!\!/_2^\star \, \mathrm{d}\!\!/\mathrm{iv} \, \overset{(1)}{\hat{\chi}})\|^2_{L^\infty_v L^2(C_v)} + \mathbb{D}^{[5]}[\overset{(1)}{Z}, \overset{(1)}{Y}] \lesssim \mathbb{E}_0, \tag{248}$$

the first one being already immediate from Theorem 2.

Moreover, the initial data energy (246) *controls in addition*:

(1) *Weighted* $L^\infty_{u,v} L^2(S^2_{u,v})$*-norms for up to five angular derivatives of the metric coefficients* $(\overset{(1)}{\hat{\not g}}, \overset{(1)}{\sqrt{\not g}}/\sqrt{\not g}, \overset{(1)}{b}, \Omega^{-1}\overset{(1)}{\Omega})$ *as in Proposition* 13.5.12.

(2) *Weighted* $L^\infty_{u,v} L^2(S^2_{u,v})$*-norms and weighted* L^2*-fluxes on null cones for*

- *up to five angular derivatives of* $\overset{(1)}{\underline{\hat{\chi}}}$ *as in Corollaries* 13.3 *and* 13.4;
- *up to five angular derivatives of* $\overset{(1)}{\hat{\chi}}$ *as in Propositions* 13.3.1, 13.3.3, 13.3.4 *and* 13.5.7;
- *up to five angular derivatives of* $\overset{(1)}{\eta}$ *as in Propositions* 13.5.3 *and* 13.5.10;
- *up to five angular derivatives of* $\overset{(1)}{\underline{\eta}}$ *as in Propositions* 13.5.3 *and* 13.5.11;
- *up to five angular derivatives for* $(\Omega \overset{(1)}{\mathrm{tr}}\, \chi)$ *as in Corollaries* 13.3 *and* 13.10;
- *up to five angular derivatives of* $(\Omega \overset{(1)}{\mathrm{tr}}\, \underline{\chi})$ *as in Proposition* 13.5.5;
- *up to five angular derivatives of* $\overset{(1)}{\omega}$ *and* $\overset{(1)}{\underline{\omega}}$ *as in Proposition*13.5.4.

(3) *Weighted* $L^\infty_{u,v} L^2(S^2_{u,v})$*-norms for four angular derivatives and weighted flux estimates for five angular derivatives of the curvature components* $(\overset{(1)}{\alpha}, \overset{(1)}{\beta}, \overset{(1)}{\varrho}, \overset{(1)}{\sigma}, \overset{(1)}{\underline{\beta}}, \overset{(1)}{\underline{\alpha}})$ *provided the non-degenerate initial energies* $\mathbb{F}^2_0[\overset{(1)}{\Psi}]$ *and* $\mathbb{F}^2_0[\overset{(1)}{\underline{\Psi}}]$ *are finite and added* (*to* (248)) *on the right-hand side. See Propositions* 13.5.1, 13.5.8 *and* 12.3.1 *for the flux estimates and Propositions* 13.5.2 *and* 12.3.3 *for the* $L^\infty_{u,v} L^2(S^2_{u,v})$ *estimates.*

Finally, let $\mathscr{K}_{\mathfrak{m},s_i}$ *be the initial data normalised Kerr solution as in Theorem 9.2 such that* $\underset{\vee}{\mathscr{S}}'=\underset{\vee}{\mathscr{S}}-\mathscr{K}_{\mathfrak{m},s_i}$ *has support outside* $\ell=0,1$. *Then, the initial norm (246) coincides for* $\underset{\vee}{\mathscr{S}}'$, *and the above statements of the theorem all hold applied to* $\underset{\vee}{\mathscr{S}}'$ *in place of* $\underset{\vee}{\mathscr{S}}$, *where now the derived energy bounds are coercive on all quantities (130) of* $\underset{\vee}{\mathscr{S}}'$.

10.3.3. Remarks and uniform pointwise boundedness

We give a number of remarks concerning the statement of Theorem 3.

Remark 10.2. If $\underset{\vee}{\mathscr{S}}$ in Theorem 3 indeed arises through Theorem 9.1 from smooth asymptotically flat seed initial data of order $\frac{1}{2}\leqslant s\leqslant 1$ and with $n\geqslant 15$, the finiteness of the initial energy (246) is seen to be a direct consequence of the estimates (439) and Propositions 9.4.3 and 9.4.2.

Remark 10.3. As we shall see, the boundedness and decay estimates proven in Theorem 3 can be proven also if the conditions (189) and (190) did not hold for $\underset{\vee}{\mathscr{S}}$. The only difference are additional boundary terms appearing on the right-hand side in the estimates; cf. Proposition 13.5.4. The horizon gauge conditions (198) and (197), the round sphere condition (191) for $\underset{\vee}{\mathscr{S}}$ and the finiteness of (246) are fundamental, however.

Remark 10.4. The propagation of the weighted norm (246) in (247) and (248) and, intimately related with it, the propagation of the round sphere condition at infinity, can be viewed as a version of propagation of asymptotic flatness for the solution $\underset{\vee}{\mathscr{S}}$, which does not lose derivatives.

Remark 10.5. In the course of the proof of Theorem 3, we shall obtain several other estimates on various derivatives of the Ricci coefficients. We have not stated these estimates explicitly above but direct the reader to the body of §13. Some of these estimates are needed to prove Corollary 10.2 below. We also emphasise that the quantities $\overset{(1)}{\beta}$ and $\overset{(1)}{\widehat{\chi}}$ can already be shown to decay to zero in time for $\underset{\vee}{\mathscr{S}}'$. This is not true for the other Ricci coefficients and curvature components (except, of course, for the gauge invariant quantities $\overset{(1)}{\alpha}$ and $\overset{(1)}{\underline{\alpha}}$ for which the conclusions of Theorem 2 hold).

Remark 10.6. The above statements focus on angular derivatives. A version of the above theorem can be obtained for *all* derivatives of curvature and Ricci coefficients up to order five, provided appropriate quantities are assumed to be finite initially. As this is standard (but lengthy) we leave it to the reader.

Simple Sobolev embedding on the spheres $S^2_{u,v}$ and using the fact that $\underset{\vee}{\mathscr{S}}'$ of Theorem 3 is supported outside of $\ell=0,1$, together with the boundedness of $\mathscr{K}_{\mathfrak{m},s_i}$, we obtain in particular the following result (see §13.5.8).

COROLLARY 10.2. *Let $\underset{\vee}{\mathscr{S}}$ and $\mathscr{K}_{\mathfrak{m},s_i}$ be as in the statement of Theorem 3. Then, all quantities (130) of $\underset{\vee}{\mathscr{S}}'=\underset{\vee}{\mathscr{S}}-\mathscr{K}_{\mathfrak{m},s_i}$ are uniformly pointwise bounded in that*

$$|r^{7/2-\varepsilon}\Omega^2\overset{(1)}{\alpha}|+|\Omega r^{7/2-\varepsilon}\overset{(1)}{\beta}|+|r^3\overset{(1)}{\varrho}|+|r^3\overset{(1)}{\sigma}|+|r^2\Omega^{-1}\overset{(1)}{\underline{\beta}}|+|r\Omega^{-2}\overset{(1)}{\underline{\alpha}}|\lesssim\sqrt{\mathbb{E}_0},$$
$$|r^2\Omega\overset{(1)}{\hat{\chi}}|+|r\Omega^{-1}\overset{(1)}{\underline{\hat{\chi}}}|+|r\overset{(1)}{\eta}|+|r^2\overset{(1)}{\underline{\eta}}|$$
$$+|r^2\Omega^{-2}(\Omega\overset{(1)}{\mathrm{tr}}\,\chi)|+|r\Omega^{-2}(\Omega\overset{(1)}{\mathrm{tr}}\,\underline{\chi})|+|r^{(5-\varepsilon)/2}\overset{(1)}{\omega}|+|\Omega^{-2}\overset{(1)}{\underline{\omega}}|\lesssim\sqrt{\mathbb{E}_0}$$
$$|\overset{(1)}{\hat{\slashed{g}}}|+\left|\frac{\overset{(1)}{\sqrt{\slashed{g}}}}{\sqrt{\slashed{g}}}\right|+r^{1/2-\varepsilon}|\overset{(1)}{b}|+|\Omega^{-1}\overset{(1)}{\Omega}|\lesssim\sqrt{\mathbb{E}_0}.$$

The same bounds hold for $\underset{\vee}{\mathscr{S}}$ in place of $\underset{\vee}{\mathscr{S}}'$ if a constant depending on

$$|\mathfrak{m}|+|s_{-1}|+|s_0|+|s_1|$$

is added to the right-hand side.

Remark 10.7. We indeed control $|r^2\Omega^{-2}(\Omega\overset{(1)}{\mathrm{tr}}\,\chi)|$ above, because the regular quantity $(\Omega\overset{(1)}{\mathrm{tr}}\,\chi)$ vanishes linearly on the event horizon for the gauge $\underset{\vee}{\mathscr{S}}$.

10.4. Theorem 4: Decay for solutions to the full system in the future-normalised gauge

We may now state our final Theorem 4 giving quantitative decay, measured in appropriate L^2 norms, for all quantities associated with the horizon-renormalised solution $\hat{\mathscr{S}}$ defined in Proposition 9.3.1. As a corollary, we shall deduce in particular pointwise polynomial decay of the metric components of $\hat{\mathscr{S}}$ to their linearised Kerr values given by $\mathscr{K}_{\mathfrak{m},s_i}$ of Theorem 9.2. The norms appearing below have already been defined in §10.1.1, §10.2.1 and §10.3.1.

THEOREM 4. *Let $\underset{\vee}{\mathscr{S}}$ be as in Theorem 3, in particular (246) holds initially. Let*

$$\hat{\mathscr{S}}=\underset{\vee}{\mathscr{S}}+\hat{\mathscr{G}}$$

be the horizon-renormalised solution defined in Proposition 9.3.1. Then, the following statements holds.

(1) *The pure gauge solution $\hat{\mathscr{G}}$ is uniformly bounded and controlled solely by the initial data energy (246) and the ingoing shear of the solution $\underset{\vee}{\mathscr{S}}$ on the initial sphere S^2_{∞,v_0}.*

In particular, the geometric quantities of $\hat{\mathscr{G}}$ satisfy the weighted boundedness estimates of Proposition 14.1.3, which are identical to the weighted boundedness estimates proven for $\underset{\vee}{\mathscr{S}}$ up to four angular derivatives of all geometric quantities.

In fact, except for a small loss([25]) *of decay towards null infinity for the highest angular derivatives of some of the* $\hat{\mathscr{G}}$ *quantities, any weighted quantity bounded in* $\underset{\vee}{\mathscr{S}}$ *by Theorem* 3 *is also bounded for* $\hat{\mathscr{G}}$, *and hence, by linearity, for* $\hat{\mathscr{S}}$.

(2) *The geometric quantities of* $\hat{\mathscr{S}}$ *satisfy the integrated decay estimates of Propositions* 14.2.1, 13.3.3, 14.2.3, 14.2.4, 14.2.5, 14.2.7 *and* 14.2.8. *In particular, we obtain a degenerate* (*near* $r=3M$) *integrated decay estimate for five angular derivatives of the linearised curvature components* $(\overset{(1)}{\alpha}, \overset{(1)}{\beta}, \overset{(1)}{\varrho}, \overset{(1)}{\sigma}, \overset{(1)}{\underline{\beta}}, \overset{(1)}{\underline{\alpha}})$ *and a non-degenerate estimate for four* (*or less*) *derivatives.*

(3) *The geometric quantities of* $\hat{\mathscr{S}}$ *satisfy the polynomial decay proven in* §14.3. *In particular, the metric coefficients satisfy the polynomial decay estimates* (*with* $\mathbb{E}_0$ *defined in* (246))

$$\|r^{-1}\cdot r^2 \mathcal{D}_2^\star \nabla \Omega^{-1}\overset{(1)}{\Omega}\|_{S^2_{u,v}} \lesssim \frac{1}{v}\sqrt{\mathbb{E}_0}, \tag{249}$$

$$\|r^{-1}\cdot r \mathcal{D}_2^\star \overset{(1)}{b}\|_{S^2_{u,v}} \lesssim \frac{1}{v^{1/2}}\sqrt{\mathbb{E}_0} \tag{250}$$

$$\|r^{-1}\cdot \mathcal{A}^{[2]} \overset{(1)}{\hat{\not g}}\|_{S^2_{u,v}} \lesssim \frac{1}{v^{1/2}}\sqrt{\mathbb{E}_0}, \tag{251}$$

$$\left\|r^{-1}\cdot r^2 \mathcal{D}_2^\star \nabla \frac{\overset{(1)}{\sqrt{\not g}}}{\sqrt{\not g}}\right\|_{S^2_{u,v}} \lesssim \frac{1}{v^{1/2}}\sqrt{\mathbb{E}_0}, \tag{252}$$

to be proven in §14.3.2.

Finally, let $\mathscr{K}_{\mathfrak{m},s_i}$ *be the reference linearised Kerr solution defined in Theorem* 3. *Then,* $\hat{\mathscr{S}}' = \hat{\mathscr{S}} - \mathscr{K}_{\mathfrak{m},s_i}$ *is supported away from* $\ell=0,1$ *and the above statements of the theorem hold as stated for* $\hat{\mathscr{S}}'$. *As in the final statement of Theorem* 3, *the energies are now coercive on all quantities* (130) *of* $\hat{\mathscr{S}}'$.

We append a remark analogous to Remark 10.4 in Theorem 3.

Remark 10.8. As one readily checks, statement (1) in Theorem 4 implies in particular estimate (248) for the geometric quantities of $\hat{\mathscr{S}}$ on the left-hand side. Therefore, analogous to Remark 10.4, we can interpret the result as a propagation of asymptotic flatness for the solution $\hat{\mathscr{S}}$.

A simple application of the Sobolev embedding theorem on the round sphere to (249)–(252) provides the following result.

Corollary 10.3. *With* $\hat{\mathscr{S}}$, $\mathscr{K}_{\mathfrak{m},s_i}$ *and* $\hat{\mathscr{S}}' = \hat{\mathscr{S}} - \mathscr{K}_{\mathfrak{m},s_i}$ *as in Theorem* 4, *the metric components* $\overset{(1)}{\Omega}[\hat{\mathscr{S}}']$, $\overset{(1)}{b}[\hat{\mathscr{S}}']$, $\overset{(1)}{\sqrt{\not g}}[\hat{\mathscr{S}}']$ *and* $\overset{(1)}{\hat{\not g}}[\hat{\mathscr{S}}']$ *of* $\hat{\mathscr{S}}'$ *satisfy the following uniform bounds*

([25]) This loss can potentially be avoided with further work. See Remark 14.1.

on $\mathcal{M}\cap\{u\geqslant u_0\}\cap\{v\geqslant v_0\}$:

$$|\Omega^{-1}\overset{(1)}{\Omega}|\lesssim\frac{1}{v}\sqrt{\mathbb{E}_0}\tag{253}$$

and

$$|\overset{(1)}{b}|+|\overset{(1)}{\sqrt{\not g}}|+|\overset{(1)}{\hat{\not g}}|\lesssim\frac{1}{v^{1/2}}\sqrt{\mathbb{E}_0}.\tag{254}$$

Thus, the metric components of $\hat{\mathscr{S}}$ *converge pointwise to the linearised Kerr values of* $\mathscr{K}_{\mathfrak{m},s_i}$.

Let us remark that one can also obtain pointwise bounds for all Ricci coefficients and curvature components from the bounds proven in this paper, but we will not state these bounds explicitly.

11. Proof of Theorem 1

The present section contains the proof of Theorem 1. As explained in §2.3, this proof follows closely previous work for the scalar wave equation (48). The reader may wish to refer to §2.3 for comparison while reading the present section.

We begin in §11.1 with the natural energy identity associated with the Regge–Wheeler equation. We then show in §11.2 a version of integrated decay which degenerates at $r=3M$, at the horizon $\mathcal{H}^+$ and at null infinity $\mathcal{I}^+$. The degeneration at $\mathcal{H}^+$ is completely removed in §11.3 using the red-shift, whereas the degeneration at $\mathcal{I}^+$ is refined in §11.4 using an r^p hierarchy. Higher-order estimates and polynomial decay estimates for the energy will be the content of §11.5.

11.1. Energy conservation for Regge–Wheeler

Let Ψ be as in the statement of Theorem 1.

From

$$\Omega\nabla\!\!\!/_3(\Omega\nabla\!\!\!/_4\Psi)-\left(1-\frac{2M}{r}\right)\Delta\!\!\!\!/\,\Psi+V\Psi=0\quad\text{with } V=\left(\frac{4}{r^2}-\frac{6M}{r^3}\right)\left(1-\frac{2M}{r}\right),\tag{255}$$

we easily derive the following identity:

$$\begin{aligned}[\Omega\nabla\!\!\!/_3+\Omega\nabla\!\!\!/_4]\int\sin\theta\,d\theta\,d\phi\left(|\Omega\nabla\!\!\!/_4\Psi|^2+|\Omega\nabla\!\!\!/_3\Psi|^2+2\frac{1-2M/r}{r^2}|r\nabla\!\!\!/\Psi|^2+2V|\Psi|^2\right)\\+[\Omega\nabla\!\!\!/_3-\Omega\nabla\!\!\!/_4]\int\sin\theta\,d\theta\,d\phi\,(|\Omega\nabla\!\!\!/_4\Psi|^2-|\Omega\nabla\!\!\!/_3\Psi|^2)=0.\end{aligned}\tag{256}$$

Using the notation $1-\mu=1-2M/r$, we define the null fluxes

$$F_u^T[\Psi](v_1,v_2)=\int_{v_1}^{v_2} dv\,\sin\theta\,d\theta\,d\phi\,(|\Omega\nabla\!\!\!/_4\Psi|^2+(1-\mu)|\nabla\!\!\!/\Psi|^2+V|\Psi|^2),$$
$$F_v^T[\Psi](u_1,u_2)=\int_{u_1}^{u_2} du\,\sin\theta\,d\theta\,d\phi\,(|\Omega\nabla\!\!\!/_3\Psi|^2+(1-\mu)|\nabla\!\!\!/\Psi|^2+V|\Psi|^2)$$

Note that

$$V=\left(1-\frac{2M}{r}\right)\left(\frac{4}{r^2}-\frac{6M}{r^3}\right)\geqslant\frac{1}{r^2}\left(1-\frac{2M}{r}\right),$$

and hence that these fluxes are manifestly coercive.

Integrating (256) with respect to $dudv$ yields a conservation law.

PROPOSITION 11.1.1. *For any $u\geqslant u_0$ and $v\geqslant v_0$, the Ψ of Theorem 1 satisfies*

$$F_u^T[\Psi](v_0,v)+F_v^T[\Psi](u_0,u)=F_{v_0}^T[\Psi](u_0,u)+F_{u_0}^T[\Psi](v_0,v). \tag{257}$$

The above is the precise analogue of the T-energy identity for solutions φ of (48).([26])

11.2. Integrated decay estimate

Let us define the operators

$$T:=\tfrac12[\Omega\nabla\!\!\!/_3+\Omega\nabla\!\!\!/_4]\quad\text{and}\quad R^\star:=\tfrac12[-\Omega\nabla\!\!\!/_3+\Omega\nabla\!\!\!/_4].$$

(We note that T above coincides with Lie-differentiation $\mathcal{L}_T$ with respect to the Killing field T of §4.2.2, but the above form will be convenient here.)

Let now $\mathfrak{f}$ be a function on $\mathcal{M}^\circ$ of $r^\star:=(v-u)$ only, i.e. $T(\mathfrak{f})=0$ and $f':=R^\star(\mathfrak{f})$. We have the identity

$$\begin{aligned}&[\Omega\nabla\!\!\!/_3+\Omega\nabla\!\!\!/_4](\mathfrak{f}\{|\Omega\nabla\!\!\!/_4\Psi|^2-|\Omega\nabla\!\!\!/_3\Psi|^2\})\\&\quad+[\Omega\nabla\!\!\!/_3-\Omega\nabla\!\!\!/_4]\left(\mathfrak{f}\left\{|\Omega\nabla\!\!\!/_4\Psi|^2+|\Omega\nabla\!\!\!/_3\Psi|^2-2\frac{1-\mu}{r^2}|r\nabla\!\!\!/\Psi|^2-2V|\Psi|^2\right\}\right)\\&\quad+2\mathfrak{f}'(|\Omega\nabla\!\!\!/_4\Psi|^2+|\Omega\nabla\!\!\!/_3\Psi|^2)-4R^\star\left(\mathfrak{f}\frac{1-\mu}{r^2}\right)|r\nabla\!\!\!/\Psi|^2-4R^\star(\mathfrak{f}V)|\Psi|^2\equiv0,\end{aligned} \tag{258}$$

where $\equiv$ means that the above becomes an equality after integration over $\int\sin\theta\,d\theta\,d\phi$, and the identity

$$\begin{aligned}&[\Omega\nabla\!\!\!/_3+\Omega\nabla\!\!\!/_4](\mathfrak{f}'\Psi\cdot[\Omega\nabla\!\!\!/_3+\Omega\nabla\!\!\!/_4]\Psi)\\&\quad-[\Omega\nabla\!\!\!/_3-\Omega\nabla\!\!\!/_4](\mathfrak{f}'\Psi\cdot[\Omega\nabla\!\!\!/_3-\Omega\nabla\!\!\!/_4]\Psi+\mathfrak{f}''|\Psi|^2)\\&\quad-2\mathfrak{f}'''|\Psi|^2-4\mathfrak{f}'\Omega\nabla\!\!\!/_3\Psi\cdot\Omega\nabla\!\!\!/_4\Psi+4\mathfrak{f}'\left(\frac{1-\mu}{r^2}|r\nabla\!\!\!/\Psi|^2+V|\Psi|^2\right)\equiv0.\end{aligned} \tag{259}$$

([26]) Let us note that one can indeed easily adapt the energy-momentum tensor formalism of §2.3.1 to Regge–Wheeler, but we here prefer to explicitly integrate by parts.

Adding (258) and (259), yields the identity

$$
\begin{aligned}
0 \equiv\, & [\Omega\nabla\!\!\!/_3+\Omega\nabla\!\!\!/_4](\mathfrak{f}\{|\Omega\nabla\!\!\!/_4\Psi|^2-|\Omega\nabla\!\!\!/_3\Psi|^2\}+\mathfrak{f}'\Psi\cdot[\Omega\nabla\!\!\!/_3+\Omega\nabla\!\!\!/_4]\Psi)\\
&+[\Omega\nabla\!\!\!/_3-\Omega\nabla\!\!\!/_4]\Bigg(\mathfrak{f}\left\{|\Omega\nabla\!\!\!/_4\Psi|^2+|\Omega\nabla\!\!\!/_3\Psi|^2-2\frac{1-\mu}{r^2}|r\nabla\!\!\!/\Psi|^2-2V|\Psi|^2\right\}\\
&\qquad\qquad -\mathfrak{f}'\Psi\cdot[\Omega\nabla\!\!\!/_3-\Omega\nabla\!\!\!/_4]\Psi-\mathfrak{f}''|\Psi|^2\Bigg)\\
&+2\mathfrak{f}'|\Omega\nabla\!\!\!/_4\Psi-\Omega\nabla\!\!\!/_3\Psi|^2+|r\nabla\!\!\!/\Psi|^2\left[-4\mathfrak{f}\left(\frac{1-\mu}{r^2}\right)'\right]+|\Psi|^2(-4\mathfrak{f}V'-2\mathfrak{f}''').
\end{aligned}
\tag{260}
$$

Note that, after integration with respect to the measure $\int du\,dv\,\sin\theta\,d\theta\,d\phi$, the term in the last line is a spacetime term, while all others are boundary terms.

11.2.1. The choice of $\mathfrak{f}$

The next lemma shows that we can choose a function $\mathfrak{f}$ in the identity (260) such that the last line of the latter is a manifestly non-negative expression. The choice below has appeared before in [36].

LEMMA 11.2.1. *If we define*

$$
\mathfrak{f}=\left(1-\frac{3M}{r}\right)\left(1+\frac{M}{r}\right),
\tag{261}
$$

then there exists a constant c such that the Ψ in Theorem 1 satisfies

$$
\begin{aligned}
\int\sin\theta\,d\theta\,d\phi\left\{|r\nabla\!\!\!/\Psi|^2\left[-\frac{1}{2}\frac{\mathfrak{f}}{1-\mu}\left(\frac{1-\mu}{r^2}\right)'\right]+|\Psi|^2\left[-\frac{1}{2}\frac{V'}{1-\mu}\mathfrak{f}-\frac{1}{4}\frac{\mathfrak{f}'''}{1-\mu}\right]\right\}&\\
\geqslant\frac{c}{r^3}\int\sin\theta\,d\theta\,d\phi\,|\Psi|^2&
\end{aligned}
\tag{262}
$$

for all (u,v) with $r\in(2M,\infty)$.

Remark 11.1. With the above choice of $\mathfrak{f}$, the square bracket multiplying the term $|r\nabla\!\!\!/\Psi|^2$ is non-negative. Since (261) implies that $\mathfrak{f}'/(1-\mu)\geqslant 2M/r^2$, the last line of (260) is indeed non-negative.

Proof. Since the term $|r\nabla\!\!\!/\Psi|^2$ is non-negative, applying Proposition 4.4.4 shows that it suffices to establish

$$
-\frac{1}{2}\frac{(V+(2/r^2)(1-\mu))'}{1-\mu}\mathfrak{f}-\frac{1}{4}\frac{\mathfrak{f}'''}{1-\mu}\geqslant\frac{c}{r^3}.
\tag{263}
$$

We compute

$$\begin{aligned}
\mathfrak{f}_r &= \frac{3M}{r^2}\left(1+\frac{M}{r}\right)+\left(1-\frac{3M}{r}\right)\left(-\frac{M}{r^2}\right)=\frac{2M}{r^2}+\frac{6M^2}{r^3},\\
\mathfrak{f}_{rr} &= -\frac{4M}{r^3}-\frac{18M^2}{r^4},\\
\mathfrak{f}_{rrr} &= \frac{12M}{r^4}+\frac{72M^2}{r^5},
\end{aligned}$$

and hence

$$\begin{aligned}
\mathfrak{f}' &= \mathfrak{f}_r(1-\mu)=\left(\frac{2M}{r^2}+\frac{6M^2}{r^3}\right)(1-\mu),\\
\mathfrak{f}'' &= \mathfrak{f}_{rr}(1-\mu)^2+\mathfrak{f}_r\frac{2M}{r^2}(1-\mu), \qquad (264)\\
\mathfrak{f}''' &= \mathfrak{f}_{rrr}(1-\mu)^3+\mathfrak{f}_{rr}\frac{6M}{r^2}(1-\mu)^2-2M\mathfrak{f}_r\frac{2}{r^3}\left(1-\frac{3M}{r}\right)(1-\mu),
\end{aligned}$$

and therefore

$$\begin{aligned}
\text{expr} &:= \frac{1}{2}\left(V+\frac{2}{r^2}(1-\mu)\right)'\mathfrak{f}+\frac{1}{4}\mathfrak{f}'''\\
&= \left[\frac{3M}{r^4}+\frac{18M^2}{r^5}\right](1-\mu)^3-\frac{3}{2}\frac{M}{r^2}(1-\mu)^2\left(\frac{4M}{r^3}+\frac{18M^2}{r^4}\right)\\
&\quad -\frac{M}{r^3}\left(\frac{2M}{r^2}+\frac{6M^2}{r^3}\right)(1-\mu)\left(1-\frac{3M}{r}\right)\\
&\quad +\frac{1}{2}\left(1-\frac{3M}{r}\right)\left(1+\frac{M}{r}\right)\left[\frac{-2}{r^3}\left(1-\frac{3M}{r}\right)(1-\mu)\left(6-\frac{6M}{r}\right)+\frac{6M}{r^4}(1-\mu)^2\right].
\end{aligned}$$

We claim that this expression is negative for $r\in(2M,\infty)$. To see this, we write the expression as follows

$$\begin{aligned}
-(1-\mu)^{-1}\,\text{expr} &= -\left[\frac{3M}{r^4}+\frac{18M^2}{r^5}\right]\left(1-\frac{4M}{r}+\frac{4M^2}{r^2}\right)\\
&\quad +\frac{3}{2}\frac{M}{r^2}\left(1-\frac{2M}{r}\right)\left(\frac{4M}{r^3}+\frac{18M^2}{r^4}\right)+\frac{M}{r^3}\left(\frac{2M}{r^2}+\frac{6M^2}{r^3}\right)\left(1-\frac{3M}{r}\right)\\
&\quad +\left(1-\frac{3M}{r}\right)\left(1+\frac{M}{r}\right)\left[\frac{1}{r^3}\left(1-\frac{3M}{r}\right)\left(6-\frac{6M}{r}\right)-\frac{3M}{r^4}\left(1-\frac{2M}{r}\right)\right],
\end{aligned}$$

which is computed to be

$$\begin{aligned}
-(1-\mu)^{-1}\,\text{expr} &= -\left[\frac{3M}{r^4}+\frac{6M^2}{r^5}-\frac{60M^3}{r^6}+\frac{72M^4}{r^7}\right]\\
&\quad +\frac{3}{2}\frac{M}{r^2}\left(\frac{4M}{r^3}+\frac{10M^2}{r^4}-\frac{36M^3}{r^5}\right)+\frac{M}{r^3}\left(\frac{2M}{r^2}-\frac{18M^3}{r^4}\right)\\
&\quad +\frac{1}{r^3}\left(1-\frac{5M}{r}+\frac{3M^2}{r^2}+\frac{9M^3}{r^3}\right)\left(6-\frac{6M}{r}\right)\\
&\quad -\frac{3M}{r^4}\left(1-\frac{4M}{r}+\frac{M^2}{r^2}+\frac{6M^3}{r^3}\right),
\end{aligned}$$

and simplifies further to

$$
\begin{aligned}
-(1-\mu)^{-1}\,\mathrm{expr} \geqslant \frac{6}{r^3}+\frac{1}{r^4}(-3M-30M-6M-3M)\\
+\frac{1}{r^5}(-6M^2+6M^2+2M^2+18M^2+30M^2+12M^2)\\
+\frac{1}{r^6}(60M^3+15M^3+54M^3-18M^3-3M^3)\\
+\frac{1}{r^7}(-72M^4-54M^4-18M^4-54M^4-18M^4).
\end{aligned}
$$

It thus suffices to establish positivity of the polynomial

$$6r^4-42Mr^3+62M^2r^2+108M^3r-216M^4, \tag{265}$$

or equivalently, upon setting $r=2Mx$, positivity of

$$p(x)=12x^4-42x^3+31x^2+27x-27 \quad \text{for } x\in[1,\infty). \tag{266}$$

Using elementary calculus, one easily shows $p(x)\geqslant 1$. □

11.2.2. The basic estimate

Upon integrating (260) with respect to $du\,dv\,\sin\theta\,d\theta\,d\phi$ over any spacetime region

$$[u_0,u]\times[v_0,v]\times S^2_{\bar u,\bar v}$$

with $\mathfrak{f}$ as chosen in Lemma 11.2.1, we see ($\mathfrak{f}$ is uniformly bounded and (264) holds) that we can estimate all boundary terms (null-fluxes) by the fluxes $F^T_u[\Psi](v_0,v)$, $F^T_v[\Psi](u_0,u)$, $F^T_{v_0}[\Psi](u_0,u)$ and $F^T_{u_0}[\Psi](v_0,v)$, which, by the conservation law (257), means that all boundary terms are controlled by a constant times $F^T_{u_0}[\Psi](v_0,v)+F^T_{u_0}[\Psi](v_0,v)$ alone. Exploiting now the statement of Lemma 11.2.1 for the term in the last line of (260), we obtain the basic Morawetz estimate

$$
\begin{aligned}
\int_{u_0}^{u}\int_{v_0}^{v}\int_{S^2_{\bar u,\bar v}} d\bar u\,d\bar v\sin\theta\,d\theta\,d\phi\,\Omega^2\left[\frac{1}{r^2}|\Omega\nabla\!\!\!/_4\Psi-\Omega\nabla\!\!\!/_3\Psi|^2+\frac{1}{r^3}|\Psi|^2\right]\\
\leqslant C[F^T_{u_0}[\Psi](v_0,v)+F^T_{v_0}[\Psi](u_0,u)].
\end{aligned}
\tag{267}
$$

The above estimate (267) can be improved immediately. First, integrating (260) again with respect to $du\,dv\,\sin\theta\,d\theta\,d\phi$ and f as in (261) we (instead of applying the Poincaré inequality to the angular term) observe that all first-order terms are actually non-negative and that the order-zero term of Ψ is controlled by (267). This gives

$$
\begin{aligned}
\int_{u_0}^{u}\int_{v_0}^{v}\int_{S^2_{\bar u,\bar v}} d\bar u\,d\bar v\sin\theta\,d\theta\,d\phi\Omega^2\left[\frac{1}{r^2}|\Omega\nabla\!\!\!/_4\Psi-\Omega\nabla\!\!\!/_3\Psi|^2+\frac{(r-3M)^2}{r^3}|\nabla\!\!\!/\Psi|^2+\frac{1}{r^3}|\Psi|^2\right]\\
\leqslant C[F^T_{u_0}[\Psi](v_0,v)+F^T_{v_0}[\Psi](u_0,u)].
\end{aligned}
$$

A standard argument allows us to recover the missing derivative: For instance, integrating the identity (258) with a bounded, monotonically increasing $\mathfrak{f}$ which vanishes to third order near $r=3M$, we obtain

$$\begin{aligned}\int_{u_0}^{u}\int_{v_0}^{v}\int_{S^2_{\bar u,\bar v}} d\bar u\, d\bar v\sin\theta\, d\theta\, d\phi\,\Omega^2\bigg[\frac{1}{r^2}|\Omega\nabla\!\!\!/_4\Psi-\Omega\nabla\!\!\!/_3\Psi|^2+\frac{1}{r^3}|\Psi|^2\\ +\frac{(r-3M)^2}{r^2}\bigg(\frac{1}{r}|\nabla\!\!\!/\psi|^2+\frac{1}{r^2}|\Omega\nabla\!\!\!/_4\Psi+\Omega\nabla\!\!\!/_3\Psi|^2\bigg)\bigg]\\ \leqslant C[F^T_{u_0}[\Psi](v_0,v)+F^T_{v_0}[\Psi](u_0,u)].\end{aligned} \tag{268}$$

The degeneration near $r=3M$ is the familiar trapping phenomenon and cannot be removed (although it can be improved to logarithmic loss; cf. [52]). The degeneration at the horizon however can be removed by exploiting the redshift. The weights near infinity can also be improved. We turn to these two refinements in §11.3 and §11.4 below.

11.3. Improving the weights near the horizon $\mathcal{H}^+$: The redshift

Given Proposition 11.1.1 and estimate (268), the argument exploiting the redshift identity, as described in §2.3.1 and §2.3.2 for the scalar wave equation (48) (cf. [25]), can be immediately adapted to Ψ.

In particular, one upgrades Proposition 11.1.1 to the non-degenerate boundedness statement

$$F_u[\Psi](v_0,v)+F_v[\Psi](u_0,u)\lesssim F_{v_0}[\Psi](u_0,u)+F_{u_0}[\Psi](v_0,v), \tag{269}$$

where these are now *non-degenerate* null-fluxes defined in (226) and (227), and the estimate (268) itself to the improved([27]) integrated decay estimate

$$\begin{aligned}\int_{u_0}^{u}\int_{v_0}^{v}\int_{S^2_{\bar u,\bar v}} d\bar u\, d\bar v\,\sin\theta\, d\theta\, d\phi\Omega^2\bigg[\frac{1}{r^2}|\Omega\nabla\!\!\!/_4\Psi-\Omega\nabla\!\!\!/_3\Psi|^2+\frac{1}{r^3}|\Psi|^2\\ +\frac{(r-3M)^2}{r^2}\bigg(\frac{1}{r}|\nabla\!\!\!/\Psi|^2+\frac{1}{r^2}|\Omega\nabla\!\!\!/_4\Psi|^2+\frac{1}{r^2}|\Omega^{-1}\nabla\!\!\!/_3\Psi|^2\bigg)\bigg]\\ \lesssim F_{u_0}[\Psi](v_0,v)+F_{v_0}[\Psi](u_0,u).\end{aligned} \tag{270}$$

Note that, taking the limit $u,v\to\infty$, the left-hand side is precisely $\mathbb{I}_{\mathrm{deg}}[\Psi]$. Hence, (269) and (270) prove the following result.

([27]) In the sense that the regular transversal derivative $(1/\Omega)\nabla\!\!\!/_3\Psi$ is also controlled near the horizon–the degeneracy near $r=3M$ remains.

PROPOSITION 11.3.1. *The Ψ in Theorem 1 satisfies the boundedness estimate*

$$\sup_u F_u[\Psi](v_0,\infty)+\sup_v F_v[\Psi](u_0,\infty)\lesssim F_{v_0}[\Psi](u_0,\infty)+F_{u_0}[\Psi](v_0,\infty), \tag{271}$$

and the integrated decay estimate

$$\mathbb{I}_{\mathrm{deg}}[\Psi]\lesssim F_{v_0}[\Psi](u_0,\infty)+F_{u_0}[\Psi](v_0,\infty), \tag{272}$$

provided the initial energies on the right-hand side are finite.

Note that higher-order versions of the above proposition are immediate from Lie differentation with the Killing fields of §4.2.2, i.e. $\mathcal{L}_T$ as well as $\mathcal{L}_{\Omega_i}$. We note also the following result.

COROLLARY 11.1. *The Ψ in Theorem 1 satisfies*

$$\mathbb{I}_{\mathrm{ndeg}}[\Psi]\lesssim \sum_{i=0}^{1} F_{v_0}[T^i\Psi](u_0,\infty)+\sum_{i=0}^{1} F_{u_0}[T^i\Psi](v_0,\infty), \tag{273}$$

provided the initial energies on the right-hand side are finite. Here, the left-hand side denotes the non-degenerate (near $3M$) integrated decay energy

$$\mathbb{I}_{\mathrm{ndeg}}[\Psi]:=\int_{u_0}^{\infty}\int_{v_0}^{\infty}\int_{S^2_{\bar{u},\bar{v}}} d\bar{u}\, d\bar{v}\, d\mathrm{vol}_{S^2}\,\Omega^2\left[\frac{1}{r^3}|\Psi|^2+\frac{1}{r}|\nabla\!\!\!/\Psi|^2+\frac{1}{r^2}|\Omega\nabla\!\!\!/_4\Psi|^2+\frac{1}{r^2}|\Omega^{-1}\nabla\!\!\!/_3\Psi|^2\right].$$

Proof. By the remark following Proposition 11.3.1, the right-hand side of (273) controls $\mathbb{I}_{\mathrm{deg}}[\Psi]+\mathbb{I}_{\mathrm{deg}}[T\Psi]$. In particular, both $|T\Psi|^2$ and $|R^\star\Psi|^2$ (and of course $|\Psi|^2$) are now controlled without degeneration at $3M$. To control also the term $|\nabla\!\!\!/\Psi|^2$ non-degenerately near $3M$ integrate the multiplier identity (259) with $f=1/r$ and use Proposition 11.3.1 to estimate the boundary terms that appear. □

11.4. Improving the weights near null infinity $\mathcal{I}^+$: The r^p hierarchy

The r^p hierarchy of [22] recalled in §2.3.3 in the context of the scalar wave equation (48) can also now be adapted to Ψ.

From the Regge–Wheeler equation for Ψ we derive the identity (for $1\leqslant p\leqslant 2$ and $k\geqslant 1$)

$$\begin{aligned}\partial_u\left[\frac{r^p}{(1-\mu)^k}|\Omega\nabla\!\!\!/_4\Psi|^2\right]+\partial_v\left[\frac{r^p}{(1-\mu)^{k-1}}|\nabla\!\!\!/\Psi|^2\right]+\partial_v\left[r^p\frac{V}{(1-\mu)^k}|\Psi|^2\right]\\ -\partial_u\left(\frac{r^p}{(1-\mu)^k}\right)|\Omega\nabla\!\!\!/_4\Psi|^2-\partial_v\left(\frac{Vr^p}{(1-\mu)^k}\right)|\Psi|^2\\ +\left[(2-p)r^{p-1}(1-\mu)^{1-k}+r^p(k-1)(1-\mu)^{-k}\frac{2M}{r^2}r_v\right]|\nabla\!\!\!/\Psi|^2\equiv 0,\end{aligned} \tag{274}$$

where we have used

$$-2\nabla\!\!\!/_4\Psi\cdot\Delta\!\!\!/\Psi \equiv \nabla\!\!\!/_4|\nabla\!\!\!/\psi|^2-2[\nabla\!\!\!/_4,\nabla\!\!\!/]\Psi\cdot\nabla\!\!\!/\Psi=\nabla\!\!\!/_4|\nabla\!\!\!/\Psi|^2+\operatorname{tr}\chi|\nabla\!\!\!/\Psi|^2,$$

and $\equiv$ indicates that (274) becomes an identity after integration against $\sin\theta\,d\theta\,d\phi$.

For our current purposes, it will be sufficient to integrate (274) for $1\leqslant p\leqslant 2$ with respect to the measure $du\,dv\,\sin\theta\,d\theta\,d\phi$ in a region

$$\mathcal{R}=\{(u,v)\in\mathcal{M}:r(u,v)\geqslant R,\ u_0\leqslant u\leqslant u_{\text{final}} \text{ and } v_0\leqslant v\leqslant v_{\text{final}}\},$$

for sufficiently large R, and u_{final} and v_{final} arbitrarily large. Precisely, we choose R sufficiently large (depending only on M) such that

$$-\partial_u\left(\frac{r^p}{(1-\mu)^k}\right)\geqslant\frac{1}{2}r^{p-1}\quad\text{for all } 1\leqslant p\leqslant 2 \text{ and } k\leqslant 5.$$

Note also that

$$\begin{aligned}
-\partial_v&\left(\frac{Vr^p}{(1-\mu)^k}\right)\\
&=-\partial_v\left(\frac{4r^{p-2}-6Mr^{p-3}}{(1-\mu)^{k-1}}\right)\\
&=\frac{r_v}{(1-\mu)^k}\left[(4(2-p)r^{p-3}-6M(3-p)r^{p-4})(1-\mu)+(k-1)\frac{2M}{r^2}(4r^{p-2}-6Mr^{p-3})\right]\\
&=\frac{r_v}{(1-\mu)^k}[4(2-p)r^{p-3}+Mr^{p-4}(8k+14p-42)+12M^2r^{p-5}(4-p-k)]
\end{aligned}$$

holds, which means that, given any $1\leqslant p\leqslant 2$, the choice $k=4$ ensures that also the estimate

$$-\partial_v\left(\frac{Vr^p}{(1-\mu)^k}\right)\geqslant 2Mr^{p-4}$$

holds in $\mathcal{R}$, for sufficiently large R (depending only on M). Therefore, integrating (274) for $p=2$ with respect to the measure $du\,dv\,\sin\theta\,d\theta\,d\phi$, we first obtain the estimate

$$\begin{aligned}
\int_{\mathcal{R}}&du\,dv\,\sin\theta\,d\theta\,d\phi\,(r|\Omega\nabla\!\!\!/_4\Psi|^2+r^{1-\varepsilon}|\nabla\!\!\!/\Psi|^2+r^{-1-\varepsilon}|\Psi|^2)\\
&\leqslant C\int_{v_0}^{\infty}dv\int_{S^2}\sin\theta\,d\theta\,d\phi\,(r^2|\Omega\nabla\!\!\!/_4\Psi|^2)(u_0,v)+C(F_{u_0}[\Psi](v_0,v)+F_{v_0}[\Psi](u_0,u)),
\end{aligned}\tag{275}$$

for $\varepsilon=1$, where the last two terms on the right-hand side account for the terms arising on the timelike hypersurface at $r=R$, which can be controlled by the Morawetz estimate (270), after averaging in R. To show that the estimate (275) holds for our fixed $0<\varepsilon<\frac{1}{8}$, we integrate (274) for $p=2-\varepsilon$ with respect to the measure $du\,dv\,\sin\theta\,d\theta\,d\phi$ and add it to the $p=2$ estimate. Note that the constant in (275) is independent of both u_{final} and v_{final}, and that the estimate hence holds for $\mathcal{R}$ replaced by $\mathcal{M}\cap\{u\geqslant u_0\}\cap\{v\geqslant v_0\}\cap\{r\geqslant R\}$.

At the same time, the integration of (274) over $\mathcal{R}$ produces good boundary terms (fluxes) on $u=u_{\text{final}}$ and $v=v_{\text{final}}$ from the terms in the first line of (274). Taking suprema, we deduce both (238) and the $n=0$ part of (240), after recalling the shorthand notation (230), (231) for the energies.

With these bounds established, we deduce the following result.

Corollary 11.2. *Under the assumptions of Theorem 1, we also have the estimate*

$$\sup_{\substack{u\geqslant u_0\\ v\geqslant v_0}} \|r^{-1}\cdot\Psi\|^2_{S^2_{u,v}} \lesssim F^{\mathcal{I}}_{u_0}[\Psi](v_0,\infty)+F^{\mathcal{I}}_{v_0}[\Psi](u_0,\infty). \tag{276}$$

Proof. The fundamental theorem of calculus in the $\nabla\!\!\!/_4$-direction and the Cauchy–Schwarz inequality using the flux (228) gives this bound with an additional (initial) term $\sup_u \|r^{-1}\cdot\Psi\|^2_{S^2_{u,v_0}}$ on the right-hand side. Applying 1-dimensional Sobolev embedding on $v=v_0$, shows that this initial term is controlled by $F^{\mathcal{I}}_{v_0}[\Psi](u_0,\infty)$. $\square$

We finally note that integrating (274) with $p=1$ and $k=4$ (instead of $p=2$ and $k=4$, as done to derive (275)) leads to additional estimates which together with the choice $p=0$ and $k=0$ (for which the identity (274) also holds) constitute the Regge–Wheeler analogue of the r^p-hierarchy for the wave equation in [22].

11.5. Higher-order estimates and polynomial decay

In this section we will extend the above weighted estimates to higher order and then infer polynomial decay.

We note the trivial fact that the Regge–Wheeler equation (255) commutes with Lie differentation with the Killing fields of §4.2.2, i.e. $\mathcal{L}_T$ as well as $\mathcal{L}_{\Omega_i}$.(28) Recalling the commuted energies (233), we hence immediately conclude the following corollary, which provides the estimate (239) and the $n>0$ part of the estimate (240) in Theorem 1.

Corollary 11.3. *If the Ψ in Theorem 1 satisfies $\mathbb{F}^{n,T,\nabla\!\!\!/}_0[\Psi]<\infty$ for some integer $n\geqslant 0$, then we have the estimate*

$$\mathbb{I}^{n,T,\nabla\!\!\!/}_{\mathcal{I},\varepsilon}[\Psi]+\mathbb{I}^{n,T,\nabla\!\!\!/}_{\text{deg}}[\Psi]+\mathbb{F}^{n,T,\nabla\!\!\!/}[\Psi]\lesssim \mathbb{F}^{n,T,\nabla\!\!\!/}_0[\Psi]. \tag{277}$$

(28) As in [18], we could alternatively commute "tensorially" with $r\nabla\!\!\!/_A$ and estimate the lower-order terms inductively.

As an immediate consequence of Corollary 11.2, we also have

$$\sup_{\substack{u\geqslant u_0\\ v\geqslant v_0}} \|r^{-1}\cdot\mathcal{A}^{[n]}\Psi\|^2_{S^2_{u,v}} \lesssim \mathbb{F}_0^{n,T,\nabla\!\!\!\!/}[\Psi].$$

We can in fact show an analogue of the above for an nth-order *non-degenerate* energy, where higher derivatives have moreover additional weights in v. This is a straightforward adaptation of the procedure appearing in [68] and [57] to Ψ, and follows by commuting the equation with the redshift operator $\Omega^{-1}\nabla\!\!\!\!/_3$ near the horizon and with the weighted operator $r\Omega\nabla\!\!\!\!/_4$ near null infinity, and observing that the terms non-controllable by (277) occur with favourable signs. We will simply state the estimate arising. Define the energy

$$\begin{aligned}\mathbb{F}^n[\Psi] := &\sum_{i+j+k\leqslant n} \sup_u F_u^{\mathcal{I}}[(\Omega^{-1}\nabla\!\!\!\!/_3)^i(r\Omega\nabla\!\!\!\!/_4)^j(r\nabla\!\!\!\!/_A)^k\Psi](v_0,\infty)\\ &+\sum_{i+j+k\leqslant n} \sup_v F_v^{\mathcal{I}}[(\Omega^{-1}\nabla\!\!\!\!/_3)^i(r\Omega\nabla\!\!\!\!/_4)^j(r\nabla\!\!\!\!/_A)^k\Psi](u_0,\infty),\end{aligned} \tag{278}$$

with initial energy

$$\begin{aligned}\mathbb{F}_0^n[\Psi] := &\sum_{i+j+k\leqslant n} F_{u_0}^{\mathcal{I}}[(\Omega^{-1}\nabla\!\!\!\!/_3)^i(r\Omega\nabla\!\!\!\!/_4)^j(r\nabla\!\!\!\!/_A)^k\Psi](v_0,\infty)\\ &+\sum_{i+j+k\leqslant n} F_{v_0}^{\mathcal{I}}[(\Omega^{-1}\nabla\!\!\!\!/_3)^i(r\Omega\nabla\!\!\!\!/_4)^j(r\nabla\!\!\!\!/_A)^k\Psi](u_0,\infty).\end{aligned} \tag{279}$$

We have the following result.

COROLLARY 11.4. *If the Ψ in Theorem 1 satisfies $\mathbb{F}_0^n[\Psi]<\infty$, then we have, for any $n\geqslant 0$ and non-negative integers i, j and k with $i+j+k\leqslant n$, the estimate*

$$\begin{aligned}\mathbb{F}^n[\Psi] &\lesssim \mathbb{F}_0^n[\Psi],\\ \mathbb{I}_{\mathrm{deg}}[(\Omega^{-1}\nabla\!\!\!\!/_3)^i(r\Omega\nabla\!\!\!\!/_4)^j(r\nabla\!\!\!\!/_A)^k\Psi]+\mathbb{I}^{\mathcal{I}}_{\varepsilon}[(\Omega^{-1}\nabla\!\!\!\!/_3)^i(r\Omega\nabla\!\!\!\!/_4)^j(r\nabla\!\!\!\!/_A)^k\Psi] &\lesssim \mathbb{F}_0^n[\Psi].\end{aligned}$$

We will in fact only use Corollary 11.4 later to optimise decay statements already obtained.

Exploiting the r^p-hierarchy for the Regge–Wheeler equation discussed in §11.4, polynomial decay estimates can be obtained for Ψ exactly as in [22] for the case of the scalar wave equation (cf. the discussion in §2.3.3). We only give the most elementary statement here.

Let us fix $r_0:=r(u_0,v_0)>2M$ and denote by $u(v,r_0)$ the u-value corresponding to the sphere of intersection between the $r=r_0$ hypersurface and the constant v hypersurface. Note that $v\sim u(v,r_0)$, for large v. Applying step by step the method of [22] (for details on the method of [22] in more general settings see [68], [57]), we obtain the following result.

PROPOSITION 11.5.1. *Fix $r_0 = r(u_0, v_0)$ and $v \geqslant v_0$ and suppose the Ψ in Theorem 1 satisfies $\mathbb{F}_0^{2,T}[\Psi] < \infty$ initially. Then, for any $V \geqslant v$ and any $U \geqslant u(v, r_0)$, we have*

$$F_U[\Psi](v,\infty)+F_V[\Psi](u(v,r_0),\infty)\lesssim \frac{1}{v^2}\cdot \mathbb{F}_0^{2,T}[\Psi].$$

The constant implicit in $\lesssim$ depends on r_0, and we recall the non-degenerate energy fluxes (226) and (227).

Proof. Since the proof is entirely analogous to that in [22], we only provide a sketch from which the reader can easily fill in the details. Adding to (275) the estimate of Corollary 11.1 (and their T-commuted analogues), we find

$$\begin{aligned}
&\int_{u_0}^{\infty}\int_{v_0}^{\infty}\int_{S^2_{\bar u,\bar v}}^{d} \bar u\, d\bar v\, d\mathrm{vol}_{S^2}\,\Omega^2\left[\frac{1}{r^2}|\Psi|^2+|\nabla\!\!\!\!/\,\Psi|^2+r|\Omega\nabla\!\!\!\!/_4\Psi|^2+\frac{1}{r^2}|\Omega^{-1}\nabla\!\!\!\!/_3\Psi|^2\right]\\
&\quad+\int_{u_0}^{\infty}\int_{v_0}^{\infty}\int_{S^2_{\bar u,\bar v}} d\bar u\, d\bar v\, d\mathrm{vol}_{S^2}\,\Omega^2\left[\frac{1}{r^2}|T\Psi|^2+|\nabla\!\!\!\!/\,T\Psi|^2+r|\Omega\nabla\!\!\!\!/_4T\Psi|^2+\frac{1}{r^2}|\Omega^{-1}\nabla\!\!\!\!/_3T\Psi|^2\right]\\
&\qquad\lesssim \mathbb{F}_0^{2,T}[\Psi].
\end{aligned}$$

From this, we extract a dyadic sequence $(v_i)_i$ with associated ingoing cone

$$\underline{\widetilde{C}}_{v_i}=[u(v_i,r_0),\infty)\times\{v_i\}\times S^2$$

and outgoing cone $\widetilde{C}_{u(v_i,r_0)}=\{u(v_i,r_0)\}\times[v_i,\infty)\times S^2$ such that for each v_i we have

$$\begin{aligned}
&\int_{u(v_i,r_0)}^{\infty} d\bar u\, d\mathrm{vol}_{S^2}\,\Omega^2\bigg(|\Omega^{-1}\nabla\!\!\!\!/_3\Psi|^2+|\nabla\!\!\!\!/\,\Psi|^2+\frac{1}{r^2}|\Psi|^2\\
&\qquad\qquad+|\Omega^{-1}\nabla\!\!\!\!/_3T\Psi|^2+|\nabla\!\!\!\!/\,T\Psi|^2+\frac{1}{r^2}|T\Psi|^2\bigg)(\bar u,v_i)\\
&\quad+\int_{v_i}^{\infty} d\bar v\, d\mathrm{vol}_{S^2}\bigg(r|\Omega\nabla\!\!\!\!/_4\Psi|^2+|\nabla\!\!\!\!/\,\Psi|^2+\frac{1}{r^2}|\Psi|^2\\
&\qquad\qquad+r|\Omega\nabla\!\!\!\!/_4T\Psi|^2+|\nabla\!\!\!\!/\,T\Psi|^2+\frac{1}{r^2}|T\Psi|^2\bigg)(u(v_i,r_0),\bar v)\lesssim\frac{\mathbb{F}_0^{2,T}[\Psi]}{v_i}.
\end{aligned}$$

Note that, in particular, the ingoing non-degenerate energy (of both Ψ and $T\Psi$) on $\underline{\widetilde{C}}_{v_i}$ and the outgoing energy (of both Ψ and $T\Psi$) on $\widetilde{C}_{u(v_i,r_0)}$ are decaying. Hence, applying Proposition 11.3.1, now from each dyadic pair of cones $\underline{\widetilde{C}}_{v_i}\cup\widetilde{C}_{u(v_i,r_0)}$ (instead of from $\underline{\widetilde{C}}_{v_0}\cup\widetilde{C}_{u_0}$) yields, using the previous estimate for the right-hand side in Proposition 11.3.1

for any $v \geqslant v_0$, the estimate

$$\begin{aligned}\int_{u(v,r_0)}^{\infty} d\bar{u}\, d\mathrm{vol}_{S^2}\, \Omega^2 \bigg(& |\Omega^{-1}\nabla\!\!\!/_3\Psi|^2 + |\nabla\!\!\!/\Psi|^2 \\ & + \frac{|\Psi|^2}{r^2} + |\Omega^{-1}\nabla\!\!\!/_3 T\Psi|^2 + |\nabla\!\!\!/ T\Psi|^2 + \frac{1}{r^2}|T\Psi|^2 \bigg)(\bar{u}, v) \\ + \int_v^{\infty} d\bar{v}\, d\mathrm{vol}_{S^2} \bigg(& \boxed{r}\, |\Omega\nabla\!\!\!/_4\Psi|^2 + |\nabla\!\!\!/\Psi|^2 + \frac{|\Psi|^2}{r^2} \\ & + \boxed{r}\, |\Omega\nabla\!\!\!/_4 T\Psi|^2 + |\nabla\!\!\!/ T\Psi|^2 + \frac{1}{r^2}|T\Psi|^2 \bigg)(u(v,r_0), \bar{v}) \lesssim \frac{\mathbb{F}_0^{2,T}[\Psi]}{v}\end{aligned} \tag{280}$$

without the boxed r-weight. In addition, we obtain (cf. (270))

$$\int_v^{\infty} d\bar{v} \int_{u(v,r_0)}^{\infty} d\bar{u} \int_{S^2_{\bar{u},\bar{v}}} d\mathrm{vol}_{S^2}\, \Omega^2 [e[\Psi] + e[T\Psi]] \lesssim \frac{\mathbb{F}_0^{2,T}[\Psi]}{v},$$

for all $v \geqslant v_0$, where

$$e[\Psi] = \frac{1}{r^2}|\Omega\nabla\!\!\!/_4\Psi - \Omega\nabla\!\!\!/_3\Psi|^2 + \frac{1}{r^3}|\Psi|^2 + \frac{(r-3M)^2}{r^2}\bigg(\frac{1}{r}|\nabla\!\!\!/\Psi|^2 + \frac{1}{r^2}|\Omega\nabla\!\!\!/_4\Psi|^2 + \frac{1}{r^2}|\Omega^{-1}\nabla\!\!\!/_3\Psi|^2\bigg).$$

We now integrate (274) (and its T-commuted version) with $p=1$ and $k=4$ in $\mathcal{R}$ from each $C_{u(v_i,r_0)}$ to improve (280) to *include* the boxed r-weight. In addition, we obtain a good spacetime term in the region $\mathcal{R}$ which can be combined with the estimate of Corollary 11.1 (exchange the initial cones $\underline{\widetilde{C}}_{v_0} \cup \widetilde{C}_{u_0}$ by the dyadic pair of cones $\underline{\widetilde{C}}_{v_i} \cup \widetilde{C}_{u(v_i,r_0)}$, and use (280) without the boxed r for the right-hand side in Corollary 11.1) to obtain

$$\int_v^{\infty} d\bar{v} \int_{u(v,r_0)}^{\infty} d\bar{u} \int_{S^2_{\bar{u},\bar{v}}} d\mathrm{vol}_{S^2}\, \Omega^2 \bigg[\frac{1}{r^2}|\Psi|^2 + |\nabla\!\!\!/\Psi|^2 + |\Omega\nabla\!\!\!/_4\Psi|^2 + \frac{1}{r^2}|\Omega^{-1}\nabla\!\!\!/_3\Psi|^2\bigg] \lesssim \frac{\mathbb{F}_0^{2,T}[\Psi]}{v} \tag{281}$$

for all $v \geqslant v_0$. From this, we find a (potentially different) dyadic sequence $(v_i)_i$ along which

$$\begin{aligned}\int_{u(v_i,r_0)}^{\infty} & d\bar{u}\, d\mathrm{vol}_{S^2}\, \Omega^2 (|\Omega^{-1}\nabla\!\!\!/_3\Psi|^2 + |\nabla\!\!\!/\Psi|^2 + \frac{|\Psi|^2}{r^2})(\bar{u}, v_i) \\ & + \int_{v_i}^{\infty} d\bar{v}\, d\mathrm{vol}_{S^2} (|\Omega\nabla\!\!\!/_4\Psi|^2 + |\nabla\!\!\!/\Psi|^2 + \frac{|\Psi|^2}{r^2})(u(v_i,r_0), \bar{v}) \lesssim \frac{\mathbb{F}_0^{2,T}[\Psi]}{(v_i)^2}.\end{aligned}$$

Finally, applying Proposition 11.3.1 from each of these dyadic pair of cones $\underline{\widetilde{C}}_{v_i} \cup \widetilde{C}_{u(v_i,r_0)}$ towards the future yields the statement of the proposition. □

Corollary 11.5. *Under the assumptions of the previous proposition, we have the integrated decay estimate*

$$\int_v^\infty d\bar{v}\int_{u(v,r_0)}^\infty d\bar{u}\int_{S^2_{\bar{u},\bar{v}}} d\mathrm{vol}_{S^2}\,\Omega^2\bigg[\frac{1}{r^2}|\Omega\nabla\!\!\!/_4\Psi-\Omega\nabla\!\!\!/_3\Psi|^2+\frac{1}{r^3}|\Psi|^2 + \frac{(r-3M)^2}{r^2}\left(\frac{1}{r}|\nabla\!\!\!/\Psi|^2+\frac{1}{r^2}|\Omega\nabla\!\!\!/_4\Psi|^2+\frac{1}{r^2}|\Omega^{-1}\nabla\!\!\!/_3\Psi|^2\right)\bigg]\lesssim\frac{\mathbb{F}_0^{2,T}[\Psi]}{v^2}.$$

Proof. Apply Proposition 11.5.1 to (270) replacing v_0 by the fixed v and u_0 by the fixed $u(v,r_0)$ in (270). ☐

12. Proof of Theorem 2

In this section we exploit the transformation formulas of §7.3 together with Theorem 1 to prove Theorem 2. The reader can refer to the overview of §2.4.2.

We begin in §12.1 with the most basic estimates for ψ, $\underline{\psi}$, α and $\underline{\alpha}$ that follow straight from the transport structure of equations (178)– (181), in conjunction with Theorem 1.

In §12.2 we obtain higher-derivative estimates for ψ, $\underline{\psi}$, α and $\underline{\alpha}$ that follow from commuting the aforementioned transport equations. Sometimes pointwise algebraic identities (§12.2.1) can be used to avoid the loss of derivatives that is encountered in using the transport equations. Combining these results, we finally complete the proof of the first three statements of Theorem 2 in §12.2.6. The final subsection (§12.3) presents some refinements of the previous results, including higher-order estimates and polynomial decay statements for solutions of the Teukolsky equation, in particular the last statement in Theorem 2.

12.1. Ascending the hierarchy: basic transport estimates

We will prove the statements of Theorem 2 regarding spin $+2$ and -2 Teukolsky equations in parallel. Let α and $\underline{\alpha}$ be as in the two parts of Theorem 2.

We first define the derived quantities ψ and P from α, and $\underline{\psi}$ and $\underline{P}$ from $\underline{\alpha}$, from formulas (178)–(179) and (180)–(181), respectively, of §7.3. By Proposition 7.3.1, it follows that both P and $\underline{P}$ satisfy the Regge–Wheeler equation. We may thus apply Theorem 1 to both P and $\underline{P}$, noting that, by the assumptions of Theorem 2, the corresponding initial energies appearing in Theorem 1 are finite.

In the subsections that follow, we will show how from control of P and $\underline{P}$ we can control ψ and $\underline{\psi}$, and then α and $\underline{\alpha}$, by estimating transport equations.

12.1.1. Estimates for ψ and $\underline{\psi}$

We begin by estimating ψ and $\underline{\psi}$.

PROPOSITION 12.1.1. *The derived quantity ψ associated with the solution α of Theorem 2 satisfies the following estimates. Along any null-hypersurface of constant $u\geqslant u_0$, including the event horizon,*

$$\int_{v_0}^{\infty} dv\, \|r^{-1}\psi\|^2_{S^2_{u,v}} r^{8-\varepsilon}\Omega^2(u,v) \lesssim \mathbb{F}_0[\Psi,\psi]. \tag{282}$$

In addition, we have the integrated decay estimate

$$\int_{u_0}^{\infty}\int_{v_0}^{\infty} du\, dv\, \|r^{-1}\psi\|^2_{S^2_{u,v}} r^{7-\varepsilon}\Omega^4 \lesssim \mathbb{F}_0[\Psi,\psi], \tag{283}$$

where the constant in $\lesssim$ depends on ε.

The derived quantity $\underline{\psi}$ associated with the solution $\underline{\alpha}$ of Theorem 2 satisfies the following estimates. Along any null hypersurface of constant $v\geqslant v_0$, including in the limit on null infinity,

$$\int_{u_0}^{\infty} du\, \|r^{-1}\underline{\psi}\|^2_{S^2_{u,v}} r^6(u,v) \lesssim \mathbb{F}_0[\,\underline{\Psi},\underline{\psi}].$$

In addition, we have the integrated decay estimate

$$\int_{u_0}^{\infty}\int_{v_0}^{\infty} du\, dv\, \|r^{-1}\underline{\psi}\|^2_{S^2_{u,v}} r^{5-\varepsilon} \lesssim \mathbb{F}_0[\underline{\Psi},\underline{\psi}], \tag{284}$$

where the constant in $\lesssim$ depends on ε.

Proof. From (179) we derive

$$\partial_u[|\psi|^2 r^6\Omega^2] = 2r^6\Omega^3(P,\psi), \tag{285}$$

or, multiplying by r^n and using $r_u = -\Omega^2$,

$$\begin{aligned}\partial_u(|\psi|^2 r^6\Omega^2\cdot r^n) + |\psi|^2 r^6\Omega^4 n r^{n-1} &= 2r^{6+n}\Omega^3 P\psi \\ &\leqslant \frac{1}{2}|\psi|^2 r^6\Omega^4 n r^{n-1} + \frac{2}{n} r^{7+n}|P|^2\Omega^2,\end{aligned} \tag{286}$$

which simplifies to

$$\partial_u(|\psi|^2 r^6\Omega^2\cdot r^n) + \frac{1}{2}|\psi|^2 r^6\Omega^4 n r^{n-1} \leqslant \frac{2}{n} r^{7+n}|P|^2\Omega^2. \tag{287}$$

The analogue of (285) is

$$\partial_v[|\underline{\psi}|^2 r^6\Omega^2] = -2r^6\Omega^3(\underline{P},\underline{\psi}). \tag{288}$$

We can multiply this by $1/\Omega^2$, which satisfies $\partial_v\Omega^{-2} = -(1/\Omega^2)2M/r^2$:

$$\partial_v\left[\frac{1}{\Omega^2}|\underline{\psi}|^2 r^6\Omega^2\right] + 2Mr^4|\underline{\psi}|^2 = -2r^6\Omega(\underline{P},\underline{\psi}), \tag{289}$$

and hence

$$\partial_v\left[\frac{1}{\Omega^2}|\underline{\psi}|^2 r^6\Omega^2\right] + Mr^4|\underline{\psi}|^2 \leqslant \frac{1}{M}r^8|\underline{P}|^2\Omega^2. \tag{290}$$

Multiplying instead by $1/r^\varepsilon$, we find similarly

$$\partial_v\left[\frac{1}{r^\varepsilon}|\underline{\psi}|^2 r^6\Omega^2\right] + \varepsilon\cdot r^{-1-\varepsilon}r^6\Omega^4|\underline{\psi}|^2 \leqslant C_\varepsilon\cdot r^{7-\varepsilon}|\underline{P}|^2\Omega^2. \tag{291}$$

For the right-hand sides in (287) (applied with $n=2-\varepsilon$ for $\varepsilon>0$), as well as (290) and (291), we already have an integrated decay estimate, i.e. the right-hand side remains controlled from initial data when integrated over the spacetime region $[u_0,u]\times[v_0,v]\times S^2$ (for any $u\geqslant u_0$ and $v\geqslant v_0$), with respect to the measure $du\,dv\,\sin\theta\,d\theta\,d\phi$; see (240). Upon this integration, the left-hand sides of (287), (290) and (291) will provide estimates for fluxes of ψ and $\underline{\psi}$, as well as integrated decay as stated in Proposition 12.1.1. $\square$

Corollary 12.1. *In addition to the bounds of Proposition 12.1.1, we have, for fixed $u\geqslant u_0$ (including the horizon $u=\infty$) and any $v\geqslant v_0$,*

$$\|r^{-1}\cdot\underline{\psi}\Omega^{-1}r^3\|^2_{S^2_{u,v}} + \int_{v_0}^{v} d\bar v\,\frac{1}{r^2}\|r^{-1}\underline{\psi}\Omega^{-1}r^3\|^2_{S^2_{u,\bar v}} \lesssim \|r^{-1}\cdot\underline{\psi}\Omega^{-1}r^3\|^2_{S^2_{u,v_0}} + \mathbb{F}_0[\underline{\Psi}].$$

Proof. Write (289) as $\partial_v[|\underline{\psi}|^2\Omega^{-2}r^6] + (4M/r^2)r^6|\underline{\psi}|^2\Omega^{-2} = -2r^6\Omega^{-1}(\underline{P},\underline{\psi})$, then integrate using Cauchy's inequality and control on the $\underline{P}$-flux from Theorem 1. $\square$

Corollary 12.2. *In addition to the bounds of Proposition 12.1.1, we have, for $n=1,2,3$ and any fixed $v\geqslant v_0$ and any $u\geqslant u_0$ (including the horizon $u=\infty$),*

$$r^n\|r^{-1}\cdot\psi\Omega r^3\|^2_{S^2_{u,v}} + \int_{u_0}^{u} d\bar u\,\Omega^2 r^{n-1}\|r^{-1}\psi\Omega r^3\|^2_{S^2_{\bar u,v}} \lesssim r^n\|r^{-1}\cdot\psi\Omega r^3\|^2_{S^2_{u_0,v}} + \mathbb{F}_0[\Psi].$$

Proof. Fix arbitrary v and integrate (286) for $n=1,2,3$ from $u=u_0$ to arbitrary u, the P-flux being controlled from Theorem 1. $\square$

12.1.2. Estimates for α and $\underline{\alpha}$

Now let us obtain decay estimates for α and $\underline{\alpha}$. The following proposition (in conjunction with Proposition 12.1.1) proves the estimate (241) of Theorem 2.

PROPOSITION 12.1.2. *The solution α of Theorem 2 satisfies the following estimates: Along any null hypersurface of constant $u\geqslant u_0$, including the event horizon, we have*

$$\int_{v_0}^{\infty} dv\, \|r^{-1}\alpha\|^2_{S^2_{u,v}} r^{6-\varepsilon}\Omega^4(u,v) \lesssim \mathbb{F}_0[\Psi,\psi,\alpha]. \tag{292}$$

The solution $\underline{\alpha}$ of Theorem 2 satisfies the following estimates: Along any null hypersurface of constant $v\geqslant v_0$, including in the limit on null infinity, we have

$$\int_{u_0}^{\infty} du\, \|r^{-1}\underline{\alpha}\|^2_{S^2_{u,v}} r^{2}\Omega^{-2}(u,v) \lesssim \mathbb{F}_0[\underline{\Psi},\underline{\psi},\underline{\alpha}]. \tag{293}$$

Finally, we have the integrated decay estimates

$$\int_{u_0}^{\infty}\int_{v_0}^{\infty} du\, dv\, r^{1-\varepsilon}\Omega^{-2}\|r^{-1}\underline{\alpha}\|^2_{S^2_{u,v}} \lesssim \mathbb{F}_0[\underline{\Psi},\underline{\psi},\underline{\alpha}],$$
$$\int_{u_0}^{\infty}\int_{v_0}^{\infty} du\, dv\, r^{5-\varepsilon}\Omega^{6}\|r^{-1}\alpha\|^2_{S^2_{u,v}} \lesssim \mathbb{F}_0[\Psi,\psi,\alpha].$$

Proof. Observe that we can write

$$\nabla\!\!\!\!/_3(r\Omega^2\alpha) = -2\psi\cdot r\Omega^2. \tag{294}$$

It follows that

$$\partial_u(r^2\Omega^4|\alpha|^2) = -4r^2\Omega^5(\psi,\alpha),$$

which, upon multiplication by r^n, becomes

$$\begin{aligned}\partial_u(r^n\cdot r^2\Omega^4|\alpha|^2)+nr^{n-1}r^2\Omega^6|\alpha|^2 &= -4r^3\Omega^5(\psi,\alpha)r^{n-1}\\ &\leqslant \frac{n}{2}r^{n-1}r^2\Omega^6|\alpha|^2+\frac{4}{n}r^{n-1}r^4\Omega^4|\psi|^2.\end{aligned}$$

From the resulting

$$\partial_u(r^n\cdot r^2\Omega^4|\alpha|^2)+\frac{n}{2}r^{n-1}r^2\Omega^6|\alpha|^2 \leqslant \frac{4}{n}r^{n-1}r^4\Omega^4|\psi|^2 \tag{295}$$

we see from (240) that, choosing $n=4-\varepsilon$, after integration with respect to $du\,dv\,\sin\theta\,d\theta\,d\phi$ over the region $[u_0,u]\times[v_0,v]\times S^2$, the right-hand side is controlled by Proposition 12.1.1 for arbitrary $u\geqslant u_0$ and $v\geqslant v_0$. Therefore, we obtain the analogue of Proposition 12.1.1 for α. The argument for $\underline{\alpha}$ is similar. From

$$\nabla\!\!\!\!/_4(r\Omega^2\underline{\alpha}) = 2\underline{\psi}\cdot r\Omega^2, \tag{296}$$

we derive ($\partial_v = \Omega \nabla\!\!\!/_4$)

$$\partial_v((1+r^{-\varepsilon})\Omega^{-6}|r\Omega^2\underline{\alpha}|^2)+\left[\varepsilon\frac{\Omega^2}{r^{1+\varepsilon}}+3\frac{2M}{r^2}(1+r^{-\varepsilon})\right]\Omega^{-6}|r\Omega^2\underline{\alpha}|^2=4r\Omega^{-1}r(\underline{\psi},\underline{\alpha})(1+r^{-\varepsilon}). \tag{297}$$

Applying Cauchy's inequality on the right yields

$$\partial_v((1+r^{-\varepsilon})\Omega^{-6}|r\Omega^2\underline{\alpha}|^2)+\left[\varepsilon\frac{\Omega^2}{r^{1+\varepsilon}}+\frac{2M}{r^2}(1+r^{-\varepsilon})\right]\Omega^{-6}|r\Omega^2\underline{\alpha}|^2\lesssim\frac{1}{r^2}|\underline{\psi}r^3|^2, \tag{298}$$

and integrating as above yields the result for $\underline{\alpha}$. □

12.2. Higher-derivative estimates

In the following subsections we estimate higher derivatives of the quantities ψ, $\underline{\psi}$, α and $\underline{\alpha}$ from our control on P and $\underline{P}$.

12.2.1. Some useful identities

We begin by computing some useful identities, which we shall write in regular form so that the behaviour at the horizon can be assessed directly.

Lemma 12.2.1. *Consider the solution α of Theorem 2 and the derived quantities ψ and P defined via (178) and (179). The following identities hold true:*

$$\nabla\!\!\!/_4(r\psi\Omega)+(2\,\mathrm{tr}\,\chi-2\hat{\omega})(r\psi\Omega)=r\Omega \mathcal{D}\!\!\!/_2^\star\,\mathrm{d}\!\!\!/\mathrm{iv}\,\alpha+\frac{3M}{r^2}\Omega\alpha, \tag{299}$$

$$\Omega\nabla\!\!\!/_4(r^5P)=-2r^5\mathcal{D}\!\!\!/_2^\star\,\mathrm{d}\!\!\!/\mathrm{iv}(\psi\Omega)+6Mr^2\psi\Omega-2r^3\psi\Omega+3rM\Omega^2\alpha, \tag{300}$$

$$\begin{aligned}\Omega\nabla\!\!\!/_4(r^2\Omega\nabla\!\!\!/_4(r^5P)) &= r\Omega\nabla\!\!\!/_4(r\Omega\nabla\!\!\!/_4(r^5P))+\Omega^2\cdot r\Omega\nabla\!\!\!/_4(r^5P)\\ &= -2r^4\mathcal{D}\!\!\!/_2^\star\,\mathrm{d}\!\!\!/\mathrm{iv}\,\mathcal{D}\!\!\!/_2^\star\,\mathrm{d}\!\!\!/\mathrm{iv}(r^3\alpha\Omega^2)-4Mr^2\mathcal{D}\!\!\!/_2^\star\,\mathrm{d}\!\!\!/\mathrm{iv}(r^3\psi\Omega)\\ &\quad -6Mr^2\mathcal{D}\!\!\!/_2^\star\,\mathrm{d}\!\!\!/\mathrm{iv}(r^2\alpha\Omega^2)+\left(-2+\frac{6M}{r}\right)\Omega\nabla\!\!\!/_4(r^5\psi\Omega)\\ &\quad -18M\Omega^2r\psi\Omega+3M\Omega\nabla\!\!\!/_4(r^3\Omega^2\alpha).\end{aligned} \tag{301}$$

Consider the solution $\underline{\alpha}$ of Theorem 2 and the derived quantities $\underline{\psi}$ and $\underline{P}$ defined *via* (180) *and* (181). *Then, the following identities hold true*:

$$\underline{\nabla}_3\left(r\frac{\underline{\psi}}{\Omega}\right)+2\operatorname{tr}\underline{\chi}\left(r\frac{\underline{\psi}}{\Omega}\right)=-r\Omega^{-1}\mathcal{D}_2^\star \,\mathrm{d}\!\!/\mathrm{iv}\,\underline{\alpha}-\frac{3M}{r^2}\frac{\underline{\alpha}}{\Omega}, \tag{302}$$

$$\Omega^{-1}\underline{\nabla}_3(r^5\underline{P})=2r^5\mathcal{D}_2^\star \,\mathrm{d}\!\!/\mathrm{iv}(\underline{\psi}\Omega^{-1})-6Mr^2\underline{\psi}\Omega^{-1}+2r^3\underline{\psi}\Omega^{-1}+3rM\underline{\alpha}, \tag{303}$$

$$\begin{aligned}\Omega^{-1}\underline{\nabla}_3(\Omega^{-1}\underline{\nabla}_3(r^5\underline{P}))=&-2r^4\mathcal{D}_2^\star \,\mathrm{d}\!\!/\mathrm{iv}\,\mathcal{D}_2^\star \,\mathrm{d}\!\!/\mathrm{iv}\left(\frac{r\underline{\alpha}}{\Omega^2}\right)+\frac{4}{r}r^2\mathcal{D}_2^\star \,\mathrm{d}\!\!/\mathrm{iv}\left(\frac{\underline{\psi}r^3}{\Omega}\right)\\&-6Mr^2\mathcal{D}_2^\star \,\mathrm{d}\!\!/\mathrm{iv}(\frac{\underline{\alpha}r}{\Omega})\left(-\frac{6M}{r}+2\right)\Omega^{-1}\underline{\nabla}_3\left(\frac{\underline{\psi}r^3}{\Omega}\right)\\&-\frac{6M}{r^2}\left(\frac{\underline{\psi}r^3}{\Omega}\right)+3M\Omega^{-1}\underline{\nabla}_3(r\underline{\alpha}),\end{aligned} \tag{304}$$

where the last identity could be simplified further by reinserting (303).

Proof. Note that (299) and (302) are just rewritings of the Teukolsky equation of spin $+2$ and spin -2 respectively. The identities (300) and (303) follow from inserting the definitions of P and $\underline{P}$, commuting derivatives and inserting the relevant Teukolsky equation. The identities (301) and (304) follow by combining the estimates already obtained. □

12.2.2. Angular derivatives of ψ and $\underline{\psi}$

From (300) and (303) we directly conclude using the bounds of Proposition 12.1.1 and Theorem 1 the following result.

PROPOSITION 12.2.1. *The quantity ψ associated with the solution α of Theorem 2 through (178) satisfies the estimate*

$$\sup_{u\geqslant u_0}\int_{v_0}^{v} d\bar{v}\, r^{8-\varepsilon}\|r^{-1}\cdot r^2\mathcal{D}_2^\star \,\mathrm{d}\!\!/\mathrm{iv}(\psi\Omega)\|^2_{S^2_{\bar{u},v}}\lesssim \mathbb{F}_0[\Psi,\psi,\alpha], \tag{305}$$

and the degenerate integrated decay estimate

$$\int_{u_0}^{\infty}du\int_{v_0}^{\infty}dv\,\frac{\Omega^2}{r^3}\left(1-\frac{3M}{r}\right)^2\|r^{-1}\cdot r^2\mathcal{D}_2^\star \,\mathrm{d}\!\!/\mathrm{iv}(r^3\psi\Omega)\|^2_{S^2_{u,v}}\lesssim \mathbb{F}_0[\Psi,\psi,\alpha].$$

The quantity $\underline{\psi}$ associated with the solution $\underline{\alpha}$ of Theorem 2 through (180) satisfies the estimate

$$\sup_{v\geqslant v_0}\int_{u_0}^{u} d\bar{u}\, \|r^{-1}\cdot r^2\mathcal{D}_2^\star \,\mathrm{d}\!\!/\mathrm{iv}(r^3\underline{\psi}\Omega^{-1})\|^2_{S^2_{\bar{u},v}}\Omega^2(\bar{u},v)\lesssim \mathbb{F}_0[\underline{\Psi},\underline{\psi},\underline{\alpha}], \tag{306}$$

and the degenerate integrated decay estimate

$$\int_{u_0}^{\infty}du\int_{v_0}^{\infty}dv\,\frac{\Omega^2}{r^3}\left(1-\frac{3M}{r}\right)^2\|r^{-1}\cdot r^2\mathcal{D}_2^\star \,\mathrm{d}\!\!/\mathrm{iv}(r^3\underline{\psi}\Omega^{-1})\|^2_{S^2_{u,v}}\lesssim \mathbb{F}_0[\underline{\Psi},\underline{\psi},\underline{\alpha}].$$

Remark 12.1. Note that, in view of (111), we control in particular the flux of first angular derivatives of $\underline{\psi}$ on null infinity $\mathcal{I}^+$, a fact that will be exploited in the next proposition.

We now look at the commuted equation

$$\nabla\!\!\!/_4(r\,\mathrm{d}\!\!/\mathrm{iv}\,\underline{\psi}r^3\Omega) = -r^3\Omega\cdot r\,\mathrm{d}\!\!/\mathrm{iv}\,\underline{P}. \tag{307}$$

From this, we derive

$$\begin{aligned}
&\Omega\nabla\!\!\!/_4\left(\left(\frac{1}{\Omega^2}-\frac{1}{\Omega^2}\bigg|_{r=3M}\right)|r\,\mathrm{d}\!\!/\mathrm{iv}\,\underline{\psi}r^3\Omega|^2\right)+\frac{2M}{r^2}|r\,\mathrm{d}\!\!/\mathrm{iv}\,\underline{\psi}r^3|^2\\
&\quad= -2r^3\Omega^2\left(\frac{1}{\Omega^2}-\frac{1}{\Omega^2}\bigg|_{r=3M}\right)(r\,\mathrm{d}\!\!/\mathrm{iv}\,\underline{P}, r\,\mathrm{d}\!\!/\mathrm{iv}\,\underline{\psi})r^3\Omega\\
&\quad= 4r^3\left(1-\frac{3M}{r}\right)(r\,\mathrm{d}\!\!/\mathrm{iv}\,\underline{P}, r\,\mathrm{d}\!\!/\mathrm{iv}\,\underline{\psi})r^3\Omega\\
&\quad\leqslant \frac{M}{r^2}|r\,\mathrm{d}\!\!/\mathrm{iv}\,\underline{\psi}r^3|^2+\frac{4}{M}\left(1-\frac{3M}{r}\right)^2 r^8|r\,\mathrm{d}\!\!/\mathrm{iv}\,\underline{P}|^2\Omega^2,
\end{aligned}$$

which yields

$$\Omega\nabla\!\!\!/_4\left(\left(\frac{1}{\Omega^2}-\frac{1}{\Omega^2}\bigg|_{r=3M}\right)|r\,\mathrm{d}\!\!/\mathrm{iv}\,\underline{\psi}r^3\Omega|^2\right)+\frac{M}{r^2}|r\,\mathrm{d}\!\!/\mathrm{iv}\,\underline{\psi}r^3|^2 \leqslant \frac{4}{M}\left(1-\frac{3M}{r}\right)^2 r^8|r\,\mathrm{d}\!\!/\mathrm{iv}\,\underline{P}|^2\Omega^2.$$

Integrating this over the spacetime region $[u_0,u]\times[v_0,v]$ for arbitrary $u\geqslant u_0$ and $v\geqslant v_0$, we observe the following facts:

- the right-hand side is controlled by initial data from (240) for $n=0$;
- the future boundary term (flux) on the constant-v hypersurface is unsigned. However, we already control it from initial data through Proposition 12.2.1.

We conclude the following result.

PROPOSITION 12.2.2. *The quantity* ψ *associated with the solution* α *of Theorem* 2 *through* (178) *satisfies the non-degenerate integrated decay estimate*

$$\int_{u_0}^{\infty}\int_{v_0}^{\infty}\int_{S^2_{\bar u,\bar v}} d\bar u\, d\bar v\, \sin\theta\, d\theta\, d\phi\, r^{7-\varepsilon}\Omega^4|r\,\mathrm{d}\!\!/\mathrm{iv}\,\psi|^2 \lesssim \mathbb{F}_0[\Psi,\mathfrak{D}\psi,\alpha].$$

The quantity $\underline{\psi}$ *associated with the solution* $\underline{\alpha}$ *of Theorem* 2 *through* (180) *satisfies the non-degenerate integrated decay estimate*

$$\int_{u_0}^{\infty}\int_{v_0}^{\infty}\int_{S^2_{\bar u,\bar v}} d\bar u\, d\bar v\, \sin\theta\, d\theta\, d\phi\, r^{5-\varepsilon}|r\,\mathrm{d}\!\!/\mathrm{iv}\,\underline{\psi}|^2 \lesssim \mathbb{F}_0[\underline{\Psi},\mathfrak{D}\underline{\psi},\underline{\alpha}].$$

Proof. The second bound follows with weight r^4 instead of $r^{5-\varepsilon}$ directly from the computation below (307). One easily improves the weight a posteriori replacing

$$\left(\frac{1}{\Omega^2}-\frac{1}{\Omega^2}\bigg|_{r=3M}\right)$$

in the computation below (307) by $(1+r^{-\varepsilon})\chi$, with χ being a cut-off function which is 1 near infinity and vanishes for $r\leqslant 8M$.

For the first bound, one repeats this argument for the commuted equation

$$\underline{\nabla}_3(r\,\mathrm{d\!/\!iv}\,\psi r^3\Omega)=r^3\Omega\cdot r\,\mathrm{d\!/\!iv}\,P,$$

now contracting with $[r^{2-\varepsilon}-(3M)^{2-\varepsilon}]\cdot r\,\mathrm{d\!/\!iv}\,\psi r^3\Omega$ and using that the flux arising on the horizon is a priori controlled by Proposition 12.2.1. $\square$

Similarly, we can directly integrate the equations

$$\Omega\underline{\nabla}_4\left|r\,\mathrm{d\!/\!iv}\,\frac{\underline{\psi}r^3}{\Omega}\right|^2+\frac{2M}{r^2}\left|r\,\mathrm{d\!/\!iv}\,\frac{\underline{\psi}r^3}{\Omega}\right|^2=-2r^3r\left(\mathrm{d\!/\!iv}\,\underline{P},r\,\mathrm{d\!/\!iv}\,\frac{\underline{\psi}r^3}{\Omega}\right)$$

and (for any $1\geqslant\delta\geqslant 0$)

$$\Omega\underline{\nabla}_3[r^{3-\delta}|r\,\mathrm{d\!/\!iv}(r^3\Omega\psi)|^2]+\frac{3-\delta}{r}\Omega^2[r^3|r\,\mathrm{d\!/\!iv}(r^3\Omega\psi)|^2]=r^{6-\delta}\Omega^2r(\mathrm{d\!/\!iv}\,P,r\,\mathrm{d\!/\!iv}(\psi r^3\Omega))$$

from initial data (integrating also over the angular variables) to obtain, after applying Cauchy–Schwarz to the right-hand side and using the P and $\underline{P}$-fluxes the following proposition.

Proposition 12.2.3. *In addition to the estimates of Proposition 12.2.2, we have, for any $u\geqslant u_0$ and $v\geqslant v_0$,*

$$\left\|r^{-1}\cdot r\,\mathrm{d\!/\!iv}\,\frac{\underline{\psi}r^3}{\Omega}\right\|^2_{S^2_{u,v}}+\int_{v_0}^{\infty}\frac{1}{r^2}\left\|r^{-1}\cdot r\,\mathrm{d\!/\!iv}\,\frac{\underline{\psi}r^3}{\Omega}\right\|^2_{S^2_{u,v}}dv\lesssim\left\|r^{-1}\cdot r\,\mathrm{d\!/\!iv}\,\frac{\underline{\psi}r^3}{\Omega}\right\|^2_{S^2_{u,v_0}}+\mathbb{F}_0[\underline{\Psi}]$$

and, for any $1\geqslant\delta\geqslant 0$,

$$\begin{aligned}r^{3-\delta}\|r^{-1}\cdot r\,\mathrm{d\!/\!iv}(\psi\Omega r^3)\|^2_{S^2_{u,v}}&+\int_{u_0}^{\infty}\Omega^2r^{2-\delta}\|r^{-1}\cdot r\,\mathrm{d\!/\!iv}(\psi\Omega r^3)\|^2_{S^2_{u,v}}\,du\\&\lesssim r^{3-\delta}\|r^{-1}\cdot r\,\mathrm{d\!/\!iv}(\psi\Omega r^3)\|^2_{S^2_{u_0,v}}+\mathbb{F}_0[\Psi].\end{aligned}$$

Moreover, the first estimate remains true if we replace $r\,\mathrm{d\!/\!iv}\,\underline{\psi}$ by $\mathcal{A}^{[i+1]}\underline{\psi}$ and $\mathbb{F}_0[\underline{\Psi}]$ by $\mathbb{F}_0^{i,T,\underline{\nabla}}[\underline{\Psi}]$ for $i=1,2$. Similarly, the second estimate remains true if we replace $r\,\mathrm{d\!/\!iv}\,\psi$ by $\mathcal{A}^{[i+1]}\psi$ and $\mathbb{F}_0[\Psi]$ by $\mathbb{F}_0^{i,T,\underline{\nabla}}[\Psi]$.

We remark that the higher-order statement of Proposition 12.2.3 follows immediately from the fact that we can repeat the above transport argument for the commuted equation

$$\not\nabla_4(\mathcal{A}^{[i]}\underline{\psi}r^3\Omega)=-r^3\Omega\cdot\mathcal{A}^{[i]}\underline{P}.$$

Corollary 12.3. *In addition to the estimates of Proposition 12.2.2, we also have the estimates*

$$\begin{aligned}\sup_{u,v}\|r^{-1}r\,\mathrm{d}\!\!/\mathrm{iv}\,\underline{\alpha}\Omega^{-2}\|^2_{S^2_{u,v}}&\lesssim\sup_u\|r^{-1}r\,\mathrm{d}\!\!/\mathrm{iv}\,\underline{\alpha}\Omega^{-2}\|^2_{S^2_{u,v_0}}\\&\quad+\sup_u\left\|r^{-1}\cdot r\,\mathrm{d}\!\!/\mathrm{iv}\,\frac{\underline{\psi}r^3}{\Omega}\right\|^2_{S^2_{u,v_0}}+\mathbb{F}_0[\underline{\Psi}],\\ \sup_{u,v}\;r^{7-\delta}\|r^{-1}r\,\mathrm{d}\!\!/\mathrm{iv}\,\alpha\Omega^2\|^2_{S^2_{u,v}}&\lesssim\sup_v r^{7-\delta}\|r^{-1}r\,\mathrm{d}\!\!/\mathrm{iv}\,\alpha\Omega^2\|^2_{S^2_{u_0,v}}\\&\quad+\sup_v\;r^{3-\delta}\|r^{-1}r\,\mathrm{d}\!\!/\mathrm{iv}(\psi\Omega r^3)\|^2_{S^2_{u_0,v}}+\mathbb{F}_0[\Psi].\end{aligned}$$

Proof. For the second estimate, apply the once angular commuted (295) with $n=5-\delta$ and use the flux in the proposition. For the first one, use (297) (with an additional (Ω^{-2} weight)) and the flux of the proposition. □

12.2.3. Estimating all first derivatives of ψ and $\underline{\psi}$

Note that we already control the $\not\nabla_3\psi$ and $\not\nabla_4\underline{\psi}$ derivatives directly from the transport equation they satisfy, (179). To estimate the remaining first derivative we commute these equations by $2\not\nabla_{R^\star}:=-\Omega\not\nabla_3+\Omega\not\nabla_4$ and recall that $\not\nabla_{R^\star}\Psi$ satisfies a *non-degenerate* (near $r=3M$) integrated decay estimate; cf. (270). We compute

$$\Omega\not\nabla_4(\not\nabla_{R^\star}\underline{\psi}r^3\Omega)=-\not\nabla_{R^\star}(r^3\Omega^2\underline{P}),\tag{308}$$

since $\Omega\not\nabla_3$ and $\Omega\not\nabla_4$ commute. Since the right-hand side again satisfies a *non-degenerate* integrated decay estimate (270) and (240), we can actually repeat the estimate for the uncommuted equation (leading from (287) to (290)) to immediately obtain the analogue of Proposition 12.1.1,([29]) as the same considerations hold for the $\not\nabla_{R^\star}$ commuted $\not\nabla_3(r^3\Omega\psi)$-equation.

([29]) Note that this would not be possible if the right-hand side satisfied an estimate which degenerated near $r=3M$. The reason is that such a right-hand side would (just as for the angular derivatives in the previous section) force us to multiply with a weight that changes sign near $3M$ which gives the future boundary term the *wrong* sign. In the angular case, this flux of the wrong sign was controlled a priori. Here this flux is not available yet.

PROPOSITION 12.2.4. *Consider the quantity ψ associated with the solution α of Theorem 2 through (178). Then, along any null-hypersurface of constant $u\geqslant u_0$ (including the event horizon), we have*

$$\int_{v_0}^{\infty} dv\, \|r^{-1}\nabla\!\!\!/_{R^\star}(\Omega\psi)\|^2_{S^2_{u,v}} r^{8-\varepsilon}(u,v) \lesssim \mathbb{F}_0[\Psi, \mathfrak{D}\psi, \alpha]. \tag{309}$$

In addition, we have the integrated decay estimate

$$\int_{v_0}^{\infty}\int_{u_0}^{\infty} dvdu \|r^{-1}\nabla\!\!\!/_{R^\star}(\Omega\psi)\|^2_{S^2_{u,v}} \Omega^2 r^{7-\varepsilon} \lesssim \mathbb{F}_0[\Psi, \mathfrak{D}\psi, \alpha]. \tag{310}$$

Consider the quantity $\underline{\psi}$ associated with the solution $\underline{\alpha}$ of Theorem 2 through (180). Then, along any null hypersurface of constant $v\geqslant v_0$, we have

$$\int_{u_0}^{\infty} du\, \|r^{-1}\nabla\!\!\!/_{R^\star}(\Omega^{-1}\underline{\psi})\|^2_{S^2_{u,v}} r^6\Omega^2(u,v) \lesssim \mathbb{F}_0[\,\underline{\Psi}, \mathfrak{D}\underline{\psi}, \underline{\alpha}\,]. \tag{311}$$

In addition, we have the integrated decay estimate

$$\int_{v_0}^{\infty}\int_{u_0}^{\infty} dv\, du\, \|r^{-1}\nabla\!\!\!/_{R^\star}(\Omega^{-1}\underline{\psi})\|^2_{S^2_{u,v}} \Omega^2 r^{5-\varepsilon} \lesssim \mathbb{F}_0[\,\underline{\Psi}, \mathfrak{D}\underline{\psi}, \underline{\alpha}\,]. \tag{312}$$

We now exploit the relations (179) and (181) to obtain estimates for *all* first derivatives of ψ and $\underline{\psi}$. From

$$\Omega\nabla\!\!\!/_4(\Omega\psi) = \Omega\nabla\!\!\!/_3(\Omega\psi) + 2\nabla\!\!\!/_{R^\star}(\Omega\psi) = \Omega^2 P + 3\frac{\Omega^2}{r}(\Omega\psi) + 2\nabla\!\!\!/_{R^\star}(\Omega\psi)$$

(and similarly for $\underline{\psi}$), it is immediate that we can obtain a non-degenerate integrated decay estimate for all first derivatives of ψ and $\underline{\psi}$. In particular, we can replace $\nabla\!\!\!/_{R^\star}$ by both $\Omega\nabla\!\!\!/_4$ or $\Omega\nabla\!\!\!/_3$ in the estimates of Proposition 12.2.4. The estimate for $\Omega\nabla\!\!\!/_4(\Omega\psi)$ will then still be non-optimal in terms of r-weights at infinity and the estimate for $\Omega\nabla\!\!\!/_3(\Omega^{-1}\underline{\psi})$ will not be optimal near the horizon. However, this is easily remedied using r-weighted estimates and the redshift respectively. We indicate this for the r-weight before stating the final proposition. We have

$$\begin{aligned}&\Omega\nabla\!\!\!/_3[\xi(r)r^{4-\varepsilon}|\Omega\nabla\!\!\!/_4(r^3\Omega\psi)|^2] + \frac{4-\varepsilon}{r}\Omega^2\xi(r)[r^{4-\varepsilon}|\Omega\nabla\!\!\!/_4(r^3\Omega\psi)|^2] - \xi_r\Omega^2\, r^{4-\varepsilon}|\Omega\nabla\!\!\!/_4(r^3\Omega\psi)|^2\\ &\quad= \xi r^{7-\varepsilon}\Omega^2\Omega\nabla\!\!\!/_4 P\cdot\Omega\nabla\!\!\!/_4(\psi r^3\Omega)\end{aligned}$$

for a radial cut-off function ξ which we choose to be 1 for $r\geqslant 8M$ and 0 for $r\leqslant 6M$. Using that one already has a non-optimal spacetime estimate for $r\leqslant 8M$, integrating the above over a spacetime region improves the weights.(30) We summarise the conclusion in the following proposition.

(30) Note again that this argument does not apply globally, because $\nabla\!\!\!/_4 P$ is only controlled in a degenerate norm near $r=3M$.

Proposition 12.2.5. *Consider the quantity ψ associated with the solution α of Theorem 2 through (178). We have the non-degenerate integrated decay estimate*

$$\int_{u_0}^{\infty}\int_{v_0}^{\infty}\int_{S^2_{\bar{u},\bar{v}}} d\bar{u}\, d\bar{v}\, \sin\theta\, d\theta\, d\phi\, r^{7-\varepsilon}[|\nabla\!\!\!/_3(\Omega\psi)|^2+\Omega^2|r\cdot\Omega\nabla\!\!\!/_4(\Omega\psi)|^2]\lesssim\mathbb{F}_0[\Psi,\mathfrak{D}\psi,\alpha],$$

as well as, for any $u\geqslant u_0$, the flux estimate

$$\int_{v_0}^{\infty} dv\, \sin\theta\, d\theta\, d\phi\, r^{4-\varepsilon}|\Omega\nabla\!\!\!/_4(\Omega\psi r^3)|^2\lesssim\mathbb{F}_0[\Psi,\mathfrak{D}\psi,\alpha].$$

Consider the quantity $\underline{\psi}$ associated with the solution $\underline{\alpha}$ of Theorem 2 through (180). We have the non-degenerate integrated decay estimate

$$\int_{u_0}^{\infty}\int_{v_0}^{\infty}\int_{S^2_{\bar{u},\bar{v}}} d\bar{u}\, d\bar{v}\, \sin\theta\, d\theta\, d\phi[r^4\Omega^2(|\Omega^{-1}\nabla\!\!\!/_3(\Omega^{-1}\underline{\psi})|^2+|r\Omega\nabla\!\!\!/_4(\underline{\psi}\Omega^{-1})|^2)]\lesssim\mathbb{F}_0[\,\underline{\Psi},\mathfrak{D}\underline{\psi},\underline{\alpha}\,]$$

as well as, for any $v\geqslant v_0$, the flux estimate

$$\int_{u_0}^{\infty} du\, \sin\theta\, d\theta\, d\phi\, \Omega^2 r^6|\Omega^{-1}\nabla\!\!\!/_3(\Omega^{-1}\underline{\psi})|^2\lesssim\mathbb{F}_0[\underline{\Psi},\mathfrak{D}\underline{\psi},\underline{\alpha}\,].$$

12.2.4. Estimating second angular derivatives of α and $\underline{\alpha}$

Proposition 12.2.5 will directly imply control over two angular derivatives of α and $\underline{\alpha}$. This follows from the identities (302) and (299). We observe that, by Propositions 12.2.5 and 12.1.1, we already control the flux (on constant v) of the left-hand side of (302) and the flux (on constant u) of the left-hand side of (299). Moreover, by the same propositions, we have integrated decay for the left hand sides of both (302) and (299). As α and $\underline{\alpha}$ itself are already controlled from Proposition 12.1.2, we immediately conclude the following result.

Proposition 12.2.6. *Consider the solution α of Theorem 2. We have the flux estimate*

$$\sum_{i=0}^{2}\int_{v_0}^{\infty} dv\, \sin\theta\, d\theta\, d\phi\, r^{6+2i-\varepsilon}|\nabla\!\!\!/^i(\alpha\Omega^2)|^2\lesssim\mathbb{F}_0[\Psi,\mathfrak{D}\psi,\alpha] \tag{313}$$

for any hypersurface of constant u, and the integrated decay estimate

$$\sum_{i=0}^{2}\int_{v_0}^{\infty}\int_{u_0}^{\infty} du\, dv \int_{S^2_{u,v}} \sin\theta\, d\theta\, d\phi\, \Omega^2[\Omega^4 r^{5-\varepsilon+2i}|\nabla\!\!\!/^i\alpha|^2]\lesssim\mathbb{F}_0[\Psi,\mathfrak{D}\psi,\alpha].$$

Consider the solution $\underline{\alpha}$ of Theorem 2. We have the flux estimate

$$\sum_{i=0}^{2}\int_{u_0}^{\infty} du\, \sin\theta\, d\theta\, d\phi\, \Omega^2 r^{2+2i} |\nabla\!\!\!/^{\,i}(\underline{\alpha}\Omega^{-2})|^2 \lesssim \mathbb{F}_0[\underline{\Psi}, \mathfrak{D}\underline{\psi}, \underline{\alpha}]$$

on any constant v hypersurface, and the integrated decay estimate

$$\sum_{i=0}^{2}\int_{v_0}^{\infty}\int_{u_0}^{\infty} du\, dv \int_{S^2_{u,v}} \sin\theta\, d\theta\, d\phi\, \Omega^2[\Omega^{-4} r^{2i} |\nabla\!\!\!/^{\,i}\underline{\alpha}|^2] \lesssim \mathbb{F}_0[\underline{\Psi}, \mathfrak{D}\underline{\psi}, \underline{\alpha}].$$

Remark 12.2. The estimates of Proposition 12.2.6 still "loses derivatives", in that the norm on the right-hand side involves three derivatives of α (one derivative of Ψ which itself is two derivatives of α). An improved estimate which does not lose in this sense will be provided in Proposition 12.3.1 below.

12.2.5. Estimating all first derivatives of α and $\underline{\alpha}$

We now establish estimates for (all) *first* derivatives of α and $\underline{\alpha}$. This is very straightforward, as we can commute $\nabla\!\!\!/_3(r\Omega^2\alpha) = -2\psi r\Omega^2$ and the corresponding $\nabla\!\!\!/_4(r\Omega^2\underline{\alpha}) = 2\underline{\psi} r\Omega^2$ equation by any derivative from $r\nabla\!\!\!/_4$, $\nabla\!\!\!/_3$ and $r\nabla\!\!\!/$, and observe that the right-hand side satisfies a non-degenerate integrated decay estimate just as the uncommuted equation did (cf. Propositions 12.2.2 and 12.2.5). Repeating the proof of Proposition 12.1.2 therefore immediately provides the analogue of that proposition.

Proposition 12.2.7. *Consider the solution α of Theorem 2. Along any null hypersurface of constant $u \geqslant u_0$, including the event horizon, we have*

$$\int_{v_0}^{\infty} dv\, \|r^{-1}\mathfrak{D}\alpha\|^2_{S^2_{u,v}} r^{6-\varepsilon}\Omega^4(u,v) \lesssim \mathbb{F}_0[\Psi, \mathfrak{D}\psi, \mathfrak{D}\alpha]. \tag{314}$$

In addition, we have the integrated decay estimate

$$\int_{v_0}^{\infty}\int_{u_0}^{\infty} du\, dv\, r^{5-\varepsilon}\Omega^6 \|r^{-1}\mathfrak{D}\alpha\|^2_{S^2_{u,v}} \lesssim \mathbb{F}_0[\Psi, \mathfrak{D}\psi, \mathfrak{D}\alpha].$$

Consider the solution $\underline{\alpha}$ of Theorem 2. Along any null hypersurface of constant $v \geqslant v_0$, including in the limit on null infinity, we have

$$\int_{u_0}^{\infty} du\, \|r^{-1}\mathfrak{D}\underline{\alpha}\|^2_{S^2_{u,v}} r^2\Omega^{-2}(u,v) \lesssim \mathbb{F}_0[\underline{\Psi}, \mathfrak{D}\underline{\psi}, \mathfrak{D}\underline{\alpha}]. \tag{315}$$

In addition, we have the integrated decay estimate

$$\int_{v_0}^{\infty}\int_{u_0}^{\infty} du\, dv\, du\, dv\, r^{1-\varepsilon}\Omega^{-2} \|r^{-1}\mathfrak{D}\underline{\alpha}\|^2_{S^2_{u,v}} \lesssim \mathbb{F}_0[\,\underline{\Psi}, \mathfrak{D}\underline{\psi}, \mathfrak{D}\underline{\alpha}\,].$$

Remark 12.3. For α (and similarly for $\underline{\alpha}$) it should be possible to apply energy estimates on the (commuted) Teukolsky equation (173) itself to control all second derivatives, using the fact that two angular derivatives and all first derivatives are already under control by Propositions 12.2.7 and 12.2.6. Such estimates can alternatively be derived directly from the Bianchi equations.

12.2.6. Completing the proof of (242) and (243)

We complete the proof of the statements (1)–(3) of Theorem 2 by proving the estimates (242) and (243), recalling that (241) was proven already in §12.1.2.

Combining the flux and integrated decay estimates of Propositions 12.2.7 and 12.2.5, and the estimates of Propositions 12.2.3 and 12.2.1, we have established (242) and (243) for $n=0$ (in fact also integrated decay and flux estimates on second angular derivatives of α and $\underline{\alpha}$ in Proposition 12.2.6 and flux estimates for *second* angular derivatives of ψ and $\underline{\psi}$ in Proposition 12.2.1). Since the Regge–Wheeler equation commutes trivially with T and angular momentum operators Ω_i (cf. §4.2.2), the higher-order statements are immediate.

12.3. Refinements and polynomial decay estimates

We now obtain the necessary refinements to obtain the final statement (4) of Theorem 2, namely Propositions 12.3.4–12.3.7 and the estimate of Corollary 12.6 below.

The statement of the first four propositions is contained in §12.3.2 below. To prove them, we shall first show certain higher-order statements in §12.3.1. Finally, in §12.3.3, we shall prove Corollary 12.6, which provides an L^1-estimate that will be useful in the proof of Theorem 3.

12.3.1. Top-order statements

Using the previous bounds in conjunction with the identities (304) and (301), we can conclude a boundedness (of energy) statement for α and $\underline{\alpha}$ which does not lose derivatives in the sense that two derivatives of $\Psi=r^5P$ and $\underline{\Psi}=r^5\underline{P}$ control four derivatives of α and $\underline{\alpha}$. This statement does however require the higher-order estimates of Corollary 11.4, which we recall include the commuted redshift near the horizon and weighted commutation near null infinity. Recall the notation (103) and the ellipticity of these operators; cf. (111).

PROPOSITION 12.3.1. *Consider the solution α of Theorem 2. Then, we have the flux bound*

$$\sup_{v\geqslant v_0}\int_{u_0}^{\infty} du\,\sin\theta\,d\theta\,d\phi\,\Omega^2\left|\mathcal{A}^{[4]}\left(\frac{r\underline{\alpha}}{\Omega^2}\right)\right|^2 \lesssim \mathbb{F}_0^1[\underline{\Psi}]+\mathbb{F}_0[\underline{\Psi},\mathfrak{D}\underline{\psi},\mathfrak{D}\underline{\alpha}]. \tag{316}$$

Consider the solution $\underline{\alpha}$ of Theorem 2. Then, we have the flux bound

$$\sup_{u\geqslant u_0}\int_{v_0}^{\infty} dv\,\sin\theta\,d\theta\,d\phi\,r^{6-\varepsilon}|\mathcal{A}^{[4]}(\alpha\Omega^2)|^2 \lesssim \mathbb{F}_0^1[\Psi]+\mathbb{F}_0[\Psi,\mathfrak{D}\psi,\mathfrak{D}\alpha]. \tag{317}$$

Proof. Apply the flux bounds of Propositions 12.2.1, 12.2.5, 12.2.6 and Corollary 11.4 to the identities (304) and (301). □

PROPOSITION 12.3.2. *Consider the solution α of Theorem 2. We have, for $i\geqslant 4$, the integrated decay estimate*

$$\int_{u_0}^{\infty} du\int_{v_0}^{\infty} dv\frac{\Omega^2}{r^{1+\varepsilon}}\left(1-\frac{3M}{r}\right)^2 r^{-1}\|r^{-1}\cdot\mathcal{A}^{[i]}(r^3\alpha\Omega^2)\|^2_{S^2_{u,v}} \lesssim \mathbb{F}_0^{i-3}[\Psi]+\mathbb{F}_0^{i-4,T,\nabla\!\!\!\!/}[\Psi,\mathfrak{D}\psi,\mathfrak{D}\alpha]$$

Consider the solution $\underline{\alpha}$ of Theorem 2.

$$\int_{u_0}^{\infty} du\int_{v_0}^{\infty} dv\,\frac{\Omega^2}{r^{1+\varepsilon}}\left(1-\frac{3M}{r}\right)^2 \|r^{-1}\cdot\mathcal{A}^{[i]}(r\underline{\alpha}\Omega^{-2})\|^2_{S^2_{u,v}} \lesssim \mathbb{F}_0^{i-3}[\underline{\Psi}]+\mathbb{F}_0^{i-4,T,\nabla\!\!\!\!/}[\underline{\Psi},\mathfrak{D}\underline{\psi},\mathfrak{D}\underline{\alpha}].$$

Moreover, for both estimates, if for any $i\geqslant 5$ we replace $\mathcal{A}^{[i]}$ by $\mathcal{A}^{[i-1]}$ on the left-hand side (while keeping the right-hand side fixed), the degeneration factor $(1-3M/r)^2$ can be dropped.

Proof. Apply the integrated decay estimates of Propositions 12.2.1, 12.2.5, 12.2.6, 12.2.2 and Corollary 11.4 to the identities (304) and (301). □

We also derive some boundedness estimates on the spheres $S^2_{u,v}$:

PROPOSITION 12.3.3. *The solution α of Theorem 2 satisfies, for any $u\geqslant u_0$ and $v\geqslant v_0$, the estimates*

$$r^3\|r^{-1}\cdot\mathcal{A}^{[3]}r^3\psi\Omega\|^2_{S^2_{u,v}} \lesssim \mathbb{F}_0^2[\Psi]+\mathbb{F}_0^{2,T,\nabla\!\!\!\!/}[\Psi,\mathfrak{D}\psi,\mathfrak{D}\alpha], \tag{318}$$

$$r\|r^{-1}\cdot\mathcal{A}^{[4]}r^3\alpha\Omega^2\|^2_{S^2_{u,v}} \lesssim \mathbb{F}_0^2[\Psi]+\mathbb{F}_0^{2,T,\nabla\!\!\!\!/}[\Psi,\mathfrak{D}\psi,\mathfrak{D}\alpha]. \tag{319}$$

The solution $\underline{\alpha}$ of Theorem 2 satisfies, for any $u\geqslant u_0$ and $v\geqslant v_0$, the estimates

$$\|r^{-1}\cdot\mathcal{A}^{[3]}r^3\underline{\psi}\Omega^{-1}\|^2_{S^2_{u,v}} \lesssim \mathbb{F}_0^2[\underline{\Psi}]+\mathbb{F}_0^{2,T,\nabla\!\!\!\!/}[\underline{\Psi},\mathfrak{D}\underline{\psi},\mathfrak{D}\underline{\alpha}], \tag{320}$$

$$\|r^{-1}\cdot\mathcal{A}^{[4]}r\underline{\alpha}\Omega^{-2}\|^2_{S^2_{u,v}} \lesssim \mathbb{F}_0^2[\underline{\Psi}]+\mathbb{F}_0^{2,T,\nabla\!\!\!\!/}[\underline{\Psi},\mathfrak{D}\underline{\psi},\mathfrak{D}\underline{\alpha}]. \tag{321}$$

Moreover, if we replace $\mathcal{A}^{[3]}$ by $\mathcal{A}^{[2]}$ on the left in (318) and (320), the first term on the right can be dropped.

Proof. The statements for ψ and $\underline{\psi}$ follow from Proposition 12.2.3 and the fact that the term on the data $u{=}u_0$ ($v{=}v_0$) can be controlled by $\mathbb{F}^2_0[\Psi]$ ($\mathbb{F}^2_0[\underline{\Psi}]$) using 1-dimensional Sobolev embedding. By the same argument, if $\mathcal{A}^{[3]}$ is replaced by $\mathcal{A}^{[2]}$, we see that $\mathbb{F}^{2,T,\nabla\!\!\!\!/}_0[\Psi,\mathfrak{D}\psi,\mathfrak{D}\alpha]$ ($\mathbb{F}^{2,T,\nabla\!\!\!\!/}_0[\underline{\Psi},\mathfrak{D}\underline{\psi},\mathfrak{D}\underline{\alpha}]$) is sufficient.

Similarly, the bounds on α and $\underline{\alpha}$ follow from the identities (304) and (301), Corollary 11.4, 1-dimensional Sobolev embedding and the previous estimates. Observe that the last term in (304) only requires a bound on $S^2_{u,v}$ for $Tr\underline{\alpha}/\Omega^2$ and $r\underline{\alpha}/\Omega^2$, which is again easily derived in terms of the right-hand side from the aforementioned transport equations. □

12.3.2. Polynomial decay for ψ, $\underline{\psi}$, α and $\underline{\alpha}$

We finally record how the refined decay estimates for Ψ of §11.5 are inherited by the derived quantities ψ, $\underline{\psi}$, α and $\underline{\alpha}$. We remark that we are not aiming for the optimal or exhaustive statement here in terms of rates or regularity.

Proposition 12.3.4. *Fix r_0 as in Proposition 11.5.1, let $v\geqslant v_0$ and recall the notation $u(v,r_0)$. Consider the solution α of Theorem 2 and the derived quantity ψ defined via (178). Then, ψ and α satisfy the following integrated decay estimates*:

$$\int_v^\infty d\bar{v}\int_{u(v,r_0)}^\infty d\bar{u}\,(\|r^{-1}\cdot\psi\Omega\|^2_{S^2_{\bar{u},\bar{v}}}\Omega^2 r^{5-\varepsilon}+\|r^{-1}\cdot\alpha\Omega^2\|^2_{S^2_{\bar{u},\bar{v}}}\Omega^2 r^{3-\varepsilon})$$
$$\lesssim\frac{1}{v^2}(\mathbb{F}^{2,T}_0[\Psi]+\mathbb{F}_0[\Psi,\psi,\alpha]).$$

Consider the solution $\underline{\alpha}$ of Theorem 2 and the derived quantity $\underline{\psi}$ defined via (180). Then, $\underline{\psi}$ and $\underline{\alpha}$ satisfy the following integrated decay estimates:

$$\int_v^\infty d\bar{v}\int_{u(v,r_0)}^\infty d\bar{u}\,(\|r^{-1}\cdot\underline{\psi}\Omega^{-1}\|^2_{S^2_{\bar{u},\bar{v}}}\Omega^2 r^{5-\varepsilon}+\|r^{-1}\cdot\underline{\alpha}\Omega^{-2}\|^2_{S^2_{\bar{u},\bar{v}}}\Omega^2 r^{1-\varepsilon})$$
$$\lesssim\frac{1}{v^2}(\mathbb{F}^{2,T}_0[\underline{\Psi}]+\mathbb{F}_0[\underline{\Psi},\underline{\psi},\underline{\alpha}]).$$

Proof. We begin with the ψ-part of the first estimate. We pick a dyadic v-sequence, $v_{i+1}{=}2v_i$, with associated u-sequence, $u_i{=}u(v_i,r_0)$. The spacetime integral of Proposition 12.1.1 allows us to find in each dyadic interval $[u_i,u_{i+1}]$ a slice $\tilde{u}_i$ with

$$\int_{v_0}^\infty d\bar{v}\,r^{7-\varepsilon}\Omega^2\|r^{-1}\cdot\psi\Omega\|^2_{S^2_{\tilde{u}_i,\bar{v}}}\lesssim\frac{1}{\tilde{u}_i}\cdot\mathbb{F}_0[\Psi,\psi,\alpha],$$

and hence in particular

$$\int_{\tilde{v}_i}^\infty d\bar{v}\,r^{7-\varepsilon}\|r^{-1}\cdot\psi\Omega\|^2_{S^2_{\tilde{u}_i,\bar{v}}}\lesssim\frac{1}{\tilde{v}_i}\cdot\mathbb{F}_0[\Psi,\psi,\alpha],$$

where $\tilde{v}_i$ defined implicitly by $\tilde{u}_i = u(\tilde{v}_i, r_0)$. Fix now an arbitrary $u \geqslant u_0$. Pick the i such that $\tilde{u}_i < u \leqslant \tilde{u}_{i+1}$. Applying the inequality (287) with $n=1-\varepsilon$ and integrated over the spacetime region $[\tilde{u}_i, u] \times [\tilde{v}_i, \infty] \times S^2$, we export the decay to the slice C_u and finally obtain

$$\begin{aligned}\int_v^\infty d\bar{v}\, r^{7-\varepsilon} \|r^{-1} \cdot \psi\Omega\|^2_{S^2_{u,v}} + \int_v^\infty d\bar{v} \int_{u(v,r_0)}^\infty d\bar{u}\, \|r^{-1} \cdot \psi\Omega\|^2_{S^2_{\bar{u},\bar{v}}} \Omega^2 r^{6-\varepsilon} \\ \lesssim \frac{1}{v} (\mathbb{F}_0^{2,T}[\Psi] + \mathbb{F}_0[\Psi, \psi, \alpha])\end{aligned} \tag{322}$$

for any $v \geqslant v_0$ and $u \geqslant u(v, r_0)$. The method to establish decay for α given the decay for ψ is entirely analogous to the one seen above establishing decay for ψ from P. Hence,

$$\begin{aligned}\int_v^\infty d\bar{v}\, r^{5-\varepsilon} \|r^{-1} \cdot \alpha\Omega^2\|^2_{S^2_{u,v}} + \int_v^\infty d\bar{v} \int_{u(v,r_0)}^\infty d\bar{u}\, \|r^{-1} \cdot \alpha\Omega^2\|^2_{S^2_{\bar{u},\bar{v}}} \Omega^2 r^{4-\varepsilon} \\ \lesssim \frac{1}{v} (\mathbb{F}_0^{2,T}[\Psi] + \mathbb{F}_0[\Psi, \psi, \alpha])\end{aligned} \tag{323}$$

for any $v \geqslant v_0$ and $u \geqslant u(v, r_0)$. It is clear that, repeating the iteration once more, one obtains the first estimate claimed in the proposition. Note that, once one has generated a dyadic sequence of slices with the weighted ψ-energy decaying like v^{-2}, one can apply (287) with $n=-\varepsilon$ between two such slices and get the desired integrated decay estimate, despite the fact that the future boundary terms has the wrong sign.

For the underlined quantities (i.e. the second estimate of the proposition), one proceeds analogously, except that in this direction we do not have to lose powers of r.

From the spacetime estimate of Proposition 12.1.1, we find in each dyadic interval $[v_i, v_{i+1}]$ a $\tilde{v}_i$ such that

$$\int_{u_0}^\infty d\bar{u}\, r^{5-\varepsilon} \Omega^2 \|r^{-1} \cdot \underline{\psi} \Omega^{-1}\|^2_{S^2_{\bar{u},\tilde{v}_i}} \lesssim \frac{1}{\tilde{v}_i} \cdot \mathbb{F}_0[\,\underline{\Psi}, \underline{\psi}, \underline{\alpha}\,].$$

Fix now an arbitrary $v \geqslant v_0$. Pick the $\tilde{v}_i$ such that $\tilde{v}_i < v \leqslant \tilde{v}_{i+1}$. Integrating (290) over the region $[\tilde{u}_i, \infty] \times [v_i, v] \times S^2 \cap \{r \leqslant r_0\}$ (where weights in r are irrelevant) and using Corollary 11.5 eventually yields

$$\begin{aligned}\int_{u(v,r_0)}^\infty d\bar{u}\, \Omega^2 \|r^{-1} \cdot \underline{\psi} \Omega^{-1}\|^2_{S^2_{u,V}} \iota_{r \leqslant r_0} + \int_v^\infty d\bar{v} \int_{u(v,r_0)}^\infty d\bar{u}\, \Omega^2 \|r^{-1} \cdot \underline{\psi} \Omega^{-1}\|^2_{S^2_{\bar{u},\bar{v}}} \iota_{r \leqslant r_0} \\ \lesssim \frac{1}{v} (\mathbb{F}_0^{2,T}[\underline{\Psi}] + \mathbb{F}_0[\,\underline{\Psi}, \underline{\psi}, \underline{\alpha}\,])\end{aligned}$$

for any $V \geqslant v \geqslant v_0$ and $u \geqslant u(v, r_0)$ with $\iota_{r \leqslant r_0}$ denoting the indicator function. Repeating the dyadic argument finally improves the power on the right-hand side from $1/v$ to $1/v^2$. To extend the estimate to the region $r \geqslant r_0$, we use the estimate (291) in connection with the previous bound and again Corollary 11.5. This proves the estimate for $\underline{\psi}$ claimed in the proposition.

For $\underline{\alpha}$ one repeats the above argument now using (298) in conjunction with the fact that polynomial decay has already been established for the (spacetime integrated) right-hand side of (298) in the previous step. $\square$

The above proof also generates control of various fluxes, in particular.

Corollary 12.4. *Consider the solution α of Theorem 2 and the derived quantity ψ defined via (178). Then, on the event horizon $\mathcal{H}^+$, we have the flux bound*

$$\int_v^\infty d\bar{v}\,[\|\alpha\Omega^2\|^2_{S^2_{\infty,\bar{v}}}+\|\psi\Omega\|^2_{S^2_{\infty,\bar{v}}}]\lesssim\frac{1}{v^2}(\mathbb{F}_0^{2,T}[\Psi]+\mathbb{F}_0[\Psi,\psi,\alpha]). \tag{324}$$

Corollary 12.5. *Consider the solution $\underline{\alpha}$ of Theorem 2 and the derived quantity $\underline{\psi}$ defined via (180). Then, for any $V\geqslant v\geqslant v_0$, we have the flux bound*

$$\int_{u(v,r_0)}^\infty d\bar{u}\,\Omega^2\left(\frac{1}{r^\varepsilon}\|r^{-1}\cdot r^3\underline{\psi}\Omega^{-1}\|^2_{S^2_{u,V}}+\|r^{-1}\cdot\underline{\alpha}r\Omega^{-2}\|^2_{S^2_{u,V}}\right)\lesssim\frac{1}{v^2}(\mathbb{F}_0^{2,T}[\Psi]+\mathbb{F}_0[\,\underline{\Psi},\underline{\psi},\underline{\alpha}\,]).$$

Remark 12.4. We will refine Corollary 12.5 in §12.3.3, and in particular remove ε.

Polynomial decay for ψ and $\underline{\psi}$ on spheres $S^2_{u,v}$

For later purposes, we also note estimates for ψ and $\underline{\psi}$ on spheres $S^2_{u,v}$.

Proposition 12.3.5. *Consider the solution α of Theorem 2 and the derived quantity ψ defined via (178). The following estimate holds for any $u\geqslant u_0$ and $v\geqslant v_0$ including the spheres $S^2_{\infty,v}$:*

$$\|r^{-1}\cdot\Omega\psi r^3\|^2_{S^2_{u,v}}\lesssim\frac{1}{v^2}\left(\mathbb{F}_0^{2,T}[\Psi]+\sup_v\|r^{-1}\cdot\Omega\psi r^4\|^2_{S^2_{u_0,v}}\right).$$

Proof. Fix $v\geqslant v_0\geqslant 2u_0$. From the defining equation (179) we derive the estimate

$$\|r^{-1}\cdot\Omega r^3\psi\|_{S^2_{u,v}}\lesssim\|r^{-1}\cdot\Omega r^3\psi\|_{S^2_{u_0,v}}+\int_{u_0}^{u=v/2}du\,\Omega^2\|r^{-1}\cdot Pr^3\|_{S^2_{u,v}}$$
$$+\int_{u=v/2}^{\max(v,v/2)}du\,\Omega^2\|r^{-1}\cdot Pr^3\|_{S^2_{u,v}}.$$

Applying the Cauchy–Schwarz inequality on the integrals, we obtain

$$\|r^{-1}\cdot\Omega r^3\psi\|_{S^2_{u,v}}\lesssim\frac{1}{v}\|r^{-1}\cdot\Omega r^4\psi\|_{S^2_{u_0,v}}+\sqrt{\int_{u_0}^{u=v/2}du\,\Omega^2\|r^{-1}\cdot\Psi\|^2_{S^2_{u,v}}}\,\frac{1}{v^{3/2}}$$
$$+\sqrt{\int_{u=v/2}^{\max(v,v/2)}du\,\frac{\Omega^2}{r^2}\|r^{-1}\cdot\Psi\|^2_{S^2_{u,v}}},$$

which yields the result after observing that the first integral is bounded and the second controlled by Proposition 11.5.1. $\square$

PROPOSITION 12.3.6. *Consider the solution* $\underline{\alpha}$ *of Theorem 2 and the derived quantity* $\underline{\psi}$ *defined via (180). Then, for* $(u,v)\times S^2_{u,v}$ *in* $\mathcal{M}\cap\{u\geqslant u_0\}\cap\{v\geqslant v_0\}\cap\{r\leqslant r_0\}$ *including the event horizon, we have*

$$s\|r^{-1}\cdot\Omega^{-1}r^3\underline{\psi}\|^2_{S^2_{u,v}} \lesssim \frac{1}{v^2}(\mathbb{F}^{2,T}_0[\underline{\Psi}]+\sup_u\|r^{-1}\cdot\Omega^{-1}r^3\underline{\psi}\|^2_{S^2_{u,v_0}}), \tag{325}$$

while for $(u,v)\times S^2_{u,v}$ *in* $\mathcal{M}\cap\{u\geqslant u_0\}\cap\{v\geqslant v_0\}\cap\{r\geqslant r_0\}$ *the above estimate holds replacing* $1/v^2$ *by* $1/u^2$ *on the right-hand side.*

Proof. For the region $r\geqslant r_0$ one integrates the estimate (290) and uses a dyadic argument together with the fact that the flux arising on the right-hand side of (290) satisfies a polynomial decay estimate from Proposition 11.5.1. For the region $r\geqslant r_0$ one notes that weights of Ω can be ignored and that $u\sim v$ on $r=r_0$. One then integrates as above using again the polynomial decay estimate from Proposition 11.5.1. □

Polynomial decay for $\underline{\alpha}$ and α on spheres $S^2_{u,v}$

Completely analogously to Propositions 12.3.5 and 12.3.6, one proves the following proposition, now starting from the defining equations for α and $\underline{\alpha}$, (178) and (180).

PROPOSITION 12.3.7. *Consider the solution* α *of Theorem 2 and the derived quantity* ψ *defined via (178). The following estimate holds for any* $u\geqslant u_0$ *and* $v\geqslant v_0$*, including the sphere* S^2_{∞,v_0}:

$$\|r^{-1}\cdot\Omega^2\alpha r^2\|^2_{S^2_{u,v}} \lesssim \frac{1}{v^2}\Big(\mathbb{F}^{2,T}_0[\Psi]+\sup_v\|r^{-1}\cdot\Omega^2\alpha r^3\|^2_{S^2_{u_0,v}}\Big). \tag{326}$$

Consider now the solution $\underline{\alpha}$ *of Theorem 2. For* $(u,v)\times S^2_{u,v}$ *in* $\mathcal{M}\cap\{u\geqslant u_0\}\cap\{v\geqslant v_0\}\cap\{r\leqslant r_0\}$ *including the event horizon, we have*

$$\|r^{-1}\cdot\Omega^{-2}r\underline{\alpha}\|^2_{S^2_{u,v}} \lesssim \frac{1}{v^2}\Big(\mathbb{F}^{2,T}_0[\underline{\Psi}]+\sup_u\|r^{-1}\cdot\Omega^{-2}r\underline{\alpha}\|^2_{S^2_{u,v_0}}\Big), \tag{327}$$

while for $(u,v)\times S^2_{u,v}$ *in* $\mathcal{M}\cap\{u\geqslant u_0\}\cap\{v\geqslant v_0\}\cap\{r\geqslant r_0\}$ *(327) holds replacing* $1/v^2$ *by* $1/u^2$ *on the right-hand side.*

Let us note that from the above estimates and additional commutations by Ω_i, one obtains trivially also *pointwise* decay estimates on α, with the corresponding commuted initial data norm on the right-hand side of the estimate.

12.3.3. Some auxiliary decay estimates

We collect here an auxiliary decay estimate for the derived quantity $\underline{\psi}$, which will be useful later when we consider the full system of gravitational perturbations.

Proposition 12.3.8. *Consider the solution $\underline{\alpha}$ of Theorem 2 and the derived quantity $\underline{\psi}$ defined via (180). Then, for any $V \geqslant v \geqslant v_0$, we have the flux bound*

$$\int_{u(v,r_0)}^{\infty} d\bar{u}\, \Omega^2 (\|r^{-1} \cdot \mathcal{A}^{[2]}(r^3 \underline{\psi} \Omega^{-1})\|^2_{S^2_{\bar{u},V}} + \|r^{-1} \cdot \nabla\!\!\!/_{R^\star}(r^3 \underline{\psi}\Omega^{-1})\|^2_{S^2_{\bar{u},V}}) \lesssim \frac{1}{v^2} < (\mathbb{F}_0^{2,T}[\Psi] + \mathbb{F}_0[\underline{\Psi}, \mathfrak{D}\underline{\psi}, \underline{\alpha}]).$$

In view of the relation (181) and Proposition 11.5.1, the estimate remains true replacing $\nabla\!\!\!/_{R^\star}$ by $\Omega \nabla\!\!\!/_3$.

Proof. We write the identity (303) as

$$2r^5 \mathcal{D}\!\!\!/_2^\star \, \mathrm{d}\!\!\!/\mathrm{iv}(\underline{\psi}\Omega^{-1}) + 2r^3 \underline{\psi}\Omega^{-1} = \Omega^{-1} \nabla\!\!\!/_3 (r^5 \underline{P}) + \frac{6M}{r} r^3 \underline{\psi}\Omega^{-1} - 3rM\underline{\alpha}. \tag{328}$$

Taking the $\|\cdot\|_{S^2_{u,v}}$ norm (using an integration by parts on the left) yields after integrating in u and applying the decay estimate of Propositions 11.5.1 and Corollary 12.5 for the terms on the right the desired estimate for the $\mathcal{A}^{[2]}$-flux of $\underline{\psi}$. To obtain the estimate for the $\nabla\!\!\!/_{R^\star}$-flux, we repeat the proof of Proposition 12.3.4 now starting from Proposition 12.2.4 to extract a dyadic sequence of good slices in $r \leqslant r_0$:

$$\int_{u_0}^{\infty} d\bar{u}\, \iota_{r\leqslant r_0} \Omega^2 \|r^{-1} \cdot \nabla\!\!\!/_{R^\star}(\underline{\psi} r^3 \Omega^{-1})\|^2_{S^2_{\bar{u},\tilde{v}_i}} \lesssim \frac{1}{\tilde{v}_i} \cdot \mathbb{F}_0[\underline{\Psi}, \mathfrak{D}\underline{\psi}, \underline{\alpha}]. \tag{329}$$

The decay will then be exported to any slice as in the proof of Proposition 12.3.4, now using the equation

$$\frac{1}{2}\partial_v [\Omega^2 |\nabla\!\!\!/_{R^\star}(\underline{\psi} r^3 \Omega^{-1})|^2] + \Omega^2 \frac{M}{r^2} |\nabla\!\!\!/_{R^\star}(\underline{\psi} r^3 \Omega^{-1})|^2 \tag{330}$$

$$= \Omega^2 \left(\frac{2M}{r^3} \Omega^2 (\underline{\psi} r^3 \Omega^{-1}) - \nabla\!\!\!/_{R^\star} \underline{P} r^3 - 3r^2 \Omega^2 \underline{P}, \nabla\!\!\!/_{R^\star}(\underline{\psi} r^3 \Omega^{-1}) \right), \tag{331}$$

which is the commuted version of (289)). Indeed, after applying the Cauchy–Schwarz inequality on the right-hand side, integrating and using the decay estimates of Propositions 12.3.4 and 12.3.8, we deduce in particular the desired flux statement (first with v^{-1} and after another iteration with v^{-2}, again completely analogous to the proof of Proposition 12.3.4. □

Corollary 12.6. *Consider the solution $\underline{\alpha}$ of Theorem 2 and the derived quantity $\underline{\psi}$ defined via (180). We have, for any $v \geqslant v_0$, the estimate*

$$\int_{u_0}^{\infty} du\, \Omega^2 (\|r^{-1} \cdot \mathcal{A}^{[2]}(r^3 \underline{\psi}\Omega^{-1})\|_{S^2_{u,v}} + \|r^{-1} \cdot \mathcal{A}^{[2]}(r\underline{\alpha}\Omega^{-2})\|_{S^2_{u,v}}) \lesssim \sqrt{\mathbb{F}_0[\underline{\Psi}, \mathfrak{D}\underline{\psi}, \underline{\alpha}]} + \sqrt{\mathbb{F}_0^{2,T}[\underline{\Psi}]}.$$

Proof. We first show the estimate for $\underline{\psi}$. We fix $v \geqslant v_0$. Note that it is sufficient to restrict the integral to $r \geqslant r_0$ as otherwise the bound follows directly from the Cauchy–Schwarz inequality and Proposition 12.3.8. We dyadically decompose the remaining u-interval, $u_{i+1}=2^i u_0$, and estimate

$$\begin{aligned}\int_{u_0}^{\infty} du\, \Omega^2 \iota_{r\geqslant r_0} \|r^{-1}\cdot \mathcal{A}^{[2]}(r^3\underline{\psi}\Omega^{-1})\|_{S^2_{u,v}} \\ &= \sum_i \int_{u_i}^{u_{i+1}} du\, \Omega^2 \|r^{-1}\cdot \mathcal{A}^{[2]}(r^3\underline{\psi}\Omega^{-1})\|_{S^2_{u,v}} \\ &\lesssim \sum_i \sqrt{\int_{u_i}^{u_{i+1}} du\, \Omega^2 \|r^{-1}\cdot \mathcal{A}^{[2]}(r^3\underline{\psi}\Omega^{-1})\|^2_{S^2_{u,v}}}\sqrt{u_i}.\end{aligned}$$

Using that the integral under the square root decays like u_i^{-2} by Proposition 12.3.8, we deduce the result.

Obviously this argument can be repeated replacing $\mathcal{A}^{[2]}$ by $\Omega \nabla\!\!\!/_3$ using again the estimate Proposition 12.3.8. The L^1-estimate for $\mathcal{A}^{[2]}\underline{\alpha}$ then follows directly from the identity (302) and the previous bounds. $\square$

13. Proof of Theorem 3

In this section we shall prove Theorem 3. The reader can refer to the overview of §2.4.3.

The first ingredient in the proof will be the bounds on gauge invariant quantities which follow from applying Theorem 2 to the curvature components $\overset{(1)}{\alpha}$ and $\overset{(1)}{\underline{\alpha}}$, respectively, of our solution to the system of linearised gravity. We shall collect these gauge invariant estimates in §13.1. We shall then obtain estimates for certain fluxes on the horizon in §13.2. These, together with a red-shift commutation argument, will be used in §13.3 to obtain decay estimates for the outgoing shear $\overset{(1)}{\hat{\chi}}$. We then obtain boundedness estimates for the ingoing shear $\overset{(1)}{\underline{\hat{\chi}}}$ in §13.4, and finally boundedness for all remaining quantities in §13.5.

13.1. Gauge-invariant estimates from Theorem 2

Let

$$\mathscr{S} = \Big(\overset{(1)}{\hat{\slashed{g}}}, \overset{(1)}{\sqrt{\slashed{g}}}, \overset{(1)}{\Omega}, \overset{(1)}{b}, \overset{(1)}{(\Omega \operatorname{tr}\chi)}, \overset{(1)}{(\Omega \operatorname{tr}\underline{\chi})}, \overset{(1)}{\hat{\chi}}, \overset{(1)}{\underline{\hat{\chi}}}, \overset{(1)}{\underline{\eta}}, \overset{(1)}{\eta}, \overset{(1)}{\omega}, \overset{(1)}{\underline{\omega}}, \overset{(1)}{\alpha}, \overset{(1)}{\beta}, \overset{(1)}{\varrho}, \overset{(1)}{\sigma}, \overset{(1)}{\underline{\beta}}, \overset{(1)}{\underline{\alpha}}, \overset{(1)}{K}\Big)$$

be as in the statement of Theorem 3.

The linearised curvature components $\overset{(1)}{\alpha}$ and $\underline{\overset{(1)}{\alpha}}$ satisfy the spin ± 2 Teukolsky equations, respectively, by Proposition 7.4.1 in §7.4. Thus, Theorem 2 applies to yield Corollary 10.1. Note also that the assumption (246) of Theorem 3 implies that the initial norms of Theorem 2, applied to $\overset{(1)}{\alpha}$ and $\underline{\overset{(1)}{\alpha}}$, respectively, are indeed finite for $n=2$.

From Proposition 7.4.1, we see also that the gauge-invariant estimates on $\overset{(1)}{\psi}$, $\underline{\overset{(1)}{\psi}}$, $\overset{(1)}{P}$, and $\underline{\overset{(1)}{P}}$ arising from Theorem 2 can immediately be re-interpreted as estimates for the right-hand sides of (182)–(184). These estimates will also be useful in what follows.

13.2. Fluxes on the horizon $\mathcal{H}^+$

In this section we exploit the horizon gauge conditions (193) and (194) to control additional (not necessarily gauge invariant) fluxes *in terms of the gauge invariant quantities* $\overset{(1)}{\Psi}$, $\overset{(1)}{\psi}$ *and* $\overset{(1)}{\alpha}$.

In §13.2.1 we obtain bounds for the linearised outgoing shear $\overset{(1)}{\hat{\chi}}$ itself, derived from the fact that $\overset{(1)}{\psi}$ is controlled on the horizon. In §13.2.2 we obtain similar bounds for the transversal derivative $\Omega^{-1}\slashed{\nabla}_3(\Omega\overset{(1)}{\hat{\chi}})$, derived from the fact that $\overset{(1)}{\Psi}$ is controlled on the horizon. Higher-derivative fluxes are obtained in §13.2.3 from the fact that the energy of $\overset{(1)}{\Psi}$ (cf. Theorem 1) controls also $\slashed{\nabla}\overset{(1)}{\Psi}$ and $\slashed{\nabla}_4\overset{(1)}{\Psi}$ on the horizon. Finally, polynomial decay bounds for $\overset{(1)}{\hat{\chi}}$ and its transversal derivatives (which are inherited from polynomial decay bounds of $\overset{(1)}{\psi}$ and $\overset{(1)}{\Psi}$) are stated in §13.2.4.

13.2.1. Obtaining the $\overset{(1)}{\hat{\chi}}$-flux on $\mathcal{H}^+$

PROPOSITION 13.2.1. *The geometric quantities associated with $\underset{\vee}{\mathscr{S}}$ satisfy on any sphere on the horizon*

$$\int_{S^2_{\infty,v}} \sin\theta\, d\theta\, d\phi \left[|\slashed{\mathcal{D}}_2^\star \overset{(1)}{\beta}\Omega|^2 + \frac{9}{4}\varrho^2 |\Omega\overset{(1)}{\hat{\chi}}|^2 + \frac{6M}{r^3}|\Omega\overset{(1)}{\beta}|^2 + |\slashed{\mathcal{D}}_2^\star \,\slashed{\mathrm{div}}\, \Omega\overset{(1)}{\hat{\chi}}|^2 \right]$$
$$\lesssim \sup_v \|r^{-1/2}\cdot\overset{(1)}{\psi}\Omega r^3\|^2_{S^2_{u_0,v}} + \mathbb{F}_0[\overset{(1)}{\Psi}].$$

We also control the horizon flux

$$\int_{\mathcal{H}(v_0,\infty)} dv\, \sin\theta\, d\theta\, d\phi \left[|\slashed{\mathcal{D}}_2^\star \overset{(1)}{\beta}\Omega|^2 + \frac{9}{4}\varrho^2 |\overset{(1)}{\hat{\chi}}|^2 + \frac{6M}{r^3}|\overset{(1)}{\beta}\Omega|^2 + |\slashed{\mathcal{D}}_2^\star \,\slashed{\mathrm{div}}\, \overset{(1)}{\hat{\chi}}\Omega|^2 \right] \lesssim \mathbb{F}_0[\overset{(1)}{\Psi}, \overset{(1)}{\psi}].$$

Proof. The bounds follow from Proposition 12.1.1 and Corollary 12.2 with $n=1$. We compute from (182)

$$|\overset{(1)}{\psi}|^2 = |\slashed{\mathcal{D}}_2^\star \overset{(1)}{\beta}|^2 + 3\varrho\overset{(1)}{\hat{\chi}}\cdot\slashed{\mathcal{D}}_2^\star\overset{(1)}{\beta} + \frac{9}{4}\varrho^2|\overset{(1)}{\hat{\chi}}|^2,$$

which, since the adjoint of $\not{\mathcal{D}}_2^\star$ is $\operatorname{d\!\!/iv}$, yields

$$\int \sin\theta\, d\theta\, d\phi\, |\overset{(1)}{\psi}|^2 = \int \sin\theta\, d\theta\, d\phi \left(|\not{\mathcal{D}}_2^\star \overset{(1)}{\beta}|^2 - \frac{6M}{r^3} \operatorname{d\!\!/iv} \overset{(1)}{\widehat{\chi}} \cdot \overset{(1)}{\beta} + \frac{9}{4} \varrho^2 |\overset{(1)}{\widehat{\chi}}|^2 \right). \tag{332}$$

Multiplying by Ω^2 (to obtain expressions regular at the horizon $\mathcal{H}^+$) and inserting the Codazzi equation (145) restricted to the horizon ($\operatorname{d\!\!/iv} \overset{(1)}{\widehat{\chi}} \Omega = -\overset{(1)}{\beta}\Omega$; here, we use that $(\Omega \overset{(1)}{\operatorname{tr}\chi}) = 0$ on $\mathcal{H}^+$ for $\underset{\vee}{\mathscr{S}}$; see Proposition 9.4.1) we obtain the first bound from Corollary 12.2 and the second from (282). □

Remark 13.1. Note that the above provides a bound on $\Omega\overset{(1)}{\beta}$ itself. The underlying reason is that, by Theorem 9.2, $\overset{(1)}{\beta}_{\ell=1}$ actually vanishes on the horizon, since it vanishes there for any reference Kerr solution $\mathscr{K}$.

Remark 13.2. By an elliptic estimate (cf. (111)), one obtains Proposition 13.2.1 for all angular derivatives up to order 2 of $\Omega\overset{(1)}{\widehat{\chi}}$.

13.2.2. Obtaining the $(1/\Omega)\not\nabla_3(\Omega\overset{(1)}{\widehat{\chi}})$-flux on $\mathcal{H}^+$

Proposition 13.2.2. *The geometric quantities associated with $\underset{\vee}{\mathscr{S}}$ satisfy on any sphere on the horizon*

$$\int_{S^2_{\infty,v}} \sin\theta\, d\theta\, d\phi \times \left[|\not{\mathcal{D}}_2^\star \overset{(1)}{\eta}|^2 + |\operatorname{d\!\!/iv} \not{\mathcal{D}}_2^\star \overset{(1)}{\eta}|^2 + |\not{\mathcal{D}}_2^\star \operatorname{d\!\!/iv} \not{\mathcal{D}}_2^\star \overset{(1)}{\eta}|^2 + \left| \frac{1}{\Omega} \not\nabla_3 (\Omega \overset{(1)}{\widehat{\chi}}) \right|^2 + \left| \not{\mathcal{D}}_2^\star \operatorname{d\!\!/iv} \frac{1}{\Omega} \not\nabla_3 (\Omega \overset{(1)}{\widehat{\chi}}) \right|^2 \right] \lesssim \sup_v \| r^{-1/2} \cdot \overset{(1)}{\psi} \Omega r^3 \|^2_{S^2_{u_0,v}} + \mathbb{F}_0[\overset{(1)}{\Psi}].$$

We also control the horizon flux

$$\int_{\mathcal{H}(v_0,\infty)} dv \sin\theta\, d\theta\, d\phi \times \left[|\not{\mathcal{D}}_2^\star \overset{(1)}{\eta}|^2 + |\operatorname{d\!\!/iv} \not{\mathcal{D}}_2^\star \overset{(1)}{\eta}|^2 + |\not{\mathcal{D}}_2^\star \operatorname{d\!\!/iv} \not{\mathcal{D}}_2^\star \overset{(1)}{\eta}|^2 + \left| \frac{1}{\Omega} \not\nabla_3 (\Omega \overset{(1)}{\widehat{\chi}}) \right|^2 + \left| \not{\mathcal{D}}_2^\star \operatorname{d\!\!/iv} \frac{1}{\Omega} \not\nabla_3 (\Omega \overset{(1)}{\widehat{\chi}}) \right|^2 \right] \lesssim \mathbb{F}_0[\overset{(1)}{\Psi}, \overset{(1)}{\psi}].$$

Remark 13.3. Note that the above implies that we control $\overset{(1)}{\eta}$ up to its $\ell=1$ modes; cf. Corollary 4.2.

Proof. By our gauge conditions on $\underset{\vee}{\mathscr{S}}$, we have that $\operatorname{d\!/\!iv} \overset{(1)}{\eta} = -\overset{(1)}{\varrho} + \overset{(1)}{\varrho}_{\ell=0}$ holds on the horizon (cf. Proposition 9.4.1). This implies that

$$\|\not{\mathcal{D}}_2^\star \not{\mathcal{D}}_1^\star (\operatorname{d\!/\!iv} \overset{(1)}{\eta}, \operatorname{c\!/\!url} \overset{(1)}{\eta})\|^2_{S^2_{\infty,v}} = \|\not{\mathcal{D}}_2^\star \not{\mathcal{D}}_1^\star (-\overset{(1)}{\varrho}, \overset{(1)}{\sigma})\|^2_{S^2_{\infty,v}} \lesssim \|\overset{(1)}{P}\|^2_{S^2_{\infty,v}} + \|\Omega \overset{(1)}{\widehat{\chi}}\|^2_{S^2_{\infty,v}}, \tag{333}$$

by Proposition 7.4.1. Note that the term proportional to $\Omega^{-1}\overset{(1)}{\underline{\widehat{\chi}}}$ in the definition of P vanishes on the horizon. Using Proposition 13.2.1 as well as Theorem 1 on the right-hand side and Lemma 4.4.3 on the left, we derive the desired estimates except for the $\not\nabla_3(\Omega\widehat{\chi})$-terms. To obtain the latter, note that equation (140), when restricted to the horizon, yields

$$\frac{1}{\Omega}\not\nabla_3(\Omega\overset{(1)}{\widehat{\chi}}) = -2\not{\mathcal{D}}_2^\star \overset{(1)}{\eta} + \frac{1}{2M}\Omega\overset{(1)}{\widehat{\chi}}, \tag{334}$$

hence control on $\overset{(1)}{\widehat{\chi}}$ (again Proposition 13.2.1) and the $\not{\mathcal{D}}_2^\star \overset{(1)}{\eta}$-fluxes implies control on the normal derivative. □

13.2.3. Obtaining higher-order fluxes on $\mathcal{H}^+$

We collect some additional flux estimates on the horizon that follow directly from the horizon fluxes of $\overset{(1)}{P}$.

PROPOSITION 13.2.3. *The geometric quantities associated with $\underset{\vee}{\mathscr{S}}$ satisfy the following estimates on the horizon for any $v \geqslant v_0$:*

$$\sum_{i=0}^{3} \int_{v_0}^{v} d\bar{v}\, \|\not\nabla^i (\Omega\overset{(1)}{\beta})\|^2_{S^2_{\infty,\bar{v}}} + \sum_{i=0}^{4} \int_{v_0}^{v} d\bar{v}\, \|\not\nabla^i (\Omega\overset{(1)}{\widehat{\chi}})\|^2_{S^2_{\infty,\bar{v}}} \lesssim \mathbb{F}_0[\overset{(1)}{\Psi}, \overset{(1)}{\psi}, \overset{(1)}{\alpha}].$$

Moreover,

$$\int_{v_0}^{v} d\bar{v}\, \|\operatorname{d\!/\!iv} \not{\mathcal{D}}_2^\star \not{\mathcal{D}}_1^\star (\overset{(1)}{\varrho}, \overset{(1)}{\sigma})\|^2_{S^2_{\infty,\bar{v}}} \lesssim \mathbb{F}_0[\overset{(1)}{\Psi}, \overset{(1)}{\psi}], \tag{335}$$

as well as

$$\int_{v_0}^{v} d\bar{v}\, \|\mathcal{A}^{[3]} \not{\mathcal{D}}_2^\star \overset{(1)}{\eta}\|^2_{S^2_{\infty,\bar{v}}} \lesssim \mathbb{F}_0[\overset{(1)}{\Psi}, \overset{(1)}{\psi}]. \tag{336}$$

Proof. Note that we control, for any $v \geqslant v_0$, the horizon flux

$$\int_{v_0}^{v} d\bar{v}\, (\|\Omega\not\nabla_4 \overset{(1)}{P}\|^2_{S^2_{\infty,\bar{v}}} + \|\not\nabla \overset{(1)}{P}\|^2_{S^2_{\infty,\bar{v}}} + \|\overset{(1)}{P}\|^2_{S^2_{\infty,\bar{v}}}) \lesssim \mathbb{F}_0[\overset{(1)}{\Psi}] \tag{337}$$

from Theorem 1. Now *restricted to the horizon* we have

$$\Omega\not\nabla_4 \overset{(1)}{P} = \not{\mathcal{D}}_2^\star \not{\mathcal{D}}_1^\star (-\operatorname{d\!/\!iv}\, \Omega\overset{(1)}{\beta}, -\Omega \operatorname{c\!/\!url} \overset{(1)}{\beta}) + \frac{3M}{(2M)^4}\Omega^2 \overset{(1)}{\alpha} - \frac{6M}{(2M)^6}\Omega\overset{(1)}{\widehat{\chi}},$$

which, in view of Propositions 13.2.1 and 12.1.2, yields the estimate for $\overset{(1)}{\beta}$ after using elliptic estimates on S^2 and the fact that the order-zero term of $\overset{(1)}{\beta}$ is controlled from Proposition 13.2.1. The estimate involving $\overset{(1)}{\widehat{\chi}}$ is then an immediate consequence of the $\mathrm{d}\!\!/\mathrm{iv}\,\overset{(1)}{\widehat{\chi}}\Omega = -\overset{(1)}{\beta}\Omega$ holding on $\mathcal{H}^+$. For (335), note that, restricted to the horizon,

$$\mathrm{d}\!\!/\mathrm{iv}\,\overset{(1)}{P} = \mathrm{d}\!\!/\mathrm{iv}\,\mathcal{D}\!\!\!/_2^\star \mathcal{D}\!\!\!/_1^\star(-\overset{(1)}{\varrho}, \overset{(1)}{\sigma}) - \frac{3M}{(2M)^4}\,\mathrm{d}\!\!/\mathrm{iv}\,\Omega\overset{(1)}{\widehat{\chi}},$$

and use (337) and Proposition 13.2.1. Finally, for (336), recall the identity (333). Using the identity (112) and the previous bound (335) and Proposition 13.2.2, the desired estimate follows. □

13.2.4. Obtaining polynomial decay bounds on $\mathcal{H}^+$

Using the decay statements for the horizon fluxes in Proposition 11.5.1 and Corollary 12.4, we can obtain the following result.

Proposition 13.2.4. *We have the following flux bounds along the event horizon $\mathcal{H}^+$:*

$$\begin{aligned}\int_v^\infty d\bar{v}\Big[\|\mathcal{A}^{[4]}\overset{(1)}{\widehat{\chi}}\Omega\|^2_{S^2_{\infty,\bar{v}}} + \|\mathcal{A}^{[3]}\mathcal{D}\!\!\!/_2^\star\overset{(1)}{\eta}\|^2_{S^2_{\infty,\bar{v}}} + \|\mathcal{A}^{[3]}\Omega^{-1}\nabla\!\!\!/_3(\Omega\overset{(1)}{\widehat{\chi}})\|_{S^2_{\infty,\bar{v}}}\Big]\\ \lesssim \frac{1}{v^2}(\mathbb{F}_0^{2,T}[\overset{(1)}{\Psi}] + \mathbb{F}_0[\overset{(1)}{\Psi}, \overset{(1)}{\psi}, \overset{(1)}{\alpha}]).\end{aligned}$$

Proof. From Corollary 12.4, we have decay of the fluxes for $\overset{(1)}{\psi}$ and $\overset{(1)}{\alpha}$. Therefore, the identity (332) restricted to the horizon $\mathcal{H}^+$ (and using $\mathrm{d}\!\!/\mathrm{iv}\,\overset{(1)}{\widehat{\chi}}\Omega = -\overset{(1)}{\beta}\Omega$) already produces the desired estimate for $\mathcal{A}^{[2]}\overset{(1)}{\widehat{\chi}}\Omega$ instead of $\mathcal{A}^{[4]}\overset{(1)}{\widehat{\chi}}\Omega$. Repeating the proof of Proposition 13.2.2 now using the decay estimate on $\overset{(1)}{\widehat{\chi}}$ just obtained and Proposition 11.5.1 produces a decay estimate for the flux of three derivatives of $\overset{(1)}{\eta}$. Repeating the proof of Proposition 13.2.3 provides the statement for $\mathcal{A}^{[4]}\overset{(1)}{\widehat{\chi}}$ and four derivatives of $\overset{(1)}{\eta}$. The remaining estimate finally follows from the $\mathcal{A}^{[3]}$-commuted (334). □

Proposition 13.2.5. *We have the following bounds along the event horizon $\mathcal{H}^+$:*

$$\begin{aligned}\|\mathcal{A}^{[2]}\overset{(1)}{\widehat{\chi}}\Omega\|^2_{S^2_{\infty,v}} + \|\mathcal{A}^{[2]}\mathcal{D}\!\!\!/_2^\star\overset{(1)}{\eta}\|^2_{S^2_{\infty,v}} + \|\mathcal{A}^{[2]}\Omega^{-1}\nabla\!\!\!/_3(\Omega\overset{(1)}{\widehat{\chi}})\|_{S^2_{\infty,v}}\\ \lesssim \frac{1}{v^2}(\mathbb{F}_0^{2,T}[\overset{(1)}{\Psi}] + \mathbb{F}_0[\overset{(1)}{\Psi}, \overset{(1)}{\psi}, \overset{(1)}{\alpha}] + \sup_v \|r^{-1}\cdot\overset{(1)}{\psi}\Omega r^4\|^2_{S^2_{u_0,v}}).\end{aligned} \tag{338}$$

Proof. Use Proposition 12.3.5 and the identity (332) restricted to the horizon to obtain the estimate on $\mathcal{A}^{[2]}\overset{(1)}{\widehat{\chi}}$. For the estimate on $\overset{(1)}{\eta}$ we use the identity (333) in conjunction with the estimate on $\mathcal{A}^{[2]}\overset{(1)}{\widehat{\chi}}$ just obtained and (337) plus 1-dimensional Sobolev embedding. The remaining estimate now follows directly from (334). □

13.3. Decay estimates for the outgoing shear $\overset{(1)}{\widehat{\chi}}$

The main result of this section is the following.

PROPOSITION 13.3.1. *Consider the solution $\overset{\vee}{\mathscr{S}}$ of Theorem 3. The following estimate holds for any $\varepsilon>0$, and any $u\geqslant u_0$ and $v\geqslant v_0$:*

$$\int_{v_0}^{v} d\bar{v}\int_{u_0}^{u} d\bar{u}\int_{S^2_{\bar{u},\bar{v}}} \sin\theta\, d\theta\, d\phi\, \frac{\Omega^2}{r^{1+\varepsilon}}\left(\left|\frac{1}{\Omega}\nabla\!\!\!/_3\left(\frac{1}{\Omega}\nabla\!\!\!/_3(r^2\overset{(1)}{\widehat{\chi}}\Omega)\right)\right|^2+\left|\frac{1}{\Omega}\nabla\!\!\!/_3(r^2\overset{(1)}{\widehat{\chi}}\Omega)\right|^2+|r^2\overset{(1)}{\widehat{\chi}}\Omega|^2\right)$$
$$\lesssim \|(\nabla\!\!\!/_3)^2\overset{(1)}{\widehat{\chi}}\|^2_{L^2(C_{v_0})}+\mathbb{F}_0[\overset{(1)}{\Psi},\overset{(1)}{\psi},\overset{(1)}{\alpha}].$$

We also control for any $v\geqslant v_0$ the flux

$$\int_{u_0}^{\infty} d\bar{u}\int_{S^2_{\bar{u},v}} \sin\theta\, d\theta\, d\phi\, \Omega^2\left(\left|\frac{1}{\Omega}\nabla\!\!\!/_3\left(\frac{1}{\Omega}\nabla\!\!\!/_3(r^2\overset{(1)}{\widehat{\chi}}\Omega)\right)\right|^2+\left|\frac{1}{\Omega}\nabla\!\!\!/_3(r^2\overset{(1)}{\widehat{\chi}}\Omega)\right|^2+\frac{1}{r^{\varepsilon}}\left|r^2\overset{(1)}{\widehat{\chi}}\Omega\right|^2\right)$$
$$\lesssim \|(\nabla\!\!\!/_3)^2\overset{(1)}{\widehat{\chi}}\|^2_{L^2(C_{v_0})}+\mathbb{F}_0[\overset{(1)}{\Psi},\overset{(1)}{\psi},\overset{(1)}{\alpha}]. \tag{339}$$

We have already discussed in §2.4.3 the main difficulty of estimating $\overset{(1)}{\widehat{\chi}}$ directly: In equation (139), there is a blue-shift factor. As explained in §2.4.3, this difficulty can be corrected by commuting twice by the "red-shift" operator $(1/\Omega)\nabla\!\!\!/_3$ (exploiting the improvement discussed in §2.3.1 in the context of the scalar wave equation), coupled with our a-priori bound on the flux of the transversal derivative, $(1/\Omega)\nabla\!\!\!/_3(\Omega\overset{(1)}{\widehat{\chi}})$, on the horizon–now established in Proposition 13.2.2.

We give a brief outline of this section. In §13.3.1, we will derive the commutation formulas with the operator $(1/\Omega)\nabla\!\!\!/_3$. The proof of Proposition 13.3.1 will then be carried out in §13.3.2 (which will obtain bounds near the horizon) and §13.3.3 (which will extend the bounds globally). Some higher-order estimates which follow from commuting and repeating the proof of Proposition 13.3.1 will be stated in §13.3.4.

13.3.1. Commuting the $\nabla\!\!\!/_4\overset{(1)}{\widehat{\chi}}$-equation

We write the transport equation for $\overset{(1)}{\widehat{\chi}}$, equation (139), as

$$\Omega\nabla\!\!\!/_4(r^2\overset{(1)}{\widehat{\chi}}\Omega)-2\Omega\widehat{\omega}(r^2\overset{(1)}{\widehat{\chi}}\Omega)=-\overset{(1)}{\alpha}\Omega^2r^2. \tag{340}$$

Note that $r^2\overset{(1)}{\widehat{\chi}}\Omega$ is regular both at the horizon and null infinity and that the second term in the above is a blue–shift term, i.e. its sign is negative. We easily deduce from §4.3.2 the commutation formulas

$$[\nabla\!\!\!/_3,\Omega\nabla\!\!\!/_4]=\widehat{\omega}\Omega\nabla\!\!\!/_3 \quad \text{as well as} \quad \left[\frac{1}{\Omega}\nabla\!\!\!/_3,\Omega\nabla\!\!\!/_4\right]=2\widehat{\omega}\Omega\cdot\frac{1}{\Omega}\nabla\!\!\!/_3.$$

Hence, commuting (340) with $(1/\Omega)\nabla\!\!\!/_3$ removes the blue-shift term in (340) and we obtain

$$\Omega\nabla\!\!\!/_4\left(\frac{1}{\Omega}\nabla\!\!\!/_3(r^2\overset{(1)}{\widehat{\chi}}\Omega)\right)-\frac{1}{\Omega}\nabla\!\!\!/_3(2\Omega\widehat{\omega})(r^2\overset{(1)}{\widehat{\chi}}\Omega)=-\frac{1}{\Omega}\nabla\!\!\!/_3(\overset{(1)}{\alpha}\Omega^2r^2),$$

which simplifies to

$$\Omega\nabla\!\!\!/_4\left(\frac{1}{\Omega}\nabla\!\!\!/_3(r^2\overset{(1)}{\widehat{\chi}}\Omega)\right)-\frac{4M}{r^3}(r^2\overset{(1)}{\widehat{\chi}}\Omega)=2\frac{1}{r}\overset{(1)}{\psi}r^3\Omega+\overset{(1)}{\alpha}r\Omega^2. \tag{341}$$

Let us commute again with $\nabla\!\!\!/_3$ to obtain

$$\begin{aligned}\Omega\nabla\!\!\!/_4\left(\nabla\!\!\!/_3\left(\frac{1}{\Omega}\nabla\!\!\!/_3(r^2\overset{(1)}{\widehat{\chi}}\Omega)\right)\right)+\widehat{\omega}\Omega\left(\nabla\!\!\!/_3\left(\frac{1}{\Omega}\nabla\!\!\!/_3(r^2\overset{(1)}{\widehat{\chi}}\Omega)\right)\right)-\frac{4M}{r^3}\nabla\!\!\!/_3(r^2\overset{(1)}{\widehat{\chi}}\Omega)-\frac{12M}{r^4}\Omega(r^2\overset{(1)}{\widehat{\chi}}\Omega)\\ =-\nabla\!\!\!/_3\left(\frac{1}{\Omega}\nabla\!\!\!/_3(\overset{(1)}{\alpha}\Omega^2r^2)\right)=2r^2\Omega\overset{(1)}{P}.\end{aligned} \tag{342}$$

To derive the above, we have used the identities

$$-\frac{1}{\Omega}\nabla\!\!\!/_3(\overset{(1)}{\alpha}\Omega^2r\cdot r)=-\frac{r}{\Omega}\nabla\!\!\!/_3(\overset{(1)}{\alpha}r\Omega^2)+\overset{(1)}{\alpha}r\Omega^2=2\frac{1}{r}\overset{(1)}{\psi}r^3\Omega+\overset{(1)}{\alpha}r\Omega^2, \tag{343}$$

and

$$-\nabla\!\!\!/_3\left(\frac{1}{\Omega}\nabla\!\!\!/_3(\overset{(1)}{\alpha}\Omega^2r^2)\right)=2\overset{(1)}{\psi}\Omega^2r+2r^2\Omega\overset{(1)}{P}-2\overset{(1)}{\psi}r\Omega^2=2r^2\Omega\overset{(1)}{P}.$$

13.3.2. The main estimate near the horizon

We shall first prove an unconditional estimate for $\overset{(1)}{\widehat{\chi}}$ in a region $r\leqslant r_1$ for some $r_1>2M$ close to the horizon, which one may think of as Proposition 13.3.1 restricted to a region near the horizon. Refer to the diagram in §2.4.3.

PROPOSITION 13.3.2. *Consider the solution $\overset{\vee}{\mathscr{S}}$ of Theorem 3. There exists an r_1, with $5/2M>r_1>2M$, such that the following estimate holds for any $v\geqslant v_0$:*

$$\begin{aligned}\int_{v_0}^{v}d\bar v\int_{u(r_1,\bar v)}^{\infty}d\bar u\int_{S^2_{\bar u,\bar v}}\sin\theta\,d\theta\,d\phi\,\Omega^2\left(\left|\frac{1}{\Omega}\nabla\!\!\!/_3\left(\frac{1}{\Omega}\nabla\!\!\!/_3(r^2\overset{(1)}{\widehat{\chi}}\Omega)\right)\right|^2+\left|\frac{1}{\Omega}\nabla\!\!\!/_3(r^2\overset{(1)}{\widehat{\chi}}\Omega)\right|^2+|r^2\overset{(1)}{\widehat{\chi}}\Omega|^2\right)\\ \lesssim\mathbb{F}_0[\overset{(1)}{\Psi},\overset{(1)}{\psi}]+\frac{1}{2}\int_{u(r_1,v_0)}^{\infty}d\bar u\int_{S^2_{\bar u,v_0}}\sin\theta\,d\theta\,d\phi\,\Omega^2\left|\frac{1}{\Omega}\nabla\!\!\!/_3(\frac{1}{\Omega}\nabla\!\!\!/_3(r^2\overset{(1)}{\widehat{\chi}}\Omega))\right|^2.\end{aligned} \tag{344}$$

Here $u(r,v)$ denotes the u-value of the intersection of the hypersurfaces of constant v and those of constant r. Moreover, the same estimate holds replacing $\int_{v_0}^{v}d\bar v$ by $\sup_{v\in(v_0,\infty)}$.

Proof. We consider the region $r\leqslant r_1$ and $v\geqslant v_0$, for some $r_1>2M$ close to $2M$ chosen below. We let $u(r_1,v)$ denote the u-value where the hypersurface of constant v intersects $r=r_1$ and similarly for $v(u,r_1)$.

The following lemma expresses the fact that, in the region $r\leqslant r_1$, we control the spacetime integral of $|\Omega r^2 \overset{(1)}{\hat{\chi}}|^2$ in a neighborhood of the horizon ("a small region in physical space") by an ε times the horizon flux and the spacetime integral of the transversal $\Omega^{-1}\nabla\!\!\!\!/_3$-derivative.

LEMMA 13.3.1. *Let $\overset{(1)}{\hat{\chi}}$ be a symmetric traceless $S^2_{u,v}$-tensor. The following estimate holds in $\mathcal{M}\cap\{r\leqslant r_1\}$:*

$$\begin{aligned}
&\int_{v_0}^{v} d\bar v \int_{u(r_1,\bar v)}^{\infty} d\bar u \int_{S^2_{\bar u,\bar v}} \sin\theta\, d\theta\, d\phi\ \Omega^2 |\overset{(1)}{\hat{\chi}}\Omega r^2|^2(\bar u,\bar v,\theta,\phi) \\
&\quad\leqslant 2|r_1-2M| \int_{v_0}^{v} d\bar v \int_{S^2_{\infty,\bar v}} \sin\theta\, d\theta\, d\phi\, |\overset{(1)}{\hat{\chi}}\Omega r^2|^2(\infty,\bar v,\theta,\phi) \\
&\qquad + 4|r_1-2M|^2 \int_{v_0}^{v} d\bar v \int_{u(r_1,\bar v)}^{\infty} d\bar u \int_{S^2_{\bar u,\bar v}} \sin\theta\, d\theta\, d\phi\, \Omega^2 \left|\frac{1}{\Omega}\nabla\!\!\!\!/_3(\overset{(1)}{\hat{\chi}}\Omega r^2)\right|^2(\bar u,\bar v,\theta,\phi).
\end{aligned}\tag{345}$$

Proof. By the fundamental theorem of calculus, for any $v\geqslant v_0$ we have

$$\begin{aligned}
&\int_{S^2_{u,v}} |\overset{(1)}{\hat{\chi}}\Omega r^2|^2(u,v,\theta,\phi)\sin\theta\, d\theta\, d\phi \\
&\quad = \int_{S^2_{\infty,v}} |\overset{(1)}{\hat{\chi}}\Omega r^2|^2(\infty,v,\theta,\phi)\sin\theta\, d\theta\, d\phi \\
&\qquad - \int_{u}^{\infty} d\bar u \int_{S^2_{\bar u,v}} \Omega\nabla\!\!\!\!/_3(|\overset{(1)}{\hat{\chi}}\Omega r^2|^2)(\bar u,v,\theta,\phi)\sin\theta\, d\theta\, d\phi.
\end{aligned}$$

We can estimate the last term (LT) for any $\lambda>0$ by

$$\begin{aligned}
|LT| &\leqslant \frac{1}{\lambda}\int_{u}^{\infty} d\bar u \int_{S^2_{\bar u,v}} \Omega^2 |\overset{(1)}{\hat{\chi}}\Omega r^2|^2(\bar u,v,\theta,\phi)\sin\theta\, d\theta\, d\phi \\
&\quad + \lambda \int_{u}^{\infty} d\bar u \int_{S^2_{\bar u,v}} \Omega^2 \left|\frac{1}{\Omega}\nabla\!\!\!\!/_3(\overset{(1)}{\hat{\chi}}\Omega r^2)\right|^2(\bar u,v,\theta,\phi)\sin\theta\, d\theta\, d\phi \\
&\leqslant \frac{|r_1-2M|}{\lambda} \sup_{\bar u\in(u(r_1,v),\infty)} \int_{S^2_{\bar u,v}} |\overset{(1)}{\hat{\chi}}\Omega r^2|^2(\bar u,v,\theta,\phi) \\
&\quad + \lambda \int_{u}^{\infty} d\bar u \int_{S^2_{\bar u,v}} \Omega^2 \left|\frac{1}{\Omega}\nabla\!\!\!\!/_3(\overset{(1)}{\hat{\chi}}\Omega r^2)\right|^2(\bar u,v,\theta,\phi)\sin\theta\, d\theta\, d\phi.
\end{aligned}$$

Choosing $\lambda=2|r_1-2M|$ yields, for any $u(r_1,v)\leqslant u\leqslant\infty$,

$$\begin{aligned}\int_{S^2_{u,v}} &|\overset{(1)}{\widehat{\chi}}\Omega r^2|^2(u,v,\theta,\phi)\sin\theta\,d\theta\,d\phi\\ &\leqslant 2\int_{S^2_{\infty,v}}|\overset{(1)}{\widehat{\chi}}\Omega r^2|^2(\infty,v,\theta,\phi)\sin\theta\,d\theta\,d\phi\\ &\qquad+4|r_1-2M|\int_u^\infty d\bar{u}\int_{S^2_{\bar{u},v}}\Omega^2\left|\frac{1}{\Omega}\nabla\!\!\!/_3(\overset{(1)}{\widehat{\chi}}\Omega r^2)\right|^2(\bar{u},v,\theta,\phi)\sin\theta\,d\theta\,d\phi.\end{aligned}$$

Multiplying this by $\Omega^2=-r_u$ and integrating in u from the horizon, we deduce also

$$\begin{aligned}\int_{u(r_1,v)}^\infty &d\bar{u}\,\Omega^2\int_{S^2_{\bar{u},v}}|\overset{(1)}{\widehat{\chi}}\Omega r^2|^2(\bar{u},v,\theta,\phi)\sin\theta\,d\theta\,d\phi\\ &\leqslant 2|r_1-2M|\int_{S^2_{\infty,v}}|\overset{(1)}{\widehat{\chi}}\Omega r^2|^2(\infty,v,\theta,\phi)\sin\theta\,d\theta\,d\phi\\ &\qquad+4|r_1-2M|^2\int_{u(r_1,v)}^\infty d\bar{u}\int_{S^2_{\bar{u},v}}\Omega^2\left|\frac{1}{\Omega}\nabla\!\!\!/_3(\overset{(1)}{\widehat{\chi}}\Omega r^2)\right|^2(\bar{u},v,\theta,\phi)\sin\theta\,d\theta\,d\phi,\end{aligned}\tag{346}$$

and, after integration in v, we obtain (345). □

We now obtain the estimate (344) from the doubly commuted equation (342). Upon contraction of (342) with $[\nabla\!\!\!/_3((1/\Omega)\nabla\!\!\!/_3(r^2\overset{(1)}{\widehat{\chi}}\Omega))]$ and integration over the region $r\leqslant r_1$, the terms from the first line of (342) yield

$$\begin{aligned}\int_{v_0}^v &d\bar{v}\int_{u(r_1,\bar{v})}^\infty d\bar{u}\int_{S^2_{\bar{u},\bar{v}}}\sin\theta\,d\theta\,d\phi\left((\text{first line of (342)}),\nabla\!\!\!/_3\left(\frac{1}{\Omega}\nabla\!\!\!/_3(r^2\overset{(1)}{\widehat{\chi}}\Omega)\right)\right)\\ &=\frac12\int_{u(v,r_1)}^\infty d\bar{u}\int_{S^2_{\bar{u},v}}\sin\theta\,d\theta\,d\phi\,\Omega^2\left|\frac{1}{\Omega}\nabla\!\!\!/_3\left(\frac{1}{\Omega}\nabla\!\!\!/_3(r^2\overset{(1)}{\widehat{\chi}}\Omega)\right)\right|^2+\text{positive term on } r=r_1\\ &\quad-\frac12\int_{u(v_0,r_1)}^\infty d\bar{u}\int_{S^2_{\bar{u},v_0}}\sin\theta\,d\theta\,d\phi\,\Omega^2\left|\frac{1}{\Omega}\nabla\!\!\!/_3\left(\frac{1}{\Omega}\nabla\!\!\!/_3(r^2\overset{(1)}{\widehat{\chi}}\Omega)\right)\right|^2\\ &\quad+\int_{v_0}^v d\bar{v}\int_{u(r_1,\bar{v})}^\infty d\bar{u}\int_{S^2_{\bar{u},\bar{v}}}\sin\theta\,d\theta\,d\phi\,\Omega^2\widehat{\omega}\Omega\left|\frac{1}{\Omega}\nabla\!\!\!/_3\left(\frac{1}{\Omega}\nabla\!\!\!/_3(r^2\overset{(1)}{\widehat{\chi}}\Omega)\right)\right|^2(\bar{u},\bar{v},\theta,\phi).\end{aligned}\tag{347}$$

Recall that $\widehat{\omega}\Omega=M/r^2$. The right-hand side of (342) after contraction with

$$\nabla\!\!\!/_3\left(\left(\frac{1}{\Omega}\right)\nabla\!\!\!/_3(r^2\overset{(1)}{\widehat{\chi}}\Omega)\right)$$

can be estimated

$$2r^2\Omega\overset{(1)}{P}\cdot\left(\nabla\!\!\!/_3\left(\frac{1}{\Omega}\nabla\!\!\!/_3(r^2\overset{(1)}{\widehat{\chi}}\Omega)\right)\right)\leqslant\frac{1}{16}\frac{M}{r^2}\left|\nabla\!\!\!/_3(\frac{1}{\Omega}\nabla\!\!\!/_3(r^2\overset{(1)}{\widehat{\chi}}\Omega))\right|^2+16\Omega^2\frac{r^6}{M}|\overset{(1)}{P}|^2,$$

so that, after integration over the region $r\leqslant r_1$, the first term can be absorbed by the good term in (347), while the last term is controlled by $\mathbb{F}_0[\overset{(1)}{\Psi}]$ from the integrated decay estimate for $\overset{(1)}{P}=r^{-5}\overset{(1)}{\Psi}$ (Theorem 1).

The two remaining terms arising from contraction of (342) can be controlled as follows. For the second term in the second line of (342) we simply note

$$\begin{aligned}
-\frac{12M}{r^4}&\Omega(r^2\overset{(1)}{\widehat{\chi}}\Omega)\cdot\left(\nabla\!\!\!/_3\left(\frac{1}{\Omega}\nabla\!\!\!/_3(r^2\overset{(1)}{\widehat{\chi}}\Omega)\right)\right)\\
&\leqslant\frac{\widehat{\omega}\Omega}{2}\left|\nabla\!\!\!/_3\left(\frac{1}{\Omega}\nabla\!\!\!/_3(r^2\overset{(1)}{\widehat{\chi}}\Omega)\right)\right|^2+\frac{1}{2}\frac{144}{\widehat{\omega}\Omega}\frac{M^2}{r^8}\cdot\Omega^2|r^2\overset{(1)}{\widehat{\chi}}\Omega|^2\\
&\leqslant\frac{\widehat{\omega}\Omega}{2}\left|\nabla\!\!\!/_3\left(\frac{1}{\Omega}\nabla\!\!\!/_3(r^2\overset{(1)}{\widehat{\chi}}\Omega)\right)\right|^2+\frac{72M}{r^6}\cdot\Omega^2|r^2\overset{(1)}{\widehat{\chi}}\Omega|^2.
\end{aligned}\tag{348}$$

The first term on the right-hand side of (348) will again be absorbed by the good terms in (347), while for the second we will eventually apply Lemma 13.3.1. For the first term in the second line of (342), we note that

$$\begin{aligned}
-\frac{4M}{r^3}&\Omega\frac{1}{\Omega}\nabla\!\!\!/_3(r^2\overset{(1)}{\widehat{\chi}}\Omega)\cdot\left(\nabla\!\!\!/_3\left(\frac{1}{\Omega}\nabla\!\!\!/_3(r^2\overset{(1)}{\widehat{\chi}}\Omega)\right)\right)\\
&=-\frac{2M}{r^3}\partial_u\left|\frac{1}{\Omega}\nabla\!\!\!/_3(r^2\overset{(1)}{\widehat{\chi}}\Omega)\right|^2\\
&=-\partial_u\left(\frac{2M}{r^3}\left|\frac{1}{\Omega}\nabla\!\!\!/_3(r^2\overset{(1)}{\widehat{\chi}}\Omega)\right|^2\right)+\frac{6M}{r^4}\Omega^2\left|\frac{1}{\Omega}\nabla\!\!\!/_3(r^2\overset{(1)}{\widehat{\chi}}\Omega)\right|^2.
\end{aligned}\tag{349}$$

Upon integration over the spacetime region, the second term has a good sign, while the first has a bad sign on the horizon and a good sign on the timelike boundary $r=r_1$.

We summarise the resulting estimate as

$$\begin{aligned}
&\frac{1}{2}\int_{u(v,r_1)}^{\infty}d\bar{u}\int_{S^2_{\bar{u},v}}\sin\theta\,d\theta\,d\phi\,\Omega^2\left|\frac{1}{\Omega}\nabla\!\!\!/_3\left(\frac{1}{\Omega}\nabla\!\!\!/_3(r^2\overset{(1)}{\widehat{\chi}}\Omega)\right)\right|^2\\
&\quad+\int_{v_0}^{v}d\bar{v}\int_{u(r_1,\bar{v})}^{\infty}d\bar{u}\int_{S^2_{\bar{u},\bar{v}}}\sin\theta\,d\theta\,d\phi\,\Omega^2\frac{1}{4}\frac{M}{r^2}\left|\frac{1}{\Omega}\nabla\!\!\!/_3\left(\frac{1}{\Omega}\nabla\!\!\!/_3(r^2\overset{(1)}{\widehat{\chi}}\Omega)\right)\right|^2\\
&\quad+\int_{v_0}^{v}d\bar{v}\int_{u(r_1,\bar{v})}^{\infty}d\bar{u}\int_{S^2_{\bar{u},\bar{v}}}\sin\theta\,d\theta\,d\phi\,\Omega^2\frac{6M}{r^4}\left|\frac{1}{\Omega}\nabla\!\!\!/_3(r^2\overset{(1)}{\widehat{\chi}}\Omega)\right|^2\\
&\qquad\leqslant\frac{1}{2}\int_{u(v_0,r_1)}^{\infty}d\bar{u}\int_{S^2_{\bar{u},v_0}}\sin\theta\,d\theta\,d\phi\,\Omega^2\left|\frac{1}{\Omega}\nabla\!\!\!/_3\left(\frac{1}{\Omega}\nabla\!\!\!/_3(r^2\overset{(1)}{\widehat{\chi}}\Omega)\right)\right|^2\\
&\qquad\quad+\int_{v_0}^{v}d\bar{v}\int_{S^2_{\infty,v}}\sin\theta\,d\theta\,d\phi\,\frac{2M}{r^3}\left|\left(\frac{1}{\Omega}\nabla\!\!\!/_3(r^2\overset{(1)}{\widehat{\chi}}\Omega)\right)\right|^2\\
&\qquad\quad+\int_{v_0}^{v}d\bar{v}\int_{u(r_1,\bar{v})}^{\infty}d\bar{u}\int_{S^2_{\bar{u},\bar{v}}}\sin\theta\,d\theta\,d\phi\,\frac{72M^2}{r^6}\cdot\Omega^2|r^2\overset{(1)}{\widehat{\chi}}\Omega|^2+C\cdot\mathbb{F}_0[\overset{(1)}{\Psi}].
\end{aligned}\tag{350}$$

Finally, applying Lemma 13.3.1 will allow us to absorb the last term of (350) by the term in the third line (for r_1-2M sufficiently small; we now *fix* r_1, depending only on m, such that this is possible) at the cost of another flux-term on the horizon.

$$\begin{aligned}
&\frac{1}{2}\int_{u(v,r_1)}^{\infty} d\bar{u}\int_{S^2_{\bar{u},v}} \sin\theta\, d\theta\, d\phi\, \Omega^2\left|\frac{1}{\Omega}\nabla\!\!\!/_3\left(\frac{1}{\Omega}\nabla\!\!\!/_3(r^2\overset{(1)}{\hat{\chi}}\Omega)\right)\right|^2\\
&\quad+\int_{v_0}^{v} d\bar{v}\int_{u(r_1,\bar{v})}^{\infty} d\bar{u}\int_{S^2_{\bar{u},\bar{v}}} \sin\theta\, d\theta\, d\phi\, \Omega^2\frac{1}{4}\frac{M}{r^2}\left|\frac{1}{\Omega}\nabla\!\!\!/_3\left(\frac{1}{\Omega}\nabla\!\!\!/_3(r^2\overset{(1)}{\hat{\chi}}\Omega)\right)\right|^2\\
&\quad+\int_{v_0}^{v} d\bar{v}\int_{u(r_1,\bar{v})}^{\infty} d\bar{u}\int_{S^2_{\bar{u},\bar{v}}} \sin\theta\, d\theta\, d\phi\, \Omega^2\frac{3M}{r^4}\left|\frac{1}{\Omega}\nabla\!\!\!/_3(r^2\overset{(1)}{\hat{\chi}}\Omega)\right|^2\\
&\qquad\leqslant\frac{1}{2}\int_{u(v_0,r_1)}^{\infty} d\bar{u}\int_{S^2_{\bar{u},v_0}} \sin\theta\, d\theta\, d\phi\, \Omega^2\left|\frac{1}{\Omega}\nabla\!\!\!/_3\left(\frac{1}{\Omega}\nabla\!\!\!/_3(r^2\overset{(1)}{\hat{\chi}}\Omega)\right)\right|^2+C\mathbb{F}_0[\overset{(1)}{\Psi}]\\
&\qquad\quad+\int_{v_0}^{v} d\bar{v}\int_{S^2_{\infty,v}} \sin\theta\, d\theta\, d\phi\left[\frac{1}{4M^2}\left|\left(\frac{1}{\Omega}\nabla\!\!\!/_3(r^2\overset{(1)}{\hat{\chi}}\Omega)\right)\right|^2+|r^2\Omega\overset{(1)}{\hat{\chi}}|^2\right].
\end{aligned}\tag{351}$$

To obtain the estimate of Proposition 13.3.2, note that the flux term on the horizon in the last line is controlled from Lemma 13.2.2.

To obtain the last statement of Proposition 13.3.2 concerning the sup, we use the positive first term in (351) for the highest derivative flux. For the lower-order terms, we use this flux together with the estimate (346), where in the latter we replace $\overset{(1)}{\hat{\chi}}\Omega r^2$ by $\Omega^{-1}\nabla\!\!\!/_3(\overset{(1)}{\hat{\chi}}\Omega r^2)$ such that the horizon term in (346) is controlled by Lemma 13.2.2, while the flux term is controlled by the first term in (351). This gives the $\Omega^{-1}\nabla\!\!\!/_3(\overset{(1)}{\hat{\chi}}\Omega r^2)$ flux on any $v\geqslant v_0$. To obtain the $\overset{(1)}{\hat{\chi}}\Omega r^2$-flux on any $v\geqslant v_0$, one again uses (346), now in its original form, and Lemma 13.2.2. □

13.3.3. Completing the proof of Proposition 13.3.1

Proposition 13.3.2 provides integrated decay (and fluxes) in $r\leqslant r_1$ from quantities purely at the level of initial data. One can now generalise this to integrated decay (and fluxes) globally. Let ξ be a smooth radial cut-off function equal to 0 in $\left[2M, 2M+\frac{1}{2}(r_1-2M)\right)$ and equal to 1 in $[r_1,\infty)$. From (139), we have $\Omega\nabla\!\!\!/_4(\overset{(1)}{\hat{\chi}}\Omega^{-1}r^2)=\overset{(1)}{\alpha}r^2$ and derive

$$\frac{1}{2}\partial_v\left(\xi r^{-\varepsilon}\left|\frac{\overset{(1)}{\hat{\chi}}}{\Omega}r^2\right|^2\right)-\frac{1}{2}(\partial_r\xi)\Omega^2\left|\frac{\overset{(1)}{\hat{\chi}}}{\Omega}r^2\right|^2+\frac{\varepsilon}{4}\xi\frac{\Omega^2}{r^{1+\varepsilon}}\left|\frac{\overset{(1)}{\hat{\chi}}}{\Omega}r^2\right|^2\leqslant C_\varepsilon\cdot\xi\cdot\Omega^2 r^{5-\varepsilon}|\overset{(1)}{\alpha}\Omega^2|^2,\tag{352}$$

where we have used that Ω^2 is bounded uniformly below in the support of ξ by a constant depending on r_1 only. Integrating over a spacetime region $[u_0,u]\times[v_0,v]\times S^2$, we observe that the second term on the left is supported for $r\in\left[12r_1+M, r_1\right]$, and can hence be absorbed by Proposition 13.3.2. Moreover, the future boundary term on constant v is positive([31]) and the term on the right is controlled by $\mathbb{F}_0[\overset{(1)}{\Psi}, \overset{(1)}{\psi}, \overset{(1)}{\alpha}]$. Combining this with Proposition 13.3.2 results in the estimate

$$\begin{aligned}
&\sup_{\bar v\in(v_0,v)} \int_{u_0}^{u} d\bar u \int_{S^2_{\bar u,\bar v}} \sin\theta\, d\theta\, d\phi\, \frac{\Omega^2}{r^\varepsilon} |r^2 \overset{(1)}{\hat\chi}\Omega|^2 \\
&\quad + \int_{v_0}^{v} d\bar v \int_{u_0}^{u} d\bar u \int_{S^2_{\bar u,\bar v}} \sin\theta\, d\theta\, d\overset{(1)}{\psi}\, \frac{\Omega^2}{r^{1+\varepsilon}} |r^2 \overset{(1)}{\hat\chi}\Omega|^2 \\
&\qquad\qquad \lesssim \mathbb{F}_0[\overset{(1)}{\Psi}, \overset{(1)}{\psi}, \overset{(1)}{\alpha}] + \|(\nabla\!\!\!/_3)^2 \overset{(1)}{\hat\chi}\|^2_{L^2(C_{v_0})}.
\end{aligned} \tag{353}$$

Now that a global integrated decay (and flux) estimate for $\overset{(1)}{\hat\chi}$ has been obtained, we can revisit (341) and repeat the above argument, in particular deriving

$$\begin{aligned}
&\frac12 \partial_v \left(\xi(1+r^{-\varepsilon}) \left| \frac{1}{\Omega} \nabla\!\!\!/_3 (r^2 \overset{(1)}{\hat\chi}\Omega) \right|^2 \right) - \frac12 (\partial_r \xi) \Omega^2 (1+r^{-\varepsilon}) \left| \frac{1}{\Omega} \nabla\!\!\!/_3 (r^2 \overset{(1)}{\hat\chi}\Omega) \right|^2 \\
&\quad + \frac{\varepsilon}{4} \xi \frac{\Omega^2}{r^{1+\varepsilon}} \left| \frac{1}{\Omega} \nabla\!\!\!/_3 (r^2 \overset{(1)}{\hat\chi}\Omega) \right|^2 \\
&\qquad \leqslant C_\varepsilon \xi (\Omega^2 r^{-5+\varepsilon} |r^2 \overset{(1)}{\hat\chi}\Omega|^2 + r^{5+\varepsilon}\Omega^2 |\overset{(1)}{\psi}\Omega|^2 + r^{3+\varepsilon}\Omega^2 |\overset{(1)}{\alpha}\Omega^2|^2).
\end{aligned} \tag{354}$$

Note that the use of the multiplier $1+r^{-\varepsilon}$ instead of $r^{-\varepsilon}$ exploits the improved decay towards infinity of the right-hand side of (341). Upon integration, the second term on the left can again be absorbed by Proposition 13.3.2, the first term on the right by (353) and the last two terms on the right by $\mathbb{F}_0[\overset{(1)}{\Psi}, \overset{(1)}{\psi}, \overset{(1)}{\alpha}]$. Finally, we apply the above argument to the twice commuted equation (342), using again the control on the lower-order terms from the previous steps and the integrated decay estimate for $\overset{(1)}{P}$ (Theorem 1) on the right-hand side. This completes the proof of Proposition 13.3.1.

COROLLARY 13.1. *Consider the solution* $\underset{\vee}{\mathscr{S}}$ *of Theorem 3. We also have* $L^\infty_{u,v}L^2(S^2)$ *estimates. In particular, for any* $u\geqslant u_0$ *and* $v\geqslant v_0$,

$$\begin{aligned}
&\int_{S^2_{u,v}} \sin\theta\, d\theta\, d\phi \left[|\Omega \overset{(1)}{\hat\chi} r^2|^2 + \left| \frac{1}{\Omega} \nabla\!\!\!/_3 (\Omega \overset{(1)}{\hat\chi} r^2) \right|^2 \right] \\
&\qquad \leqslant \mathbb{F}_0[\overset{(1)}{\Psi}, \overset{(1)}{\psi}, \overset{(1)}{\alpha}] + \sup_v \|r^{-1} \overset{(1)}{\psi}\Omega r^3\|^2_{S^2_{u_0,v}} + \|(\nabla\!\!\!/_3)^2 \overset{(1)}{\hat\chi}\|^2_{L^2(C_{v_0})}.
\end{aligned}$$

([31]) This term vanishes in the limit on null infinity. To control the $\overset{(1)}{\hat\chi}$-flux on null infinity, one should choose $f\sim 1$ near infinity. However, in this case one needs to know decay in u of α in order to control the right-hand side. One can obtain $L^\infty_{u,v}L^2(S^2)$ bounds with $f=1$, however. See Corollary 13.1 below.

Proof. (Sketch) It is straightforward to obtain these bounds in a region $r\leqslant 4M$ (or globally with weaker r-weights) using the $L^2_{\infty,v}(S^2)$ bounds on these quantities on the horizon (see Propositions 13.2.1 and 13.2.2, accounting for the second term on the right), and the fluxes of Proposition 13.3.1 together with the fundamental theorem of calculus. For $r\geqslant 4M$ one integrates the transport equations $\nabla\!\!\!/_4(\overset{(1)}{\hat{\chi}}\Omega^{-1}r^2)=\overset{(1)}{\alpha}\Omega^{-1}r^2$ and (341), respectively, towards infinity using the fluxes (282) and (292). □

Using the horizon flux of $\overset{(1)}{\hat{\chi}}$ (Proposition 13.2.1) in conjunction with the integrated decay estimate on $|(1/\Omega)\nabla\!\!\!/_3(\Omega\overset{(1)}{\hat{\chi}}r^2)|^2$ of Proposition 13.3.1, one also proves the following.

COROLLARY 13.2. *Consider the solution $\underset{\vee}{\mathscr{S}}$ of Theorem 3. For any $u\geqslant u_0$,*

$$\int_{v_0}^{\infty} dv \int_{S^2_{u,v}} \sin\theta\, d\theta\, d\phi\, r^{-1-\varepsilon}|\Omega\overset{(1)}{\hat{\chi}}r^2|^2 \leqslant \mathbb{F}_0[\overset{(1)}{\Psi},\overset{(1)}{\psi},\overset{(1)}{\alpha}]+\|(\nabla\!\!\!/_3)^2\overset{(1)}{\hat{\chi}}\|^2_{L^2(C_{v_0})}.$$

13.3.4. Higher-order estimates and summary

The arguments of the previous sections can be repeated for angular commuted equations. We have, in particular, the following result.

PROPOSITION 13.3.3. *Proposition 13.3.1 holds replacing $\overset{(1)}{\hat{\chi}}$ on the left by $\mathcal{A}^{[3]}\overset{(1)}{\hat{\chi}}$ and replacing the right-hand side by $\big\|\nabla\!\!\!/_3^2(\mathcal{A}^{[3]}\overset{(1)}{\hat{\chi}})\big\|^2_{L^2(C_{v_0})}+\mathbb{F}_0^{2,T,\nabla\!\!\!/}[\overset{(1)}{\Psi},\mathfrak{D}\overset{(1)}{\psi},\mathfrak{D}\overset{(1)}{\alpha}]$.*

Proof. We outline how to repeat the proof of Proposition 13.3.1 for the angular commuted equations (340)–(342), noting that $\mathcal{A}^{[i]}$ commutes trivially with those equations.

Recall that, to prove Proposition 13.3.1, we first proved Proposition 13.3.2, i.e. the estimates for $r\leqslant r_1$. The key ingredients in the proof of Proposition 13.3.2 were (1) an integrated decay estimate for $\overset{(1)}{P}$, $\overset{(1)}{\psi}$ and $\overset{(1)}{\alpha}$ in $r\leqslant r_1$ and (2) control of the fluxes of $(\overset{(1)}{\hat{\chi}}\Omega)$ and $\Omega^{-1}\nabla\!\!\!/_3(\overset{(1)}{\hat{\chi}}\Omega)$ on the event horizon. Since $\mathbb{F}_0^{2,T,\nabla\!\!\!/}[\overset{(1)}{\Psi},\mathfrak{D}\overset{(1)}{\psi},\mathfrak{D}\overset{(1)}{\alpha}]$ provides an integrated decay estimate for $\mathcal{A}^{[3]}\overset{(1)}{P}$ (away from $r=3M$), $\mathcal{A}^{[3]}\overset{(1)}{\psi}$ and $\mathcal{A}^{[3]}\overset{(1)}{\alpha}$, the first ingredient is present for the $\mathcal{A}^{[3]}$ commuted equations. The second ingredient is present as well, because the fluxes of $\mathcal{A}^{[3]}(\overset{(1)}{\hat{\chi}}\Omega)$ and $\mathcal{A}^{[3]}\Omega^{-1}\nabla\!\!\!/_3(\overset{(1)}{\hat{\chi}}\Omega)$ on the event horizon are controlled by the once angular commuted version of Propositions 13.2.1 and 13.2.2. This proves the $\mathcal{A}^{[3]}$-commuted version of Proposition 13.3.2.

To complete the proof of Proposition 13.3.3, we finally repeat the proof of §13.3.3. While the proof would go through unchanged for the $\mathcal{A}^{[2]}$ commuted equation (replacing $\mathbb{F}_0[\overset{(1)}{\Psi},\overset{(1)}{\psi},\overset{(1)}{\alpha}]$ by $\mathbb{F}_0^{2,T,\nabla\!\!\!/}[\overset{(1)}{\Psi},\overset{(1)}{\psi},\overset{(1)}{\alpha}]$ on the right), an additional renormalisation argument is required when repeating the argument for the $\mathcal{A}^{[3]}$-commuted equation (342), since

$\mathbb{F}_0^{2,T,\nabla\!\!\!/}[\overset{(1)}{\Psi}, \mathfrak{D}\overset{(1)}{\psi}, \mathfrak{D}\overset{(1)}{\alpha}]$ does *not* provide a *non-degenerate* (near $r=3M$) integrated decay estimate for three angular derivatives of $\overset{(1)}{P}$. We present this argument now. By commuting (139) and using the identity (299), we derive

$$\Omega\nabla\!\!\!/_4[\mathcal{A}^{[i+2]}\overset{(1)}{\widehat{\chi}}\Omega^{-1}r^2+r^4\mathcal{A}^{[i]}\overset{(1)}{\psi}\Omega^{-1}]=-r^3\mathcal{A}^{[i]}\overset{(1)}{\psi}\Omega+3Mr\mathcal{A}^{[i]}\overset{(1)}{\alpha}. \tag{355}$$

Commuting (355) once with $\Omega\nabla\!\!\!/_3$ yields

$$\begin{aligned}\Omega\nabla\!\!\!/_4[\Omega\nabla\!\!\!/_3(\mathcal{A}^{[i+2]}\overset{(1)}{\widehat{\chi}}\Omega^{-1}r^2)+r^4\mathcal{A}^{[i]}\overset{(1)}{P}-\mathcal{A}^{[i]}\overset{(1)}{\psi}r^3\Omega+2Mr\Omega^{-1}\mathcal{A}^{[i]}\overset{(1)}{\psi}]\\ =-\mathcal{A}^{[i]}\overset{(1)}{P}r^3\Omega^2-6Mr\mathcal{A}^{[i]}\overset{(1)}{\psi}+\frac{6M^2}{r}\mathcal{A}^{[i]}\overset{(1)}{\alpha}\Omega^2.\end{aligned} \tag{356}$$

We leave the final $\Omega\nabla\!\!\!/_3$ commutation of (356) to the reader noting that it is already clear that at most second (one being angular, one being $\Omega\nabla\!\!\!/_3$) derivatives of $\overset{(1)}{P}$, $\overset{(1)}{\psi}$ and $\overset{(1)}{\alpha}$ will appear both in the boundary and in the terms on the right. For such terms, fluxes on hypersurfaces of constant v, as well as *non-degenerate* integrated decay estimates, are available([32]) through Theorem 2 and controlled by $\mathbb{F}_0^{2,T,\nabla\!\!\!/}[\overset{(1)}{\Psi}, \mathfrak{D}\overset{(1)}{\psi}, \mathfrak{D}\overset{(1)}{\alpha}]$ (noting also that weights in Ω^2 are irrelevant in this part of the argument). We can hence contract (355) with ξ/r^{ε} times the expression in the square bracket, (356) with $\xi(1+1/r^{\varepsilon})$ times the expression in the square bracket and similarly for the double commuted equation, just as done in §13.3.3 for the uncommuted equations. This completes the proof. □

The analogues of Corollaries 13.1 and 13.2 are now deduced just as before. Note that, from the commuted version of Propositions 13.2.1 and 13.2.2, controlling $\mathcal{A}^{[3]}\overset{(1)}{\widehat{\chi}}\Omega$ and $\mathcal{A}^{[3]}\Omega^{-1}\nabla\!\!\!/_3(\overset{(1)}{\widehat{\chi}}\Omega)$ in $L^2(S^2)$ on the horizon (or the fluxes on the horizon, respectively) requires only one angular derivative of ψ and one derivative of P which is clearly controlled by $\mathbb{F}_0^{2,T,\nabla\!\!\!/}[\overset{(1)}{\Psi}, \mathfrak{D}\overset{(1)}{\psi}, \mathfrak{D}\overset{(1)}{\alpha}]$. This proves the following result.

Proposition 13.3.4. *We have, for any $u\geqslant u_0$ and $v\geqslant v_0$,*

$$\begin{aligned}\|r^{-1}\cdot\mathcal{A}^{[3]}\Omega\overset{(1)}{\widehat{\chi}}r^2\|^2_{S^2_{u,v}}+\left\|r^{-1}\cdot\mathcal{A}^{[3]}\frac{1}{\Omega}\nabla\!\!\!/_3(\Omega\overset{(1)}{\widehat{\chi}}r^2)\right\|^2_{S^2_{u,v}}\\ \lesssim\|\nabla\!\!\!/_3^2(\mathcal{A}^{[3]}\overset{(1)}{\widehat{\chi}})\|_{L^2(C_{v_0})}+\mathbb{F}_0^{2,T,\nabla\!\!\!/}[\overset{(1)}{\Psi}, \mathfrak{D}\overset{(1)}{\psi}, \mathfrak{D}\overset{(1)}{\alpha}],\end{aligned}$$

and, for any $u\geqslant u_0$,

$$\int_{v_0}^{\infty}dv\int_{S^2_{u,v}}\sin\theta\,d\theta\,d\phi\,r^{-1-\varepsilon}|\mathcal{A}^{[3]}\Omega\overset{(1)}{\widehat{\chi}}r^2|^2\lesssim\|\nabla\!\!\!/_3^2(\mathcal{A}^{[3]}\overset{(1)}{\widehat{\chi}})\|_{L^2(C_{v_0})}+\mathbb{F}_0^{2,T,\nabla\!\!\!/}[\overset{(1)}{\Psi}, \mathfrak{D}\overset{(1)}{\psi}, \mathfrak{D}\overset{(1)}{\alpha}].$$

([32]) Note in particular that the $\mathrm{d}\!\!\!/\mathrm{iv}\, R^{\star}\overset{(1)}{\Psi}$-derivative is controlled non-degenerately already in $\mathbb{F}^{1,T,\nabla\!\!\!/}[\overset{(1)}{\Psi}]$, while the $\mathrm{d}\!\!\!/\mathrm{iv}\, T\overset{(1)}{\psi}$-derivative is non-degenerately controlled in $\mathbb{F}^{2,T,\nabla\!\!\!/}[\overset{(1)}{\Psi}]$.

13.4. Boundedness estimates for the ingoing shear $\overset{(1)}{\underline{\hat{\chi}}}$

In this section we will prove *boundedness* of the quantity $\overset{(1)}{\underline{\hat{\chi}}}$ and derivatives thereof. Key to the boundedness proof is the auxiliary geometric quantity $\overset{(1)}{Y}$ introduced in (218), which satisfies a propagation equation with integrable gauge invariant right-hand side.

Remark 13.4. With the redshift estimates on $\overset{(1)}{\hat{\chi}}$ of §13.3 in mind, one might hope that a similar argument near null infinity will produce *decay* estimates for the geometric quantity $\overset{(1)}{\underline{\hat{\chi}}}$ of $\check{\mathscr{S}}$. However, this is not the case and we shall only be able to prove boundedness. It is only for the horizon-renormalised $\hat{\mathscr{S}}$, where the geometric quantity $\overset{(1)}{\underline{\hat{\chi}}}$ decays. See Theorem 4.

13.4.1. Control on angular derivatives of $\overset{(1)}{\underline{\hat{\chi}}}$

We recall the quantity $\overset{(1)}{Y}$, a symmetric traceless $S^2_{u,v}$-tensor, from (218).

LEMMA 13.4.1. *We have the propagation equations*

$$\nabla\!\!\!/_3 \overset{(1)}{Y} = \frac{1}{2} r \operatorname{tr} \underline{\chi} \frac{r^3 \overset{(1)}{\underline{\psi}}}{\Omega} + \frac{3M \overset{(1)}{\underline{\alpha}} r}{\Omega}, \tag{357}$$

and

$$\nabla\!\!\!/_3 (\mathcal{A}^{[2]} \overset{(1)}{Y}) = \nabla\!\!\!/_3 \Big(-\frac{1}{2} \overset{(1)}{\underline{\Psi}} - 3Mr^3 \overset{(1)}{\underline{\psi}} \Omega^{-1} \Big) + r^3 \overset{(1)}{\underline{\psi}} + 3Mr^2 \overset{(1)}{\underline{\psi}} + \frac{3M}{2} \Omega \overset{(1)}{\underline{\alpha}} r - 9M^2 \frac{\overset{(1)}{\underline{\alpha}}}{\Omega}, \tag{358}$$

$$\begin{aligned} \nabla\!\!\!/_3 (\mathcal{A}^{[4]} \overset{(1)}{Y}) = \nabla\!\!\!/_3 \Big(& -\frac{1}{2} r^2 \mathcal{D}\!\!\!/_2^\star \operatorname{d\!\!\!/iv} \overset{(1)}{\underline{\Psi}} - 3Mr^5 \mathcal{D}\!\!\!/_2^\star \operatorname{d\!\!\!/iv} \overset{(1)}{\underline{\psi}} \Omega^{-1} \\ & + \frac{1}{2} \overset{(1)}{\underline{\Psi}} + \frac{3M}{2r} \overset{(1)}{\underline{\Psi}} - \frac{3M}{2} r^3 \overset{(1)}{\underline{\psi}} \Omega + \frac{9M^2}{r} r^3 \overset{(1)}{\underline{\psi}} \Omega^{-1} \Big) + J_2, \end{aligned} \tag{359}$$

where J_2 denotes a linear combination of terms $r^{-2} \overset{(1)}{\underline{\Psi}} \Omega$, $r^3 \overset{(1)}{\underline{\psi}}$ and $r \overset{(1)}{\underline{\alpha}} / \Omega$, with uniformly bounded coefficients depending only on M.

Remark 13.5. Recall that the operators $\mathcal{A}^{[2]} = r^2 \mathcal{D}\!\!\!/_2^\star \operatorname{d\!\!\!/iv}$ and $\mathcal{A}^{[4]} = r^4 \mathcal{D}\!\!\!/_2^\star \operatorname{d\!\!\!/iv} \mathcal{D}\!\!\!/_2^\star \operatorname{d\!\!\!/iv}$ acting on symmetric traceless tensors are uniformly elliptic; cf. (111) and Remark 4.2.

Proof. To derive the above, note that, by (302),

$$\nabla\!\!\!/_3 (r^4 \overset{(1)}{\underline{\psi}} \Omega^{-1}) + \tfrac{1}{2} \operatorname{tr} \underline{\chi} (r^4 \overset{(1)}{\underline{\psi}} \Omega^{-1}) = -r^4 \Omega^{-1} \mathcal{D}\!\!\!/_2^\star \operatorname{d\!\!\!/iv} \overset{(1)}{\underline{\alpha}} - 3Mr \overset{(1)}{\underline{\alpha}} \Omega^{-1}, \tag{360}$$

while

$$\nabla\!\!\!/_3(r^2\overset{(1)}{\underline{\hat{\chi}}}\Omega^{-1}) = -r^2\Omega^{-1}\overset{(1)}{\underline{\alpha}} \quad \text{and hence} \quad \nabla\!\!\!/_3(r^4 \mathcal{D}\!\!\!/_2^\star \,\mathrm{d}\!\!\!/\mathrm{iv}\, \overset{(1)}{\underline{\hat{\chi}}}\Omega^{-1}) = -r^4\Omega^{-1}\mathcal{D}\!\!\!/_2^\star \,\mathrm{d}\!\!\!/\mathrm{iv}\, \overset{(1)}{\underline{\alpha}}.$$

Subtracting (360) from the last identity yields (357). For the second identity, the computation is straightforward commuting the first identity with $\mathcal{A}^{[2]}$ and using the identities (302) and (303). The third identity is produced by another commutation with $\mathcal{A}^{[2]}$ and using again the identities (302) and (303). $\square$

The point of the identity (358) is that the highest derivatives on the right-hand side appear as a boundary term, while the remaining terms are essentially as in (357), i.e. loosely speaking commuting with two angular derivatives does not "lose" regularity.

PROPOSITION 13.4.1. *Consider the solution $\underset{\vee}{\mathscr{S}}$ of Theorem 3. We have, for any $u \geqslant u_0$ and $v \geqslant v_0$, the estimate*

$$\|r^{-1}\cdot\overset{(1)}{Y}\|_{S^2_{u,v}} \lesssim \|r^{-1}\cdot\overset{(1)}{Y}\|_{S^2_{u_0,v}} + \sqrt{\mathbb{F}^{2,T}_0[\overset{(1)}{\underline{\Psi}}]} + \sqrt{\mathbb{F}_0[\overset{(1)}{\underline{\Psi}}, \mathfrak{D}\overset{(1)}{\underline{\psi}}, \overset{(1)}{\underline{\alpha}}]} \tag{361}$$

and

$$\begin{aligned}\|r^{-1}\cdot\mathcal{A}^{[3]}\overset{(1)}{Y}\|_{S^2_{u,v}} \lesssim \|r^{-1}\cdot\mathcal{A}^{[3]}\overset{(1)}{Y}\|_{S^2_{u_0,v}} + \sqrt{\mathbb{F}^{2,T,\nabla\!\!\!/}_0[\overset{(1)}{\underline{\Psi}}]} + \sqrt{\mathbb{F}_0[\overset{(1)}{\underline{\Psi}}, \mathfrak{D}\overset{(1)}{\underline{\psi}}, \overset{(1)}{\underline{\alpha}}]} \\ + \|r^{-1}\cdot r\,\mathrm{d}\!\!\!/\mathrm{iv}\,\overset{(1)}{\underline{\psi}}\Omega^{-1}r^3\|_{S^2_{u,v_0}} + \|r^{-1}\cdot r\,\mathrm{d}\!\!\!/\mathrm{iv}\,\overset{(1)}{\underline{\Psi}}\|_{S^2_{u,v_0}},\end{aligned} \tag{362}$$

as well as

$$\begin{aligned}\|r^{-1}\cdot\mathcal{A}^{[4]}\overset{(1)}{Y}\|_{S^2_{u,v}} \lesssim \|r^{-1}\cdot\mathcal{A}^{[4]}\overset{(1)}{Y}\|_{S^2_{u_0,v}} + \sqrt{\mathbb{F}^{2,T,\nabla\!\!\!/}_0[\overset{(1)}{\underline{\Psi}}]} + \sqrt{\mathbb{F}_0[\overset{(1)}{\underline{\Psi}}, \mathfrak{D}\overset{(1)}{\underline{\psi}}, \overset{(1)}{\underline{\alpha}}]} \\ + \|r^{-1}\cdot\mathcal{A}^{[2]}\overset{(1)}{\underline{\psi}}\Omega^{-1}r^3\|_{S^2_{u,v_0}} + \|r^{-1}\cdot\mathcal{A}^{[2]}\overset{(1)}{\underline{\Psi}}\|_{S^2_{u,v_0}},\end{aligned} \tag{363}$$

where we recall (234) for the definition of $\mathbb{F}^{2,T}_0[\overset{(1)}{\underline{\Psi}}]$. Note that, by assumption (246), all right-hand sides are finite.

Remark 13.6. We can actually drop the last term on the right-hand side of (362) and (363), as it is controlled by $\mathbb{F}^{2,T,\nabla\!\!\!/}_0[\overset{(1)}{\underline{\Psi}}]$. We can also drop the penultimate term in (362) and (363), provided we replace $\mathbb{F}_0[\overset{(1)}{\underline{\Psi}}, \overset{(1)}{\underline{\psi}}, \overset{(1)}{\underline{\alpha}}]$ by $\mathbb{F}^{2,T,\nabla\!\!\!/}_0[\overset{(1)}{\underline{\Psi}}, \mathfrak{D}\overset{(1)}{\underline{\psi}}, \overset{(1)}{\underline{\alpha}}]$. Both statements are a direct consequence of 1-dimensional Sobolev embedding and the definition of the norms.

Proof. It is clear from (357) that the first estimate would follow from boundedness for the L^1-fluxes

$$\int_{u_0}^{\infty} du\,\Omega^2 \left\| r^{-1}\cdot\frac{r^3\overset{(1)}{\underline{\psi}}}{\Omega}\right\|_{S^2_{u,v}} \quad \text{and} \quad \int_{u_0}^{\infty} du\,\Omega^2 \left\| r^{-1}\cdot\frac{r\overset{(1)}{\underline{\alpha}}}{\Omega^2}\right\|_{S^2_{u,v}}, \tag{364}$$

which is guaranteed by Corollary 12.6. For the second estimate, we commute (358) with $r\,\mathrm{d\!\!/iv}$ and estimate the quantity

$$\left\| r^3\,\mathrm{d\!\!/iv}\,\mathcal{D\!\!\!/}_2^\star\,\mathrm{d\!\!/iv}\,\overset{(1)}{Y} - \tfrac{1}{2}r^5 r\,\mathrm{d\!\!/iv}\,\overset{(1)}{\underline{P}} + Mr\,\mathrm{d\!\!/iv}\,r^3\overset{(1)}{\underline{\psi}}\Omega^{-1}\right\|_{S^2_{u,v}},$$

by integrating the transport equation, using again Corollary 12.6. Afterwards, we use Corollary 11.2 and Proposition 12.2.3 to deduce the desired estimate. The third estimate is similar. $\square$

Combining Proposition 13.4.1 with the definition of $\overset{(1)}{Y}$ and Proposition 12.2.3, we deduce the following corollary whose last claim follows from Proposition 12.3.3.

Corollary 13.3. *Consider the solution $\underset{\vee}{\mathscr{S}}$ of Theorem 3. We have, for any $u\geqslant u_0$ and $v\geqslant v_0$ and $i=0,2,3$,*

$$\left\| r^{-1}\cdot\mathcal{A}^{[i+2]}\left(\frac{r\overset{(1)}{\widehat{\underline{\chi}}}}{\Omega}\right)\right\|_{S^2_{u,v}} \lesssim \|r^{-1}\cdot\mathcal{A}^{[i]}\overset{(1)}{Y}\|_{S^2_{u_0,v}} + \sqrt{\mathbb{F}_0^{2,T,\nabla\!\!\!/}[\overset{(1)}{\underline{\Psi}}]} + \sqrt{\mathbb{F}_0[\overset{(1)}{\underline{\Psi}},\mathfrak{D}\overset{(1)}{\underline{\psi}},\overset{(1)}{\underline{\alpha}}]}$$
$$+\|r^{-1}\cdot\mathcal{A}^{[i]}\overset{(1)}{\underline{\psi}}\Omega^{-1}r^3\|_{S^2_{u,v_0}}.$$

For $i=0,2$, the last three terms can all be controlled by (the square root of)

$$\mathbb{F}_0^{2,T,\nabla\!\!\!/}[\underline{\Psi},\mathfrak{D}\underline{\psi},\mathfrak{D}\underline{\alpha}].$$

Using that we can multiply both the square of (362) and (363) by Ω^2/r^2 and integrate in u, we find using the (twice angular commuted) fluxes of Proposition 12.2.1 the following corollary.

Corollary 13.4. *Consider the solution $\underset{\vee}{\mathscr{S}}$ of Theorem 3. We have, for any $v\geqslant v_0$, the flux estimates*

$$\int_{u_0}^{\infty} du\,\Omega^2\left\| r^{-1}\cdot\mathcal{A}^{[5]}\left(\frac{r\overset{(1)}{\widehat{\underline{\chi}}}}{\Omega}\right)\right\|^2_{S^2_{u,v}} \lesssim \|r^{-1}\cdot\mathcal{A}^{[3]}\overset{(1)}{Y}\|^2_{S^2_{u_0,v}} + \mathbb{F}_0^{2,T,\nabla\!\!\!/}[\overset{(1)}{\underline{\Psi}}] + \mathbb{F}_0^{2,T,\nabla\!\!\!/}[\overset{(1)}{\underline{\Psi}},\mathfrak{D}\overset{(1)}{\underline{\psi}},\overset{(1)}{\underline{\alpha}}],$$
$$\int_{u_0}^{\infty} du\,\Omega^2\left\| r^{-1}\cdot\mathcal{A}^{[6]}\left(\frac{r\overset{(1)}{\widehat{\underline{\chi}}}}{\Omega}\right)\right\|^2_{S^2_{u,v}} \lesssim \|r^{-1}\cdot\mathcal{A}^{[4]}\overset{(1)}{Y}\|^2_{S^2_{u_0,v}} + \mathbb{F}_0^{2,T,\nabla\!\!\!/}[\overset{(1)}{\underline{\Psi}}] + \mathbb{F}_0^{2,T,\nabla\!\!\!/}[\overset{(1)}{\underline{\Psi}},\mathfrak{D}\overset{(1)}{\underline{\psi}},\overset{(1)}{\underline{\alpha}}].$$

13.4.2. Control on angular derivatives of $\nabla\!\!\!/_4\overset{(1)}{\widehat{\underline{\chi}}}$

Commuting (357) with $\Omega\nabla\!\!\!/_4$ we derive

$$\nabla\!\!\!/_3(\Omega\nabla\!\!\!/_4\overset{(1)}{Y}) = r^3\Omega\overset{(1)}{\underline{P}} + 3M(2r\overset{(1)}{\underline{\psi}} - 2\widehat{\omega}\overset{(1)}{\underline{\alpha}}r) \tag{365}$$

using (179) and (296).

Proposition 13.4.2. *Consider the solution $\underset{\vee}{\mathscr{S}}$ of Theorem 3. We have, for any $u\geqslant u_0$ and $v\geqslant v_0$, the estimate*

$$\|r^{-1}\cdot r\Omega\slashed{\nabla}_4\overset{(1)}{Y}\|^2_{S^2_{u,v}}\lesssim\sup_v\|r^{-1}\cdot r\Omega\slashed{\nabla}_4\overset{(1)}{Y}\|^2_{S^2_{u_0,v}}+\mathbb{F}_0[\overset{(1)}{\underline{\Psi}},\overset{(1)}{\underline{\psi}},\overset{(1)}{\underline{\alpha}}],\tag{366}$$

as well as, for any $v\geqslant v_0$, the flux estimate

$$\int_{u_0}^{\infty}d\bar{u}\,\Omega^2r^{-1}\|r^{-1}\cdot r\Omega\slashed{\nabla}_4\overset{(1)}{Y}\|^2_{S^2_{\bar{u},v}}\lesssim\sup_v\|r^{-1}\cdot r\Omega\slashed{\nabla}_4\overset{(1)}{Y}\|^2_{S^2_{u_0,v}}+\mathbb{F}_0[\overset{(1)}{\underline{\Psi}},\overset{(1)}{\underline{\psi}},\overset{(1)}{\underline{\alpha}}].\tag{367}$$

We also have the following flux and integrated decay estimate for any $v\geqslant v_0$ and $u\geqslant u_0$:

$$\begin{aligned}\int_{v_0}^{v}d\bar{v}\,r^{-1-\varepsilon}\|r^{-1}\cdot r\Omega\slashed{\nabla}_4\overset{(1)}{Y}\|^2_{S^2_{u,\bar{v}}}+\int_{v_0}^{v}d\bar{v}\int_{u_0}^{u}d\bar{u}\,r^{-2-\varepsilon}\Omega^2\|r^{-1}\cdot r\Omega\slashed{\nabla}_4\overset{(1)}{Y}\|^2_{S^2_{\bar{u},\bar{v}}}&\\ \lesssim\frac{1}{\varepsilon}\sup_v\|r^{-1}\cdot r\Omega\slashed{\nabla}_4\overset{(1)}{Y}\|^2_{S^2_{u_0,v}}+\mathbb{F}_0[\overset{(1)}{\underline{\Psi}},\overset{(1)}{\underline{\psi}},\overset{(1)}{\underline{\alpha}}].&\end{aligned}\tag{368}$$

Proof. From (365), we deduce for $\gamma>0$ the estimate

$$\tfrac{1}{2}\Omega\slashed{\nabla}_3[(\Omega\slashed{\nabla}_4\overset{(1)}{Y})^2r^\gamma]+\tfrac{1}{4}\gamma r^{\gamma-1}\Omega^2(\Omega\slashed{\nabla}_4\overset{(1)}{Y})^2\leqslant\Omega^2r^{\gamma-3}(|\overset{(1)}{\underline{\Psi}}|^2+|r^3\overset{(1)}{\underline{\psi}}\Omega^{-1}|^2+|\overset{(1)}{\underline{\alpha}}r\Omega^{-2}|^2).\tag{369}$$

The estimates of the proposition now follow from direct integration over the angular variables and either the u direction (for the first two estimates, $\gamma=2$) or both u and v (for the last estimate, $\gamma=1-\varepsilon$). $\square$

Remark 13.7. Stronger r-weighted norms are propagated (in particular, we could apply (369) with $\gamma=3$ and $\gamma=2-\varepsilon$, respectively), but we will not make use of this here.

A simple commutation yields (using that $r\slashed{\nabla}_A$ commutes trivially) the following.

Corollary 13.5. *Consider the solution $\underset{\vee}{\mathscr{S}}$ of Theorem 3. Let $i\in\{1,2,3\}$. The three estimates (366)–(368) hold replacing $\slashed{\nabla}_4\overset{(1)}{Y}$ by $\mathcal{A}^{[i]}\slashed{\nabla}_4\overset{(1)}{Y}$ everywhere, and $\mathbb{F}_0[\overset{(1)}{\underline{\Psi}},\overset{(1)}{\underline{\psi}},\overset{(1)}{\underline{\alpha}}]$ by $\mathbb{F}_0^{2,T,\slashed{\nabla}}[\overset{(1)}{\underline{\Psi}},\mathfrak{D}\overset{(1)}{\underline{\psi}},\mathfrak{D}\overset{(1)}{\underline{\alpha}}]$ on the right.*

Note that it is indeed $\mathbb{F}_0^{2,T,\slashed{\nabla}}[\overset{(1)}{\underline{\Psi}},\mathfrak{D}\overset{(1)}{\underline{\psi}},\mathfrak{D}\overset{(1)}{\underline{\alpha}}]$ appearing on the new right-hand side which stems from the fact that the estimate invoked in the proof only requires the fluxes of $\overset{(1)}{\underline{\Psi}}$, and $\mathbb{F}_0^{2,T,\slashed{\nabla}}[\overset{(1)}{\underline{\Psi}},\mathfrak{D}\overset{(1)}{\underline{\psi}},\mathfrak{D}\overset{(1)}{\underline{\alpha}}]$ by definition already contains three angular derivatives of $\overset{(1)}{\underline{\Psi}}$.

Given the estimates for $\slashed{\nabla}_4\overset{(1)}{Y}$, we can obtain estimates for $\slashed{\nabla}_4\overset{(1)}{\underline{\hat{\chi}}}$ from the easily verified identity

$$\Omega\slashed{\nabla}_4\overset{(1)}{Y}=\frac{\Omega^2}{r}\overset{(1)}{Y}+r\cdot r^2\slashed{\mathcal{D}}_2^\star\,\slashed{\mathrm{div}}\,\Omega\slashed{\nabla}_4\left(\frac{r\overset{(1)}{\underline{\hat{\chi}}}}{\Omega}\right)-r^4\overset{(1)}{\underline{P}}-2Mr^2\frac{\overset{(1)}{\underline{\psi}}}{\Omega}.\tag{370}$$

In particular, multiplying the above by r, squaring and integrating over the angular variable, we deduce the following.

Corollary 13.6. *Consider the solution* $\underset{\vee}{\mathscr{S}}$ *of Theorem* 3. *We have, for any* $u\geqslant u_0$ *and* $v\geqslant v_0$, *the estimate*

$$\left\| r^{-1}\cdot r^2\cdot r^2 \mathcal{D}_2^\star \,\mathrm{d\!/\!iv}\, \Omega \nabla\!\!\!/_4 \left(\frac{r\overset{(1)}{\underline{\hat{\chi}}}}{\Omega}\right)\right\|^2_{S^2_{u,v}} \lesssim \sup_v \|r^{-1}\cdot r\Omega\nabla\!\!\!/_4 \overset{(1)}{Y}\|^2_{S^2_{u_0,v}} + \sup_v \|r^{-1}\cdot \overset{(1)}{Y}\|^2_{S^2_{u_0,v}} + \mathbb{F}_0[\overset{(1)}{\underline{\Psi}}, \overset{(1)}{\underline{\psi}}, \overset{(1)}{\underline{\alpha}}] + \sup_u \|r^{-1}\cdot \overset{(1)}{\underline{\psi}}\Omega^{-1} r^3\|^2_{S^2_{u,v_0}} + \sup_u \|r^{-1}\cdot \overset{(1)}{\underline{\Psi}}\Omega^{-1}\|^2_{S^2_{u,v_0}}.$$

The last three terms on the right-hand side could be replaced by $\mathbb{F}_0[\overset{(1)}{\underline{\Psi}}, \mathfrak{D}\overset{(1)}{\underline{\psi}}, \overset{(1)}{\underline{\alpha}}]$, using Sobolev embedding in dimension 1. Using Corollary 13.5, we also have the following commuted version.

Corollary 13.7. *Consider the solution* $\underset{\vee}{\mathscr{S}}$ *of Theorem* 3. *We have, for any* $u\geqslant u_0$ *and* $v\geqslant v_0$, *the estimate*

$$\left\| r^{-1}\cdot r^2\cdot \mathcal{A}^{[4]} \Omega \nabla\!\!\!/_4 \left(\frac{r\overset{(1)}{\underline{\hat{\chi}}}}{\Omega}\right)\right\|^2_{S^2_{u,v}} \lesssim \sup_v \|r^{-1}\cdot r^3 \mathcal{D}_2^\star \,\mathrm{d\!/\!iv}\, \Omega\nabla\!\!\!/_4 \overset{(1)}{Y}\|^2_{S^2_{u_0,v}} + \sup_v \|r^{-1}\cdot r^2 \mathcal{D}_2^\star \,\mathrm{d\!/\!iv}\, \overset{(1)}{Y}\|^2_{S^2_{u_0,v}} + \mathbb{F}_0^{2,T,\nabla\!\!\!/}[\overset{(1)}{\underline{\Psi}}, \mathfrak{D}\overset{(1)}{\underline{\psi}}, \overset{(1)}{\underline{\alpha}}]. \tag{371}$$

The horizon flux estimate of the next corollary follows directly from the (twice angular commuted) flux of (368) (applied with $u\to\infty$), and suggests that, while $\overset{(1)}{\underline{\hat{\chi}}}\Omega^{-1}$ itself does not decay, applying a T-derivative gives rise to a decaying quantity.

Corollary 13.8. *Consider the solution* $\underset{\vee}{\mathscr{S}}$ *of Theorem* 3. *We control the horizon flux*

$$\int_{v_0}^\infty d\bar{v}\, \|\mathcal{A}^{[4]}\Omega\nabla\!\!\!/_4 (r\overset{(1)}{\underline{\hat{\chi}}}\Omega^{-1})\|^2_{S^2_{\infty,\bar{v}}} \lesssim \sup_v \|r^{-1}\cdot \mathcal{A}^{[2]} r\Omega\nabla\!\!\!/_4 \overset{(1)}{Y}\|^2_{S^2_{u_0,v}} + \mathbb{F}_0^{2,T,\nabla\!\!\!/}[\overset{(1)}{\underline{\Psi}}, \overset{(1)}{\underline{\psi}}, \overset{(1)}{\underline{\alpha}}]. \tag{372}$$

13.4.3. A polynomial decay estimate for $\nabla\!\!\!/_4(\Omega^{-1}\overset{(1)}{\underline{\hat{\chi}}})$ on the horizon

For later purposes, we derive here a simple polynomial decay estimate for the quantity $\nabla\!\!\!/_4(\Omega^{-1}\overset{(1)}{\underline{\hat{\chi}}})$.

Proposition 13.4.3. *Consider the solution* $\underset{\vee}{\mathscr{S}}$ *of Theorem* 3. *We have, along the event horizon, the estimate*

$$\|\mathcal{A}^{[2]}\Omega\nabla\!\!\!/_4 (r\overset{(1)}{\underline{\hat{\chi}}}\Omega^{-1})\|^2_{S^2_{\infty,\bar{v}}} \lesssim \frac{1}{v^2}\mathbb{E}_0,$$

where we recall (246).

Proof. Starting from equation (365), one repeats the argument (splitting integrals when integrating the transport equations) of the proof of Proposition 12.3.5 using also the flux bound of Proposition 12.3.8 (Corollary 12.5 is also sufficient). This yields in particular

$$\|\Omega \nabla\!\!\!/_4 \overset{(1)}{Y}\|^2_{S^2_{\infty,\bar{v}}} \lesssim \frac{1}{v^2}\mathbb{E}_0. \tag{373}$$

One now revisits the identity (370) evaluated on the horizon and uses the fact that v^{-2}-decay for $\underline{\overset{(1)}{\psi}}$ is implied by Proposition 12.3.6, while v^{-2}-decay for $\underline{\overset{(1)}{\Psi}}$ (on spheres on the horizon) is an immediate consequence of Proposition 11.5.1. □

13.5. Boundedness for all remaining quantities

In this section we conclude the proof of Theorem 3 by exploiting the estimates derived on the outgoing linearised shear $\hat{\chi}$ and the ingoing linearised shear $\underline{\overset{(1)}{\hat{\chi}}}$ in the previous two subsections to bound all remaining linearised Ricci and curvature components of the solution $\underset{\vee}{\mathscr{S}}$.

13.5.1. $L^\infty_{u,v}$-estimates on $S^2_{u,v}$ and fluxes on constant v-hypersurfaces

The estimates obtained thus far are sufficient to obtain flux bounds for five angular derivatives of the curvature components $\underline{\overset{(1)}{\beta}}$ and $(\overset{(1)}{\varrho}, \overset{(1)}{\sigma})$, Note that we already control the flux of five angular derivatives of $\underline{\overset{(1)}{\alpha}}$ by Proposition 12.3.1, as well as six angular derivatives of $\underline{\overset{(1)}{\hat{\chi}}}$ from Corollary 13.4.

PROPOSITION 13.5.1. *Consider the solution* $\underset{\vee}{\mathscr{S}}$ *of Theorem* 3*. We have the following flux estimates*:

$$\sup_v \int_{u_0}^{\infty} du\, \Omega^2 \|r^{-1}\cdot \mathcal{A}^{[3]} r^2 \mathcal{D}\!\!\!/_2^\star \mathcal{D}\!\!\!/_1^\star (r^3 \overset{(1)}{\varrho}, r^3 \overset{(1)}{\sigma})\|^2_{S^2_{u,v}} \lesssim \mathbb{E}_0,$$
$$\sup_v \int_{u_0}^{\infty} du\, \Omega^2 \|r^{-1}\cdot \mathcal{A}^{[4]} r \mathcal{D}\!\!\!/_2^\star (r^2 \underline{\overset{(1)}{\beta}}\Omega^{-1})\|^2_{S^2_{u,v}} \lesssim \mathbb{E}_0.$$

Proof. The estimates follow from the identities

$$\mathcal{A}^{[i]} r^2 \mathcal{D}\!\!\!/_2^\star \mathcal{D}\!\!\!/_1^\star (r^3 \overset{(1)}{\varrho}, r^3 \overset{(1)}{\sigma}) = \mathcal{A}^{[i]}(r^5 \overset{(1)}{P}) + 3M\Omega r(\mathcal{A}^{[i]}\overset{(1)}{\hat{\chi}} - \mathcal{A}^{[i]}\underline{\overset{(1)}{\hat{\chi}}}), \tag{374}$$
$$\mathcal{A}^{[i]} \mathcal{D}\!\!\!/_2^\star (\underline{\overset{(1)}{\beta}}\Omega^{-1}) = \mathcal{A}^{[i]}(\Omega^{-1}\underline{\overset{(1)}{\psi}}) + \tfrac{3}{2}\varrho \mathcal{A}^{[i]}(\Omega^{-1}\underline{\overset{(1)}{\hat{\chi}}}), \tag{375}$$

the flux-estimates on P obtained in Theorem 1 and the estimates on $\overset{(1)}{\hat{\chi}}$ and $\underline{\overset{(1)}{\hat{\chi}}}$ obtained in Corollary 13.4 and Proposition 13.3.4. □

The next proposition concerns L^2-estimates on spheres. Note that below for $\overset{(1)}{\beta}$ and $\underline{\overset{(1)}{\beta}}$ we need the redshift (and $r\nabla\!\!\!/_4$-) commuted energy $\mathbb{F}_0^2[\overset{(1)}{\Psi}]$ and $\mathbb{F}_0^2[\underline{\overset{(1)}{\Psi}}]$ on the right-hand side; see Corollary 11.4. The same is true for the analogous estimates for $\overset{(1)}{\alpha}$ and $\underline{\overset{(1)}{\alpha}}$ obtained previously in Proposition 12.3.3.

Proposition 13.5.2. *Consider the solution $\underset{\vee}{\mathscr{S}}$ of Theorem 3. We have, for any $u\geqslant u_0$ and any $v\geqslant v_0$, the estimates*

$$\|r^{-1}\cdot\mathcal{A}^{[2]}r^2\mathcal{D}\!\!\!\!/_2^\star\mathcal{D}\!\!\!\!/_1^\star(r^3\overset{(1)}{\varrho},r^3\overset{(1)}{\sigma})\|^2_{S^2_{u,v}}\lesssim\mathbb{E}_0 \tag{376}$$

and

$$\|r^{-1}\cdot\mathcal{A}^{[3]}r\mathcal{D}\!\!\!\!/_2^\star(r^2\underline{\overset{(1)}{\beta}}\Omega^{-1})\|^2_{S^2_{u,v}}+\|r^{-1}\cdot\mathcal{A}^{[3]}r\mathcal{D}\!\!\!\!/_2^\star(r^{7/2}\overset{(1)}{\beta}\Omega)\|^2_{S^2_{u,v}}\lesssim\mathbb{E}_0+\mathbb{F}_0^2[\overset{(1)}{\Psi}]+\mathbb{F}_0^2[\underline{\overset{(1)}{\Psi}}],$$

provided $\mathbb{F}_0^2[\overset{(1)}{\Psi}]+\mathbb{F}_0^2[\underline{\overset{(1)}{\Psi}}]<\infty$. Moreover, if on the left-hand side of the second estimate $\mathcal{A}^{[3]}$ is replaced by $\mathcal{A}^{[2]}$ and the exponent $\frac{7}{2}$ by $\frac{7}{2}-\varepsilon$, then the last term on the right-hand side can be dropped.

Proof. Use the identities (374), (375) and (395) now with Corollary 13.3, applied with $i=0$ and $i=2$, and Propositions 13.3.4 and 12.3.3. For the final remark, use the once angular commuted Proposition 12.2.3 with $\delta=\varepsilon$ and observe that the right-hand side of the latter is controlled by the $\mathbb{E}_0$ alone via 1-dimensional Sobolev embedding. □

13.5.2. Estimates for four angular derivatives of $\overset{(1)}{\eta}$ and $\underline{\overset{(1)}{\eta}}$

With the estimates at our disposal we can already prove the following result.

Proposition 13.5.3. *Consider the solution $\underset{\vee}{\mathscr{S}}$ of Theorem 3. For any $u\geqslant u_0$ and $v\geqslant v_0$, we have, for $i=1,2,3$, the estimate*

$$\begin{aligned}&\int_{S^2_{u,v}}\sin\theta\,d\theta\,d\phi\cdot[r^6|\mathcal{A}^{[i]}\mathcal{D}\!\!\!\!/_2^\star\underline{\overset{(1)}{\eta}}|^2+r^4|\mathcal{A}^{[i]}\mathcal{D}\!\!\!\!/_2^\star\overset{(1)}{\eta}|^2]\\&\lesssim\|\nabla\!\!\!/_3^2(r^3\,\mathrm{d\!\!/iv}\,\mathcal{D}\!\!\!\!/_2^\star\,\mathrm{d\!\!/iv}\,\overset{(1)}{\hat{\chi}})\|_{L^2(C_{v_0})}+\mathbb{D}_0^{[5]}[\overset{(1)}{Y},\overset{(1)}{Z}]+\mathbb{F}_0^{2,T,\nabla\!\!\!/}[\underline{\overset{(1)}{\Psi}},\mathfrak{D}\underline{\overset{(1)}{\psi}},\mathfrak{D}\underline{\overset{(1)}{\alpha}}]+\mathbb{F}_0^{2,T,\nabla\!\!\!/}[\overset{(1)}{\Psi},\mathfrak{D}\overset{(1)}{\psi},\mathfrak{D}\overset{(1)}{\alpha}]\\&\lesssim\mathbb{E}_0.\end{aligned}\tag{377}$$

For $\overset{(1)}{\eta}$ we have in addition the flux estimate[33]

$$\int_{u_0}^\infty du\int_{S^2_{u,v}}\sin\theta\,d\theta\,d\phi\,\Omega^2r^4|\mathcal{A}^{[3]}\mathcal{D}\!\!\!\!/_2^\star\overset{(1)}{\eta}|^2\lesssim\mathbb{E}_0. \tag{378}$$

[33] Note that this flux estimate does not lose in r compared with the estimate (377) on spheres.

Proof. Consider the estimate

$$r^2|\mathcal{A}^{[i]}r\mathcal{D}_2^\star\overset{(1)}{\eta}|^2\lesssim\left|\mathcal{A}^{[i]}\frac{1}{\Omega}\nabla_3(r^2\overset{(1)}{\hat{\chi}}\Omega)\right|^2+\frac{1}{r^2}|\mathcal{A}^{[i]}r^2\overset{(1)}{\hat{\chi}}\Omega|^2+|\mathcal{A}^{[i]}r\overset{(1)}{\underline{\hat{\chi}}}\Omega|^2, \tag{379}$$

which is an easy consequence of (140) and trivial angular commutation with the $\mathcal{A}^{[i]}$ defined in (103), and

$$r^4|\mathcal{A}^{[i]}r\mathcal{D}_2^\star\overset{(1)}{\underline{\eta}}|^2\lesssim\left|\mathcal{A}^{[i]}r^2\Omega\nabla_4\left(\frac{r\overset{(1)}{\underline{\hat{\chi}}}}{\Omega}\right)\right|^2+|\mathcal{A}^{[i]}r^2\overset{(1)}{\hat{\chi}}\Omega|^2+|\mathcal{A}^{[i]}r\overset{(1)}{\underline{\hat{\chi}}}\Omega^{-1}|^2, \tag{380}$$

which is easily derived from (141). For $i=1,2,3$, the terms on the right-hand sides are controlled by Proposition 13.3.4 and Corollaries 13.3 and 13.7. For the flux estimate, we need to use in addition Proposition 13.3.3 and Corollaries 13.4 and 13.8. □

13.5.3. Control on five angular derivatives of $\overset{(1)}{\omega}$ and $\overset{(1)}{\underline{\omega}}$

We turn to estimates for $\overset{(1)}{\omega}$ and $\overset{(1)}{\underline{\omega}}$.

PROPOSITION 13.5.4. *Consider the solution $\underset{\vee}{\mathscr{S}}$ of Theorem 3. For any $u\geqslant u_0$ and $v\geqslant v_0$, we have the estimate*

$$\|r^{-1}\cdot\mathcal{A}^{[3]}r^2\mathcal{D}_2^\star\nabla_A\overset{(1)}{\omega}\|_{S^2_{u,v}}\lesssim r^{-(5-\varepsilon)/2}(u,v)\cdot\mathbb{E}_0. \tag{381}$$

We also have the estimate

$$\|r^{-1}\cdot\mathcal{A}^{[2]}r^2\mathcal{D}_2^\star\nabla_A\overset{(1)}{\underline{\omega}}\Omega^{-2}\|_{S^2_{u,v}}\lesssim\mathbb{E}_0. \tag{382}$$

Proof. Commuting equation (144) we can write, for $i=0,1,2,3$,

$$\begin{aligned}&\Omega\nabla_3(\mathcal{A}^{[i]}r^2\mathcal{D}_2^\star\nabla_A\overset{(1)}{\omega})\\&\quad=\Omega^2\left(\mathcal{A}^{[i]}r^2\overset{(1)}{P}-\mathcal{A}^{[i]}r^2\overset{(1)}{\underline{P}}+\frac{6M}{r^2}\mathcal{A}^{[i]}(\Omega\overset{(1)}{\hat{\chi}}-\Omega\overset{(1)}{\underline{\hat{\chi}}})\right)+\Omega^2\frac{2M}{r}\mathcal{A}^{[i]}\mathcal{D}_2^\star(\overset{(1)}{\eta}+\overset{(1)}{\underline{\eta}}).\end{aligned}$$

Integrating this transport equation using the Cauchy–Schwarz inequality in conjunction with Theorem 1 and Corollary 11.3, Proposition 13.3.4 and Corollary 13.4, as well as Proposition 13.5.3, yields the result. Note that the initial term on $u=u_0$ vanishes in view of $\underset{\vee}{\mathscr{S}}$ satisfying the gauge condition (189). It is easy to see how to incorporate a non-vanishing boundary term if (189) did not hold on the data; cf. Remark 10.3.

For $\overset{(1)}{\underline{\omega}}$ the argument is similar, now integrating in the 4-direction and starting from (144) written in the red-shifted form

$$\Omega\nabla_4(\overset{(1)}{\underline{\omega}}\Omega^{-2})+\frac{2M}{r^2}\overset{(1)}{\underline{\omega}}\Omega^{-2}=-\overset{(1)}{\varrho}+\frac{4M}{r^3}\Omega^{-1}\overset{(1)}{\Omega},$$

from which after angular commutation as above, one proves the boundedness statement. □

13.5.4. Control on five angular derivatives of $(\Omega \overset{(1)}{\operatorname{tr}} \underline{\chi})$

The control obtained on four angular derivatives of $\overset{(1)}{\eta}$ is sufficient to estimate all angular derivatives of $(\Omega \overset{(1)}{\operatorname{tr}} \underline{\chi})$. For this, we write the Codazzi equation (145) as

$$\mathcal{A}^{[i]} \mathcal{D}_2^\star \operatorname{d\!/iv} \Omega^{-1} \overset{(1)}{\underline{\hat{\chi}}} = \frac{1}{r} \mathcal{A}^{[i]} \mathcal{D}_2^\star \overset{(1)}{\eta} + \mathcal{A}^{[i]} \Omega^{-1} \overset{(1)}{\underline{\psi}} + \frac{3}{2} \rho \mathcal{A}^{[i]} \overset{(1)}{\underline{\hat{\chi}}} \Omega^{-1} + \frac{1}{2\Omega^2} \mathcal{A}^{[i]} \mathcal{D}_2^\star \nabla\!\!\!/ (\Omega \overset{(1)}{\operatorname{tr}} \underline{\chi}). \tag{383}$$

PROPOSITION 13.5.5. *Consider the solution* $\overset{\vee}{\mathscr{S}}$ *of Theorem 3. For any* $u \geqslant u_0$ *and* $v \geqslant v_0$, *we have the estimate*

$$\left\| r^{-1} \cdot r^3 \mathcal{A}^{[3]} \mathcal{D}_2^\star \nabla\!\!\!/ \frac{(\Omega \overset{(1)}{\operatorname{tr}} \underline{\chi})}{\Omega^2} \right\|^2_{S^2_{u,v}} \lesssim \mathbb{E}_0. \tag{384}$$

Moreover, we have the flux estimate

$$\sup_v \int_{u_0}^{\infty} du\, \Omega^2 \left\| r^{-1} \cdot r^3 \mathcal{A}^{[3]} \mathcal{D}_2^\star \nabla\!\!\!/ \frac{(\Omega \overset{(1)}{\operatorname{tr}} \underline{\chi})}{\Omega^2} \right\|^2_{S^2_{u,v}} \lesssim \mathbb{E}_0. \tag{385}$$

Proof. This is a consequence of the identity (383), Corollaries 13.3 and 13.4, Proposition 13.5.3 and the estimates on $\overset{(1)}{\underline{\psi}}$ obtained in Propositions 12.2.1 and 12.3.3 (the latter only for the first bound). □

13.5.5. Top-order estimates for angular derivatives of $\overset{(1)}{\hat{\chi}}$ and $(\Omega \overset{(1)}{\operatorname{tr}} \chi)$: The role of $\overset{(1)}{Z}$

At this point we have estimated all Ricci coefficients except $(\Omega \overset{(1)}{\operatorname{tr}} \chi)$. Estimates for the latter could be obtained directly from the estimates on $\overset{(1)}{\hat{\chi}}$ of §13.3 (cf. Propositions 13.3.3 and 13.3.4) and the Codazzi equation (389). However, the estimates of §13.3 "lose" $\nabla\!\!\!/_3$ derivatives, in the sense that we could only estimate $\overset{(1)}{\hat{\chi}}$ after commutation (twice!) with the redshift vectorfield. In this section we show how to avoid this loss of derivatives and how to estimate five angular derivatives of $\overset{(1)}{\hat{\chi}}$.

Key to the argument is the auxiliary quantity $\overset{(1)}{Z}$ defined in (219) which essentially allows to prove that, given estimates on $\overset{(1)}{\eta}$ and $\overset{(1)}{\underline{\eta}}$, the quantity $(\Omega \overset{(1)}{\operatorname{tr}} \chi)$ can be estimated without a loss near the horizon. The Codazzi equation (389) can then be used to estimate angular derivatives of $\overset{(1)}{\hat{\chi}}$. The weights near infinity obtained in the process are not optimal. However, once estimates for angular derivatives of $\overset{(1)}{\hat{\chi}}$ are available near the horizon, one can use the angular commuted transport equation for $\overset{(1)}{\hat{\chi}}$, (139), to optimise the weights for (angular derivatives of) $\overset{(1)}{\hat{\chi}}$ and then use once again (389) to optimise them for (angular derivatives of) $(\Omega \overset{(1)}{\operatorname{tr}} \chi)$.

Angular derivatives of $(\Omega\, \overset{(1)}{\mathrm{tr}}\, \chi)$. Observe that we can write

$$\Omega \nabla\!\!\!\!/_4\Big((\Omega\, \overset{(1)}{\mathrm{tr}}\, \chi)\frac{r^2}{\Omega^2} - 4r\Omega^{-1}\overset{(1)}{\Omega}\Big) = -4\Omega^2\Omega^{-1}\overset{(1)}{\Omega}. \tag{386}$$

Recall the quantity $\overset{(1)}{Z} = (r^3/\Omega^2)\nabla\!\!\!\!/(\Omega\, \overset{(1)}{\mathrm{tr}}\, \chi) - 2r^2(\overset{(1)}{\eta} + \overset{(1)}{\underline{\eta}})$ defined in (219). Commuting, we can write

$$\Omega \nabla\!\!\!\!/_4(\mathcal{A}^{[i]} r \mathcal{D}\!\!\!\!/_2^\star \overset{(1)}{Z}) = -2\Omega^2 \mathcal{A}^{[i]} r^2 \mathcal{D}\!\!\!\!/_2^\star(\overset{(1)}{\eta} + \overset{(1)}{\underline{\eta}}), \tag{387}$$

where we recall the definition of $\mathcal{A}^{[i]}$ in (103). Contracting (387) with $(r^{-n}/\Omega^2)\mathcal{A}^{[i]} r \mathcal{D}\!\!\!\!/_2^\star \overset{(1)}{Z}$ (with the appropriate i) we find([34])

$$\partial_v\left[\frac{|\mathcal{A}^{[i]} r \mathcal{D}\!\!\!\!/_2^\star \overset{(1)}{Z}|^2}{r^n\Omega^2}\right] + \left(\frac{n\Omega^2}{r^1} + \frac{2M}{r^2}\right)\frac{|\mathcal{A}^{[i]} r \mathcal{D}\!\!\!\!/_2^\star \overset{(1)}{Z}|^2}{r^n\Omega^2} = -2r^{2-n}(\mathcal{A}^{[i]}\mathcal{D}\!\!\!\!/_2^\star(\overset{(1)}{\eta} + \overset{(1)}{\underline{\eta}}), \mathcal{A}^{[i]} r \mathcal{D}\!\!\!\!/_2^\star \overset{(1)}{Z}).$$

Integrating and using the Cauchy–Schwarz inequality (note that, by Proposition 9.4.3, the quantity $r\mathcal{D}\!\!\!\!/_2^\star \overset{(1)}{Z}$ vanishes like Ω^2 on the sphere S^2_{∞,v_0} in our gauge), one finds the following.

Proposition 13.5.6. *Consider the solution $\underset{\vee}{\mathscr{S}}$ of Theorem 3. We have, for $n \geqslant 0$, any $i \in \mathbb{N}$ and any $u \geqslant u_0$ and $v \geqslant v_0$, the estimate*

$$\begin{aligned}
&\int_{S^2_{u,v}} \sin\theta\, d\theta\, d\phi\, \frac{|\mathcal{A}^{[i]} r \mathcal{D}\!\!\!\!/_2^\star \overset{(1)}{Z}|^2}{r^n\Omega^2} + \int_{v_0}^{v} d\bar v \int_{S^2_{u,\bar v}} \sin\theta\, d\theta\, d\phi\, \frac{1}{r}\cdot\frac{|\mathcal{A}^{[i]} r \mathcal{D}\!\!\!\!/_2^\star \overset{(1)}{Z}|^2}{r^n\Omega^2} \\
&\quad\lesssim \int_{S^2_{u,v_0}} \sin\theta\, d\theta\, d\phi\, \frac{|\mathcal{A}^{[i]} r \mathcal{D}\!\!\!\!/_2^\star \overset{(1)}{Z}|^2}{r^n\Omega^2} + \int_{v_0}^{v} d\bar v \int_{S^2_{u,\bar v}} \sin\theta\, d\theta\, d\phi\, r^{5-n}\Omega^2|\mathcal{A}^{[i]}\mathcal{D}\!\!\!\!/_2^\star(\overset{(1)}{\eta} + \overset{(1)}{\underline{\eta}})|^2.
\end{aligned}$$

Applying the proposition we can reinsert the definition of $\overset{(1)}{Z}$ and obtain an estimate for i angular derivatives of $(\Omega\, \overset{(1)}{\mathrm{tr}}\, \chi)$ in terms of $i-1$ angular derivatives of $\overset{(1)}{\eta} + \overset{(1)}{\underline{\eta}}$.

Corollary 13.9. *Consider the solution $\underset{\vee}{\mathscr{S}}$ of Theorem 3. We have, for $n \geqslant 0$, any $i \in \mathbb{N}$ and any $u \geqslant u_0$ and $v \geqslant v_0$, the estimate*

$$\begin{aligned}
&\int_{S^2_{u,v}} \sin\theta\, d\theta\, d\phi\, r^{8-n}\left|\mathcal{A}^{[i]}\mathcal{D}\!\!\!\!/_2^\star \nabla\!\!\!\!/ \frac{(\Omega\, \overset{(1)}{\mathrm{tr}}\, \chi)}{\Omega^2}\right|^2 \\
&\quad\lesssim \int_{S^2_{u,v_0}} \sin\theta\, d\theta\, d\phi \frac{|\mathcal{A}^{[i]} r \mathcal{D}\!\!\!\!/_2^\star \overset{(1)}{Z}|^2}{r^n\Omega^2} + \sup_v \int_{S^2_{u,v}} \sin\theta\, d\theta\, d\phi\, r^{6-n}|\mathcal{A}^{[i]}\mathcal{D}\!\!\!\!/_2^\star(\overset{(1)}{\eta} + \overset{(1)}{\underline{\eta}})|^2 \\
&\qquad + \int_{v_0}^{v} d\bar v \int_{S^2_{u,\bar v}} \sin\theta\, d\theta\, d\phi\, r^{5-n}\Omega^2|\mathcal{A}^{[i]}\mathcal{D}\!\!\!\!/_2^\star(\overset{(1)}{\eta} + \overset{(1)}{\underline{\eta}})|^2.
\end{aligned}$$

([34]) In principle, we could contract with $(1/\Omega^4)\mathcal{A}^{[i]} r \mathcal{D}\!\!\!\!/_2^\star \overset{(1)}{Z}$ to estimate a stronger norm, but since in our gauge $\overset{(1)}{\eta}$ and $\overset{(1)}{\underline{\eta}}$ do not decay along the event horizon, we need an additional Ω^2 to perform the v integration in the estimate below.

This estimate is crucial, as it allows us to estimate $i+1$ angular derivatives of $(\Omega \overset{(1)}{\mathrm{tr}}\, \chi)$ in terms of i angular derivatives of $\overset{(1)}{\eta}$ and $\overset{(1)}{\underline{\eta}}$. We now apply Corollary 13.9 with $i=3$ and $n=2+\varepsilon$ (to make the right-hand side integrable), using the bounds of Proposition 13.5.3 to conclude that

$$\sup_{u,v} r^{2-\varepsilon} \left\| r^{-1} \cdot \Omega^{-2} \cdot \mathcal{A}^{[3]} r^2 \not{\mathcal{D}}_2^\star \not\nabla (\Omega \overset{(1)}{\mathrm{tr}}\, \chi) \right\|^2_{S^2_{u,v}} \lesssim \mathbb{E}_0. \tag{388}$$

Note the factor of Ω^{-2}. The estimate is clearly not optimal near infinity, as $2-\varepsilon$ should be replaced by 4. This will be achieved below.

Higher angular derivatives of $\overset{(1)}{\widehat{\chi}}$. We now write the Codazzi equation (145) as

$$\mathcal{A}^{[i]} \not{\mathcal{D}}_2^\star \,\mathrm{d\!\!/iv}\, \Omega \overset{(1)}{\widehat{\chi}} = -\frac{\Omega^2}{r} \mathcal{A}^{[i]} \not{\mathcal{D}}_2^\star \overset{(1)}{\underline{\eta}} - \mathcal{A}^{[i]} \Omega \overset{(1)}{\psi} - \frac{3}{2} \varrho \mathcal{A}^{[i]} \overset{(1)}{\widehat{\chi}} \Omega + \frac{1}{2} \mathcal{A}^{[i]} \not{\mathcal{D}}_2^\star \not\nabla (\Omega \overset{(1)}{\mathrm{tr}}\, \chi). \tag{389}$$

Using the estimates available for the terms on the right-hand side we conclude the following.

Proposition 13.5.7. *Consider the solution $\overset{\vee}{\mathscr{S}}$ of Theorem 3. We have, for any $u \geqslant u_0$ and $v \geqslant v_0$, the estimate*

$$\| r^{-1} \cdot \mathcal{A}^{[2]} r^2 \not{\mathcal{D}}_2^\star \,\mathrm{d\!\!/iv} (\Omega \overset{(1)}{\widehat{\chi}} r^2) \|^2_{S^2_{u,v}} \lesssim \mathbb{E}_0, \tag{390}$$

and

$$\sup_{u \geqslant u_0} \int_{v_0}^{\infty} dv r^{-1-\varepsilon} \| r^{-1} \cdot \mathcal{A}^{[2]} r^2 \not{\mathcal{D}}_2^\star \,\mathrm{d\!\!/iv} (\Omega \overset{(1)}{\widehat{\chi}} r^2) \|^2_{S^2_{u,v}} \lesssim \mathbb{E}_0. \tag{391}$$

Suppose now also that the initial norm $\mathbb{F}^2_0[\overset{(1)}{\Psi}] < \infty$. Then, we can replace $\mathcal{A}^{[2]}$ by $\mathcal{A}^{[3]}$ in both (390) and (391) provided we add the expression $\mathbb{F}^2_0[\overset{(1)}{\Psi}]$ on the right.

Proof. A weaker version of (390), namely with weight $r^{-2-\varepsilon}$ on the left-hand side, follows directly from the identity (389) after applying the estimates (388), Propositions 13.3.4, 12.2.3 and 12.2.1 and 13.5.3. To optimise the weight near infinity, one recalls (355)

$$\Omega \not\nabla_4 \left(\mathcal{A}^{[i+2]} \overset{(1)}{\widehat{\chi}} \Omega^{-1} r^2 + \frac{r^4 \mathcal{A}^{[i]} \overset{(1)}{\psi}}{\Omega} \right) = -r^3 \mathcal{A}^{[i]} \overset{(1)}{\psi} \Omega + 3 M r \mathcal{A}^{[i]} \overset{(1)}{\alpha}.$$

Note that the quantity in brackets is not regular on the horizon. However, we can integrate from the hypersurface of constant $r_0 = r(u_0, v_0)$ (where we know that

$$\| \mathcal{A}^{[4]} \overset{(1)}{\widehat{\chi}} \Omega^{-1} r^2 \|_{S^2_{u,v}} + \| r^2 \mathcal{A}^{[2]} \overset{(1)}{\psi} \Omega^{-1} \|_{S^2_{u,v}} \lesssim \sqrt{\mathbb{E}_0}$$

holds, by the estimate just established and Proposition 12.3.3) forward using the fluxes on $\mathcal{A}^{[2]}\overset{(1)}{\psi}$ and $\mathcal{A}^{[2]}\overset{(1)}{\alpha}$ of Theorem 2. To obtain (390) with $\mathcal{A}^{[3]}$, one follows the same argument with $i=3$. Proposition 12.3.3 will now lead to the extra term.

To prove the inequality (391), note first that restricting the integral to $r\geqslant r_0$, (391) follows immediately from (390). For $r\leqslant r_0$, use (389) and apply Proposition 13.3.4, the fluxes for ψ of Theorem 2, and observe that (388) gains a power in Ω^2. □

Revisiting again (389) immediately improves the estimate (388).

Corollary 13.10. *Consider the solution $\underset{\vee}{\mathscr{S}}$ of Theorem 3. We have*

$$\sup_{u,v}\left\|r^{-1}\cdot\Omega^{-2}\cdot\mathcal{A}^{[2]}r^4\mathcal{D}_2^\star\nabla\!\!\!/(\Omega\overset{(1)}{\mathrm{tr}}\,\chi)\right\|^2_{S^2_{u,v}}\lesssim\mathbb{E}_0. \tag{392}$$

Moreover, provided the initial norm $\mathbb{F}^2_0[\overset{(1)}{\Psi}]$ is finite, the estimate remains true if we replace $\mathcal{A}^{[2]}$ by $\mathcal{A}^{[3]}$ on the left and add $\mathbb{F}^2_0[\overset{(1)}{\Psi}]$ on the right.

Proposition 13.5.7 without the improvement mentioned in the second part of its statement is already sufficient to prove the analogue of Proposition 13.5.1, i.e. flux estimates on constant u-hypersurfaces for the quantities $\overset{(1)}{\varrho}$ and $\overset{(1)}{\beta}$.

Proposition 13.5.8. *Consider the solution $\underset{\vee}{\mathscr{S}}$ of Theorem 3. We have the following flux estimates*:

$$\sup_u\int_{v_0}^\infty dv\,\frac{1}{r^2}\|r^{-1}\cdot\mathcal{A}^{[3]}r^2\mathcal{D}_2^\star\mathcal{D}_1^\star(r^3\overset{(1)}{\varrho},r^3\overset{(1)}{\sigma})\|^2_{S^2_{u,v}}\lesssim\mathbb{E}_0, \tag{393}$$

$$\sup_u\int_{v_0}^\infty dv\,\frac{1}{r^2}\|r^{-1}\cdot\mathcal{A}^{[4]}r\mathcal{D}_2^\star(r^3\overset{(1)}{\beta}\Omega)\|^2_{S^2_{u,v}}\lesssim\mathbb{E}_0. \tag{394}$$

Proof. The estimates follow from the identities

$$\begin{gathered}\mathcal{A}^{[i]}r^2\mathcal{D}_2^\star\mathcal{D}_1^\star(r^3\overset{(1)}{\varrho},r^3\overset{(1)}{\sigma})=\mathcal{A}^{[i]}(r^5\overset{(1)}{P})+3M\Omega r(\mathcal{A}^{[i]}\overset{(1)}{\widehat{\chi}}-\mathcal{A}^{[i]}\overset{(1)}{\underline{\widehat{\chi}}}),\\ \mathcal{A}^{[i]}\mathcal{D}_2^\star(\overset{(1)}{\beta}\Omega)=\mathcal{A}^{[i]}(\Omega\overset{(1)}{\psi})-\tfrac{3}{2}\varrho\mathcal{A}^{[i]}(\Omega\overset{(1)}{\widehat{\chi}}),\end{gathered} \tag{395}$$

and the estimate on $\overset{(1)}{\widehat{\chi}}$ obtained in (391), as well as Corollary 13.3. □

13.5.6. Refined estimates for higher angular derivatives

We finally prove some refined estimates for higher angular derivatives which will eventually allow us to prove the estimate (248), i.e. to show the propagation of the $\mathbb{D}^{[5]}$-norm in §13.5.8 below.

Higher angular derivatives of $\boldsymbol{\nabla\!\!\!/}_3(\Omega \overset{(1)}{\mathrm{tr}}\, \chi)$ and $\boldsymbol{\nabla\!\!\!/}_3(\overset{(1)}{\hat{\chi}}\Omega)$. Now that we have estimated five angular derivatives of $\overset{(1)}{\hat{\chi}}$, we can apply the identity (380) with i=4 to estimate five angular derivatives of $\overset{(1)}{\underline{\eta}}$. This is Proposition 13.5.11 below. To estimate five derivatives of $\overset{(1)}{\eta}$, we first estimate $\mathcal{A}^{[4]}\Omega^{-1}\nabla\!\!\!/_3(\Omega \overset{(1)}{\mathrm{tr}}\, \chi)$, and then revisit (389) with i=2 and one $\Omega\nabla\!\!\!/_3^{-1}$ derivative applied to it, to obtain a bound on $\mathcal{A}^{[4]}\Omega^{-1}\nabla\!\!\!/_3\overset{(1)}{\hat{\chi}}$. Finally, revisiting (379) now with i=4 will then control five derivatives of $\overset{(1)}{\eta}$.

PROPOSITION 13.5.9. *Consider the solution $\underset{\vee}{\mathscr{S}}$ of Theorem 3. We have the estimate*

$$\sup_{u,v}\left\| r^{-1}\cdot r\nabla\!\!\!/_3\left(\mathcal{A}^{[2]}r^2\mathcal{D}\!\!\!/_2^\star\nabla\!\!\!/\frac{r(\Omega \overset{(1)}{\mathrm{tr}}\, \chi)}{\Omega^2}\right)\right\|^2_{S^2_{u,v}} \lesssim \mathbb{E}_0. \tag{396}$$

The estimate also holds with $\mathcal{A}^{[3]}$ replacing $\mathcal{A}^{[2]}$, provided the term $\mathbb{F}^2_0[\overset{(1)}{\Psi}]<\infty$ is added on the right. Finally,

$$\sup_{u,v}\|r^{-1}\cdot\Omega^{-1}\nabla\!\!\!/_3(\mathcal{A}^{[2]}r^2\mathcal{D}\!\!\!/_2^\star\, \mathrm{d}\!\!\!/\mathrm{iv}(\Omega\overset{(1)}{\hat{\chi}}r^2))\|^2_{S^2_{u,v}} \lesssim \mathbb{E}_0. \tag{397}$$

Proof. Starting from (136) and using (137), (135), (142) and (151), we derive

$$\begin{aligned}\Omega\nabla\!\!\!/_4&\left(r\Omega^{-1}\nabla\!\!\!/_3\left[\mathcal{A}^{[i]}r^2\mathcal{D}\!\!\!/_2^\star\nabla\!\!\!/\frac{r(\Omega \overset{(1)}{\mathrm{tr}}\, \chi)}{\Omega^2}\right]\right)+\frac{2M}{r^2}\left(r\Omega^{-1}\nabla\!\!\!/_3\left[\mathcal{A}^{[i]}r^2\mathcal{D}\!\!\!/_2^\star\nabla\!\!\!/\frac{r(\Omega \overset{(1)}{\mathrm{tr}}\, \chi)}{\Omega^2}\right]\right)\\ &=\frac{8M}{r^2}\mathcal{A}^{[i]}r^2\mathcal{D}\!\!\!/_2^\star(\overset{(1)}{\eta}+\overset{(1)}{\underline{\eta}})-4\mathcal{A}^{[i]}r^3\mathcal{D}\!\!\!/_2^\star\nabla\!\!\!/\overset{(1)}{\varrho}+\left(\frac{2M}{r}-\Omega^2\right)\mathcal{A}^{[i]}r^2\mathcal{D}\!\!\!/_2^\star\nabla\!\!\!/\frac{(\Omega \overset{(1)}{\mathrm{tr}}\, \chi)}{\Omega^2},\end{aligned}$$

from which the redshift is manifest. Since for $\underset{\vee}{\mathscr{S}}$ the quantities $\overset{(1)}{\eta}$ and $\overset{(1)}{\underline{\eta}}$ are not expected to decay, we contract the above by

$$\Omega^2\cdot\left(r\Omega^{-1}\nabla\!\!\!/_3\left[\mathcal{A}^{[i]}r^2\mathcal{D}\!\!\!/_2^\star\nabla\!\!\!/\frac{r(\Omega \overset{(1)}{\mathrm{tr}}\, \chi)}{\Omega^2}\right]\right).$$

Using the flux of Proposition 13.5.8 and $L^\infty_{u,v}L^2(S^2_{u,v})$ bounds on $\overset{(1)}{\eta}$ and $\overset{(1)}{\underline{\eta}}$ of Proposition 13.5.3, as well as Corollary 13.10, we conclude the first estimate.

The estimate (397) now follows from the identity arising from (389) with i=2, multiplied by r^4 and $(1/\Omega)\nabla\!\!\!/_3$, applied to it. After inserting (142), (140) and (179) (and noting the relation (182)) all terms on the right-hand side can be controlled on spheres by Corollary 11.3, Propositions 12.3.3 and 13.5.3, Corollary 13.3 (with i=2) and the bounds (390), (392) and (396). □

Top-order angular derivatives of $\overset{(1)}{\eta}$ and $\overset{(1)}{\underline{\eta}}$ With the results above, we immediately obtain estimates for the highest derivatives of $\overset{(1)}{\eta}$ and $\overset{(1)}{\underline{\eta}}$.

PROPOSITION 13.5.10. *Consider the solution $\underset{\vee}{\mathscr{S}}$ of Theorem 3. For any $u \geqslant u_0$ and $v \geqslant v_0$ we have*

$$\int_{S^2_{u,v}} \sin\theta\, d\theta\, d\phi\, r^6 |\mathcal{A}^{[4]} \not{\mathcal{D}}_2^\star \overset{(1)}{\underline{\eta}}|^2 \lesssim \mathbb{E}_0. \tag{398}$$

Proof. Apply (380) with $i=4$ and use (390) and (371), as well as Corollary 13.3 with $i=2$. □

PROPOSITION 13.5.11. *Consider the solution $\underset{\vee}{\mathscr{S}}$ of Theorem 3. For any $u \geqslant u_0$ and $v \geqslant v_0$, we have the estimate*

$$\int_{S^2_{u,v}} \sin\theta\, d\theta\, d\phi\, r^4 |\mathcal{A}^{[4]} \not{\mathcal{D}}_2^\star \overset{(1)}{\eta}|^2 \lesssim \mathbb{E}_0. \tag{399}$$

Proof. Apply (379) with $i=4$ and use (390), Corollary 13.3 with $i=2$ and (397). □

13.5.7. Boundedness of the metric components

PROPOSITION 13.5.12. *Consider the solution $\underset{\vee}{\mathscr{S}}$ of Theorem 3. We have, for any $v \geqslant v_0$ and $u \geqslant u_0$ and $i=0,1,2,3$,*

$$\left\| r^{-1} \cdot \sqrt{r} \mathcal{A}^{[i]} r^2 \not{\mathcal{D}}_2^\star \not\nabla \frac{\overset{(1)}{\sqrt{\not g}}}{\sqrt{\not g}} \right\|^2_{S^2_{u,v}} \lesssim \left\| r^{-1} \cdot \sqrt{r} \mathcal{A}^{[i]} r^2 \not{\mathcal{D}}_2^\star \not\nabla \frac{\overset{(1)}{\sqrt{\not g}}}{\sqrt{\not g}} \right\|^2_{S^2_{u_0,v}} + \mathbb{E}_0, \tag{400}$$

and also

$$\| r^{-1} \cdot \sqrt{r} \mathcal{A}^{[i+2]} \overset{(1)}{\hat{\not g}} \|^2_{S^2_{u,v}} \lesssim \| r^{-1} \cdot \sqrt{r} \mathcal{A}^{[i+2]} \overset{(1)}{\hat{\not g}} \|^2_{S^2_{u_0,v}} + \mathbb{E}_0. \tag{401}$$

For the metric component $\overset{(1)}{b}$ we have, for $i=0,1,2,3$, the estimate

$$r \| r^{-1} \cdot \mathcal{A}^{[i]} r \not{\mathcal{D}}_2^\star \overset{(1)}{b} \|^2_{S^2_{u,v}} \lesssim \mathbb{E}_0, \tag{402}$$

while, for $i=4$, the estimate (402) holds replacing the first r by $r^{-\varepsilon}$. For $\Omega^{-1} \overset{(1)}{\Omega}$ we have, for $i=0,1,2,3,4$,

$$\| r^{-1} \cdot \mathcal{A}^{[i]} r^2 \not{\mathcal{D}}_2^\star \not\nabla \Omega^{-1} \overset{(1)}{\Omega} \|^2_{S^2_{u,v}} \lesssim \mathbb{E}_0. \tag{403}$$

Remark 13.8. Note that the initial term vanishes for (402).

We remark also that the first term on the right-hand side of (400) and the first term on the right-hand side of (401) are in fact also controlled by $\mathbb{E}_0$, and could hence be dropped. This follows easily from the round sphere conditions (191) and (192), and the boundedness of the initial energy (248): Integrate (131) and (132) from infinity.

The estimates above hence show in particular that the round sphere conditions (191) and (192) are preserved in evolution.

Proof. From (131) we derive, for any $n\in\mathbb{R}$,

$$\frac{1}{2}\partial_u\left[r^n\left|\mathcal{A}^{[i]}r^2\not{\mathcal{D}}_2^\star\not\nabla\frac{\overset{(1)}{\sqrt{\not g}}}{\sqrt{\not g}}\right|^2\right]+\frac{\Omega^2 n}{2r}\left[r^n\left|\mathcal{A}^{[i]}r^2\not{\mathcal{D}}_2^\star\not\nabla\frac{\overset{(1)}{\sqrt{\not g}}}{\sqrt{\not g}}\right|^2\right]$$
$$=r^n\mathcal{A}^{[i]}r^2\not{\mathcal{D}}_2^\star\not\nabla(\Omega\overset{(1)}{\operatorname{tr}}\underline{\chi})\cdot\mathcal{A}^{[i]}r^2\not{\mathcal{D}}_2^\star\not\nabla\frac{\overset{(1)}{\sqrt{\not g}}}{\sqrt{\not g}},$$

which we apply with $n=1$ from initial data, use the Cauchy–Schwarz inequality on the right-hand side and the flux of Proposition 13.5.5. Similarly, from (132) we have

$$\frac{1}{2}\partial_u(r^n|\mathcal{A}^{[i+2]}\overset{(1)}{\hat{\not g}}|^2)+\frac{1}{2}\frac{n\Omega^2}{r}(r^n|\mathcal{A}^{[i+2]}\overset{(1)}{\hat{\not g}}|^2)=2r^{n-1}\Omega^2\mathcal{A}^{[i+2]}(r\overset{(1)}{\hat{\underline{\chi}}}\Omega^{-1})\cdot\mathcal{A}^{[i+2]}\overset{(1)}{\hat{\not g}},$$

which we apply with $n=1$ from initial data, use the Cauchy–Schwarz inequality on the right-hand side and the flux of Corollary 13.4. From the transport equation (133), we derive

$$\frac{1}{2}\partial_u\left(r^n\left|\frac{\mathcal{A}^{[i]}r\not{\mathcal{D}}_2^\star\overset{(1)}{b}}{r}\right|^2\right)+\frac{n\Omega^2}{2r}\left(r^n\left|\frac{\mathcal{A}^{[i]}r\not{\mathcal{D}}_2^\star\overset{(1)}{b}}{r}\right|^2\right)=2\Omega^2\frac{1}{r}\mathcal{A}^{[i]}r\not{\mathcal{D}}_2^\star(\overset{(1)}{\eta}-\overset{(1)}{\underline{\eta}})\cdot\frac{\mathcal{A}^{[i]}r\not{\mathcal{D}}_2^\star\overset{(1)}{b}}{r}\cdot r^n, \tag{404}$$

which, for $i=0,1,2,3$, we apply with $n=3$, use the Cauchy–Schwarz inequality on the right-hand side and Proposition 13.5.3 the flux estimate (378) and the estimate (377). For $i=4$ we can still apply the above with $n=2-\varepsilon$, and use the higher-order estimates of Propositions 13.5.10 and 13.5.11. Note the loss arises from not having the top-order *flux* estimate (378) for η.

The last estimate follows directly from (134) and Propositions 13.5.10 and 13.5.11. □

Remark 13.9. The loss in r-weight in the top-order estimate for $\overset{(1)}{b}$ can be removed, once one derives the top-order analogue of the flux estimate (378). The latter in turn requires a flux estimate on four angular derivatives $(1/\Omega)\not\nabla_3\overset{(1)}{\widehat{\chi}}$. This is most easily done from an estimate on $(1/\Omega)\not\nabla_3(\Omega\overset{(1)}{\operatorname{tr}}\chi)$ in the context of the future gauge of Theorem 4. We will not pursue this further here.

13.5.8. Proof of (248) and Corollary 10.2

The statement (248) in the boundedness theorem now follows from (362) and Corollary 13.5 with $i=2$ for the $\overset{(1)}{Y}$-part in the $\mathbb{D}[\overset{(1)}{Y},\overset{(1)}{Z}]$-norm, from Proposition 13.5.6 applied with $n=2+\varepsilon$ for the $\overset{(1)}{Z}$-part in the $\mathbb{D}[\overset{(1)}{Y},\overset{(1)}{Z}]$-norm, and finally from Proposition 13.5.9 for the remaining part in (244).

For Corollary 10.2, one uses the classical Sobolev embedding on $S^2_{u,v}$ in conjunction with the estimates of Corollary 13.3 for $\underline{\widehat{\chi}}$, Proposition 13.3.4 for $\overset{(1)}{\widehat{\chi}}$, Proposition 13.5.3 for $\overset{(1)}{\eta}$ and $\overset{(1)}{\underline{\eta}}$ (recall that the $\ell=0,1$ modes vanish for $\underset{\vee}{\mathscr{S}}'$), Proposition 13.5.12 for all metric quantities, Proposition 13.5.5 for $(\Omega \overset{(1)}{\mathrm{tr}} \underline{\chi})$, Corollary 13.10 for $(\Omega \overset{(1)}{\mathrm{tr}} \chi)$, Proposition 13.5.2 for $\overset{(1)}{\varrho}$, $\overset{(1)}{\sigma}$, $\overset{(1)}{\beta}$ and $\overset{(1)}{\underline{\beta}}$, and Corollary 12.3 for $\overset{(1)}{\alpha}$ and $\overset{(1)}{\underline{\alpha}}$. The pointwise bounds for $\underset{\vee}{\mathscr{S}}$ itself follow from the identity $\underset{\vee}{\mathscr{S}}=\underset{\vee}{\mathscr{S}}'+\mathscr{K}_{\mathfrak{m},s_i}$, together with the fact that, as is checked by direct computation, reference Kerr solutions $\mathscr{K}_{\mathfrak{m},s_i}$ indeed satisfy the boundedness property of the corollary with right-hand side controlled by a constant depending only on the parameters $\mathfrak{m}$ and s_i.

14. Proof of Theorem 4

In this final section of the paper, we turn to the proof of Theorem 4. The reader can again refer to the overview in §2.4.4.

In §14.1, we shall show that the pure gauge solution $\hat{\mathscr{G}}$, and thus also $\hat{\mathscr{S}}$, satisfies a uniform boundedness statement and an asymptotic flatness statement. This gives statement (1) of Theorem 4. In §14.2, we obtain statement (2) of the theorem concerning integrated local energy decay. Finally, we prove the final statement (3) of the theorem concerning polynomial decay in §14.3.

14.1. Boundedness of the pure gauge solution $\hat{\mathscr{G}}$

Let $\hat{\mathscr{G}}$ denote the pure gauge solution in the statement of Theorem 4. The goal of this section is to prove the boundedness of $\hat{\mathscr{G}}$, from which a similar statement will follow for $\hat{\mathscr{S}}=\underset{\vee}{\mathscr{S}}+\hat{\mathscr{G}}$, in view of Theorem 3 applied to $\underset{\vee}{\mathscr{S}}$.

We will begin in §14.1.1 below with certain preliminary estimates for $\hat{\mathscr{S}}$ on the horizon. We shall then use these in §14.1.2 to infer estimates for the function f defining $\hat{\mathscr{G}}$. The precise boundedness statements that follow will be given in §14.1.3.

To distinguish between the quantities (129) associated with $\underset{\vee}{\mathscr{S}}$ or $\hat{\mathscr{S}}$, we agree on the following *convention*: The geometric quantities of the solution $\underset{\vee}{\mathscr{S}}$ will, from now on, be denoted with an additional $[\underset{\vee}{\mathscr{S}}]$ next to them, while those of $\hat{\mathscr{S}}$ will appear without any additional notation, unless there is potential confusion, in which case we add $[\hat{\mathscr{S}}]$. The general rationale is to always write an estimate for a quantity of $\hat{\mathscr{S}}$ on the left in terms of initial quantities of $\underset{\vee}{\mathscr{S}}$ on the right.

For the geometric quantities associated with the pure gauge solution $\hat{\mathscr{G}}$, we shall always add $[\hat{\mathscr{G}}]$.

14.1.1. Decay bounds on the ingoing shear $\overset{(1)}{\hat{\underline{\chi}}}$ at the horizon

Recall from §13.4 (part of the proof of Theorem 3) that the "first" obstruction to proving decay for $\underset{\vee}{\mathscr{S}}$ arose from the quantity $\overset{(1)}{\hat{\underline{\chi}}}[\underset{\vee}{\mathscr{S}}]$. We will show in this section that our choice of $\hat{\mathscr{G}}$ ensures that $\overset{(1)}{\hat{\underline{\chi}}}=\overset{(1)}{\hat{\underline{\chi}}}[\hat{\mathscr{S}}]$ does indeed decay along the event horizon $\mathcal{H}^+$. The estimates obtained will then allow us in the next section to infer bounds for the gauge function f defining $\hat{\mathscr{G}}$.

First, some preliminary remarks: We note that the pure gauge solution $\hat{\mathscr{G}}$ has vanishing linearised shear $\overset{(1)}{\hat{\chi}}[\hat{\mathscr{G}}]=0$. Therefore, in addition to the estimates on the gauge invariant quantities, also the estimates on $\overset{(1)}{\hat{\chi}}$ proven in §13.3 remain valid as stated for $\overset{(1)}{\hat{\chi}}=\overset{(1)}{\hat{\chi}}[\hat{\mathscr{S}}]$. We also recall from Proposition 9.3.1 that $\Omega^{-2}(\Omega\overset{(1)}{\mathrm{tr}}\,\underline{\chi})[\hat{\mathscr{G}}]=0$ and $\Omega^{-1}\overset{(1)}{\hat{\underline{\chi}}}[\hat{\mathscr{G}}]=0$ on S^2_{∞,v_0}, and thus

$$\Omega^{-2}(\Omega\overset{(1)}{\mathrm{tr}}\,\underline{\chi})[\hat{\mathscr{S}}]=\Omega^{-2}(\Omega\overset{(1)}{\mathrm{tr}}\,\underline{\chi})[\underset{\vee}{\mathscr{S}}] \text{ and } \Omega^{-1}\overset{(1)}{\hat{\underline{\chi}}}[\hat{\mathscr{S}}]=\Omega^{-1}\overset{(1)}{\hat{\underline{\chi}}}[\underset{\vee}{\mathscr{S}}] \quad \text{on } S^2_{\infty,v_0}. \tag{405}$$

Finally, again by Proposition 9.3.1 and Lemma 6.1.1, we have

$$\overset{(1)}{\eta}[\hat{\mathscr{G}}]=0,\ \overset{(1)}{\varrho}[\hat{\mathscr{G}}]=0 \quad \text{on the event horizon } \mathcal{H}^+. \tag{406}$$

Hence, in particular, the gauge condition (194) holds for both $\underset{\vee}{\mathscr{S}}$ and $\hat{\mathscr{S}}$. Since (406) and (212) hold on the horizon $\mathcal{H}^+$, we conclude

$$\mathcal{D}_2^\star\overset{(1)}{\underline{\eta}}[\hat{\mathscr{S}}]=-\mathcal{D}_2^\star\overset{(1)}{\eta}[\hat{\mathscr{S}}]=-\mathcal{D}_2^\star\overset{(1)}{\eta}[\underset{\vee}{\mathscr{S}}]. \tag{407}$$

We now deduce the following flux bounds on the horizon.

Proposition 14.1.1. *On the horizon $\mathcal{H}^+$, the geometric quantities of $\hat{\mathscr{S}}$ in Theorem 4 satisfy, for $i\geqslant 3$ and any $v\geqslant v_0$,*

$$\|\mathcal{A}^{[i]}\Omega^{-1}\overset{(1)}{\hat{\underline{\chi}}}\|^2_{S^2_{\infty,v}}+\int_{v_0}^{v}d\bar{v}\|\mathcal{A}^{[i]}\Omega^{-1}\overset{(1)}{\hat{\underline{\chi}}}\|^2_{S^2_{\infty,\bar{v}}}\lesssim\|\mathcal{A}^{[i]}\Omega^{-1}\overset{(1)}{\hat{\underline{\chi}}}[\underset{\vee}{\mathscr{S}}]\|^2_{S^2_{\infty,v_0}}+\mathbb{F}_0^{i-3,T,\nabla\!\!\!\!/}[\overset{(1)}{\Psi},\overset{(1)}{\psi},\overset{(1)}{\alpha}]$$

and

$$\left\|\mathcal{A}^{[i-2]}\slashed{\mathcal{D}}_2^\star\slashed{\nabla}\frac{\overset{(1)}{(\Omega\,\mathrm{tr}\,\underline{\chi})}}{\Omega^2}\right\|^2_{S^2_{\infty,v}}+\int_{v_0}^{v}d\bar{v}\left\|\mathcal{A}^{[i-2]}\slashed{\mathcal{D}}_2^\star\slashed{\nabla}\frac{\overset{(1)}{(\Omega\,\mathrm{tr}\,\underline{\chi})}}{\Omega^2}\right\|^2_{S^2_{\infty,\bar{v}}}$$
$$\lesssim\left\|\mathcal{A}^{[i-2]}\slashed{\mathcal{D}}_2^\star\slashed{\nabla}\frac{\overset{(1)}{(\Omega\,\mathrm{tr}\,\underline{\chi})}[\underset{\vee}{\mathscr{S}}]}{\Omega^2}\right\|^2_{S^2_{\infty,v_0}}+\mathbb{F}_0^{i-3,T,\slashed{\nabla}}[\overset{(1)}{\Psi}].$$

Proof. Restricting the angular commuted (141) to the horizon, we have, on $\mathcal{H}^+$,

$$\Omega\slashed{\nabla}_4\left(\mathcal{A}^{[i]}\frac{\overset{(1)}{\underline{\hat{\chi}}}}{\Omega}\right)+\frac{1}{2M}\mathcal{A}^{[i]}\frac{\overset{(1)}{\underline{\hat{\chi}}}}{\Omega}=\frac{\Omega}{2M}\mathcal{A}^{[i]}\overset{(1)}{\hat{\chi}}-2\mathcal{A}^{[i]}\slashed{\mathcal{D}}_2^\star\overset{(1)}{\underline{\eta}}. \tag{408}$$

Hence, contracting with $\mathcal{A}^{[i]}\overset{(1)}{\underline{\hat{\chi}}}/\Omega$ and applying the Cauchy–Schwarz inequality on the right, in particular

$$\frac{1}{2}\partial_v\left|\mathcal{A}^{[i]}\frac{\overset{(1)}{\underline{\hat{\chi}}}}{\Omega}\right|^2+\frac{1}{4M}\left|\mathcal{A}^{[i]}\frac{\overset{(1)}{\underline{\hat{\chi}}}}{\Omega}\right|^2\lesssim|\mathcal{A}^{[i]}(\Omega\overset{(1)}{\hat{\chi}})|^2+|\mathcal{A}^{[i]}\slashed{\mathcal{D}}_2^\star\overset{(1)}{\underline{\eta}}|^2. \tag{409}$$

Taking into account (405) on the sphere S^2_{∞,v_0}, integration yields

$$\left\|\mathcal{A}^{[i]}\frac{\overset{(1)}{\underline{\hat{\chi}}}}{\Omega}\right\|^2_{S^2_{\infty,V}}+\int_{v_0}^{V}d\bar{v}\left\|\mathcal{A}^{[i]}\frac{\overset{(1)}{\underline{\hat{\chi}}}}{\Omega}\right\|^2_{S^2_{\infty,\bar{v}}}$$
$$\lesssim\left\|\mathcal{A}^{[i]}\frac{\overset{(1)}{\underline{\hat{\chi}}}[\underset{\vee}{\mathscr{S}}]}{\Omega}\right\|^2_{S^2_{\infty,v_0}}+\int_{v_0}^{V}d\bar{v}\,[\|\mathcal{A}^{[i]}(\Omega\overset{(1)}{\hat{\chi}})\|^2_{S^2_{\infty,\bar{v}}}+\|\mathcal{A}^{[i]}\slashed{\mathcal{D}}_2^\star\overset{(1)}{\underline{\eta}}\|^2_{S^2_{\infty,\bar{v}}}].$$

We now use (407), recall $\overset{(1)}{\hat{\chi}}[\hat{\mathscr{S}}]=\overset{(1)}{\hat{\chi}}[\underset{\vee}{\mathscr{S}}]$ and use Proposition 13.2.3 (recalling $\slashed{\mathrm{div}}\,\overset{(1)}{\hat{\chi}}=-\overset{(1)}{\beta}$ holds on $\mathcal{H}^+$ from (145)) to obtain the first estimate.

For the second, we proceed similarly. Commuting (135) and restricting to the horizon $\mathcal{H}^+$ yields

$$\Omega\slashed{\nabla}_4\left[\mathcal{A}^{[i-3]}\slashed{\mathcal{D}}_2^\star\slashed{\nabla}_A\frac{\overset{(1)}{(\Omega\,\mathrm{tr}\,\underline{\chi})}}{\Omega^2}\right]+\frac{1}{2M}\mathcal{A}^{[i-3]}\slashed{\mathcal{D}}_2^\star\slashed{\nabla}_A\frac{\overset{(1)}{(\Omega\,\mathrm{tr}\,\underline{\chi})}}{\Omega^2}=4\mathcal{A}^{[i-3]}\slashed{\mathcal{D}}_2^\star\slashed{\nabla}_A\overset{(1)}{\varrho}, \tag{410}$$

where we have used $\slashed{\mathcal{D}}_2^\star\slashed{\nabla}_A\Omega^{-1}\overset{(1)}{\Omega}[\hat{\mathscr{S}}](\infty,v,\theta,\phi)=0$ by (407) and $\slashed{\mathcal{D}}_2^\star\slashed{\nabla}_A(\slashed{\mathrm{div}}\,\overset{(1)}{\eta}+\overset{(1)}{\varrho})=0$ on $\mathcal{H}^+$ (for both $\underset{\vee}{\mathscr{S}}$ and $\hat{\mathscr{S}}$). Integrating the identity (410) as in the previous case yields the second estimate after applying Proposition 13.2.3 to control the flux on the right-hand side (recall (406)). $\square$

COROLLARY 14.1. *On the horizon $\mathcal{H}^+$, the geometric quantities of $\hat{\mathscr{S}}$ in Theorem 4 satisfy, for $i\geqslant 3$,*

$$\int_{v_0}^{\infty} d\bar{v} \left\| \Omega \nabla\!\!\!/_4 \mathcal{A}^{[i]} \frac{\overset{(1)}{\underline{\hat{\chi}}}}{\Omega} \right\|^2_{S^2_{\infty,\bar{v}}} \lesssim \left\| \mathcal{A}^{[i]} \frac{\overset{(1)}{\underline{\hat{\chi}}}[\mathscr{S}]}{\Omega} \right\|^2_{S^2_{\infty,v_0}} + \mathbb{F}_0^{i-3,T,\nabla\!\!\!/}[\overset{(1)}{\Psi}, \overset{(1)}{\psi}, \overset{(1)}{\alpha}].$$

Proof. Follows directly from (408), recalling (407) and using the flux bounds of Propositions 14.1.1 and 13.2.3. $\square$

COROLLARY 14.2. *On the horizon $\mathcal{H}^+$, the geometric quantities of $\hat{\mathscr{S}}$ in Theorem 4 satisfy in addition the $L^\infty_v L^2(S^2_{\infty,v})$-bound, for $i\geqslant 2$,*

$$\sup_v \left\| \Omega \nabla\!\!\!/_4 \mathcal{A}^{[i]} \frac{\overset{(1)}{\underline{\hat{\chi}}}}{\Omega} \right\|^2_{S^2_{\infty,v}} \lesssim \left\| \mathcal{A}^{[i]} \frac{\overset{(1)}{\underline{\hat{\chi}}}[\mathscr{S}]}{\Omega} \right\|^2_{S^2_{\infty,v_0}} + \sup_v \| r^{-1/2} \cdot \mathcal{A}^{[i-2]} \overset{(1)}{\psi} \Omega r^3 \|^2_{S^2_{u_0,v}} + \mathbb{F}_0^{i-2,T,\nabla\!\!\!/}[\overset{(1)}{\Psi}, \overset{(1)}{\psi}, \overset{(1)}{\alpha}].$$

Proof. Revisit (408) and use the $L^\infty_{u,v}$-bound of Proposition 14.1.1, the $L^\infty_{u,v}$-bound on $\mathcal{A}^{[i]}\overset{(1)}{\underline{\hat{\chi}}} = \mathcal{A}^{[i]}\overset{(1)}{\underline{\hat{\chi}}}[\mathscr{S}]$ of Proposition 13.2.1 and the $L^\infty_{u,v}$ bound on $\mathcal{A}^{[i]}\mathcal{D}\!\!\!/_2^\star \overset{(1)}{\underline{\eta}} = -\mathcal{A}^{[i]}\mathcal{D}\!\!\!/_2^\star \overset{(1)}{\eta}[\mathscr{S}]$ of Proposition 13.2.2. $\square$

If we use the polynomial decay estimates of Propositions 11.5.1 and 12.3.4, we also have, using Proposition 14.1.1 in conjunction with a pigeonhole principle, the following corollary.

COROLLARY 14.3. *On the horizon $\mathcal{H}^+$, the geometric quantities of $\hat{\mathscr{S}}$ in Theorem 4 satisfy the decay estimate*

$$\left\| \mathcal{A}^{[3]} \frac{\overset{(1)}{\underline{\hat{\chi}}}}{\Omega} \right\|^2_{S^2_{\infty,v}} + \left\| \mathcal{A}^{[2]} \Omega \nabla\!\!\!/_4 \frac{\overset{(1)}{\underline{\hat{\chi}}}}{\Omega} \right\|^2_{S^2_{\infty,v}} \lesssim \frac{1}{v^2} \left(\left\| \mathcal{A}^{[3]} \frac{\overset{(1)}{\underline{\hat{\chi}}}[\mathscr{S}]}{\Omega} \right\|^2_{S^2_{\infty,v}} + \mathbb{F}_0^{2,T}[\overset{(1)}{\Psi}] + \mathbb{F}_0[\overset{(1)}{\Psi}, \overset{(1)}{\psi}, \overset{(1)}{\alpha}] \right).$$

Proof. For the bound on $\overset{(1)}{\underline{\hat{\chi}}}$, we combine Proposition 14.1.1 with a simple dyadic argument. In particular, we use the fact that the fluxes appearing on the horizon on the right-hand side of (409) (after integration) satisfy the polynomial decay estimates of Proposition 13.2.4. To derive the bound for $\nabla\!\!\!/_4 \overset{(1)}{\underline{\hat{\chi}}}$, we revisit the identity (408) with $i=2$ and show that all other terms have the desired decay. The bound for two angular derivatives of $\overset{(1)}{\underline{\hat{\chi}}}$ has just been obtained. From Proposition 12.3.5 and the identity (332) restricted to the horizon, we see we have the decay bound for two angular derivatives of $\overset{(1)}{\hat{\chi}}$ on $S^2_{\infty,v}$. Finally, to estimate the term involving three derivatives of $\overset{(1)}{\underline{\eta}} = -\overset{(1)}{\eta} = -\overset{(1)}{\eta}[\mathscr{S}]$ in (408), we use the identity (333) and the fact that an estimate for $\|\Psi\|_{S^2_{\infty,v}}$ follows directly from Proposition 11.5.1 and 1-dimensional Sobolev embedding. $\square$

14.1.2. Controlling the gauge function

With Proposition 14.1.1 controlling $\overset{(1)}{\underline{\hat{\chi}}}$ on the horizon (from data in $\underset{\vee}{\mathscr{S}}$) and Corollary 13.3 controlling $\overset{(1)}{\underline{\hat{\chi}}}[\underset{\vee}{\mathscr{S}}]$ on the horizon (also from data in $\underset{\vee}{\mathscr{S}}$), we can infer boundedness of the gauge function.

Proposition 14.1.2. *The gauge function $f=f(v,\theta,\phi)$ associated with the pure gauge solution $\hat{\mathscr{G}}$ in Theorem 4 via Proposition 9.3.1 satisfies, for $i=5$,*

$$\int_{S^2} \sin\theta\, d\theta\, d\phi\, |\mathcal{A}^{[i]} r^2 \mathcal{D}_2^\star \nabla f|^2 \lesssim \|\mathcal{A}^{[i]}\Omega^{-1}\overset{(1)}{\underline{\hat{\chi}}}[\underset{\vee}{\mathscr{S}}]\|^2_{S^2_{\infty,v_0}} + \mathbb{E}_0, \tag{411}$$

$$\int_{S^2} \sin\theta\, d\theta\, d\phi\, |\mathcal{A}^{[i-1]} r^2 \mathcal{D}_2^\star \nabla \partial_v f|^2 \lesssim \|\mathcal{A}^{[i-1]}\Omega^{-1}\overset{(1)}{\underline{\hat{\chi}}}[\underset{\vee}{\mathscr{S}}]\|^2_{S^2_{\infty,v_0}} + \mathbb{E}_0, \tag{412}$$

where we have introduced the shorthand notation $\mathcal{D}_2^\star \nabla f = \mathcal{D}_2^\star \mathcal{D}_1^\star(-f,0)$. We also have the flux bound

$$\int_{v_0}^{\infty} d\bar{v} \int_{S^2} \sin\theta\, d\theta\, d\phi\, |\mathcal{A}^{[i-1]} r^2 \mathcal{D}_2^\star \nabla \partial_v f|^2 \lesssim \|\mathcal{A}^{[i-1]}\Omega^{-1}\overset{(1)}{\underline{\hat{\chi}}}[\underset{\vee}{\mathscr{S}}]\|^2_{S^2_{\infty,v_0}} + \mathbb{E}_0 \tag{413}$$

and the decay bound

$$v^2 \int_{S^2} \sin\theta\, d\theta\, d\phi\, |\mathcal{A}^{[2]} r^2 \mathcal{D}_2^\star \nabla \partial_v f|^2 \lesssim \|\mathcal{A}^{[3]}\Omega^{-1}\overset{(1)}{\underline{\hat{\chi}}}[\underset{\vee}{\mathscr{S}}]\|^2_{S^2_{\infty,v_0}} + \mathbb{E}_0. \tag{414}$$

Proof. We have, from Lemma 6.1.1,

$$r\cdot\Omega^{-1} r \overset{(1)}{\underline{\hat{\chi}}}[\hat{\mathscr{S}}] - r\cdot\Omega^{-1} r \overset{(1)}{\underline{\hat{\chi}}}[\underset{\vee}{\mathscr{S}}] = r\cdot\Omega^{-1} r \overset{(1)}{\underline{\hat{\chi}}}[\hat{\mathscr{G}}] = -2r^2 \mathcal{D}_2^\star \nabla f, \tag{415}$$

which, when restricted to the horizon $u=\infty$ (where $r=2M$), leads to (411) after using Proposition 14.1.1 and Corollary 13.3. For the second estimate, we commute the defining equation (214) with $\mathcal{A}^{[i-1]} r^2 \mathcal{D}_2^\star \nabla$, and estimate f by (411) and the quantity $\mathcal{A}^{i-1} r \mathcal{D}_2^\star(\overset{(1)}{\eta} + \overset{(1)}{\underline{\eta}})$ from Proposition 13.5.11 and (the twice angular commuted $S^2_{\infty,v}$-estimate of) Proposition 13.2.2. For the third estimate, we use again Lemma 6.1.1 to conclude that, on the horizon $\mathcal{H}^+$,

$$\Omega \nabla_4 (r \overset{(1)}{\underline{\hat{\chi}}}[\hat{\mathscr{S}}]\Omega^{-1}) - \Omega \nabla_4 (r \overset{(1)}{\underline{\hat{\chi}}}[\underset{\vee}{\mathscr{S}}]\Omega^{-1}) = -\frac{2}{2M} r^2 \mathcal{D}_2^\star \nabla \partial_v f.$$

The flux estimates of Corollary 14.1 (with $i=4$) and Corollary 13.8 produce (413). Combining Proposition 13.4.3 with Corollary 14.3 yields the bound (414). $\square$

14.1.3. Boundedness of the pure gauge and horizon-renormalised solution

Combining the estimates of Proposition 14.1.2 with Lemma 6.1.1, we can easily deduce the uniform boundedness of $\hat{\mathscr{G}}$ in Theorem 4, as well as deduce a uniform boundedness statement for $\hat{\mathscr{S}}$ from the estimate on $\underset{\vee}{\mathscr{S}}$ and $\hat{\mathscr{G}}$.

PROPOSITION 14.1.3. *The curvature components* $\overset{(1)}{\rho}$, $\overset{(1)}{\sigma}$, $\overset{(1)}{\beta}$ *and* $\overset{(1)}{\underline{\beta}}$ *of the pure gauge solution* $\hat{\mathscr{G}}$ *in Theorem 4 satisfy the same boundedness estimates as these quantities for* $\underset{\vee}{\mathscr{S}}$ *in Proposition 13.5.2, provided the term*

$$\left\|\mathcal{A}^{[5]}\Omega^{-1}\overset{(1)}{\underline{\hat{\chi}}}[\underset{\vee}{\mathscr{S}}]\right\|^2_{S^2_{\infty,v_0}}$$

is added on all right-hand sides of that proposition. Furthermore, the Ricci and metric coefficients of the solution $\hat{\mathscr{G}}$ *satisfy, for all* u *and* v,

$$\begin{aligned}
&r\|r^{-1}\cdot\mathcal{A}^{[4]}r\mathcal{D}_2^{\star}\overset{(1)}{\underline{\eta}}\|_{S^2_{u,v}}+r^2\|r^{-1}\cdot\mathcal{A}^{[2]}r\mathcal{D}_2^{\star}\overset{(1)}{\underline{\eta}}\|_{S^2_{u,v}}+r\|r^{-1}\cdot\mathcal{A}^{[4]}r\mathcal{D}_2^{\star}\overset{(1)}{\eta}\|_{S^2_{u,v}}\\
&\quad+\|r^{-1}\cdot\mathcal{A}^{[5]}r\Omega^{-1}\overset{(1)}{\underline{\hat{\chi}}}\|_{S^2_{u,v}}+\|r^{-1}\cdot r^3\mathcal{A}^{[3]}\mathcal{D}_2^{\star}\nabla\!\!\!/\,\Omega^{-2}(\Omega\overset{(1)}{\mathrm{tr}}\,\underline{\chi})\|_{S^2_{u,v}}\\
&\quad+\|r^{-1}\cdot\Omega^{-2}\mathcal{A}^{[2]}r^4\mathcal{D}_2^{\star}\nabla\!\!\!/\,(\Omega\overset{(1)}{\mathrm{tr}}\,\chi)\|_{S^2_{u,v}}+\left\|r^{-1}\cdot\sqrt{r}\mathcal{A}^{[3]}r^2\mathcal{D}_2^{\star}\nabla\!\!\!/\,\frac{\overset{(1)}{\sqrt{\not g}}}{\sqrt{\not g}}\right\|_{S^2_{u,v}}\\
&\quad+\|r^{-1}\cdot\sqrt{r}\mathcal{A}^{[5]}\overset{(1)}{\hat{\not g}}\|_{S^2_{u,v}}+r\|r^{-1}\cdot\mathcal{A}^{[2]}r\mathcal{D}_2^{\star}\overset{(1)}{b}\|_{S^2_{u,v}}+r^{-\varepsilon}\|r^{-1}\cdot\mathcal{A}^{[4]}r\mathcal{D}_2^{\star}\overset{(1)}{b}\|_{S^2_{u,v}}\\
&\qquad\lesssim\|\mathcal{A}^{[5]}\Omega^{-1}\overset{(1)}{\underline{\hat{\chi}}}[\underset{\vee}{\mathscr{S}}]\|_{S^2_{\infty,v_0}}+\sqrt{\mathbb{E}_0}.
\end{aligned}$$

Proof. Use Lemma 6.1.1 in conjunction with Corollary 14.1.2. □

We leave stating the estimate for five angular derivatives of $(\Omega\overset{(1)}{\mathrm{tr}}\,\chi)$ arising from Corollary 13.10 and more refined estimates for the metric coefficients to the reader. Note that the estimate for $\overset{(1)}{\hat{\chi}}$ is unchanged, as $\hat{\mathscr{G}}$ has $\overset{(1)}{\hat{\chi}}=0$. We finally remark that, for $\overset{(1)}{b}$, stronger estimates hold, but Corollary 14.4 would not be true.

In view of $\hat{\mathscr{S}}=\underset{\vee}{\mathscr{S}}+\hat{\mathscr{G}}$, we immediately conclude the following.

COROLLARY 14.4. *The estimates of Proposition 14.1.3 hold also for the solution* $\hat{\mathscr{S}}$.

Proof. Compare the estimates with

- Propositions 13.5.3 and 13.5.10 for $\overset{(1)}{\underline{\eta}}$;
- Propositions 13.5.3 and 13.5.11 for $\overset{(1)}{\eta}$;
- Corollary 13.3 for $\overset{(1)}{\underline{\hat{\chi}}}$;
- Proposition 13.5.5 for $(\Omega\overset{(1)}{\mathrm{tr}}\,\underline{\chi})$;

- Corollary 13.10 for $(\Omega \overset{(1)}{\mathrm{tr}}\, \chi)$;
- Proposition 13.5.12 for the metric coefficients. Note that the initial terms appearing in (400) and (401) can be estimated by $\mathbb{E}_0$ using the round sphere condition (191) and (192). $\square$

Remark 14.1. Proposition 14.1.3 and Corollary 14.4 can be paraphrased by saying that the solution in the future gauge $\hat{\mathscr{S}} = \underset{\vee}{\mathscr{S}} + \hat{\mathscr{G}}$ satisfies the same boundedness estimates as the solution $\underset{\vee}{\mathscr{S}}$ in the initial data gauge. In particular, there is no loss of derivatives at the level of flux bounds. It is important to note, however, that the estimates we obtain for five angular derivatives of $\overset{(1)}{\underline{\eta}}$ and $\Omega^{-1}\overset{(1)}{\Omega}$ are slightly weaker in terms of their r-decay towards null infinity than those for $\overset{(1)}{\eta}[\underset{\vee}{\mathscr{S}}]$ and $\Omega^{-1}\overset{(1)}{\Omega}[\underset{\vee}{\mathscr{S}}]$. This is because the estimate establishing decay of f_v (413) is not optimal in terms of regularity. Compare the estimates (412) and (414).

14.2. Integrated local energy decay

We now turn to show integrated decay statements for the quantities associated with $\hat{\mathscr{S}}$.

Recall from our comments above that, in addition to the estimates for the gauge-invariant quantities, the results of §13.3, in particular the integrated decay statement Proposition 13.3.1, remain valid for $\overset{(1)}{\hat{\chi}}[\hat{\mathscr{S}}]$.

The first quantity for which one could not obtain integrated decay in the initial data normalised gauge was the quantity $\overset{(1)}{\underline{\hat{\chi}}}[\underset{\vee}{\mathscr{S}}]$. In §14.2.1 below, we will succeed to obtain an integrated local energy decay statement for $\overset{(1)}{\underline{\hat{\chi}}}[\hat{\mathscr{S}}]$, obtaining also a similar estimate for $\nabla\!\!\!/_4 \overset{(1)}{\underline{\hat{\chi}}}$ in §14.2.2. We will then finally unravel the decay hierarchy, proving successively integrated local energy decay for $\overset{(1)}{\rho}$, $\overset{(1)}{\sigma}$ and $\overset{(1)}{\underline{\beta}}$ in §14.2.3, $\overset{(1)}{\eta}$ and $\overset{(1)}{\underline{\eta}}$ in §14.2.4 and $(\Omega \overset{(1)}{\mathrm{tr}}\, \underline{\chi})$ in §14.2.5. Finally, we will obtain various refined statements for higher angular derivatives in §14.2.6, which allow to infer integrated local energy decay for $\overset{(1)}{\beta}$.

14.2.1. Integrated decay for angular derivatives of $\overset{(1)}{\underline{\hat{\chi}}}$

Using the new horizon fluxes obtained in the previous section, we will now obtain global control on angular derivatives of $\overset{(1)}{\underline{\hat{\chi}}}$.

PROPOSITION 14.2.1. *We have the following integrated decay estimate for the ingoing shear $\underline{\hat{\chi}}$ of the solution $\hat{\mathscr{S}}$ in Theorem 4. For any $v \geqslant v_0$ and $i=4,5$, we have*

$$\int_{v_0}^{v} d\bar{v} \int_{u}^{\infty} d\bar{u} \int_{S^2_{u,\bar{v}}} \sin\theta\, d\theta\, d\phi\, \frac{\Omega^2}{r^{1+\varepsilon}} \left| \mathcal{A}^{[i]} \left(\frac{r \overset{(1)}{\underline{\hat{\chi}}}}{\Omega} \right) \right|^2 \lesssim \|\mathcal{A}^{[i-2]} \overset{(1)}{\underline{\psi}} \Omega^{-1}\|^2_{S^2_{\infty,v_0}} + \mathbb{E}_0.$$

In view of Proposition 12.3.3, the first term can be dropped for $i=4$.

Proof. To show the statement for $i=5$, we contract the $r\not{\mathcal{D}}^\star_2$-commuted (358) by $r^{-2-\varepsilon}\mathcal{A}^{[3]}\overset{(1)}{Y}$, to obtain

$$\begin{aligned}
&\frac{1}{2}\partial_u[-r^{-2-\varepsilon}|\mathcal{A}^{[3]}\overset{(1)}{Y}|^2] + \frac{2+\varepsilon}{2r^{3+\varepsilon}}\Omega^2|\mathcal{A}^{[3]}\overset{(1)}{Y}|^2 \\
&\quad \leqslant \frac{1}{2}\frac{1}{r^{3+\varepsilon}}\Omega^2|\mathcal{A}^{[3]}\overset{(1)}{Y}|^2 \\
&\qquad + C\cdot\frac{\Omega^2}{r^{1+\varepsilon}}(|\Omega^{-1}\not{\nabla}_3(r\,\not{\mathrm{div}}\,\overset{(1)}{\underline{\Psi}})|^2 + |\Omega^{-1}\not{\nabla}_3(r\,\not{\mathrm{div}}\,\Omega^{-1}\overset{(1)}{\underline{\psi}}r^3)|^2 \\
&\qquad\qquad + |r\,\not{\mathrm{div}}\,\Omega^{-1}\overset{(1)}{\underline{\psi}}r^3|^2 + |r\,\not{\mathrm{div}}\,\Omega^{-2}\overset{(1)}{\underline{\alpha}}r|^2).
\end{aligned}$$

Absorbing the first term on the right by the left-hand side, and using the estimate (243) of Theorem 2 (for the underlined quantities), we obtain an integrated decay estimate for $\mathcal{A}^{[3]}\overset{(1)}{Y}$ after integrating the estimate over spacetime, observing that the flux term on the horizon has the wrong sign, but is controlled from Propositions 12.2.3 and 14.1.1 using the definition of $\overset{(1)}{Y}$, (218). On the left-hand side, we use again (218) and that we control $\mathcal{A}^{[3]}\overset{(1)}{\underline{\psi}}$ from the integrated decay statement of Theorem 2 applied with $n=2$ to descend to the desired estimate for angular derivatives of $\overset{(1)}{\underline{\hat{\chi}}}$. This establishes the estimate claimed with an additional term $\|\mathcal{A}^{[5]}\Omega^{-1}\overset{(1)}{\underline{\hat{\chi}}}[\overset{\vee}{\mathscr{S}}]\|^2_{S^2_{\infty,v_0}}$ on the right-hand side, which entered when we applied Proposition 14.1.1. Corollary 13.3 allows to replace it by $\|\mathcal{A}^{[3]}\overset{(1)}{\underline{\psi}}\Omega^{-1}\|^2_{S^2_{\infty,v_0}}$. The statement for $i=4$ is proven entirely analogously directly from (358), without an additional $r\not{\mathcal{D}}^\star_2$-commutation. $\square$

14.2.2. Integrated decay for angular derivatives of $\not{\nabla}_4\overset{(1)}{\underline{\hat{\chi}}}$

PROPOSITION 14.2.2. *We have the following integrated decay estimate for the quantity $\not{\nabla}_4\underline{\hat{\chi}}$ of the solution $\hat{\mathscr{S}}$ in Theorem 4. For any $v \geqslant v_0$,*

$$\int_{v_0}^{v} d\bar{v} \int_{u}^{\infty} d\bar{u} \int_{S^2_{u,\bar{v}}} \sin\theta\, d\theta\, d\phi\, \Omega^2 r^{-1-\varepsilon} \left| \mathcal{A}^{[4]} r\Omega\not{\nabla}_4 \left(\frac{r\overset{(1)}{\underline{\hat{\chi}}}}{\Omega} \right) \right|^2 \lesssim \mathbb{E}_0.$$

Proof. We commute (365),

$$\nabla\!\!\!/_3(\mathcal{A}^{[2]}\Omega\nabla\!\!\!/_4\overset{(1)}{Y})=r^3\Omega\mathcal{A}^{[2]}\overset{(1)}{\underline{P}}+3M(2r\mathcal{A}^{[2]}\overset{(1)}{\underline{\psi}}-2\widehat{\omega}\mathcal{A}^{[2]}\overset{(1)}{\underline{\alpha}}r),$$

which we contract with $r^{-\varepsilon}\mathcal{A}^{[2]}\Omega\nabla\!\!\!/_4\overset{(1)}{Y}$, to obtain

$$\begin{aligned}&\frac{1}{2}\partial_u[-r^{-\varepsilon}|\mathcal{A}^{[2]}\Omega\nabla\!\!\!/_4\overset{(1)}{Y}|^2]+\frac{\varepsilon}{2r^{1+\varepsilon}}\Omega^2|\mathcal{A}^{[2]}\Omega\nabla\!\!\!/_4\overset{(1)}{Y}|^2\\ &\quad\leqslant\frac{\varepsilon}{4}\frac{1}{r^{1+\varepsilon}}\Omega^2|\mathcal{A}^{[2]}\Omega\nabla\!\!\!/_4\overset{(1)}{Y}|^2\\ &\qquad+C_\varepsilon\Omega^2r^{1-\varepsilon}r^{-4}(|\mathcal{A}^{[2]}\overset{(1)}{\underline{\Psi}}|^2+|\mathcal{A}^{[2]}\overset{(1)}{\underline{\psi}}r^3\Omega^{-1}|^2+|\mathcal{A}^{[2]}\overset{(1)}{\underline{\alpha}}r\Omega^{-2}|^2).\end{aligned}$$

We absorb the first term on the right by the left-hand side. Upon integration over a spacetime region, for the last term on the right we use the integrated decay estimate on $\overset{(1)}{\underline{\Psi}}$ of Theorem 1, and the integrated decay estimates on $\overset{(1)}{\underline{\psi}}$ and $\overset{(1)}{\underline{\alpha}}$ of Theorem 2. The flux term on the horizon (which has a bad sign) arising from the first term on the left is controlled by inserting the commuted (370), which reads

$$\mathcal{A}^{[2]}\Omega\nabla\!\!\!/_4\overset{(1)}{Y}=\frac{\Omega^2}{r}\mathcal{A}^{[2]}\overset{(1)}{Y}+r\cdot\mathcal{A}^{[4]}\Omega\nabla\!\!\!/_4\left(\frac{r\overset{(1)}{\widehat{\underline{\chi}}}}{\Omega}\right)-r^4\mathcal{A}^{[2]}\overset{(1)}{\underline{P}}-2Mr^2\mathcal{A}^{[2]}\frac{\overset{(1)}{\underline{\psi}}}{\Omega}.\tag{416}$$

Note that, when considering the horizon flux, the first term on the right vanishes, the second is controlled by Corollary 14.1 with i=4, and the last two by (242) of Theorem 2 for n=1. This produces an integrated decay for $\nabla\!\!\!/_4\overset{(1)}{Y}$. To descend to the desired quantity $\mathcal{A}^{[4]}r\Omega\nabla\!\!\!/_4(r\overset{(1)}{\widehat{\underline{\chi}}}/\Omega)$, we use once more the identity (416), now expressing the second term on the right in terms of everything else, in conjunction with the integrated decay estimate of Proposition 14.2.1, the definition of $\overset{(1)}{Y}$ (see (218)), and the estimate (243) of Theorem 2 (again n=1 is actually sufficient). This establishes the estimate with the additional term $\|\mathcal{A}^{[4]}\overset{(1)}{\widehat{\underline{\chi}}}[\mathscr{S}_\vee]\Omega^{-1}\|^2_{S^2_{\infty,v_0}}$ on the right (having entered through Corollary 14.1). Using (218) to re-express $\overset{(1)}{\widehat{\underline{\chi}}}$ in terms of $\overset{(1)}{Y}$ and $\overset{(1)}{\underline{\psi}}$, and applying Propositions 12.3.3 and 13.4.1 (note Remark 13.6) we deduce $\|\mathcal{A}^{[4]}\overset{(1)}{\widehat{\underline{\chi}}}[\mathscr{S}_\vee]\Omega^{-1}\|^2_{S^2_{\infty,v_0}}\lesssim\mathbb{E}_0$. □

14.2.3. Integrated decay for $\overset{(1)}{\varrho}$, $\overset{(1)}{\sigma}$ and $\overset{(1)}{\underline{\beta}}$

With the integrated decay estimates on $\overset{(1)}{\widehat{\underline{\chi}}}$, we can easily show decay of all curvature components. Recall that, for $\overset{(1)}{\alpha}$ and $\overset{(1)}{\underline{\alpha}}$, we already have these statements from Proposition 12.3.2. The estimate for $\overset{(1)}{\beta}$ will be proven in Proposition 14.2.8 after we have estimated four angular derivatives of $\overset{(1)}{\widehat{\chi}}$. (The i=2 non-degenerate version for $\overset{(1)}{\beta}$ can be proven at this point already; cf. the proof of Proposition 14.2.8.)

PROPOSITION 14.2.3. *For $i=3$ the following holds for the geometric quantities of $\hat{\mathscr{S}}$ in Theorem 4:*

$$\int_{v_0}^{\infty} dv \int_{u_0}^{\infty} du \, \frac{\Omega^2}{r^{1+\varepsilon}} \left(1-\frac{3M}{r}\right)^2 \\ \times (\|r^{-1}\cdot \mathcal{A}^{[i]} \mathcal{D}_2^\star \mathcal{D}_1^\star (r^3 \overset{(1)}{\varrho}, r^3 \overset{(1)}{\sigma})\|^2_{S^2_{u,v}} + \|r^{-1}\cdot \mathcal{A}^{[i+1]} \mathcal{D}_2^\star (\overset{(1)}{\underline{\beta}} r^2 \Omega^{-1})\|^2_{S^2_{u,v}}) \lesssim \mathbb{E}_0.$$

For $i=2$, this estimate holds without the degenerating factor of $(1-3M/r)^2$.

Proof. The bound on $(r^3 \overset{(1)}{\varrho}, r^3 \overset{(1)}{\sigma})$ is a direct consequence of the identity (374) applied with $i=3$ (or $i=2$), the estimate (240) of Theorem 1 applied with $n=2$ (note this provides a degenerate (near $r=3M$) integrated decay estimate for $\mathcal{A}^{[3]}P$ and a non-degenerate estimate for $\mathcal{A}^{[2]}P$), Propositions 14.2.1 (applied with $i=4$) and 13.3.3.

For the bound on $\overset{(1)}{\underline{\beta}}$, we use the identity (375) applied with $i=4$ ($i=3$) and rewrite the term $\mathcal{A}^{[4]} \overset{(1)}{\underline{\psi}}$ ($\mathcal{A}^{[3]} \overset{(1)}{\underline{\psi}}$) using the $\mathcal{A}^{[2]}$ ($\mathcal{A}^{[1]}$) commuted (303). All terms can then be estimated by the integrated decay estimates of Theorems 1 and 2 and Proposition 14.2.1 applied with $i=4$. □

14.2.4. Integrated decay for $\mathcal{D}_2^\star \overset{(1)}{\eta}$ and $\mathcal{D}_2^\star \overset{(1)}{\underline{\eta}}$

PROPOSITION 14.2.4. *We have the following integrated decay estimate on the solution $\hat{\mathscr{S}}$ in Theorem 4:*

$$\int_{v_0}^{\infty} dv \int_{u_0}^{\infty} du \, \Omega^2 r^{1-\varepsilon} (\|r^{-1}\cdot \mathcal{A}^{[3]} r \mathcal{D}_2^\star \overset{(1)}{\underline{\eta}}\|^2_{S^2_{u,v}} + \|r^{-1}\cdot \mathcal{A}^{[3]} r \mathcal{D}_2^\star \overset{(1)}{\eta}\|^2_{S^2_{u,v}}) \lesssim \mathbb{E}_0.$$

Proof. This follows directly from (379) and (380) applied with $i=3$ using the integrated decay estimates of Propositions 14.2.1 (with $i=4$), 14.2.2 and 13.3.3. □

14.2.5. Integrated decay for angular derivatives of $(\Omega \, \overset{(1)}{\mathrm{tr}\, \underline{\chi}})$

PROPOSITION 14.2.5. *We have the following integrated decay estimate for the solution $\hat{\mathscr{S}}$ in Theorem 4: for $i=2,3$,*

$$\int_{v_0}^{\infty} dv \int_{u_0}^{\infty} du \, \frac{\Omega^2}{r^{1+\varepsilon}} \left\| r^{-1} \cdot \frac{\mathcal{A}^{[i]} r^3 \mathcal{D}_2^\star \nabla (\Omega \, \overset{(1)}{\mathrm{tr}\, \underline{\chi}})}{\Omega^2} \right\|^2_{S^2_{u,v}} \lesssim \|\mathcal{A}^{[i]} \overset{(1)}{\underline{\psi}} \Omega^{-1}\|^2_{S^2_{\infty, v_0}} + \mathbb{E}_0.$$

In view of Proposition 12.3.3, the first term can be dropped for $i=3$.

Proof. Follows directly from (383) using the integrated decay estimates of Propositions 14.2.1, 14.2.4 and (the $\mathcal{A}^{[2]}$ commuted version of) 12.2.2 (or apply Theorem 2). □

14.2.6. Refined estimates for higher angular derivatives and integrated decay for $\overset{(1)}{\beta}$

We easily derive the following analogue of Proposition 13.5.6 by integrating both in u and v (and not in v only as in the derivation of Proposition 13.5.6).

Proposition 14.2.6. *We have, for $n \geqslant 0$, any $i \in \mathbb{N}$ and any $u \geqslant u_0$, the estimate ($d\omega = \sin\theta \, d\theta \, d\phi$)*

$$\int_{u_0}^{\infty} d\bar{u} \int_{S^2_{\bar{u},v}} d\omega \frac{|\mathcal{A}^{[i]} r \not{\mathcal{D}}_2^\star \overset{(1)}{Z}|^2}{r^n \Omega^2} + \int_{u_0}^{\infty} d\bar{u} \int_{v_0}^{v} d\bar{v} \int_{S^2_{\bar{u},\bar{v}}} d\omega \, \frac{1}{r} \cdot \frac{|\mathcal{A}^{[i]} r \not{\mathcal{D}}_2^\star \overset{(1)}{Z}|^2}{r^n \Omega^2}$$
$$\lesssim \int_{u_0}^{\infty} d\bar{u} \int_{S^2_{\bar{u},v_0}} d\omega \, \frac{|\mathcal{A}^{[i]} r \not{\mathcal{D}}_2^\star \overset{(1)}{Z}[\underset{\vee}{\mathscr{S}}]|^2}{r^n \Omega^2} + \int_{u_0}^{\infty} d\bar{u} \int_{v_0}^{v} d\bar{v} \int_{S^2_{\bar{u},\bar{v}}} d\omega \, r^{5-n} \Omega^2 |\mathcal{A}^{[i]} \not{\mathcal{D}}_2^\star (\overset{(1)}{\eta} + \overset{(1)}{\underline{\eta}})|^2$$

for the solution $\hat{\mathscr{S}}$ in Theorem 4.

We remind the reader that, following our notation, $\overset{(1)}{\eta} + \overset{(1)}{\underline{\eta}}$ on the right denote the geometric quantities of $\hat{\mathscr{S}}$. In the above Proposition we have used that $\overset{(1)}{Z}[\hat{\mathscr{S}}] = \overset{(1)}{Z}[\underset{\vee}{\mathscr{S}}]$ holds on $v = v_0$. To deduce the latter, note that $\overset{(1)}{Z}[\hat{\mathscr{G}}] = 0$ holds on $v = v_0$, which follows from Proposition 9.3.1 and Lemma 6.1.1. (Indeed, note that $\eta[\hat{\mathscr{G}}] = 0$ on $v = v_0$ by (214) and that the expression $r \not{\nabla}_A (\Omega \overset{(1)}{\mathrm{tr}} \chi) - 2\Omega^2 \overset{(1)}{\underline{\eta}}$ vanishes for any pure gauge solution in Lemma 6.1.1.) Recall also that $\overset{(1)}{Z} \sim \Omega^2$ holds on $v = v_0$ near the horizon by Proposition 9.4.3, so the first term in the second line is indeed finite for smooth data. We can reinsert the definition of $\overset{(1)}{Z}$ for the second term on the left to obtain an estimate for i angular derivatives of $(\Omega \overset{(1)}{\mathrm{tr}} \chi)$ in terms of $i-1$ angular derivatives of $\overset{(1)}{\eta} + \overset{(1)}{\underline{\eta}}$.

Corollary 14.5. *We have, for $n \geqslant 0$, any $i \in \mathbb{N}$ and any $(u \geqslant u_0, v \geqslant v_0)$, the estimate*

$$\int_{u_0}^{\infty} d\bar{u} \int_{v_0}^{v} d\bar{v} \int_{S^2_{\bar{u},\bar{v}}} d\omega \, r^{7-n} \Omega^2 \left| \mathcal{A}^{[i]} \not{\mathcal{D}}_2^\star \not{\nabla} \frac{(\Omega \overset{(1)}{\mathrm{tr}} \chi)}{\Omega^2} \right|^2$$
$$\lesssim \int_{u_0}^{\infty} d\bar{u} \int_{S^2_{\bar{u},v_0}} d\omega \frac{|\mathcal{A}^{[i]} r \not{\mathcal{D}}_2^\star \overset{(1)}{Z}[\underset{\vee}{\mathscr{S}}]|^2}{r^n \Omega^2} + \int_{u_0}^{\infty} d\bar{u} \int_{v_0}^{v} d\bar{v} \int_{S^2_{\bar{u},\bar{v}}} d\omega r^{5-n} \Omega^2 |\mathcal{A}^{[i]} \not{\mathcal{D}}_2^\star (\overset{(1)}{\eta} + \overset{(1)}{\underline{\eta}})|^2$$

for the solution $\hat{\mathscr{S}}$ in Theorem 4.

Again, consistent with our notation, $\overset{(1)}{\eta} + \overset{(1)}{\underline{\eta}}$ above denotes the geometric quantities of $\hat{\mathscr{S}}$. We conclude the following result.

Proposition 14.2.7. *We have the integrated decay estimate*

$$\int_{u_0}^{\infty} d\bar{u} \int_{v_0}^{\infty} d\bar{v}\, \frac{\Omega^2}{r^{3+\varepsilon}} \left(\left\| r^{-1} \cdot \mathcal{A}^{[3]} r^2 \mathcal{D}_2^{\star} \nabla\!\!\!/ \frac{r^2 (\Omega \overset{(1)}{\operatorname{tr}} \chi)}{\Omega^2} \right\|^2_{S^2_{\bar{u},\bar{v}}} + \| r^{-1} \cdot \mathcal{A}^{[3]} r^2 \mathcal{D}_2^{\star} \operatorname{div}\!\!\!/ (\Omega \overset{(1)}{\widehat{\chi}} r^2) \|^2_{S^2_{\bar{u},\bar{v}}} \right)$$
$$\lesssim \mathbb{E}_0 + \int_{u_0}^{\infty} d\bar{u} \int_{S^2_{\bar{u},v_0}} d\omega\, \frac{|\mathcal{A}^{[i]} r \mathcal{D}_2^{\star} \overset{(1)}{Z}[\underset{\vee}{\mathscr{S}}]|^2}{r^n \Omega^2}$$

for the geometric quantities of $\hat{\mathscr{S}}$ *in Theorem* 4.

Proof. Apply Corollary 14.5 with $n=2+\varepsilon$ and $i=3$, and use Proposition 14.2.4 to obtain the first part of the estimate. Then, use identity (389) in conjunction with Proposition 14.2.4, Theorem 2 and Proposition 13.3.3 to obtain the second part. □

Remark 14.2. The weights near infinity are far from optimal. The weight near infinity for $\mathcal{A}^{[5]} \overset{(1)}{\widehat{\chi}}$ can be improved a posteriori from the transport equation (139) and the (degenerate near $r=3M$) integrated decay estimate for $\mathcal{A}^{[5]} \overset{(1)}{\alpha}$. See also Corollary 13.10.

The weights are sufficient to prove the integrated decay estimate of Proposition 14.2.3 for the missing curvature component $\overset{(1)}{\beta}$.

Proposition 14.2.8. *For the geometric quantities of* $\hat{\mathscr{S}}$ *in Theorem* 4, *we have the following integrated decay estimate for* $i=3$:

$$\int_{u_0}^{\infty} du \int_{v_0}^{\infty} dv\, \frac{\Omega^2}{r^{1+\varepsilon}} \left(1 - \frac{3M}{r}\right)^2 (\| r^{-1} \cdot \mathcal{A}^{[i+1]} r \mathcal{D}_2^{\star} (\overset{(1)}{\beta} r^3 \Omega) \|^2_{S^2_{u,v}}) \tag{417}$$
$$\lesssim \mathbb{E}_0 + \int_{u_0}^{\infty} d\bar{u} \int_{S^2_{\bar{u},v_0}} \sin\theta\, d\theta\, d\phi \frac{|\mathcal{A}^{[i]} r \mathcal{D}_2^{\star} \overset{(1)}{Z}[\underset{\vee}{\mathscr{S}}]|^2}{r^n \Omega^2}.$$

For $i=2$, *this estimate holds without the degenerating factor of* $(1-3M/r)^2$ *on the left, and without the last term on the right.*

Proof. For $i=3$, this follows from the identity (395) and application of Proposition 14.2.7 (to estimate the $\frac{3}{2}\varrho \mathcal{A}^{[4]}(\Omega \overset{(1)}{\widehat{\chi}})$ term), as well as using the identity (300) commuted with $\mathcal{A}^{[2]}$ to estimate the $\mathcal{A}^{[4]}(\Omega \overset{(1)}{\psi})$ term (through application of Theorems 1 and 2). For $i=2$, this also follows from (395), but observing that now Proposition 13.3.3 already estimates three angular derivatives of $\overset{(1)}{\widehat{\chi}}$, while (the twice angular commuted) Proposition 12.2.2 estimates (non-degenerately) three angular derivatives of $\overset{(1)}{\psi}$. □

Finally, repeating the arguments[35] leading to Propositions 13.5.9, 13.5.10 and 13.5.11, we can prove integrated decay for five derivatives of $\overset{(1)}{\eta}$ and $\overset{(1)}{\underline{\eta}}$ of the solution $\hat{\mathscr{S}}$,

([35]) All that is required is to insert an additional u-integration everywhere.

i.e. the estimate

$$\int_{v_0}^{\infty} d\bar v \int_{u_0}^{\infty} d\bar u \int_{S^2_{\bar u,\bar v}} \sin\theta\, d\theta\, d\phi\, \Omega^2 r^{3-\varepsilon}\big(|\mathcal{A}^{[4]}\mathcal{D}_2^{\star}\overset{(1)}{\eta}|^2+|\mathcal{A}^{[4]}\mathcal{D}_2^{\star}\overset{(1)}{\underline{\eta}}|^2\big) \lesssim \mathbb{E}_0+\int_{u_0}^{\infty} d\bar u \int_{S^2_{\bar u,v_0}} \sin\theta\, d\theta\, d\phi\, \frac{|\mathcal{A}^{[i]} r\mathcal{D}_2^{\star}\overset{(1)}{Z}[\underset{\vee}{\mathscr{S}}]|^2}{r^n\Omega^2}. \tag{418}$$

14.3. Polynomial decay estimates and conclusions

Finally, in this section we prove appropriate L^2 polynomial decay of all quantites associated with $\hat{\mathscr{S}}$. This corresponds to statement (3) of Theorem 4. We shall then infer Corollary 10.3 giving pointwise estimates.

We will consider first Ricci coefficients in §14.3.1, and then the metric components themselves in §14.3.2. We shall treat Corollary 10.3 in §14.3.3.

14.3.1. Polynomial decay for $\overset{(1)}{\hat{\chi}}$, $\overset{(1)}{\underline{\hat{\chi}}}$, $\overset{(1)}{\eta}$ and $\overset{(1)}{\underline{\eta}}$

PROPOSITION 14.3.1. *Fix r_0 as in Proposition 11.5.1, let $v\geqslant v_0$ and recall the notation $u(v,r_0)$. We have the following decay estimates for the geometric quantities of $\hat{\mathscr{S}}$ (and equivalently $\underset{\vee}{\mathscr{S}}$) in Theorem 4*:

$$\int_{u(v,r_0)}^{\infty} d\bar u \left(\frac{\Omega^2}{r^{2+\varepsilon}}\|r^{-1}\cdot r^2\overset{(1)}{\hat{\chi}}\Omega\|^2_{S^2_{\bar u,V}}+\frac{\Omega^2}{r^{\varepsilon}}\|r^{-1}\cdot\Omega^{-1}\nabla\!\!\!/_3(r^2\overset{(1)}{\hat{\chi}}\Omega)\|^2_{S^2_{\bar u,V}} +\frac{\Omega^2}{r^{\varepsilon}}\|r^{-1}\cdot\Omega^{-1}\nabla\!\!\!/_3(\Omega^{-1}\nabla\!\!\!/_3(r^2\overset{(1)}{\hat{\chi}}\Omega))\|^2_{S^2_{\bar u,V}}\right)\lesssim\frac{1}{v^2}\mathbb{E}_0 \tag{419}$$

for all $V\geqslant v_0$ and

$$\int_{v}^{\infty} d\bar v\int_{u(v,r_0)}^{\infty} d\bar u\left(\frac{\Omega^2}{r^{3+\varepsilon}}\|r^{-1}\cdot r^2\overset{(1)}{\hat{\chi}}\Omega\|^2_{S^2_{\bar u,\bar v}}+\frac{\Omega^2}{r^{1+\varepsilon}}\|r^{-1}\cdot\Omega\nabla\!\!\!/_3(r^2\overset{(1)}{\hat{\chi}}\Omega)\|^2_{S^2_{\bar u,\bar v}} +\frac{\Omega^2}{r^{1+\varepsilon}}\|r^{-1}\cdot\Omega^{-1}\nabla\!\!\!/_3(\Omega^{-1}\nabla\!\!\!/_3(r^2\overset{(1)}{\hat{\chi}}\Omega))\|^2_{S^2_{\bar u,\bar v}}\right)\lesssim\frac{1}{v^2}\mathbb{E}_0. \tag{420}$$

Proof. Recall that $\overset{(1)}{\hat{\chi}}[\hat{\mathscr{S}}]=\overset{(1)}{\hat{\chi}}[\underset{\vee}{\mathscr{S}}]$. By Proposition 13.2.4, we have, on the horizon,

$$\int_{v}^{\infty} d\bar v\,(\|\Omega\overset{(1)}{\hat{\chi}}\|^2_{S^2_{\infty,\bar v}}+\|\Omega^{-1}\nabla\!\!\!/_3(\Omega\overset{(1)}{\hat{\chi}})\|^2_{S^2_{\infty,\bar v}})\lesssim\frac{1}{v^2}(\mathbb{F}_0^{2,T}[\overset{(1)}{\Psi}]+\mathbb{F}_0[\overset{(1)}{\Psi},\overset{(1)}{\psi},\overset{(1)}{\alpha}]). \tag{421}$$

The *boundedness* of these fluxes was crucial in the proof of Proposition 13.3.1. We now repeat this proof using instead (421) to generate the desired decay estimates. The procedure is outlined below.

Choosing a dyadic sequence $v_i=2^i v_0$, we can, by Proposition 13.3.1 and the mean value theorem, extract a sequence $\tilde{v}_i$ of slices $v_i \leqslant \tilde{v}_i \leqslant v_{i+1}$ with the property that

$$\int_{u_0}^{\infty} d\bar{u} \left(\frac{\Omega^2}{r^{\varepsilon}} \|r^{-1} \cdot r^2 \overset{(1)}{\widehat{\chi}} \Omega\|^2_{S^2_{\bar{u},\tilde{v}_i}} + \Omega^2 \|r^{-1} \cdot \Omega^{-1} \nabla\!\!\!\!/_3 (r^2 \overset{(1)}{\widehat{\chi}} \Omega)\|^2_{S^2_{\bar{u},\tilde{v}_i}} \right. \\ \left. + \Omega^2 \|r^{-1} \cdot \Omega^{-1} \nabla\!\!\!\!/_3 (\Omega^{-1} \nabla\!\!\!\!/_3 (r^2 \overset{(1)}{\widehat{\chi}} \Omega))\|^2_{S^2_{\bar{u},\tilde{v}_i}} \right) \lesssim \frac{1}{v_i} \mathbb{E}_0 \tag{422}$$

Repeating the proof of Proposition 13.3.2 starting now from the slices $\tilde{v}_i$ and using (421) and the integrated decay estimates on Ψ, $\overset{(1)}{\psi}$ and $\overset{(1)}{\alpha}$ of Proposition 12.3.4 and Corollary 11.5 yields, for any $v \geqslant v_0$,

$$\int_{u(v,r_1)}^{\infty} d\bar{u} \left(\frac{\Omega^2}{r^{\varepsilon}} \|r^{-1} \cdot r^2 \overset{(1)}{\widehat{\chi}} \Omega\|^2_{S^2_{\bar{u},v}} + \Omega^2 \|r^{-1} \cdot \Omega^{-1} \nabla\!\!\!\!/_3 (r^2 \overset{(1)}{\widehat{\chi}} \Omega)\|^2_{S^2_{\bar{u},v}} \right. \\ \left. + \Omega^2 \|r^{-1} \cdot \Omega^{-1} \nabla\!\!\!\!/_3 (\Omega^{-1} \nabla\!\!\!\!/_3 (r^2 \overset{(1)}{\widehat{\chi}} \Omega))\|^2_{S^2_{\bar{u},v}} \right) \lesssim \frac{1}{v} \mathbb{E}_0, \tag{423}$$

and the localised integrated decay estimate

$$\int_{v}^{\infty} d\bar{v} \int_{u(\bar{v},r_1)}^{\infty} d\bar{u} \left(\frac{\Omega^2}{r^{1+\varepsilon}} \|r^{-1} \cdot r^2 \overset{(1)}{\widehat{\chi}} \Omega\|^2_{S^2_{\bar{u},\bar{v}}} + \frac{\Omega^2}{r^{1+\varepsilon}} \|r^{-1} \cdot \Omega \nabla\!\!\!\!/_3 (r^2 \overset{(1)}{\widehat{\chi}} \Omega)\|^2_{S^2_{\bar{u},\bar{v}}} \right. \\ \left. + \frac{\Omega^2}{r^{1+\varepsilon}} \|r^{-1} \cdot \Omega^{-1} \nabla\!\!\!\!/_3 (\Omega^{-1} \nabla\!\!\!\!/_3 (r^2 \overset{(1)}{\widehat{\chi}} \Omega))\|^2_{S^2_{\bar{u},V}} \right) \lesssim \frac{1}{v} \mathbb{E}_0.$$

Note that r-weights do not play any role in the region $r \leqslant r_1$ under consideration. To obtain the *global* estimate, we repeat the transport argument of §13.3.3 using now the multiplier $\xi r^{-1-\varepsilon}$ (instead of $\xi r^{-\varepsilon}$) in (352) and (as before) the multiplier $\xi r^{-\varepsilon}$ for (341) and (342). (The reason one needs $r^{-1-\varepsilon}$ is to be able to replace $r^{5+\varepsilon}$ to $r^{4+\varepsilon}$ on the right-hand side of (352) so that the decay estimate (323) can be used for the right-hand side of (352). Note that, for (354), the right-hand side can be estimated directly by (322), (323) and the previous estimate.) This gives, for any $v \geqslant v_0$ and $V \geqslant v$,

$$\int_{u(v,r_0)}^{\infty} d\bar{u} \left(\frac{\Omega^2}{r^{2+\varepsilon}} \|r^{-1} \cdot r^2 \overset{(1)}{\widehat{\chi}} \Omega\|^2_{S^2_{\bar{u},V}} + \Omega^2 \|r^{-1} \cdot \Omega^{-1} \nabla\!\!\!\!/_3 (r^2 \overset{(1)}{\widehat{\chi}} \Omega)\|^2_{S^2_{\bar{u},V}} \right. \\ \left. + \Omega^2 \|r^{-1} \cdot \Omega^{-1} \nabla\!\!\!\!/_3 (\Omega^{-1} \nabla\!\!\!\!/_3 (r^2 \overset{(1)}{\widehat{\chi}} \Omega))\|^2_{S^2_{\bar{u},V}} \right) \lesssim \frac{1}{v} \mathbb{E}_0 \tag{424}$$

and (420) with v^{-1} instead of v^{-2} on the right-hand side. It is clear that we can iterate this argument. There is an ε-loss in the second iteration (corresponding to using the multipliers $\xi r^{-\varepsilon}$ instead of $\xi r^{-1-\varepsilon}$), because Proposition 12.3.4 and Corollary 11.5 have to be applied to control the terms on the right-hand side of (354) and the two-times-commuted equation. We leave the details to the reader and end the proof, by noting that $\hat{\mathscr{G}}$ has $\overset{(1)}{\widehat{\chi}}=0$ globally, so indeed the statement holds equivalently for $\hat{\mathscr{S}}$ and $\underset{\vee}{\mathscr{S}}$. □

Repeating the proof of Corollary 13.1, using now the decay estimates of Proposition 13.2.5 and the decaying fluxes (419) of Proposition 14.3.1, we also have the following result.

COROLLARY 14.6. *Fix $v \geqslant v_0$. Then, for all $u \geqslant u(v, r_0)$ and $V \geqslant v$, we have*

$$\frac{1}{r^{1+\varepsilon}} \|\Omega r^2 \overset{(1)}{\hat{\chi}}\|^2_{S^2_{u,V}} + \frac{1}{r^{\varepsilon}} \|\Omega^{-1} \nabla\!\!\!\!/_3 (r^2 \Omega \overset{(1)}{\hat{\chi}})\|^2_{S^2_{u,V}} \lesssim \frac{1}{v^2} \mathbb{E}_0$$

for the geometric quantities of $\hat{\mathscr{S}}$ (and equivalently $\underset{\vee}{\mathscr{S}}$) in Theorem 4.

For the shear in the ingoing direction, $\overset{(1)}{\underline{\hat{\chi}}}$, we obtain the following result.

PROPOSITION 14.3.2. *Fix r_0 as in Proposition 11.5.1 and let $v \geqslant v_0$ and recall the notation $u(v, r_0)$. We have the following decay estimate for the geometric quantities of $\hat{\mathscr{S}}$ in Theorem 4:*

$$\int_v^\infty d\bar{v} \int_{u(v,r_0)}^\infty d\bar{u}\, \frac{\Omega^2}{r^{1+\varepsilon}} (\|r^{-1} \cdot \mathcal{A}^{[2]} r \overset{(1)}{\underline{\hat{\chi}}} \Omega^{-1}\|^2_{S^2_{\bar{u},\bar{v}}} + \|r^{-1} \cdot \mathcal{A}^{[2]} r \Omega \nabla\!\!\!\!/_4 (r \overset{(1)}{\underline{\hat{\chi}}} \Omega^{-1})\|^2_{S^2_{\bar{u},\bar{v}}}) \lesssim \frac{1}{v^2} \mathbb{E}_0.$$

Proof. We first establish the following estimate for the horizon flux:

$$\int_v^\infty d\bar{v}\, (\|\mathcal{A}^{[3]} \Omega^{-1} \overset{(1)}{\underline{\hat{\chi}}}\|^2_{S^2_{\infty,\bar{v}}} + \|\mathcal{A}^{[3]} \Omega \nabla\!\!\!\!/_4 (\Omega^{-1} \overset{(1)}{\underline{\hat{\chi}}})\|^2_{S^2_{\infty,\bar{v}}} + \|\Omega^{-1} \overset{(1)}{\underline{\psi}}\|^2_{S^2_{\infty,\bar{v}}}) \lesssim \frac{1}{v^2} \mathbb{E}_0. \tag{425}$$

Indeed, integrating (290) written as $\partial_v [\Omega^{-2} |\overset{(1)}{\underline{\psi}}|^2 r^6] + 3M r^4 \Omega^{-2} |\overset{(1)}{\underline{\psi}}|^2 \lesssim r^8 |\overset{(1)}{\underline{P}}|^2$ from v to ∞, using Proposition 11.5.1 for the right-hand side and Proposition 12.3.6 for the boundary terms, the estimate for $\overset{(1)}{\underline{\psi}}$ follows. Integrating (409) for $i=3$ from v to ∞, using Proposition 13.2.4 for the terms on the right-hand side and Corollary 14.3 for the boundary terms, the estimate for $\mathcal{A}^{[3]} \overset{(1)}{\underline{\hat{\chi}}}$ follows. Now use (408) pointwise with $i=3$ and the bounds just obtained to establish the desired estimate for the second term. (Observe that the term $\mathcal{A}^{[3]} \overset{(1)}{\underline{\hat{\chi}}}$ on $S^2_{u_0,\infty}$ entering through Corollary 14.3 is controlled by $\mathbb{E}_0$ from Corollary 13.3; similarly for the term involving $\overset{(1)}{\underline{\psi}}$ on S^2_{u,v_0} entering via Proposition 12.3.6).

With (425) established, we can repeat the proofs of Proposition 14.2.1 (integrating now (357) itself from the horizon and using the decay estimates of Proposition 12.3.4) and Proposition 14.2.2 (without the additional $\mathcal{A}^{[2]}$ commutation and using now the decay estimates of Proposition 12.3.4 and Corollary 11.5) to deduce the desired bounds. □

We also directly prove the following result.

PROPOSITION 14.3.3. *Fix r_0 as in Proposition 11.5.1 and fix $v \geqslant v_0$. We have, for all $V \geqslant v$, the estimate*

$$\int_{u(v,r_0)}^\infty d\bar{u}\, \Omega^2 \|r^{-1} \cdot r^2 \mathcal{D}\!\!\!\!/_2^\star \, \mathrm{d}\!\!\!/\mathrm{iv} (r \overset{(1)}{\underline{\hat{\chi}}} \Omega^{-1})\|^2_{S^2_{\bar{u},V}} \lesssim \frac{1}{v^2} \mathbb{E}_0 \tag{426}$$

for the geometric quantities of $\hat{\mathscr{S}}$ *in Theorem 4. In addition, for all* $V \geqslant v$ *and any* $u \geqslant u(v, r_0)$,

$$\|r^{-1} \cdot r^2 \not{\mathcal{D}}_2^\star \not{\mathrm{div}}(r \overset{(1)}{\hat{\underline{\chi}}} \Omega^{-1})\|^2_{S^2_{u,V}} \lesssim \frac{1}{v^2} \mathbb{E}_0.$$

Proof. Note first that, from the definition of Y, Corollary 14.3 and Proposition 12.3.6 (and Proposition 12.3.3 allowing to incorporate all terms on the right into $\mathbb{E}_0$), we can deduce for any $v \geqslant v_0$ the horizon bound

$$\int_{S^2_{\infty,v}} |\overset{(1)}{Y}|^2 \sin\theta \, d\theta \, d\phi \lesssim \frac{1}{v^2} \mathbb{E}_0.$$

Next, from (357), we have

$$-\frac{1}{2} \partial_u \frac{|\overset{(1)}{Y}|^2}{r} + \frac{1}{4} \Omega^2 \frac{|\overset{(1)}{Y}|^2}{r^2} \lesssim \Omega^2 (|\overset{(1)}{\underline{\alpha}} r \Omega^{-2}|^2 + |\overset{(1)}{\underline{\psi}} r^3 \Omega^{-1}|^2), \tag{427}$$

which we integrate from the horizon. Using the previous bound for the boundary term on the horizon, and noting that we control the right-hand side by Proposition 12.3.8 and Corollary 12.5, we deduce both a flux and a bound on spheres for $\overset{(1)}{Y}$. To convert from $\overset{(1)}{Y}$ to $r^2 \not{\mathcal{D}}_2^\star \not{\mathrm{div}}(r \overset{(1)}{\hat{\underline{\chi}}} \Omega^{-1})$, we use once more the definition of $\overset{(1)}{Y}$ and again Propositions 12.3.8 and 12.3.6, respectively, on $\overset{(1)}{\underline{\psi}}$. □

COROLLARY 14.7. *Fix* r_0 *as in Proposition 11.5.1, let* $v \geqslant v_0$ *and recall the notation* $u(v, r_0)$. *We have the following integrated decay estimate for the geometric quantities of* $\hat{\mathscr{S}}$ *in Theorem 4*:

$$\int_v^\infty d\bar{v} \int_{u(v,r_0)}^\infty d\bar{u} \, \frac{\Omega^2}{r^{1+\varepsilon}} (\|r^{-1} \cdot r^2 \not{\mathcal{D}}_2^\star \overset{(1)}{\eta}\|^2_{S^2_{\bar{u},\bar{v}}} + \|r^{-1} \cdot r^2 \not{\mathcal{D}}_2^\star \overset{(1)}{\underline{\eta}}\|^2_{S^2_{\bar{u},\bar{v}}}) \lesssim \frac{1}{v^2} \mathbb{E}_0.$$

We also have the flux estimates, for any $V \geqslant v$,

$$\int_{u(v,r_0)}^\infty d\bar{u} \, \frac{\Omega^2}{r^\varepsilon} \|r^{-1} \cdot r^2 \not{\mathcal{D}}_2^\star \overset{(1)}{\eta}\|^2_{S^2_{\bar{u},V}} \lesssim \frac{1}{v^2} \mathbb{E}_0, \tag{428}$$

$$\int_{u(v,r_0)}^\infty d\bar{u} \, \frac{\Omega^2}{r^\varepsilon} \left\| r^{-1} \cdot r^3 \not{\mathcal{D}}_2^\star \not{\nabla} \frac{\overset{(1)}{(\Omega \, \mathrm{tr} \, \underline{\chi})}}{\Omega^2} \right\|^2_{S^2_{u,V}} \lesssim \frac{1}{v^2} \mathbb{E}_0. \tag{429}$$

In addition, for any $V \geqslant v$ *and* $u \geqslant u(v, r_0)$, *we have*

$$\|r^{-1} \cdot r^2 \not{\mathcal{D}}_2^\star \overset{(1)}{\eta}\|^2_{S^2_{\bar{u},V}} \lesssim \frac{1}{v^2} \mathbb{E}_0. \tag{430}$$

Finally, the ε *in (428) and (429) can be removed if the decay rate is changed from* v^{-2} *to* v^{-1} *on the right-hand side.*

Proof. The first two and the last estimate follow directly from Propositions 14.3.1–14.3.3 and the identities (379) and (380). The remaining estimate follows similarly from (383) using (428) and Propositions 12.3.8 and 14.3.3. To avoid the ε-loss in (428) and (429), use the estimate (424) for the $\Omega^{-1}\nabla\!\!\!/_3(r^2\overset{(1)}{\hat{\underline{\chi}}}\Omega)$-flux. [A decay rate of $v^{-2+\varepsilon}$ can be shown in this case using interpolation, but this will not be done here.] □

We finally derive an estimate for $\overset{(1)}{\underline{\eta}}$ on the spheres.

PROPOSITION 14.3.4. *Fix $v\geqslant v_0$. For any $V\geqslant v$ and $u\geqslant u(v,r_0)$, we have*

$$\|r^{-1}\cdot r^2 \mathcal{D}\!\!\!/_2^\star\overset{(1)}{\underline{\eta}}\|^2_{S^2_{u,V}} \lesssim \frac{1}{v^2}\mathbb{E}_0 \tag{431}$$

for the geometric quantity $\overset{(1)}{\underline{\eta}}$ of $\hat{\mathscr{S}}$ in Theorem 4.

Proof. Recalling (407) and Proposition 13.2.5, we deduce

$$\|\mathcal{D}\!\!\!/_2^\star\overset{(1)}{\underline{\eta}}\|^2_{S^2_{\infty,v}} = \|\mathcal{D}\!\!\!/_2^\star\overset{(1)}{\eta}[\underset{\vee}{\mathscr{L}}]\,\|^2_{S^2_{\infty,v}} \lesssim \frac{1}{v^2}\mathbb{E}_0$$

after controlling the last (initial data) term in (338) from $\mathbb{E}_0$ using 1-dimensional Sobolev embedding. We now integrate

$$\Omega\nabla\!\!\!/_3(r^2\mathcal{D}\!\!\!/_2^\star\overset{(1)}{\underline{\eta}}) = -\Omega^2 r\mathcal{D}\!\!\!/_2^\star\overset{(1)}{\eta} + r^2\Omega^2\left(\overset{(1)}{\underline{\psi}}\Omega^{-1} + \tfrac{3}{2}\varrho\overset{(1)}{\hat{\underline{\chi}}}\Omega^{-1}\right) \tag{432}$$

backwards from the horizon and use (428), (426) and Corollary 12.5 to control the terms on the right. □

14.3.2. Polynomial decay of the metric coefficients: Proof of (249)–(252)

In this final subsection we prove the estimates (249)–(252) on the metric quantities of $\hat{\mathscr{S}}$.

The estimate (249) is a direct consequence of Corollary 14.7, Proposition 14.3.4 and the definition (134).

For (251) we present the proof without the (trivial) commutation with $\mathcal{A}^{[2]}$ which can be inserted into all formulas below. We write (132) in the form

$$\Omega\nabla\!\!\!/_3\hat{\not{g}}^{(1)} = 2\Omega\overset{(1)}{\hat{\underline{\chi}}}, \tag{433}$$

and derive, for any fixed $u\geqslant u_0$ and $v\geqslant v_0$,

$$\left\|r^{-1}\cdot\overset{(1)}{\hat{\not{g}}}\right\|_{S^2_{u,v}} \lesssim \left\|r^{-1}\cdot\overset{(1)}{\hat{\not{g}}}\right\|_{S^2_{u_0,v}} + \int_{u_0}^{u} d\bar{u}\,\|r^{-1}\cdot\overset{(1)}{\hat{\underline{\chi}}}\Omega\|_{S^2_{\bar{u},v}}. \tag{434}$$

Consider the first term on the right-hand side. Recall that the solution $\underset{\vee}{\mathscr{S}}$ had $\overset{(1)}{b}=0$ on C_{u_0}. By Lemma 6.1.1 and the estimates on the gauge function in Proposition 14.1.2, the pure gauge solution $\hat{\mathscr{G}}$ generates a b satisfying $|\not{\mathcal{D}}_2^\star \overset{(1)}{b}|\lesssim r^{-2}$ along C_{u_0}. Moreover, again by the boundedness estimates on f in Proposition 14.1.2, the pure gauge solution $\hat{\mathscr{G}}$ also satisfies the round sphere condition (192) at infinity. Therefore, integrating equation (132) from infinity along C_{u_0} using Corollary 13.1 and the aforementioned bound on $\overset{(1)}{b}$ yields

$$\|r^{-1}\cdot\overset{(1)}{\hat{\not{g}}}\|_{S^2_{u_0,v}}\lesssim\frac{1}{r(u_0,v)}\cdot\sqrt{\mathbb{E}_0}. \tag{435}$$

For the second term, we define $u_\star:=\min\big(u,\frac{3}{4}v\big)$ and split the integral as

$$\int_{u_0}^{u}d\bar{u}\,\|r^{-1}\cdot\overset{(1)}{\hat{\underline{\chi}}}\Omega\|_{S^2_{\bar{u},v}}=\int_{u_0}^{u_\star}d\bar{u}\,\|r^{-1}\cdot\overset{(1)}{\hat{\underline{\chi}}}\Omega\|_{S^2_{\bar{u},v}}+\int_{u_\star}^{u}d\bar{u}\,\|r^{-1}\cdot\overset{(1)}{\hat{\underline{\chi}}}\Omega\|_{S^2_{\bar{u},v}}.$$

Now, for the first integral, we have by Proposition 14.3.3 (applied with $v=v_0$)

$$\begin{aligned}\int_{u_0}^{u_\star}d\bar{u}\,\|r^{-1}\cdot\overset{(1)}{\hat{\underline{\chi}}}\Omega\|_{S^2_{\bar{u},v}}&\lesssim\sqrt{\int_{u_0}^{u_\star}d\bar{u}\,\Omega^2\|r^{-1}\cdot(r\overset{(1)}{\hat{\underline{\chi}}}\Omega^{-1})\|^2_{S^2_{\bar{u},v}}}\sqrt{\int_{u_0}^{u_\star}d\bar{u}\,\frac{\Omega^2}{r^2}}\\&\lesssim\sqrt{\mathbb{E}_0}\left[r\left(\frac{3}{4}v,v\right)\right]^{-1/2}.\end{aligned}$$

For the second integral (which vanishes if $u\leqslant\frac{3}{4}v$), we have

$$\int_{u_\star}^{u}d\bar{u}\,\|r^{-1}\cdot\overset{(1)}{\hat{\underline{\chi}}}\Omega\|_{S^2_{\bar{u},v}}\lesssim\sqrt{\int_{3/4v}^{\infty}d\bar{u}\,\Omega^2\|r^{-1}\cdot(r\overset{(1)}{\hat{\underline{\chi}}}\Omega^{-1})\|^2_{S^2_{\bar{u},v}}}\sqrt{\int_{u_0}^{\infty}d\bar{u}\,\frac{\Omega^2}{r^2}}\lesssim\sqrt{\mathbb{E}_0}\cdot v^{-1},$$

where we have used the decay estimate of Proposition 14.3.3 and boundedness of the integral in the second square root. Combining the estimates yields (251).

The argument to prove (252) is analogous now starting from (131), which reads

$$\Omega\not{\nabla}_3\frac{\overset{(1)}{\sqrt{\not{g}}}}{\sqrt{\not{g}}}=(\Omega\overset{(1)}{\operatorname{tr}\underline{\chi}}). \tag{436}$$

We use the estimate on $|\not{\mathrm{d}}\mathrm{iv}\,\overset{(1)}{b}|\lesssim r^{-2}$ for the pure gauge solution and the fact that the round sphere condition (191) in conjunction with (192) implies that $\overset{(1)}{\sqrt{\not{g}}}/\sqrt{\not{g}}$ vanishes for $\ell\geqslant2$ at infinity (see (221)) to derive the analogue of (435). Using then the bound (429) of Corollary 14.7 without the $r^{-\varepsilon}$, we derive (252) following the integration argument above.

To derive (250), we first note that (402) yields

$$\|r^{-1}\cdot r \not{\mathcal{D}}_2^\star \overset{(1)}{b}[\underset{\vee}{\mathscr{S}}]\|^2_{S^2_{u,v}} \lesssim \frac{1}{v}\mathbb{E}_0 \quad \text{for } r(u,v) \gtrsim \frac{1}{4}v.$$

Using the estimates on the gauge function of Corollary 14.1.2 and Lemma 6.1.1, we conclude

$$\|r^{-1}\cdot r \not{\mathcal{D}}_2^\star \overset{(1)}{b}[\hat{\mathscr{S}}]\|^2_{S^2_{u,v}} \lesssim \frac{1}{v}\mathbb{E}_0 \quad \text{for } r(u,v) \gtrsim \frac{1}{4}v. \tag{437}$$

Fix now $v \geqslant 2v_0$ large (for smaller v the estimate (250) is implied by Proposition 14.1.3). Define the tortoise coordinate $r^\star = v - u - (v_0 - u_0 - 1)$ so that $r^\star(r_0) = 1$ and also $r^\star \sim r$ for large r. Let $\bar{u} = u\big(r_0, \frac{3}{4}v\big)$. Then, on the hypersurface $[u_0, \bar{u}] \times \{v\}$, we have $r \sim r^\star \geqslant \frac{1}{4}v$, and hence (437) holds. From $(\bar{u}, v)$ we apply the transport estimate (404), this time for the geometric quantities of $\hat{\mathscr{S}}$ with $n = 2 - \varepsilon$. After inserting the bound (437) for the initial term and the bounds on $\overset{(1)}{\eta}$ and $\overset{(1)}{\underline{\eta}}$ (Corollary 14.7 and Proposition 14.3.4) on the right-hand side, we conclude that

$$\|r^{-1}\cdot r \not{\mathcal{D}}_2^\star \overset{(1)}{b}\|^2_{S^2_{u,v}} \lesssim \frac{1}{v}\mathbb{E}_0. \tag{438}$$

also for $[\bar{u}, \infty) \times \{v\}$.

14.3.3. Proof of Corollary 10.3

The proof of Corollary 10.3 is now immediate: We consider $\hat{\mathscr{S}}' = \hat{\mathscr{S}} - \mathscr{K}_{\mathfrak{m}, s_i}$. Using the bounds (249)–(252), we apply the classical Sobolev embedding on the spheres $S^2_{u,v}$ to the left-hand side. Note that the quantities $\overset{(1)}{\sqrt{\not{g}}}$, $\overset{(1)}{\hat{\not{g}}}$, $\overset{(1)}{b}$ and $\Omega^{-1}\overset{(1)}{\Omega}$ associated with $\hat{\mathscr{S}}'$ are supported on $\ell \geqslant 2$, and that Proposition 4.4.2 and Corollary 4.2 guarantee that all second-order derivatives are indeed controlled in $L^2(S^2_{u,v})$. The estimates (253)–(254) follow immediately.

Appendix A. Construction of data and propagation of asymptotic flatness

In this appendix we construct and estimate from a smooth seed initial data set (Definition 8.1) which is asymptotically flat with weight s to order n (Definition 8.2) all quantities of the solution $\mathscr{S}$ associated with the data set through Theorem 8.1, first on the initial cones $C_{u_0} \cup C_{v_0}$ and then globally in the spacetime. The main result is the following.

Theorem A.1. *Consider a smooth seed initial data set which is asymptotically flat with weight s to order $n\geqslant 11$, and the corresponding smooth solution $\mathscr{S}$ arising from Theorem 8.1. For an element ξ of $\mathscr{S}$, we denote*

$$|\mathfrak{D}^k\xi| = \sum_{0\leqslant j_1+j_2+j_3\leqslant k} |(\Omega^{-1}\slashed{\nabla}_3)^{j_1}(r\slashed{\nabla})^{j_2}(r\Omega\slashed{\nabla}_4)^{j_3}\xi|.$$

The solution $\mathscr{S}$ has the following property: On the initial cones $C_{u_0}\cup C_{v_0}$, the estimates

$$\begin{aligned}
|\mathfrak{D}^k(r^{3+s}\overset{(1)}{\alpha}\Omega^2)|+|\mathfrak{D}^k(r^{3+s}\overset{(1)}{\beta}\Omega)|+|\mathfrak{D}^k(r^3\overset{(1)}{\varrho})|+|\mathfrak{D}^k(r^3\overset{(1)}{\sigma})|+|\mathfrak{D}^k(r^2\overset{(1)}{\underline{\beta}}\Omega^{-1})| &\leqslant C_k,\\
|\mathfrak{D}^k(r\overset{(1)}{\underline{\alpha}}\Omega^{-2})|+|\mathfrak{D}^k(r^2\overset{(1)}{K})|+|\mathfrak{D}^k(r^2\overset{(1)}{\hat{\chi}}\Omega)|+|\mathfrak{D}^k(r\overset{(1)}{\hat{\underline{\chi}}}\Omega^{-1})|+|\mathfrak{D}^k(r\overset{(1)}{\eta})|+|\mathfrak{D}^k(r^2\overset{(1)}{\underline{\eta}})| &\leqslant C_k,\\
|\mathfrak{D}^k(r^{2+s}\overset{(1)}{\omega})|+|\mathfrak{D}^k(\Omega^{-2}\overset{(1)}{\underline{\omega}})|+|\mathfrak{D}^k(r^2(\Omega\overset{(1)}{\mathrm{tr}}\,\chi))|+|\mathfrak{D}^k(r\Omega^{-2}(\Omega\overset{(1)}{\mathrm{tr}}\,\underline{\chi}))| &\leqslant C_k,\\
|\mathfrak{D}^k(\Omega^{-1}\overset{(1)}{\Omega})|+|\mathfrak{D}^k\overset{(1)}{\hat{\slashed{g}}}|+|\mathfrak{D}^k(\mathrm{tr}_{\slashed{g}}\overset{(1)}{\slashed{g}})|+|\mathfrak{D}^k(r\overset{(1)}{b})| &\leqslant C_k
\end{aligned} \tag{439}$$

hold for any $k\leqslant n-3$ and a constant C_k which can be computed explicitly and depends only on finitely many constants $C_{\circ,n_1,n_2}$ appearing in Definition 8.2. In particular, the constants $C_{\circ,n_1,n_2}$ with $n_1+n_2\leqslant k+3$ are sufficient. For any $k\leqslant n-4$ the quantities on the left-hand side with of (439) with $s=0$ have well-defined limits on null infinity.

Moreover, given any $u_0<U<\infty$, the estimates (439) actually hold for any $k\leqslant n-10$ in any spacetime region $\mathcal{M}\cap\{u_0\leqslant u\leqslant U\}$, where C_k now also depends on the choice of U and the constants $C_{\circ,n_1,n_2}$ with $n_1+n_2\leqslant k+10$ appearing in Definition 8.2. For any $k\leqslant n-11$ the quantities on the left-hand side of (439) with $s=0$ have well-defined limits on null infinity.

We remark that the conditions $n_1+n_2\leqslant k+3$ in the first part and $n_1+n_2\leqslant k+10$ in the second part of the theorem account for the loss of derivatives in the characteristic initial value problem and losses from applying Sobolev embedding. It is of course not optimal. We also remark that we have stated (439) for quantities *regular* at the horizon (cf. (130)). Since C_k is allowed to depend on U in the second part of the theorem this is not essential.

The proof of Theorem A.1 will proceed in two steps. In §A.1 we prove that the estimates (439) hold on the initial cone $C_{u_0}\cup C_{v_0}$ by constructing from an asymptotically flat seed initial data set of Definition 8.2 all quantities of the solution $\mathscr{S}$ on $C_{u_0}\cup C_{v_0}$. In the second step, we show that these bounds are in fact propagated by the evolution. The statement about the limit at null infinity will follow from the fact that the $\Omega\slashed{\nabla}_4$-derivative of any of the quantities in the round brackets of (439) is always integrable in v. Note that $\overset{(1)}{\alpha}$ and $\overset{(1)}{\omega}$ are part of the seed data, and we gain integrability from taking the limit of $r^3\overset{(1)}{\alpha}$ and $r^2\overset{(1)}{\omega}$.

Remark A.1. Observe that all quantities in $\mathscr{S}$ except $\overset{(1)}{\underline{\omega}}$ decay at least as fast towards null infinity as their background Schwarzschild value. For $\overset{(1)}{\underline{\omega}}$ we can only propagate boundedness, while $\underline{\omega}$ decays like r^{-2} towards null infinity. This exceptional behaviour is rooted in our choice of null frame for the linearisation. If we compare the components linearised with respect to the frame $\boldsymbol{\mathcal{N}}_{EF^\star}$, where $\underline{\boldsymbol{\omega}}=0$ and hence $\overset{(1)}{\underline{\omega}}=0$ identically (cf. §5.1.4) all linearised quantities decay as fast as their Schwarzschild value.

A.1. Proof of the first part: Constructing the data

In this section we shall prove the first part of Theorem A.1, i.e. the statement that the solution is determined from seed data with the bounds (439) holding on $C_{u_0}\cup C_{v_0}$. We will focus on establishing the latter bounds for $k=0$. The statement for arbitrary angular commutations is then easily obtained from the fact that $r\nabla\!\!\!/$ commutes with $\Omega\nabla\!\!\!/_3$ and $\Omega\nabla\!\!\!/_4$ and has good commutation properties with angular derivatives in the sense that

$$|[r\nabla\!\!\!/_A, r\nabla\!\!\!/_B]\xi| \leqslant C|\xi|.$$

To obtain the remaining tangential derivatives and the transversal derivatives, we will use the null structure and Bianchi equations directly in conjunction with an inductive procedure which is outlined below.

For the remainder of the proof, we will allow ourselves to drop the $\circ$ subscript from all quantities as well as the "in" and "out" from $\overset{(1)}{g\!\!\!/}$, as it will be clear from the context which cone we are on.

Elliptic equations on the horizon sphere S^2_{∞,v_0}

We first note that the seed data determines on the horizon sphere:

- $\overset{(1)}{\sigma}$ uniquely from (146);
- $\overset{(1)}{K}$ uniquely from the fact that $\overset{(1)}{\hat{g\!\!\!/}}$ and $\overset{(1)}{\sqrt{g\!\!\!/}}/\sqrt{g\!\!\!/}$ are part of the seed data and (221);
- $\overset{(1)}{\rho}$ uniquely from (147);
- $\overset{(1)}{\underline{\beta}}$ uniquely from (145);
- $\overset{(1)}{\eta}$ uniquely from $\overset{(1)}{\underline{\eta}}=-\overset{(1)}{\eta}+2\nabla\!\!\!/_A\Omega^{-1}\overset{(1)}{\Omega}$ (equation (134)).

Transport equations along C_{v_0}: Part I

We now integrate our seed data from S^2_{∞,v_0} along the cone C_{v_0}.

Along C_{v_0}, the tensor $\hat{\overset{(1)}{\not g}}$ is part of the seed data. Note that this determines uniquely, along C_{v_0}, the quantities $\Omega^{-1}\overset{(1)}{\hat{\underline{\chi}}}$ via (131) and $\Omega^{-2}\overset{(1)}{\underline{\alpha}}$ via (139).

Next, from (137), we have

$$\partial_u\left[\frac{r^2}{\Omega^2}(\Omega\overset{(1)}{\operatorname{tr}}\underline{\chi})\right]=-4r\overset{(1)}{\underline{\omega}}.$$

Since $\overset{(1)}{\underline{\omega}}$ is prescribed along C_{v_0} as part of the seed data, as is the value of $\Omega^{-2}r^2(\Omega\overset{(1)}{\operatorname{tr}}\underline{\chi})$ on S^2_{∞,v_0}, the above ODE determines $(\Omega\overset{(1)}{\operatorname{tr}}\underline{\chi})$ uniquely along C_{v_0}. Note that now $\overset{(1)}{\sqrt{\not g}}/\sqrt{\not g}$ is now uniquely determined along C_{v_0} by (131).

Since $\overset{(1)}{\underline{\beta}}$ was already uniquely determined on S^2_{∞,v_0} above, we can integrate the Bianchi equation (156) written as

$$\Omega\not\nabla_3[r^4\Omega^{-1}\overset{(1)}{\underline{\beta}}]=-r^4\,\mathrm{d}\!\!/\mathrm{iv}\,\overset{(1)}{\underline{\alpha}}$$

along C_{v_0}, which, since the right-hand side is uniquely determined from seed data, determines $\overset{(1)}{\underline{\beta}}$ uniquely on C_{v_0}. The Bianchi equations (151) and (154) read as ODEs along C_{v_0} now uniquely determine $\overset{(1)}{\varrho}$ and $\overset{(1)}{\sigma}$, since the initial value of these quantities on S^2_{∞,v_0} were already determined by seed data above. Similarly, $\overset{(1)}{\underline{\eta}}$ is determined uniquely from (142). Using (134) we conclude that $\overset{(1)}{\eta}$ is also uniquely determined along C_{v_0}. We finally use (136) written as

$$\partial_u(r(\Omega\overset{(1)}{\operatorname{tr}}\chi))=\mathcal{Q},$$

with $\mathcal{Q}$ uniquely determined on C_{v_0} to determine uniquely $r(\Omega\overset{(1)}{\operatorname{tr}}\chi)$ along C_{v_0}. Noting that $\overset{(1)}{K}$ is uniquely determined by (147) along C_{v_0}, we conclude that we have determined all geometric quantities of a solution $\mathcal{S}$ uniquely along C_{v_0} except $\overset{(1)}{\hat{\chi}}$, $\overset{(1)}{\beta}$, $\overset{(1)}{\alpha}$, $\overset{(1)}{b}$ and $\overset{(1)}{\omega}$ along C_{v_0}.

Estimates on the sphere $S^2_{u_0,v_0}$

We note that, since $\hat{\overset{(1)}{\not g}}$, $\Omega^{-1}\overset{(1)}{\Omega}$ and $\overset{(1)}{b}$ are part of the seed data on C_{u_0},

- the quantity $\overset{(1)}{\hat{\chi}}$ is determined uniquely on $S^2_{u_0,v_0}$ by (132);
- the quantity $\overset{(1)}{\alpha}$ is determined uniquely on $S^2_{u_0,v_0}$ by (139);
- the quantity $\overset{(1)}{\omega}$ is determined uniquely on $S^2_{u_0,v_0}$ by (134).

Hence, by (145) and the fact that we already determined $\overset{(1)}{\underline{\eta}}$ and $(\Omega\overset{(1)}{\operatorname{tr}}\chi)$ uniquely on $S^2_{u_0,v_0}$ above, the quantity $\overset{(1)}{\beta}$ is also uniquely determined on $S^2_{u_0,v_0}$.

Transport equations along C_{v_0}: Part II

We can now determine the missing quantities $\overset{(1)}{\widehat{\chi}}$, $\overset{(1)}{\beta}$, $\overset{(1)}{\alpha}$, $\overset{(1)}{b}$ and $\overset{(1)}{\omega}$ along C_{v_0}, recalling that they have been determined uniquely on $S^2_{u_0,v_0}$: Use (133) to determine $\overset{(1)}{b}$, (140) to determine $\overset{(1)}{\widehat{\chi}}$, (150) to determine $\overset{(1)}{\beta}$, (148) to determine $\overset{(1)}{\alpha}$ and finally $\overset{(1)}{\omega}$ from (144) all uniquely along C_{v_0}.

We have determined all geometric quantities in terms of seed data and uniform bounds for all quantities which extend smoothly to the horizon $\mathcal{H}^+$ along C_{v_0}.

Transport equations along C_{u_0}

We finally turn to the conjugate cone C_{u_0}. Recall that all geometric quantities have been determined uniquely on the sphere $S^2_{u_0,v_0}$, and that moreover the seed data $\overset{(1)}{\hat{\not{g}}}$ along C_{u_0} determines uniquely $\overset{(1)}{\widehat{\chi}}$ by (132) and $\overset{(1)}{\alpha}$ by (139) along C_{u_0}.

We now determine all quantities along C_{u_0}. Starting with (137), we have

$$\partial_v\left(\frac{r^2}{\Omega^2}(\Omega \overset{(1)}{\mathrm{tr}}\,\chi)\right)=4r\overset{(1)}{\omega}.$$

We see that the right-hand side is integrable by the asymptotic flatness condition therefore, producing the bound

$$|r^2(\Omega \overset{(1)}{\mathrm{tr}}\,\chi)|<C,$$

where C can be computed explicitly from the seed data (and depends on $0<s\leqslant 1$). From (131) and the asymptotic flatness condition on $\overset{(1)}{b}$, we immediately conclude

$$\left|\frac{\overset{(1)}{\sqrt{\not{g}}}}{\sqrt{\not{g}}}\right|<C.$$

Of course, the same bounds hold for arbitrary many angular derivatives $r\nabla\!\!\!/$ of these quantities.

Remark A.2. One also sees that the quantities $\overset{(1)}{\hat{\not{g}}}$ and $\overset{(1)}{\sqrt{\not{g}}}/\sqrt{\not{g}}$ (as well as angular derivatives $r\nabla\!\!\!/$ of these quantities) have smooth limits at null infinity. By this, we mean that there exists a symmetric 2-tensor $\overset{(1)}{\hat{\not{g}}}_\infty$ and a scalar $\overset{(1)}{\sqrt{\not{g}}}_\infty/\sqrt{\not{g}}$ on $\mathcal{M}$ satisfying in any spherical coordinate patch

$$\partial_u\frac{(\overset{(1)}{\hat{\not{g}}}_\infty)_{AB}}{\sqrt{\not{g}}}=\partial_v\frac{(\overset{(1)}{\hat{\not{g}}}_\infty)_{AB}}{\sqrt{\not{g}}}=0,\quad \partial_u\frac{\overset{(1)}{\sqrt{\not{g}}}_\infty}{\sqrt{\not{g}}}=\partial_v\frac{\overset{(1)}{\sqrt{\not{g}}}_\infty}{\sqrt{\not{g}}}=0$$

and

$$\lim_{v\to\infty}\frac{\overset{(1)}{\hat{\not{g}}}_{AB}}{\sqrt{\not{g}}}=\frac{(\overset{(1)}{\hat{\not{g}}}_{\infty})_{AB}}{\sqrt{\not{g}}},\quad \lim_{v\to\infty}\frac{\overset{(1)}{\sqrt{\not{g}}}}{\sqrt{\not{g}}}=\frac{\overset{(1)}{\sqrt{\not{g}}}_{\infty}}{\sqrt{\not{g}}},$$

in the limit along the cone C_{u_0}.

We now note that $\overset{(1)}{\beta}$ is uniquely determined from (149) producing the uniform bound

$$|r^{3+s}\overset{(1)}{\beta}|<C.$$

Similarly, we can determine $\overset{(1)}{\eta}$ and $\overset{(1)}{\underline{\eta}}$. For this, we note the equations (following from (142) and (134)):

$$\begin{aligned}\not{\nabla}_4(r^2\overset{(1)}{\underline{\eta}})&=\not{\nabla}_4(r^2\overset{(1)}{\eta})-\not{\nabla}_4(r^2\overset{(1)}{\underline{\eta}}+r^2\overset{(1)}{\eta})\\&=2\frac{\Omega}{r}(\overset{(1)}{\eta}+\overset{(1)}{\underline{\eta}})-r^2\overset{(1)}{\beta}-2\not{\nabla}_4\not{\nabla}_A r^2(\Omega^{-1}\overset{(1)}{\Omega})=-r^2\overset{(1)}{\beta}-2r^2\not{\nabla}\overset{(1)}{\omega}.\end{aligned}\tag{440}$$

Note that the right-hand side is uniquely determined along C_{u_0} and integrable, leading to $\overset{(1)}{\underline{\eta}}$ being uniquely determined along C_{u_0} with the uniform bound

$$|r^2\overset{(1)}{\underline{\eta}}|<C.$$

We also have, by the relation (134), that $\overset{(1)}{\eta}$ is uniquely determined along C_{u_0} with the uniform bound

$$|r\overset{(1)}{\eta}|<C.$$

We turn to (151) and (153), which clearly determine $\overset{(1)}{\varrho}$ and $\overset{(1)}{\sigma}$ uniquely along C_{u_0} with the uniform bounds

$$|r^3\overset{(1)}{\varrho}|+|r^3\overset{(1)}{\sigma}|<C.$$

In fact, since we can repeat the above procedure commuting with angular derivatives, we also have in particular

$$|r\not{\nabla}(r^3\overset{(1)}{\varrho})|+|r\not{\nabla}(r^3\overset{(1)}{\sigma})|<C.$$

It is now easy to see that (155) determines $\overset{(1)}{\underline{\beta}}$ uniquely with the uniform bound

$$|r^2\overset{(1)}{\underline{\beta}}|<C,$$

and that (141) determined $\overset{(1)}{\underline{\hat{\chi}}}$ uniquely with the uniform bound

$$|r\overset{(1)}{\underline{\hat{\chi}}}|<C.$$

Similarly, (157) determines $\overset{(1)}{\underline{\alpha}}$ uniquely with the uniform bound

$$|r\overset{(1)}{\underline{\alpha}}| < C.$$

For the remaining components, note that (144) determines $\overset{(1)}{\underline{\omega}}$ uniquely with the uniform bound

$$|\overset{(1)}{\underline{\omega}}| < C.$$

The quantity $(\Omega\overset{(1)}{\mathrm{tr}}\,\chi)$ is determined uniquely from (136) with the bound

$$|r(\Omega\overset{(1)}{\mathrm{tr}}\,\chi)| < C,$$

where we have used that we can write (136) as

$$\partial_v(r(\Omega\overset{(1)}{\mathrm{tr}}\,\chi)) = \mathrm{RHS}$$

with the right-hand side uniquely determined and integrable along C_{u_0}. Finally, equation (147) determines $\overset{(1)}{K}$ uniquely along C_{u_0}, with the uniform bound

$$|r^2\overset{(1)}{K}| < C.$$

In fact, by the remark above and the fact that the right-hand side in $\nabla\!\!\!/_4(r^2\overset{(1)}{K})=\mathrm{RHS}$ is integrable, the weighted linearised Gaussian curvature $r^2\overset{(1)}{K}$ extends smoothly to null infinity.

We note once more that the same bounds hold for arbitrarily many angular derivatives $r\nabla\!\!\!/$ of the quantities estimated either by trivial commutation with angular momentum operators Ω_i or, if the reader prefers, tensorial commutation with $r\nabla\!\!\!/$ and inductively estimating lower-order terms.

We conclude the proof by estimating the remaining weighted tangential derivatives $r\nabla\!\!\!/_4$ and the transversal derivative $\nabla\!\!\!/_3$ on the cone C_{u_0}. The procedure on C_{v_0} is analogous (but easier, since there are no weights at null infinity) and is hence omitted.

For the remaining weighted tangential derivative $r\nabla\!\!\!/_4$, we use

- equation (149) pointwise for $r\nabla\!\!\!/_4\overset{(1)}{\beta}$;
- equation (151) pointwise for $r\nabla\!\!\!/_4\overset{(1)}{\varrho}$;
- equation (151) pointwise for $r\nabla\!\!\!/_4\overset{(1)}{\sigma}$;
- equation (155) pointwise for $r\nabla\!\!\!/_4\overset{(1)}{\underline{\beta}}$;
- equation (157) pointwise for $r\nabla\!\!\!/_4\overset{(1)}{\underline{\alpha}}$;

which estimates the first derivative of all linearised curvature components. Similarly one can use the null structure equations to exchange a 4-derivative by an angular derivative, to estimate all linearised Ricci coefficients and the linearised metric components. This estimates all (first) derivatives tangential to the cone, and it is easy to see how to continue inductively to estimate tangential derivatives of arbitrary high order.

To estimate the transversal derivatives on C_{u_0}, one follows a similar procedure, now using the null structure and Bianchi equations in the $\nabla\!\!\!/_3$-direction: More specifically, one

- uses the null structure equations, which express the transversal derivatives of all Ricci coefficients in terms of angular derivatives or curvature components that have already been obtained;
- uses the Bianchi equations, which express the transversal derivatives of all curvature components in term of angular derivatives of curvature and Ricci coeffcients that have already been obtained. For instance, (148) for $\overset{(1)}{\alpha}$, (150) for $\overset{(1)}{\beta}$, (152) for $\overset{(1)}{\varrho}$, (154) for $\overset{(1)}{\sigma}$ and (156) for the transversal derivatives of $\overset{(1)}{\underline{\beta}}$ along C_{u_0}. Finally, for $\nabla\!\!\!/_3\overset{(1)}{\alpha}$, one needs to commute (157) and use the fact that the transversal derivative of $\underline{\beta}$ has just been obtained.

A simple induction allows to estimate all derivatives of all quantities on $C_{u_0}\cup C_{v_0}$. Finally, counting derivatives in the above procedure, one observes that the bounds claimed in Theorem A.1 hold on the hypersurface $C_{v_0}\cup C_{u_0}$ for a constant C_k which only depends on the constants $C_{\circ,n_1,n_2}$ with $n_1+n_2\leqslant k+3$ in the definition of an asymptotically flat seed initial data set and the size of the data on the compact hypersurface C_{v_0}. One also sees that applying a $\Omega\nabla\!\!\!/_4$-derivative to any quantity in the round brackets of (439) the right-hand side is integrable, which ensures the existence of the limit at null infinity. This generally loses a derivative, e.g. (440), explaining the $k\leqslant n-4$.

A.2. Proof of the second part: Propagation of decay

Knowing that the desired bounds hold on $C_{v_0}\cup C_{u_0}$, we continue with the proof of Theorem A.1.

As noted in §7.3 and §7.4, the derived quantities $\overset{(1)}{P}$ and $\overset{(1)}{\underline{P}}$ of the solution $\mathscr{S}$, which can be expressed through (183) and (184), satisfy the Regge–Wheeler equation. It is easy to see that, for asymptotically flat seed initial data with weight s of order $n\geqslant 10$, the initial energies of Corollary 11.4, $\mathbb{F}^k_0[\overset{(1)}{\Psi}=r^5\overset{(1)}{P}]$ and $\mathbb{F}^k_0[\overset{(1)}{\underline{\Psi}}=r^5\overset{(1)}{\underline{P}}]$ are indeed finite for every $k\leqslant n-6$ with the bound depending only constants $C_{\circ,n_1,n_2}$ in the definition of asymptotically flat seed data with $n_1+n_2\leqslant k+6$. Corollary 11.2 and standard Sobolev embedding hence yield the bound

$$|\mathfrak{D}^k(\overset{(1)}{P}r^5)|+|\mathfrak{D}^k(\overset{(1)}{\underline{P}}r^5)|<C_k(U)$$

in any spacetime region $\mathcal{M}\cap\{u_0\leqslant u\leqslant U\}$ with $C_k(U)$ as claimed. We now use the definition of $\overset{(1)}{P}$ and $\overset{(1)}{\underline{P}}$ via the transport equations (178)–(181) to obtain the bounds

$$|\mathfrak{D}^k(\overset{(1)}{\psi}r^{4+s}))|+|\mathfrak{D}^k(\overset{(1)}{\underline{\psi}}r^3)|+|\mathfrak{D}^k(\overset{(1)}{\alpha}r^{3+s})|+|\mathfrak{D}^k(\overset{(1)}{\underline{\alpha}}r)|<C_k(U)$$

in any spacetime region $\mathcal{M}\cap\{u_0\leqslant u\leqslant U\}$ with $C_k(U)$ as claimed. Indeed, these bounds hold initially on $C_{u_0}\cup C_{v_0}$, and are propagated by the transport equations. Note that weights near the horizon are irrelevant, as $U<\infty$ and the constants are allowed to depend on U. We can now use the equations (139) to obtain

$$|\mathfrak{D}^k(\overset{(1)}{\hat{\chi}}r^2))|+|\mathfrak{D}^k(\overset{(1)}{\underline{\hat{\chi}}}r)|<C_k(U),$$

and the expressions for $\overset{(1)}{\psi}$ and $\overset{(1)}{\underline{\psi}}$ in (182) to deduce

$$|\mathfrak{D}^k(\not{\mathcal{D}}_2^\star\overset{(1)}{\beta}r^{4+s}))|+|\mathfrak{D}^k(\not{\mathcal{D}}_2^\star\overset{(1)}{\underline{\beta}}r^3)|<C_k(U),$$

both in any spacetime region $\mathcal{M}\cap\{u_0\leqslant u\leqslant U\}$ with $C_k(U)$ as claimed. To derive the above, note that $r(u_0,v)\leqslant r(U,v)+C(U)$ for $U<\infty$. Using (140) pointwise and then (142) in the 3-direction, as well as (154) in the 3-direction and the definition of $\overset{(1)}{P}$, we obtain the estimates

$$|\mathfrak{D}^k(\not{\mathcal{D}}_2^\star\overset{(1)}{\eta}r^2))|+|\mathfrak{D}^k(\not{\mathcal{D}}_2^\star\overset{(1)}{\underline{\eta}}r^3))|+|\mathfrak{D}^k(r^5(\not{\mathcal{D}}_2^\star\not{\nabla}_A(\overset{(1)}{\varrho},\overset{(1)}{\sigma}))|<C_k(U).$$

The Codazzi equations (145) and equation (134) now provide the bounds

$$|\mathfrak{D}^k(r^3\not{\mathcal{D}}_2^\star\not{\nabla}_A(\Omega\overset{(1)}{\operatorname{tr}}\underline{\chi}))|+|\mathfrak{D}^k(r^4\not{\mathcal{D}}_2^\star\not{\nabla}_A(\Omega\overset{(1)}{\operatorname{tr}}\chi))|+|\mathfrak{D}^k(r^2\not{\mathcal{D}}_2^\star\not{\nabla}_A\Omega^{-1}\overset{(1)}{\Omega})|<C_k(U)$$

in any spacetime region $\mathcal{M}\cap\{u_0\leqslant u\leqslant U\}$ with $C_k(U)$ as claimed. Note this implies already the desired bound for $\overset{(1)}{\underline{\omega}}=\partial_u(\Omega^{-1}\overset{(1)}{\Omega})$. For $\overset{(1)}{\omega}$ we use (144) to obtain the improved bound

$$|\mathfrak{D}^k(r^{4+s}\not{\mathcal{D}}_2^\star\not{\nabla}_A\overset{(1)}{\omega})|<C_k(U).$$

Finally, we use (131), (132) in the 3-direction and (133) to deduce also

$$|\mathfrak{D}^k\overset{(1)}{\hat{\not{g}}}|+|\mathfrak{D}^k(r^2\not{\mathcal{D}}_2^\star\not{\nabla}_A(\operatorname{tr}_{\not{g}}\overset{(1)}{\not{g}}))|+|\mathfrak{D}^k(r^2\not{\mathcal{D}}_2^\star\overset{(1)}{b})|\leqslant C_k(U)$$

in any spacetime region $\mathcal{M}\cap\{u_0\leqslant u\leqslant U\}$ with $C_k(U)$ as claimed.

In view of Corollaries 4.4.2 and 4.2, we have proven Theorem A.1 except for the $\ell=0$ and $\ell=1$ modes of the solution $\mathscr{S}$. The latter have been understood in detail in §9, Theorem 9.2. Specifically, we can now add to $\mathscr{S}$ a pure gauge solution $\mathscr{G}$ (generated by a gauge function supported on $\ell=0,1$ only) and a member of the Kerr family $\mathscr{K}$ with the following properties:

- both $\mathscr{G}$ and $\mathscr{K}$ satisfy the desired bounds of Theorem A.1;
- both $\mathscr{G}$ and $\mathscr{K}$ do not alter any of the bounds proven in this section;
- $\mathscr{S}+\mathscr{G}+\mathscr{K}$ is supported on $\ell\geqslant 2$ only

This finishes the proof, up to the claim concerning the limits on null infinity. These follow by the argument given in the first part of the proof, which can be repeated on any cone C_u with $u\leqslant U$.

A.3. Propagation of roundness at infinity

In this section we state a corollary to Theorem A.1, which can be understood as the propagation of the round sphere condition (191) at null infinity. It states that, if the linearised Gaussian curvature $\overset{(1)}{K}$ behaves like r^{-3} on the outgoing cone C_{u_0}, then $\overset{(1)}{K}r^3$ remains bounded on any cone which is a finite u distance away.

Note that Theorem 9.1 stated in particular that, given any solution $\mathscr{S}$ as in Theorem A.1, we can construct a pure gauge solution $\mathscr{G}$ such that the sum $\mathscr{S}+\mathscr{G}$ satisfies on C_{u_0} the stronger bounds in the Corollary below.[36] The corollary then shows that these stronger bounds are propagated. Of course, this is directly related to the propagation of uniform boundedness for the quantity $\overset{(1)}{Y}$ in our Theorems 3 and 4.

COROLLARY A.1. *With the assumptions of Theorem A.1, assume in addition that*

$$|(r\nabla\!\!\!/)^m(r^3\overset{(1)}{K})|\leqslant C_m \tag{441}$$

holds for all $m\leqslant n-9$ on C_{u_0}. Then, given any $u_0<U<\infty$, the estimate (441) actually holds for any $m\leqslant n-10$ in any spacetime region $\mathcal{M}\cap\{u_0\leqslant u\leqslant U\}$, where C_m now also depends on the choice of U and the constants C_m appearing in Theorem A.1.

Proof. Compute from (147)

$$\nabla\!\!\!/_3((r\nabla\!\!\!/)^m\overset{(1)}{K}r^3)=\mathcal{Q}_m,$$

and deduce that $\mathcal{Q}_m$ is pointwise uniformly bounded using the bounds (439) of Theorem A.1. □

[36] At the non-linear level this can be interpreted as refoliating the cone such that the sphere at infinity becomes round.

We finally remark that a similar corollary is easily deduced for the quantities

$$(r\nabla\!\!\!/)^m \frac{\overset{(1)}{\sqrt{g\!\!\!/}}}{\sqrt{g\!\!\!/}} \quad \text{and} \quad (r\nabla\!\!\!/)^m \overset{(1)}{\hat{g\!\!\!/}}.$$

Appendix B. Characterizing the vanishing of gauge invariant quantities

In this appendix, we discuss solutions characterized by the vanishing of gauge invariant quantities.

We first show in Appendix B.1 that the vanishing of $\overset{(1)}{\alpha}$ and $\overset{(1)}{\underline{\alpha}}$ identically implies that a solution $\mathscr{S}$ is the sum $\mathscr{S}=\mathscr{G}+\mathscr{K}$ of a pure gauge solution and a linearised Kerr solution, provided that $\mathscr{S}$ is asymptotically flat. We then consider in Appendix B.2 the larger class of solutions such that $\overset{(1)}{P}$ and $\overset{(1)}{\underline{P}}$ vanish identically. We shall show that this class corresponds precisely to the linearised Robinson–Trautman solutions, again up to the addition of a pure gauge solution.

B.1. $\overset{(1)}{\alpha}=\overset{(1)}{\underline{\alpha}}=0$ implies $\mathscr{S}=\mathscr{G}+\mathscr{K}$

In this section we prove that any solution $\mathscr{S}$ which is asymptotically flat and satisfies $\overset{(1)}{\alpha}=\overset{(1)}{\underline{\alpha}}\equiv 0$ globally is necessarily a pure gauge solution $\mathscr{G}$ plus a reference linearised Kerr solution $\mathscr{K}$.

THEOREM B.1. *Let $\mathscr{S}$ be a smooth solution of the full system of linearised gravity arising from a smooth seed initial data set on $C_{u_0}\cup C_{v_0}$ through Theorem 8.1. Assume that the data are asymptotically flat with weight s, as in Definition 8.2. Assume further that*

$$\overset{(1)}{\alpha}=\overset{(1)}{\underline{\alpha}}=0 \tag{442}$$

holds globally on $\mathcal{M}\cap\{u\geqslant 0\}\cap\{v\geqslant 0\}$. Then, $\mathscr{S}$ is the sum of a pure gauge solution $\mathscr{G}$ and a reference linearised Kerr solution $\mathscr{K}$.

We remark that the assumption (188) in Definition 8.2 can actually be deduced from (442), so the assumptions (186) and (187) on the data suffice in conjunction with (442).

Proof. We let $\underset{\vee}{\mathscr{S}}$ be as in Theorem 9.1, i.e. we put the solution $\mathscr{S}$ in the initial data gauge. Moreover, let us subtract $\mathscr{K}_{\mathfrak{m},s_i}$ of Theorem 9.2, so that $\underset{\vee}{\mathscr{S}}'\doteq\underset{\vee}{\mathscr{S}}-\mathscr{K}_{\mathfrak{m},s_i}$ is supported outside of $\ell=0,1$. Below, we shall consider quantities associated with $\underset{\vee}{\mathscr{S}}'$.

From the fact that $\overset{(1)}{Y}$ is bounded for asymptotically flat initial data and $\overset{(1)}{\underline{\psi}}=0$ globally we deduce, noting that $\overset{(1)}{\underline{\alpha}}=0$ implies $\overset{(1)}{\underline{\psi}}=0$, the bound

$$|r^2\overset{(1)}{\underline{\widehat{\chi}}}| \leqslant C \quad \text{along } C_{u_0}$$

for a constant determined purely by the seed initial data. Let $f_{\text{out}}(v,\theta,\phi)$ be a solution of the elliptic equation along C_{u_0}:

$$r^2 \mathcal{D}\!\!\!/_2^\star \nabla\!\!\!\!/_A f_{\text{out}}(v,\theta,\phi) = \frac{r^2}{2\Omega^2}(u_0,v)\cdot\overset{(1)}{\underline{\widehat{\chi}}}(u_0,v,\theta,\phi).$$

The solution is uniquely determined, once we insist that f_{out} has vanishing $\ell=0$ and $\ell=1$ modes. It is also clear that f_{out} is uniformly bounded.

We add the pure gauge solution generated by Lemma 6.1.1 through f_{out} to $\underset{\vee}{\mathscr{S}}'$, and call the resulting solution $\mathscr{S}_1$. The solution $\mathscr{S}_1$ satisfies $\overset{(1)}{\underline{\widehat{\chi}}}=0$ along C_{u_0}, and hence globally on $\mathcal{M}\cap\{u\geqslant 0\}\cap\{v\geqslant 0\}$ by the transport equation (139). Importantly, since f_{out} is uniformly bounded, the solution $\mathscr{S}_1$ also still satisfies the round sphere condition (191) and (192).

Let now $\Omega^2 f_{\text{in}}(u,\theta,\phi)$ be determined by the following elliptic equation along C_{v_0}:

$$r^2 \mathcal{D}\!\!\!/_2^\star \nabla\!\!\!\!/_A f_{\text{in}}(u,\theta,\phi)\Omega^2(u,v_0) = \tfrac{1}{2}\Omega r^2(u,v_0)\overset{(1)}{\widehat{\chi}}[\mathscr{S}_1](u,v_0,\theta,\phi).$$

We add the pure gauge solution generated by Lemma 6.1.3 through f_{in} to $\mathscr{S}_1$, and call the resulting solution $\mathscr{S}_2$. The solution $\mathscr{S}_2$ satisfies $\overset{(1)}{\widehat{\chi}}=0$ on C_{v_0}, which implies $\overset{(1)}{\widehat{\chi}}=0$ globally through the transport equation (139). Importantly, since $\Omega^2 f_{\text{in}}$ is uniformly bounded, the solution $\mathscr{S}_2$ also still satisfies the round sphere condition (191) and (192). Note also that the pure gauge solution added through Lemma 6.1.3 has $\overset{(1)}{\underline{\widehat{\chi}}}=0$, so $\mathscr{S}_2$ satisfies

$$\overset{(1)}{\alpha} = \overset{(1)}{\underline{\alpha}} = \overset{(1)}{\widehat{\chi}} = \overset{(1)}{\underline{\widehat{\chi}}} = 0 \quad \text{on } \mathcal{M}\cap\{u\geqslant 0\}\cap\{v\geqslant 0\}. \tag{443}$$

Note that both pure gauge transformation added to $\underset{\vee}{\mathscr{S}}$ do not change the conclusions of Theorem 9.2, as f_{out} and f_{in} both have vanishing projection to $\ell=0,1$. From (140) and (141) one now concludes that $\mathcal{D}\!\!\!/_2^\star\overset{(1)}{\eta}=\mathcal{D}\!\!\!/_2^\star\overset{(1)}{\underline{\eta}}=0$, and hence, since $\mathscr{S}_2$ satisfies

$$(\mathrm{d}\!\!/\mathrm{iv}\,\overset{(1)}{\eta})_{\ell=1} = (\mathrm{c}\!\!/\mathrm{url}\,\overset{(1)}{\eta})_{\ell=1} = 0$$

(and similarly for $\overset{(1)}{\underline{\eta}}$), that in fact $\overset{(1)}{\eta}=\overset{(1)}{\underline{\eta}}=0$ on $\mathcal{M}\cap\{u\geqslant 0\}\cap\{v\geqslant 0\}$. Since $\overset{(1)}{\alpha}=\overset{(1)}{\underline{\alpha}}=0$ implies $\overset{(1)}{\psi}=\overset{(1)}{\underline{\psi}}=0$, we have from the formulas (182) that $\overset{(1)}{\beta}=\overset{(1)}{\underline{\beta}}=0$, after using the fact that

$$(\mathrm{d}\!\!/\mathrm{iv}\,\overset{(1)}{\beta})_{\ell=1} = (\mathrm{c}\!\!/\mathrm{url}\,\overset{(1)}{\beta})_{\ell=1} = 0$$

(and similarly for $\overset{(1)}{\underline{\beta}}$). The Codazzi equations (145) and the fact that the $\ell{=}0,1$ modes of $(\Omega \overset{(1)}{\mathrm{tr}}\,\chi)$ and $(\Omega \overset{(1)}{\mathrm{tr}}\,\underline{\chi})$ vanish then yield $(\Omega \overset{(1)}{\mathrm{tr}}\,\chi){=}(\Omega \overset{(1)}{\mathrm{tr}}\,\underline{\chi}){=}0$ identically. Equation (134) and the vanishing of the $\ell{=}0,1$ modes allows the conclusion for $\Omega^{-1}\overset{(1)}{\Omega}$, and hence $\overset{(1)}{\omega}$ and $\overset{(1)}{\underline{\omega}}$. Finally, global vanishing of ϱ and σ is now a consequence of (135) and (146).

So far, we have shown that all linearised curvature and all linearised Ricci coefficients of the solution vanish for $\mathscr{S}_2$. To conclude also the vanishing of the linearised metric components, we need to add another pure gauge solution.

The geometric quantity $\overset{(1)}{b}$ of the solution $\mathscr{S}_2$, while having globally vanishing projection to $\ell{=}1$,(37) has a potentially non-vanishing trace on C_{u_0}, which we denote by $\tilde{b}$. It satisfies the bound $|\nabla\!\!\!/\,\tilde{b}|{\lesssim}v^{-2}$ along C_{u_0}, following from the fact that $\mathscr{S}$ satisfies it and that all pure gauge solutions added so far do.

Propositions 9.2.5 and 9.2.6 construct a pure gauge solution $\mathscr{G}$, generated by bounded $q_1(v,\theta,\phi)$ and $q_2(v,\theta,\phi)$ having vanishing projection to $\ell{=}0,1$, with the property that $\mathscr{G}$ satisfies the round sphere condition (192), and moreover $\overset{(1)}{b}[\mathscr{G}]{=}{-}\tilde{b}$ along C_{u_0}. Finally, all linearised Ricci coefficients and curvature components vanish for $\mathscr{G}$.

It is easy to see that the solution $\mathscr{S}_3{=}\mathscr{S}_2{+}\mathscr{G}$ is the trivial solution: One uses the propagation equation (132) from infinity to first conclude that $\overset{(1)}{\hat{\slashed{g}}}{=}0$ identically along C_{u_0} and then that $\overset{(1)}{\hat{\slashed{g}}}{=}0$ identically on $\mathcal{M}{\cap}\{u{\geqslant}0\}{\cap}\{v{\geqslant}0\}$ by using the propagation in the 3-direction of (132).

We have shown thus that $\underset{\vee}{\mathscr{S}}'$ is the sum of pure gauge solutions, and thus the original $\mathscr{S}{=}\underset{\vee}{\mathscr{S}}'{-}\underset{\vee}{\mathscr{G}}{+}\mathscr{K}$ is the sum of a pure gauge solution and a reference linearised Kerr. □

B.2. $\overset{(1)}{P}{=}\overset{(1)}{\underline{P}}{=}0$ implies linearised Robinson–Trautman

While global vanishing of $\overset{(1)}{\alpha}$ and $\overset{(1)}{\underline{\alpha}}$ together with asymptotic flatness implies that the solution is the sum of a pure gauge solution and linearised Kerr solution by Theorem B.1, one may ask whether vanishing of the derived quantities P and $\overset{(1)}{\underline{P}}$ is sufficient for this conclusion. As we shall see in this section, this is *not* the case. The non-trivial solutions arising can however be completely described: They are given by the linearisation of a family of algebraically special solutions to the Einstein vacuum equations, the celebrated *Robinson–Trautman metrics* [63]. These vacuum metrics can be characterized geometrically by the fact that they admit a shear-free congruence of null-geodesics which is also hypersurface orthogonal. See [63], [16] and also §10 of [18] for an introduction to this family.

(37) Recall that, by this, we mean that $(\mathrm{d\!\!/iv}\,\overset{(1)}{b})_{\ell=1}{=}(\mathrm{cu\!\!/rl}\,\overset{(1)}{b})_{\ell=1}{=}0$ hold.

We shall only sketch here the relevant computations.

B.2.1. The Calabi equation

Suppose first $\overset{(1)}{P}=0$ *or* $\overset{(1)}{\underline{P}}=0$. Then, we have one of

$$r^2 \not{\mathcal{D}}_2^\star \not{\mathcal{D}}_1^\star (r^3 \overset{(1)}{\varrho}, \mp r^3 \overset{(1)}{\sigma}) - 3Mr\overset{(1)}{\underline{\hat{\chi}}} + \mathcal{W} = 0,$$

with the upper sign in case $\overset{(1)}{P}=0$ and the lower sign if $\overset{(1)}{\underline{P}}=0$, where $\mathcal{W}$ indicates a term that vanishes in the limit on null infinity by asymptotic flatness. Taking a 3-derivative yields

$$r^2 \not{\mathcal{D}}_2^\star \not{\mathcal{D}}_1^\star (-r^3 \,\mathrm{d\!\!/iv}\, \overset{(1)}{\underline{\beta}}, \pm r^3 \,\mathrm{cu\!\!/rl}\, \overset{(1)}{\underline{\beta}}) + 3Mr\overset{(1)}{\underline{\alpha}} + \mathcal{W} = 0.$$

Application of another 3-derivative yields the equation

$$r^4 \not{\mathcal{D}}_2^\star \not{\mathcal{D}}_1^\star (\mathrm{d\!\!/iv}\,\mathrm{d\!\!/iv}(r\overset{(1)}{\underline{\alpha}}), \mp \,\mathrm{cu\!\!/rl}(\mathrm{d\!\!/iv}\, r\overset{(1)}{\underline{\alpha}})) + 3M \not{\nabla}_3 (r\overset{(1)}{\underline{\alpha}}) + \mathcal{W} = 0.$$

Therefore, if $\overset{(1)}{\underline{P}}=0$ then the fourth-order parabolic equation (cf. the Calabi equation in [63])

$$\not{\nabla}_3 (r\overset{(1)}{\underline{\alpha}})^{\mathcal{I}} = -\frac{1}{3M} r^4 \not{\mathcal{D}}_2^\star \not{\mathcal{D}}_1^\star \not{\mathcal{D}}_1 \not{\mathcal{D}}_2 (r\overset{(1)}{\underline{\alpha}})^{\mathcal{I}} \tag{444}$$

has to hold along null infinity. Here $(r\overset{(1)}{\underline{\alpha}})^{\mathcal{I}}$ is the symmetric traceless tensor obtained as the limit at null infinity $v \to \infty$ of the quantity $r\overset{(1)}{\underline{\alpha}}(u,v,\theta)$ measured in an orthonormal frame on the spheres $S^2_{u,v}$. We can interpret $(r\overset{(1)}{\underline{\alpha}})^{\mathcal{I}}$ either as a symmetric traceless spacetime $S^2_{u,v}$-tensor whose components in an orthonormal frame do not depend on v, or as a symmetric traceless S^2_u-tensor defined on the cylinder $[u_0,\infty)\times S^2$ equipped with the metric $-du^2 + d\theta^2 + \sin^2\theta\, d\phi^2$. Taking the latter point of view and considering $r^4 \not{\mathcal{D}}_2^\star \not{\mathcal{D}}_1^\star \not{\mathcal{D}}_1 \not{\mathcal{D}}_2$ in (444) as an operator on the unit sphere, equation (444) becomes a parabolic equation on the cylinder $[u_0,\infty)\times S^2$ whose solution is uniquely determined if data are prescribed on the "initial" sphere $S^2_{u_0}$.

A priori, it seems that we have the full freedom of specifying a symmetric traceless tensor. However, if in addition $P=0$, then $\not{\mathcal{D}}_2^\star \not{\mathcal{D}}_1^\star (0, \overset{(1)}{\sigma}) = \frac{1}{2}(\overset{(1)}{P} - \overset{(1)}{\underline{P}}) = 0$ everywhere, and hence $\overset{(1)}{\sigma} \equiv 0$ globally, provided the $\ell = 0, 1$ modes of $\overset{(1)}{\sigma}$ also vanish. This implies that $r^4 \not{\mathcal{D}}_2^\star \not{\mathcal{D}}_1^\star \,\mathrm{cu\!\!/rl}\,\mathrm{d\!\!/iv}(r\overset{(1)}{\underline{\alpha}})^{\mathcal{I}} = 0$ on null infinity. It follows that we can prescribe $(r\overset{(1)}{\underline{\alpha}})^{\mathcal{I}}$ initially on one sphere, subject to the condition $r^4 \not{\mathcal{D}}_2^\star \not{\mathcal{D}}_1^\star \,\mathrm{cu\!\!/rl}\,\mathrm{d\!\!/iv}(r\overset{(1)}{\underline{\alpha}})^{\mathcal{I}} = 0$ (which then propagates), which reduces the number of degrees of freedom to one function on the initial sphere, just as it is the case for the Robinson–Trautman class [63].

It is useful to scalarise equation (444) by setting $(r\overset{(1)}{\underline{\alpha}})^{\mathcal{I}}=r^2\mathcal{D}_2^\star\mathcal{D}_1^\star(f,0)$, determining uniquely (up to projection to $\ell=0,1$) a function f on $[u_0,\infty)\times S^2$. If $(r\overset{(1)}{\underline{\alpha}})^{\mathcal{I}}$ satisfies (444), then f satisfies

$$\partial_u f=-\frac{1}{6M}(\Delta_{S^2}\Delta_{S^2}+2\Delta_{S^2}f). \tag{445}$$

This should be compared with equations (2.4) and (2.5) in [16], which, upon linearisation $f=1+\varepsilon\overset{(1)}{f}+\mathcal{O}(\varepsilon^2)$ and setting $M=2m$, yields (445).

We can solve (445) mode by mode obtaining the solution

$$f_{\ell,m}(u)=e^{-(u/2M)(\ell-1)\ell(\ell+1)(\ell+2)/3}Y_m^\ell.$$

Note that, for fixed v, this behaves like an integer power of $\Omega^2(u,v)\sim e^{-u/2M}$ near the event horizon.

B.2.2. Constructing the full solution in the horizon-normalised gauge

We now outline the argument that assuming $\overset{(1)}{P}=\overset{(1)}{\underline{P}}=0$ and specifying such an $(r\overset{(1)}{\underline{\alpha}})^{\mathcal{I}}$ initially on the sphere $S^2_{u_0,\infty}$ determines a solution of the system of gravitational perturbations which is unique up to pure gauge solutions. It is important to note that (444) will produce solutions $(r\overset{(1)}{\underline{\alpha}})^{\mathcal{I}}$, which decay exponentially in u.

Suppose we have an asymptotically flat seed initial data set for which $\overset{(1)}{P}$ and $\overset{(1)}{\underline{P}}$ are zero on $C_{u_0}\cap C_{v_0}$. Since $\overset{(1)}{P}$ and $\overset{(1)}{\underline{P}}$ satisfy the Regge–Wheeler equation, $\overset{(1)}{P}\equiv 0$ and $\overset{(1)}{\underline{P}}\equiv 0$ in $\mathcal{M}\cap\{u\geqslant u_0\}\cap\{v\geqslant v_0\}$. We construct the full solution $\hat{\mathscr{S}}'=\hat{\mathscr{S}}-\mathscr{K}_{\mathfrak{m},s_i}$ directly in the horizon-normalised gauge of Theorem 4.

(1) From the discussion above, we have

$$\overset{(1)}{\sigma}=0 \quad \text{in } \mathcal{M}\cap\{u\geqslant u_0\}\cap\{v\geqslant v_0\}.$$

(2) On the horizon $\mathcal{H}^+$, we have

$$\begin{aligned}
0&=\int_{v_0}^{\infty}dv\int_{S^2}\sin\theta\,d\theta\,d\phi\,|\overset{(1)}{P}|^2\\
&=\int_{v_0}^{v}dv\int_{S^2}\sin\theta\,d\theta\,d\phi\left|\mathcal{D}_2^\star\mathcal{D}_1^\star(\overset{(1)}{\varrho},0)+\frac{3}{8M^3}\Omega\overset{(1)}{\hat{\chi}}\right|^2\\
&=\int_{v_0}^{\infty}dv\int_{S^2}\sin\theta\,d\theta\,d\phi\left[|\mathcal{D}_2^\star\mathcal{D}_1^\star(\overset{(1)}{\varrho},0)|^2+\frac{9|\Omega\overset{(1)}{\hat{\chi}}|^2}{64M^6}+\frac{3}{4M^3}\mathcal{D}_1^\star(\overset{(1)}{\varrho},0)(-\Omega\overset{(1)}{\beta})\right]\\
&=\int_{v_0}^{\infty}dv\int_{S^2}\sin\theta\,d\theta\,d\phi\left[|\mathcal{D}_2^\star\mathcal{D}_1^\star(\overset{(1)}{\varrho},0)|^2+\frac{9}{64M^6}|\Omega\overset{(1)}{\hat{\chi}}|^2+\frac{3}{4M^3}\left(-\frac{1}{2}\right)\partial_v|\overset{(1)}{\varrho}|^2\right]
\end{aligned}$$

and, since $\overset{(1)}{\varrho}\to 0$ as $v\to\infty$ by Theorem 4 (recall Corollary 14.6, the restriction of $\overset{(1)}{P}$ on the horizon and the fact that $\overset{(1)}{\varrho}_{\ell=0,1}=0$), we conclude $\Omega\overset{(1)}{\widehat{\chi}}=0$, $\Omega\overset{(1)}{\beta}=0$ (Codazzi) and $\overset{(1)}{\varrho}=0$ along $\mathcal{H}^+$. Therefore, also $\Omega^2\overset{(1)}{\alpha}=0$ and $\Omega\overset{(1)}{\psi}=0$ on $\mathcal{H}^+$. We also have $\overset{(1)}{\eta}=0$ on $\mathcal{H}^+$, since $\operatorname{c\!\!/url}\overset{(1)}{\eta}=0$ holds from $\overset{(1)}{\sigma}=0$ and $\operatorname{d\!\!/iv}\overset{(1)}{\eta}=0$ by the horizon gauge condition (194). Note that these bounds hold both for the solution $\hat{\mathscr{S}}$ of Theorem 4 and the solution $\underset{\vee}{\mathscr{S}}$ of Theorem 3.

(3) The equations $\Omega^{-1}\nabla\!\!\!/_3(\overset{(1)}{\psi}r^2\Omega)=0$ and $\Omega^{-1}\nabla\!\!\!/_3(\overset{(1)}{\alpha}r\Omega^2)=0$ following from (178) and (179) now imply

$$\overset{(1)}{\psi}\Omega=\overset{(1)}{\alpha}\Omega^2\equiv 0 \quad \text{in } \mathcal{M}\cap\{u\geqslant u_0\}\cap\{v\geqslant v_0\}. \tag{446}$$

(4) On null infinity, we know that $r\overset{(1)}{\underline{\alpha}}=(r\overset{(1)}{\underline{\alpha}})^{\mathcal{I}}(u,\theta)$ is entirely determined by the parabolic equation and exponentially decaying. The solution decays at least like $e^{-4u/M}$ (as follows from the fact that α has at least $\ell\geqslant 2$; see also [15] and compare with $\Omega^2\sim e^{-u/2M}$). Therefore (recalling $\overset{(1)}{\sigma}=0$)

$$\begin{aligned}
\lim_{v\to\infty}(r^2\overset{(1)}{\underline{\beta}}) &= (r^2\overset{(1)}{\underline{\beta}})^{\mathcal{I}}(u,\theta)=\int_u^\infty r\operatorname{d\!\!/iv}(r\overset{(1)}{\underline{\alpha}})^{\mathcal{I}}(\bar{u},\theta)\,d\bar{u},\\
\lim_{v\to\infty}(r^3\overset{(1)}{\varrho}) &= (r^3\overset{(1)}{\varrho})^{\mathcal{I}}(u,\theta)=\int_u^\infty r\operatorname{d\!\!/iv}(r^2\overset{(1)}{\underline{\beta}})^{\mathcal{I}}(\bar{u},\theta)\,d\bar{u},\\
\lim_{v\to\infty}(r^4\overset{(1)}{\beta}) &= (r^4\overset{(1)}{\beta})^{\mathcal{I}}(u,\theta)=-\int_u^\infty r\mathcal{D}\!\!\!/_1^\star(-(r^3\overset{(1)}{\varrho})^{\mathcal{I}},0)(\bar{u},\theta)\,d\bar{u}
\end{aligned}$$

are all determined on null infinity. The existence of these limits and their vanishing as $u\to\infty$ is a consequence of Theorem 4 (Propositions 12.3.6 and 14.3.3) for $(r^2\overset{(1)}{\underline{\beta}})^{\mathcal{I}}$ and $(r^3\overset{(1)}{\varrho})^{\mathcal{I}}$. For $(r^4\overset{(1)}{\beta})^{\mathcal{I}}$, it follows from $\overset{(1)}{\alpha}$ globally vanishing, (149) and the fact that $\overset{(1)}{\beta}\Omega$ vanishes on the horizon. We also see from (139)

$$\lim_{v\to\infty}(r\overset{(1)}{\underline{\widehat{\chi}}})=(r\overset{(1)}{\underline{\widehat{\chi}}})^{\mathcal{I}}=\int_u^\infty(r\overset{(1)}{\underline{\alpha}})^{\mathcal{I}}(\bar{u},\theta)\,d\bar{u}.$$

(5) Integrating backwards from null infinity (151) now yields from (446)

$$\begin{aligned}
\overset{(1)}{\beta}(v,u,\theta) &= \frac{\Omega}{r^4}[(r^4\overset{(1)}{\beta})^{\mathcal{I}}(u,\theta)],\\
\overset{(1)}{\widehat{\chi}}(v,u,\theta) &= \frac{4M}{3}\Omega r^3\mathcal{D}\!\!\!/_2^\star\overset{(1)}{\beta}=\frac{4M}{3}r^{-2}\Omega\cdot r\mathcal{D}\!\!\!/_2^\star((r^4\overset{(1)}{\beta})^{\mathcal{I}}(u,\theta)),
\end{aligned}$$

valid in an orthonormal frame on the spheres $S^2_{u,v}$.[38]

([38]) Recall that the statement that the spacetime tensor $(r^4\overset{(1)}{\beta})^{\mathcal{I}}$ does not depend on v is true only in an orthonormal frame. Otherwise factors of r appear from raising and lowering indices.

(6) Recall now the equations

$$\nabla\!\!\!/_4(\overset{(1)}{\underline{\psi}}r^3\Omega)=-r^3\Omega\overset{(1)}{\underline{P}} \quad \text{and} \quad \nabla\!\!\!/_4(r\Omega^2\overset{(1)}{\underline{\alpha}})=2r\Omega^2\overset{(1)}{\underline{\psi}},$$

from which we conclude that

$$r^3\Omega\overset{(1)}{\underline{\psi}}(v,u,\theta)=\int_u^\infty r^2\mathcal{D}\!\!\!/_2^\star \,\mathrm{d}\!\!\!/\mathrm{iv}(r\overset{(1)}{\underline{\alpha}})^{\mathcal{I}}(\bar{u},\theta)\,d\bar{u}$$

and

$$r\Omega^2\overset{(1)}{\underline{\alpha}}(v,u,\theta)=(r\overset{(1)}{\underline{\alpha}})^{\mathcal{I}}(u,\theta)-\frac{2}{r(u,v)}\int_u^\infty r^2\mathcal{D}\!\!\!/_2^\star \,\mathrm{d}\!\!\!/\mathrm{iv}(r\overset{(1)}{\underline{\alpha}})^{\mathcal{I}}(\bar{u},\theta)\,d\bar{u}, \tag{447}$$

again valid in an orthonormal frame. Note that $(r\overset{(1)}{\underline{\alpha}})^{\mathcal{I}}(u,\theta)$ needs to decay at least as fast as Ω^4 towards the event horizon for $\overset{(1)}{\underline{\alpha}}$, to extend regularly to $\mathcal{H}^+$.

(7) Since in the horizon-normalised gauge we also have $\overset{(1)}{\underline{\eta}}=0$ on $\mathcal{H}^+$, we conclude using (141) and the fact that $\Omega^{-1}\overset{(1)}{\underline{\widehat{\chi}}}\to 0$ along $\mathcal{H}^+$ that $\Omega^{-1}\overset{(1)}{\underline{\widehat{\chi}}}=0$ on $\mathcal{H}^+$. We now determine $\overset{(1)}{\underline{\widehat{\chi}}}$ globally from (139) using (447) as

$$\overset{(1)}{\underline{\widehat{\chi}}}r^2\Omega^{-1}(v,u,\theta)=\int_u^\infty r^2\overset{(1)}{\underline{\alpha}}(v,\bar{u},\theta)\,d\bar{u}.$$

With $\overset{(1)}{\widehat{\chi}}$ and $\overset{(1)}{\underline{\widehat{\chi}}}$ determined globally, (140) and (141) allow us to obtain $\overset{(1)}{\eta}$ and $\overset{(1)}{\underline{\eta}}$, since the $\ell=0,1$ modes of all quantities vanish for $\hat{\mathscr{S}}$ (cf. Corollary 4.2). Codazzi (145) implies expressions for $(\Omega\overset{(1)}{\mathrm{tr}}\chi)$ and $\Omega^{-2}(\Omega\overset{(1)}{\mathrm{tr}}\underline{\chi})$. Finally, (134) implies an expression for $\Omega^{-1}\overset{(1)}{\Omega}$.

(8) To determine the remaining metric quantities $(\overset{(1)}{b},\overset{(1)}{\sqrt{g\!\!\!/}},\overset{(1)}{\hat{g\!\!\!/}})$, it is most convenient to add to $\hat{\mathscr{S}}'$ another pure gauge solution which achieves $\overset{(1)}{b}=0$ on C_{u_0}, while preserving the condition (192) and not changing any of the quantities discussed in (1)–(7) above. The existence of such a solution follows from Proposition 9.2.5. One can then use (131)–(133) to determine explicit formulas for $(\overset{(1)}{b},\overset{(1)}{\sqrt{g\!\!\!/}},\overset{(1)}{\hat{g\!\!\!/}})$.

B.2.3. Regularity

To determine the smoothness properties of the solution constructed, it suffices to check its regularity at the level of the seed initial data, i.e. whether the following quantities extend to the horizon on C_{v_0}:

$$(e^{-u/2M}\partial_u)^n(\overset{(1)}{\underline{\widehat{\chi}}}r^2\Omega^{-1}),\quad (e^{-u/2M}\partial_u)^n((\Omega\overset{(1)}{\mathrm{tr}}\underline{\chi})\Omega^{-2}),\quad (e^{-u/2M}\partial_u)^n(\Omega^{-1}\overset{(1)}{\Omega}). \tag{448}$$

Since all quantities of the solution determined above behave like $[\Omega^2(u,v_0)]^k$ for some integer k, the solution is smooth in the extended sense.

Interestingly, the non-smoothness that was observed in [16] for the class of Robinson–Trautman metric seems to be a feature of the non-linear terms in the parabolic equation, and is not seen at the linearised level.

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