

Gauduchon metrics with prescribed volume form

by

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1. Introduction

Let M be a compact complex manifold of complex dimension n . Suppose that M admits a metric $\alpha = \sqrt{-1} \alpha_{i\bar{j}} dz^i \wedge d\bar{z}^j > 0$ which is Kähler (that is, $d\alpha = 0$). Yau's celebrated solution [43] of the Calabi conjecture says that given any smooth positive volume form σ on M with $\int_M \sigma = \int_M \alpha^n$, we can find a Kähler metric ω with this prescribed volume form:

$$\omega^n = \sigma. \tag{1.1}$$

Moreover, there exists such a metric so that $[\omega] = [\alpha]$ in $H^2(M, \mathbb{R})$, and with this cohomological constraint the metric ω is unique.

Furthermore, Yau's theorem is equivalent to a statement about the first Chern class $c_1(M)$. Namely, given any smooth representative Ψ of $c_1(M)$, there exists a unique Kähler metric ω cohomologous to α such that

$$\text{Ricci}(\omega) = \Psi, \tag{1.2}$$

where $\text{Ricci}(\omega)$ is the Ricci form of the Kähler metric ω . Indeed, this follows immediately from the definition of $c_1(M)$ and by applying the operator $-\sqrt{-1} \partial \bar{\partial} \log$ to (1.1).

It is natural to investigate whether similar results hold when M does not admit a Kähler metric, but only a Hermitian metric α . If we do not impose any constraint on

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the class of Hermitian metrics that we consider, then (1.1) can be trivially solved by a conformal change of metric. However, there is a natural class of Hermitian metrics which exist on all compact complex manifolds, namely Gauduchon metrics. A Hermitian metric α is called *Gauduchon* if

$$\partial\bar{\partial}(\alpha^{n-1}) = 0,$$

and a classical result of Gauduchon [12] says that every Hermitian metric is conformal to a Gauduchon metric (uniquely up to scaling, when $n \geq 2$). In particular, if we restrict our attention to Gauduchon metrics, then we cannot use non-trivial conformal changes.

Motivated by Yau's theorem, in 1984 Gauduchon [13, §IV.5] posed the following conjecture.

Conjecture 1.1. Let M be a compact complex manifold and Ψ be a closed real $(1, 1)$ -form on M with $[\Psi] = c_1^{\text{BC}}(M) \in H_{\text{BC}}^{1,1}(M, \mathbb{R})$. Then there is a Gauduchon metric ω on M with

$$\text{Ricci}(\omega) = \Psi. \quad (1.3)$$

To explain our notation here,

$$H_{\text{BC}}^{1,1}(M, \mathbb{R}) = \frac{\{d\text{-closed real } (1, 1)\text{-forms}\}}{\{\sqrt{-1}\partial\bar{\partial}\psi : \psi \in C^\infty(M, \mathbb{R})\}}$$

denotes the (finite-dimensional) *Bott–Chern cohomology group*, and $\text{Ricci}(\omega)$ is the Chern–Ricci form of ω , which is locally given by

$$\text{Ricci}(\omega) = -\sqrt{-1}\partial\bar{\partial} \log \det g,$$

where we write $\omega = \sqrt{-1}g_{i\bar{j}}dz^i \wedge d\bar{z}^j$. It is a closed real $(1, 1)$ -form and its first Bott–Chern cohomology class $c_1^{\text{BC}}(M) = [\text{Ricci}(\omega)] \in H_{\text{BC}}^{1,1}(M, \mathbb{R})$ is immediately seen to be independent of the choice of ω .

In the spirit of Yau's theorem, we restate Conjecture 1.1 as an equivalent statement about the existence of Gauduchon metrics with prescribed volume form.

Conjecture 1.2. Let M be a compact complex manifold and σ be a smooth positive volume form. Then, there is a Gauduchon metric ω on M with

$$\omega^n = \sigma. \quad (1.4)$$

The equivalence with Conjecture 1.1 follows by applying the operator $-\sqrt{-1}\partial\bar{\partial} \log$ to (1.4).

Our result, Theorem 1.3 below, gives a proof of Conjecture 1.1 (and hence also of Conjecture 1.2). Moreover, our result strengthens the conjecture by imposing a cohomological constraint on the solution ω . Before we state our results, we make some remarks about Conjecture 1.1.

(1) When M is Kähler, this conjecture follows from Yau's theorem.

(2) When $n=2$, the conjecture was proved by Cherrier [4] in 1987 by solving a complex Monge–Ampère equation (see also [38] and [17] for different proofs).

(3) More recently, the second- and third-named authors [40] proved Conjecture 1.1 when M admits an *astheno-Kähler* metric, i.e. a Hermitian metric α with $\partial\bar{\partial}(\alpha^{n-2})=0$ (a condition introduced in [21]).

(4) Clearly, there can be no uniqueness in Conjecture 1.1 as stated.

(5) In [37] the second- and third-named authors proved that given a Hermitian metric α , one can always find another Hermitian metric ω of the form $\omega=\alpha+\sqrt{-1}\partial\bar{\partial}u$, for $u\in C^\infty(M, \mathbb{R})$, solving (1.3). If $n=2$, then α Gauduchon implies that ω is also Gauduchon (and this equation was solved in [4]), but this is no longer the case when $n\geq 3$. Hence, the result of [37] does not help to solve Conjecture 1.1 in dimension 3 or higher.

(6) A consequence of Conjecture 1.1 is that $c_1^{\text{BC}}(M)=0$ holds if and only if there exist Chern–Ricci-flat Gauduchon metrics on M . More information about these “non-Kähler Calabi–Yau” manifolds can be found in [35].

We now state our main results. We first introduce some terminology concerning cohomology classes of $(n-1, n-1)$ -forms. Define the *Aeppli cohomology group*

$$H_A^{n-1, n-1}(M, \mathbb{R}) = \frac{\{\partial\bar{\partial}\text{-closed real } (n-1, n-1)\text{-forms}\}}{\{\partial\gamma + \bar{\partial}\bar{\gamma} : \gamma \in \Lambda^{n-2, n-1}(M)\}}.$$

This space is naturally in duality with the Bott–Chern cohomology group we considered earlier, with the non-degenerate pairing $H_A^{n-1, n-1}(M, \mathbb{R}) \otimes H_{\text{BC}}^{1, 1}(M, \mathbb{R}) \rightarrow \mathbb{R}$ given by wedge product and integration over M (see e.g. [1]). If α_0 is a Gauduchon metric, then α_0^{n-1} defines a class $[\alpha_0^{n-1}] \in H_A^{n-1, n-1}(M, \mathbb{R})$.

We prove the following result.

THEOREM 1.3. *Let M be a compact complex manifold with a Gauduchon metric α_0 , and Ψ be a closed real $(1, 1)$ -form on M with $[\Psi] = c_1^{\text{BC}}(M) \in H_{\text{BC}}^{1, 1}(M, \mathbb{R})$. Then, there exists a Gauduchon metric ω satisfying $[\omega^{n-1}] = [\alpha_0^{n-1}]$ in $H_A^{n-1, n-1}(M, \mathbb{R})$ and*

$$\text{Ricci}(\omega) = \Psi. \tag{1.5}$$

This result immediately implies Conjectures 1.1 and 1.2.

In [40], the second- and third-named authors observed that to solve Theorem 1.3 it is enough to solve a certain partial differential equation, which was also independently introduced by Popovici [30]. This equation is a variant of one introduced by Fu–Wang–Wu [10] and related to Harvey–Lawson’s notion of $(n-1)$ -plurisubharmonic functions [18], [19].

Namely, we seek a Hermitian metric ω on M with the property that

$$\omega^{n-1} = \alpha_0^{n-1} + \partial\gamma + \bar{\partial}\bar{\gamma},$$

where

$$\gamma = \frac{1}{2}\sqrt{-1}\partial\bar{\partial}u \wedge \alpha^{n-2},$$

$u \in C^\infty(M, \mathbb{R})$ and α is a background Gauduchon metric. Clearly, by construction, the metric ω is Gauduchon assuming that α_0 is Gauduchon. Substituting, we see that

$$\omega^{n-1} = \alpha_0^{n-1} + \sqrt{-1}\partial\bar{\partial}u \wedge \alpha^{n-2} + \operatorname{Re}(\sqrt{-1}\partial u \wedge \bar{\partial}(\alpha^{n-2})), \quad (1.6)$$

while if we write

$$\operatorname{Ricci}(\omega) = \Psi + \sqrt{-1}\partial\bar{\partial}F,$$

then (1.5) is equivalent to

$$\omega^n = e^{F+b}\alpha^n,$$

for some constant $b \in \mathbb{R}$. This is exactly the equation that we solve, thus resolving [40, Conjecture 1.5] and [30, Question 1.2].

THEOREM 1.4. *Let M be a compact complex manifold with $\dim_{\mathbb{C}} M = n \geq 2$, equipped with a Hermitian metric α_0 and a Gauduchon metric α . Given a smooth function F on M , we can find a unique $u \in C^\infty(M, \mathbb{R})$ with $\sup_M u = 0$, and a unique $b \in \mathbb{R}$ such that the Hermitian metric ω defined by*

$$\omega^{n-1} := \alpha_0^{n-1} + \sqrt{-1}\partial\bar{\partial}u \wedge \alpha^{n-2} + \operatorname{Re}(\sqrt{-1}\partial u \wedge \bar{\partial}(\alpha^{n-2})) > 0$$

satisfies

$$\omega^n = e^{F+b}\alpha^n. \quad (1.7)$$

Clearly, as we just described, Theorem 1.3 follows from this result if we take $\alpha_0 = \alpha$ Gauduchon. We make some remarks about Theorem 1.4.

(1) In the case when α is Kähler, or more generally if the linear term involving ∂u is removed, the equation (1.7) reduces to the Monge–Ampère equation for $(n-1)$ -plurisubharmonic functions, solved by the second- and third-named authors [39], [40] (see also [11] for earlier partial results).

(2) In the case when $n=2$, this equation reduces to the complex Monge–Ampère equation solved in [4] (see also [38]).

(3) It was shown in [40] that Theorem 1.4 can be reduced to a second-order a-priori estimate of the form (cf. [20])

$$\sup_M |\sqrt{-1}\partial\bar{\partial}u|_\alpha \leq C \left(1 + \sup_M |\nabla u|_\alpha^2 \right),$$

for solutions u of (1.7). This is precisely the estimate we prove in this paper.

(4) If in Theorem 1.4 we assume that α_0 is *strongly Gauduchon* in the sense of Popovici [29], namely that $\bar{\partial}(\alpha_0^{n-1})$ is ∂ -exact, then, by construction, so is the solution ω . Thus, we also get a Calabi–Yau-type theorem for strongly Gauduchon metrics. More applications of this theorem can be found in [30].

(5) Our method of proof of Theorem 1.4 can also be used to solve an equation introduced by Fu–Wang–Wu [10] in certain cases. Suppose we have a compact Hermitian manifold (M, α_0) and we seek a Hermitian metric ω solving (1.7) with the property that

$$\omega^{n-1} = \alpha_0^{n-1} + \sqrt{-1}\partial\bar{\partial}(u\alpha^{n-2})$$

for some $u \in C^\infty(M, \mathbb{R})$ and some Hermitian metric α . This setup is particularly interesting because if α_0 is balanced (i.e. $d(\alpha_0^{n-1})=0$, see [26]), then so is ω , and one obtains a Calabi–Yau theorem for balanced metrics (see also [35, §4]). When α is Kähler, this setup reduces to the setting of item (1). If we instead assume that α is astheno-Kähler, then we see that

$$\omega^{n-1} = \alpha_0^{n-1} + \sqrt{-1}\partial\bar{\partial}u \wedge \alpha^{n-2} + 2 \operatorname{Re}(\sqrt{-1}\partial u \wedge \bar{\partial}(\alpha^{n-2})), \tag{1.8}$$

which differs from (1.6) just for a factor of 2. Therefore, this problem falls into our general framework (see Theorem 2.2 below), and we conclude that we have uniform a-priori estimates for solutions of this equation. The exact same argument as in [40, Theorem 1.7] using the continuity method then shows that the equation is indeed solvable. In the case when we choose α_0 to be balanced, this gives a proof of [35, Conjectures 4.1 and 4.2], assuming that M admits astheno-Kähler metrics. However, we should remark that we are not aware of any example of a non-Kähler compact complex manifold which admits both balanced and astheno-Kähler metrics.⁽¹⁾

(6) The same argument as the proof of Theorem 1.4 also allows us to find a Gauduchon metric ω solving the “complex-Hessian” equation

$$\omega^k \wedge \alpha^{n-k} = e^{F+b} \alpha^n$$

⁽¹⁾ After this paper was posted, and prompted by our remark, explicit examples were constructed in [9] and [22] in all complex dimensions ≥ 4 .

for any $1 \leq k \leq n$; see also [34, Proposition 24] for the case of $(n-1)$ -plurisubharmonic functions, and [8], [20] for the standard Kähler case where $\omega = \alpha + \sqrt{-1} \partial \bar{\partial} u$.

(7) The complex setting is very different from the real analogue of (1.7), treated for example in more generality in [16]. The underlying reason is the fact that there are two different types of complex derivatives. In our case, the special structure of the gradient term in (1.7) plays a key role.

In fact, Theorem 1.4 follows from a much more general result, where we consider a large class of fully non-linear second-order elliptic equations on Hermitian manifolds. This result is analogous to the main result in [34], giving a-priori estimates in the presence of a suitable subsolution. We will state this as Theorem 2.2 in §2. This result fits into a large body of work on fully non-linear second-order elliptic equations, going back to the work of Caffarelli–Nirenberg–Spruck [3] on the Dirichlet problem on domains in \mathbb{R}^n . Some other works on this topic include [6], [7], [14]–[17], [23]–[25], [27], [28], [32], [33], [41], [42], [44]–[46].

In our proof of Theorem 2.2, we use some of the language and approaches of the recent paper of the first-named author [34]. However, if one is only interested in a direct proof of Theorem 1.4, one can equally well use the language of [40]. In any case, the key new ingredient is an understanding of the structure of the term $\operatorname{Re}(\sqrt{-1} \partial u \wedge \bar{\partial}(\alpha^{n-2}))$.

The paper is organized as follows. In §2 we will introduce some notation and state our main technical result (Theorem 2.2). The proof of this theorem will be given in §3, and in §4 we show how this implies Theorem 1.4.

As the present work neared completion, we were informed that Bo Guan and Xiaolan Nie have a work in progress on related results.

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2. Background and the general setting

Let (M, α) be a compact Hermitian manifold of complex dimension n and write

$$\alpha = \sqrt{-1} \alpha_{i\bar{j}} dz^i \wedge d\bar{z}^j > 0.$$

Fix a background $(1, 1)$ -form $\chi = \sqrt{-1} \chi_{i\bar{j}} dz^i \wedge d\bar{z}^j$ which is not necessarily positive definite. Let $W_{i\bar{j}}(\nabla u)$ be a Hermitian tensor which depends linearly on ∇u . For $u: M \rightarrow \mathbb{R}$, define a new tensor $g_{i\bar{j}}$ by

$$g_{i\bar{j}} := \chi_{i\bar{j}} + u_{i\bar{j}} + W_{i\bar{j}}. \quad (2.1)$$

Note that we do not assume that $(g_{i\bar{j}})$ is positive definite. We will study equations for g , where W has a special structure related to the equation (1.7). To define this, let us

write

$$\tilde{g}_{i\bar{j}} := P_\alpha(g_{i\bar{j}}) = \frac{1}{n-1}((\text{tr}_\alpha g)\alpha_{i\bar{j}} - g_{i\bar{j}}), \tag{2.2}$$

where P_α is an operator on tensors, depending on the fixed metric α , defined by the second equality in (2.2). Here $\text{tr}_\alpha g$ is the trace of g with respect to α . As an aside, if α is the Euclidean metric on \mathbb{C}^n , then the condition $P_\alpha(u_{i\bar{j}}) \geq 0$ is equivalent to saying that u is $(n-1)$ -plurisubharmonic, in the sense of Harvey–Lawson [18].

Observe that, writing $\Delta = \alpha^{k\bar{\ell}} \partial_k \partial_{\bar{\ell}}$,

$$\tilde{g}_{i\bar{j}} = \tilde{\chi}_{i\bar{j}} + \frac{1}{n-1}((\Delta u)\alpha_{i\bar{j}} - u_{i\bar{j}}) + Z_{i\bar{j}} \tag{2.3}$$

for Z given by

$$Z_{i\bar{j}} := P_\alpha(W_{i\bar{j}}) = \frac{1}{n-1}((\text{tr}_\alpha W)\alpha_{i\bar{j}} - W_{i\bar{j}}), \tag{2.4}$$

and similarly, $\tilde{\chi}_{i\bar{j}} = P_\alpha(\chi_{i\bar{j}})$. Note that we can also write W explicitly in terms of Z :

$$W_{i\bar{j}} = (\text{tr}_\alpha Z)\alpha_{i\bar{j}} - (n-1)Z_{i\bar{j}}. \tag{2.5}$$

A crucial assumption we make is that W depends on ∇u in the following way: we assume that the tensor Z has the form

$$Z_{i\bar{j}} = Z_{ij}^p u_p + \overline{Z_{j\bar{i}}^p u_p}, \tag{2.6}$$

for some tensor $Z_{i\bar{j}}^p$ independent of u . In addition we have the following.

Assumption for W . In orthonormal coordinates for α at any given point, the component $Z_{i\bar{j}}$ is independent of $u_{\bar{i}}$ and u_j (in other words, $Z_{i\bar{j}}^j = 0$ for all i and j), and $\nabla_i Z_{i\bar{i}}$ is independent of $u_{\bar{i}}$ (in other words, $\nabla_{\bar{i}} Z_{i\bar{i}}^i = 0$ for all i).

Here, ∇ is the Chern connection of α . This assumption expresses a certain skew-symmetry requirement for the tensor W . This assumption is satisfied for the $(n-1)$ -plurisubharmonic Monge–Ampère equation, the case of most interest to us, see (4.3) below, the key reason being that the torsion tensor is skew-symmetric.

Let us record here a few consequences of this assumption, which will be used later. Taking ∇_p of (2.6), setting $i=j$, evaluating at that point and using that $Z_{i\bar{i}}^i = 0$, we see that $\nabla_p Z_{i\bar{i}}$ is independent of u_{ii} and $u_{i\bar{i}}$ (at that point, in orthonormal coordinates for α). Here, the subscripts of u denote ordinary partial derivatives. Similarly, $\nabla_i Z_{p\bar{i}}$ is independent of u_{ii} . Taking two covariant derivatives, we have

$$\begin{aligned} \nabla_{\bar{i}} \nabla_i Z_{i\bar{i}} &= \nabla_{\bar{i}} \nabla_i Z_{i\bar{i}}^p u_p + Z_{i\bar{i}}^p \nabla_{\bar{i}} \nabla_i u_p + \nabla_i Z_{i\bar{i}}^p \nabla_{\bar{i}} u_p + \nabla_{\bar{i}} Z_{i\bar{i}}^p \nabla_i u_p \\ &\quad + \nabla_{\bar{i}} \nabla_i \overline{Z_{i\bar{i}}^p u_p} + \overline{Z_{i\bar{i}}^p} \nabla_{\bar{i}} \nabla_i u_{\bar{p}} + \nabla_i \overline{Z_{i\bar{i}}^p} \nabla_{\bar{i}} u_{\bar{p}} + \nabla_{\bar{i}} \overline{Z_{i\bar{i}}^p} \nabla_i u_{\bar{p}}, \end{aligned}$$

and evaluating at our point and using the assumptions $\nabla_{\bar{i}} Z_{i\bar{i}}^i = 0$ and $Z_{i\bar{i}}^i = 0$, we see that $\nabla_{\bar{i}} \nabla_i Z_{i\bar{i}}$ is independent of $u_{ii}, u_{\bar{i}\bar{i}}, u_{i\bar{i}\bar{i}}$ and $u_{\bar{i}\bar{i}i}$.

Given a smooth function h , we study equations of the form

$$F(A) = h,$$

where A is the endomorphism $A_j^i = \alpha^{i\bar{p}} g_{j\bar{p}}$ of the holomorphic tangent bundle, which is Hermitian with respect to the inner product defined by α , and $F(A)$ is a symmetric function of the eigenvalues $\lambda_1, \dots, \lambda_n$ of A :

$$F(A) = f(\lambda_1, \dots, \lambda_n). \tag{2.7}$$

We assume that our operator F has the special form $F(M) = \tilde{F}(P(M))$, where

$$P(M) = (n-1)^{-1}(\text{Tr}(M)I - M),$$

analogous to P_α above (here writing $\text{Tr}(M)$ for the trace of the matrix M), and

$$\tilde{F}(B) = \tilde{f}(\mu_1, \dots, \mu_n),$$

where μ_1, \dots, μ_n are the eigenvalues of B , and \tilde{f} is another symmetric function. In terms of eigenvalues, this means that

$$f(\lambda_1, \dots, \lambda_n) = (\tilde{f} \circ P)(\lambda_1, \dots, \lambda_n), \tag{2.8}$$

where we are writing P for the map $\mathbb{R}^n \rightarrow \mathbb{R}^n$ induced on diagonal matrices by the matrix map P above. Explicitly, writing $\mu = P(\lambda)$ for the corresponding n -tuples $\lambda, \mu \in \mathbb{R}^n$, we have

$$f(\lambda_1, \dots, \lambda_n) = \tilde{f}(\mu_1, \dots, \mu_n), \quad \text{for } \mu_k = \frac{1}{n-1} \sum_{i \neq k} \lambda_i. \tag{2.9}$$

Assumptions for \tilde{f} and h . We make the following assumptions on \tilde{f} , and the function h in our equation:

(i) \tilde{f} is defined on an open symmetric convex cone $\tilde{\Gamma} \subsetneq \mathbb{R}^n$, containing the positive orthant $\Gamma_n = \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_i > 0, i = 1, \dots, n\}$.

(ii) \tilde{f} is symmetric, smooth, concave and increasing, i.e. its partials satisfy $\tilde{f}_i > 0$ for all i .

(iii) $\sup_{\partial \tilde{\Gamma}} \tilde{f} < \inf_M h$.

(iv) For all $\mu \in \tilde{\Gamma}$, we have

$$\lim_{t \rightarrow \infty} \tilde{f}(t\mu) = \sup_{\tilde{\Gamma}} \tilde{f},$$

where both sides are allowed to be ∞ .

(v) h is a smooth function on M .

Define the cone $\Gamma \subset \mathbb{R}^n$ by $\Gamma = P^{-1}(\tilde{\Gamma})$. Observe that P maps Γ_n into Γ_n . It is then easy to see that the function $f = \tilde{f} \circ P: \Gamma \rightarrow \mathbb{R}$ satisfies exactly the same conditions as \tilde{f} . In particular, some of the results of [34] can be applied to the equation $F(A) = h$. We need the following definition (see [34, Remark 8] to see the equivalence with the definition there), which is a modification of a notion introduced by Guan [14].

Definition 2.1. We say that u is a \mathcal{C} -subsolution for the equation $F(A) = h$ if the following holds. Let $g_{i\bar{j}}$ be defined as in (2.1). We require that for every point $x \in M$, if $\lambda = (\lambda_1, \dots, \lambda_n)$ denote the eigenvalues of the endomorphism $\alpha^{i\bar{p}}g_{j\bar{p}}$ at x , then for all $i = 1, \dots, n$ we have

$$\lim_{t \rightarrow \infty} f(\lambda + t\mathbf{e}_i) > h(x).$$

Here \mathbf{e}_i denotes the i th standard basis vector. Note that part of the requirement is that $\lambda + t\mathbf{e}_i \in \Gamma$ for sufficiently large t , for the limit to be defined.

With this background, our main estimate is the following, analogous to the main result in [34]. We will give the proof in §3.

THEOREM 2.2. *Suppose that \underline{u} is a \mathcal{C} -subsolution for the equation $F(A) = h$, and u is a smooth solution, normalized by $\sup_M u = 0$. Suppose that F and h satisfy the assumptions above, including the assumption for the gradient term W . Then, for each $k = 0, 1, 2, \dots$, we have an estimate $\|u\|_{C^k(M, \alpha)} \leq C_k$, with constant C_k depending on k , on the background data M, α, χ, F, h , the coefficients of W and the subsolution \underline{u} .*

The case of primary interest for us is equation (1.7), which corresponds to the symmetric function

$$\tilde{f}(\mu_1, \dots, \mu_n) = \log(\mu_1 \dots \mu_n) \tag{2.10}$$

on the positive orthant $\tilde{\Gamma} = \Gamma_n$. It is straightforward to check that \tilde{f} satisfies the conditions above. Indeed, \tilde{f} converges to $-\infty$ on the boundary $\partial\Gamma_n$, so (iii) is satisfied, and for (iv) it is enough to note that $\tilde{f}(t\mu) = \tilde{f}(\mu) + n \log t$, which converges to ∞ as $t \rightarrow \infty$.

In addition, if $\mu \in \Gamma_n$, then we also have

$$\lim_{t \rightarrow \infty} \tilde{f}(\mu + t\mathbf{e}_i) = \infty$$

for all i . This means that for a function u to be a \mathcal{C} -subsolution for this equation, the only requirement is that, at each point, the eigenvalues λ of $\alpha^{i\bar{p}}g_{j\bar{p}}$ satisfy $P(\lambda) \in \Gamma_n$. In other words, the requirement is that $\tilde{g}_{i\bar{j}}$, defined in (2.2), is positive definite.

Note that, if \underline{u} is a \mathcal{C} -subsolution, then replacing χ by

$$\chi'_{i\bar{j}} = \chi_{i\bar{j}} + \underline{u}_{i\bar{j}} + W_{i\bar{j}}(\nabla \underline{u}),$$

we may assume that $\underline{u} = 0$. The important consequence of 0 being a \mathcal{C} -subsolution is the following, which follows from [34, Proposition 6 and Lemma 9].

PROPOSITION 2.3. *Suppose that 0 is a \mathcal{C} -subsolution for the equation $F(A)=h$, and u is a solution. Define $g_{i\bar{j}}$ as in (2.1). There are constants $R, \varkappa > 0$, independent of u , with the following property. Let $x \in M$, and choose orthonormal coordinates for α at x , such that g is diagonal, with eigenvalues $\lambda = (\lambda_1, \dots, \lambda_n)$. If $|\lambda| > R$, then there are two possibilities:*

(a) *we either have*

$$\sum_k f_k(\lambda) [\chi_{k\bar{k}} - \lambda_k] > \varkappa \sum_k f_k(\lambda),$$

(b) *or $f_k(\lambda) > \varkappa \sum_i f_i(\lambda)$ for all k .*

In addition $\sum_k f_k(\lambda) > \varkappa$.

We collect some other basic properties of the functions f and \tilde{f} . Suppose that $\lambda \in \Gamma$ with $\lambda_1 \geq \dots \geq \lambda_n$. Then $\mu_1 \leq \dots \leq \mu_n$, and so, by property (ii), $\tilde{f}_1 \geq \dots \geq \tilde{f}_n > 0$ (see e.g. [34, p. 12]). We have

$$f_k = \frac{1}{n-1} \sum_{i \neq k} \tilde{f}_i, \quad (2.11)$$

which implies that $0 < f_1 \leq \dots \leq f_n$. Also, for $k > 1$,

$$0 < \frac{\tilde{f}_1}{n-1} \leq f_k \leq \tilde{f}_1, \quad (2.12)$$

i.e. the f_k for $k > 1$ are all comparable, while f_1 may be relatively small. In addition, from (2.11) with $k=1$, we obtain

$$\tilde{f}_i \leq (n-1)f_1, \quad \text{for } i > 1. \quad (2.13)$$

Proposition 2.3 is easy to verify directly in the case of equation (1.7), where

$$f(\lambda) = \log(\mu_1 \dots \mu_n),$$

with μ_k defined as in (2.9). Indeed, in this case

$$\tilde{f}_i(\mu) = \frac{1}{\mu_i}$$

and

$$f_k(\lambda) = \frac{1}{n-1} \sum_{i \neq k} \frac{1}{\mu_i}.$$

The function 0 being a \mathcal{C} -subsolution means that $\tilde{\chi}$ in (2.3) is positive definite. We have

$$\sum_k f_k(\lambda) \chi_{k\bar{k}} = \sum_i \frac{1}{\mu_i} \tilde{\chi}_{i\bar{i}} > \tau \sum_i \frac{1}{\mu_i} = \tau \sum_k f_k(\lambda),$$

for some $\tau > 0$ depending on a lower bound for $\tilde{\chi}$. We also have

$$\sum_k \lambda_k f_k(\lambda) = n.$$

It follows that we have the alternative (a) in Proposition 2.3 whenever there is one sufficiently small μ_i , which by the equation $f(\lambda) = h$ is equivalent to having at least one large μ_i , i.e. at least one large λ_i . In addition,

$$\sum_k f_k(\lambda) = \sum_{i=1}^n \frac{1}{\mu_i} \geq n(\mu_1 \dots \mu_n)^{-1/n} = ne^{-h/n},$$

so that the final claim in Proposition 2.3 also holds.

3. Proof of the main estimate

In this section, we give the proof of Theorem 2.2.

First of all, a uniform bound $\|u\|_{L^\infty(M)} \leq C$ can be obtained by a simple modification of the argument in [34, Proposition 10 and Remark 12], which is itself inspired by Błocki's proof of the L^∞ estimate in Yau's theorem [2]. In the setting of equation (1.7), the L^∞ estimate of u was first proved in [40], using a different method more analogous to the arguments in [4], [37], [39], [43].

Our main goal is the following estimate:

$$\sup_M |\sqrt{-1} \partial \bar{\partial} u|_\alpha \leq C \left(\sup_M |\nabla u|_\alpha^2 + 1 \right), \quad (3.1)$$

for a constant C depending only on the fixed data of Theorem 2.2. We remark that an estimate of this form was proved in the context of the complex Hessian equations by Hou–Ma–Wu [20], making use of ideas of Chou–Wang [5]. For the $(n-1)$ -plurisubharmonic equation (namely, equation (1.7) without the linear term in ∂u), an estimate of this type was proved in [39], [40]. This was then generalized much further in [34], where the estimate was shown to hold for a large class of equations. Our proof begins along similar lines to these papers. The new difficulty comes from the linear term in ∂u , which, fortunately, has a special structure that we can exploit.

In fact, the estimate (3.1) is equivalent to the bound

$$\lambda_1 \leq CK,$$

where $K = 1 + \sup_M |\nabla u|_\alpha^2$ and λ_1 is the largest eigenvalue of $A = (A_j^i) = (\alpha^{i\bar{p}} g_{j\bar{p}})$. Indeed, our assumption on the cone Γ implies that $\sum_i \lambda_i > 0$ (see Caffarelli–Nirenberg–Spruck [3]).

Then, if λ_1 is bounded from above by CK , so is $|\lambda_i|$ for all i , giving the same bound for $\sup_M |\sqrt{-1}\partial\bar{\partial}u|_\alpha$.

We consider the function

$$H = \log \lambda_1 + \phi(|\nabla u|_\alpha^2) + \psi(u),$$

where ϕ is defined by

$$\phi(t) = -\frac{1}{2} \log\left(1 - \frac{t}{2K}\right),$$

so that $\phi(|\nabla u|_\alpha^2) \in [0, \frac{1}{2} \log 2]$ satisfies

$$\frac{1}{4K} < \phi' < \frac{1}{2K}, \quad \phi'' = 2(\phi')^2 > 0,$$

and ψ is defined by

$$\psi(t) = D_1 e^{-D_2 t}, \tag{3.2}$$

for sufficiently large uniform constants $D_1, D_2 > 0$ to be chosen later. By the L^∞ bound on u , the quantity $\psi(u)$ is uniformly bounded.

We remark that we follow [34] by computing with the largest eigenvalue λ_1 instead of the analogous quantity in [40], but in fact either quantity works, at least in the case of equation (1.7). Also note that, while the function ϕ here coincides with that in [20] (and also in [39], [40]), our choice of ψ is crucially different.

We work at a point where H achieves its maximum, in orthonormal complex coordinates for α centered at this point, such that g is diagonal and $\lambda_1 = g_{1\bar{1}}$. The quantity H need not be smooth at this maximum point, because the largest eigenvalue of A may have eigenspace of dimension larger than 1. To take care of this, we carry out a perturbation argument as in [34], choosing local coordinates such that H achieves its maximum at the origin, where A is diagonal with eigenvalues $\lambda_1 \geq \dots \geq \lambda_n$, as before. We fix a diagonal matrix B , with $B_1^1 = 0$ and $0 < B_2^2 < \dots < B_n^n$, and we define $\tilde{A} = A - B$, denoting its eigenvalues by $\tilde{\lambda}_1, \dots, \tilde{\lambda}_n$.

At the origin, we have

$$\tilde{\lambda}_1 = \lambda_1 \quad \text{and} \quad \tilde{\lambda}_i = \lambda_i - B_i^i, \quad i > 1,$$

and $\tilde{\lambda}_1 > \tilde{\lambda}_2 > \dots > \tilde{\lambda}_n$. As discussed above, our assumption on the cone Γ implies that $\sum_i \lambda_i > 0$, and we fix the matrix B small enough so that

$$\sum_i \tilde{\lambda}_i > -1.$$

We can choose this matrix B in such a way that, in addition,

$$\sum_{p>1} \frac{1}{\lambda_1 - \tilde{\lambda}_p} \leq C, \quad (3.3)$$

for some fixed constant C depending on the dimension n . Now, after possibly shrinking the chart, the quantity

$$\tilde{H} = \log \tilde{\lambda}_1 + \phi(|\nabla u|_\alpha^2) + \psi(u)$$

is smooth on the chart and achieves its maximum at the origin. We will apply the maximum principle to \tilde{H} . Our goal is to obtain the bound $\tilde{\lambda}_1 \leq CK$ at the origin, which will give us the required estimate (3.1). Hence, we may and do assume that $\tilde{\lambda}_1 \gg K$ at this point.

We now differentiate \tilde{H} at the origin and, as before, we use subscripts k and $\bar{\ell}$ to denote the partial derivatives $\partial/\partial z^k$ and $\partial/\partial \bar{z}^{\bar{\ell}}$. We have

$$\begin{aligned} \tilde{H}_k &= \frac{\tilde{\lambda}_{1,k}}{\tilde{\lambda}_1} + \phi'(\alpha^{p\bar{q}} u_p u_{\bar{q}k} + \alpha^{p\bar{q}} u_{pk} u_{\bar{q}} + (\alpha^{p\bar{q}})_k u_p u_{\bar{q}}) + \psi' u_k \\ &= \frac{\tilde{\lambda}_{1,k}}{\tilde{\lambda}_1} + \phi'(u_p u_{\bar{p}k} + u_{pk} u_{\bar{p}} + (\alpha^{p\bar{q}})_k u_p u_{\bar{q}}) + \psi' u_k = \frac{\tilde{\lambda}_{1,k}}{\tilde{\lambda}_1} + \phi' V_k + \psi' u_k, \end{aligned} \quad (3.4)$$

where $V_k := u_p u_{\bar{p}k} + u_{pk} u_{\bar{p}} + (\alpha^{p\bar{q}})_k u_p u_{\bar{q}}$. Differentiating once more,

$$\begin{aligned} \tilde{H}_{k\bar{k}} &= \frac{\tilde{\lambda}_{1,k\bar{k}}}{\lambda_1} - \frac{|\tilde{\lambda}_{1,k}|^2}{\lambda_1^2} + \phi' \left(u_p u_{\bar{p}k\bar{k}} + u_{\bar{p}} u_{pk\bar{k}} + \sum_p |u_{pk}|^2 + \sum_p |u_{\bar{p}k}|^2 \right. \\ &\quad \left. + (\alpha^{p\bar{q}})_{\bar{k}} u_p u_{\bar{q}k} + (\alpha^{p\bar{q}})_{\bar{k}} u_{pk} u_{\bar{q}} + (\alpha^{p\bar{q}})_{k\bar{k}} u_p u_{\bar{q}} + (\alpha^{p\bar{q}})_k (u_p u_{\bar{q}\bar{k}} + u_{p\bar{k}} u_{\bar{q}}) \right) \\ &\quad + \phi'' |V_k|^2 + \psi'' |u_k|^2 + \psi' u_{k\bar{k}}, \end{aligned}$$

where we use the convention that we sum in all repeated indices except the free index k .

Since $1/4K < \phi' < 1/2K$, we can absorb all the terms involving α into the squared terms up to a constant, i.e. we have

$$\begin{aligned} \tilde{H}_{k\bar{k}} &\geq \frac{\tilde{\lambda}_{1,k\bar{k}}}{\lambda_1} - \frac{|\tilde{\lambda}_{1,k}|^2}{\lambda_1^2} + \phi'(u_p u_{\bar{p}k\bar{k}} + u_{\bar{p}} u_{pk\bar{k}}) + \frac{1}{5K} \sum_p (|u_{pk}|^2 + |u_{\bar{p}k}|^2) \\ &\quad + \phi'' |V_k|^2 + \psi'' |u_k|^2 + \psi' u_{k\bar{k}} - C. \end{aligned} \quad (3.5)$$

The constant C denotes a constant that may change from line to line, but it does not depend on the parameters D_1 and D_2 that we are yet to choose.

• **Calculation of $\tilde{\lambda}_{1,k\bar{k}}$**

Let us now compute the derivatives of $\tilde{\lambda}_1$. We have the following general formulas for the derivatives of the eigenvalue λ_i of complex $n \times n$ matrices at a diagonal matrix with distinct real eigenvalues (see for instance Spruck [31] in the case of matrices with real entries):

$$\begin{aligned}\lambda_i^{pq} &= \delta_{pi}\delta_{qi} \\ \lambda_i^{pq,rs} &= (1-\delta_{ip})\frac{\delta_{iq}\delta_{ir}\delta_{ps}}{\lambda_i-\lambda_p} + (1-\delta_{ir})\frac{\delta_{is}\delta_{ip}\delta_{rq}}{\lambda_i-\lambda_r},\end{aligned}\quad (3.6)$$

where λ_i^{pq} denotes the derivative with respect to the (p, q) -entry A_q^p of the matrix A , as a complex variable.

Denoting by $\tilde{\lambda}_1$ the largest eigenvalue of the endomorphism \tilde{A} again, we have, using (3.6),

$$\tilde{\lambda}_{1,k} = \tilde{\lambda}_1^{pq}\nabla_k(\tilde{A}_q^p) = \nabla_k(\tilde{A}_1^1) = \nabla_k g_{1\bar{1}} - \nabla_k B_1^1 = g_{1\bar{1}k} + (\alpha^{1\bar{1}})_k g_{1\bar{1}}, \quad (3.7)$$

since $\nabla_k B_1^1 = 0$ at the origin. Here we computed using covariant derivatives with respect to the Chern connection of α , which makes the positivity of certain terms more apparent when we take second derivatives:

$$\tilde{\lambda}_{1,k\bar{k}} = \tilde{\lambda}_1^{pq}\nabla_{\bar{k}}\nabla_k\tilde{A}_q^p + \tilde{\lambda}_1^{pq,rs}(\nabla_k\tilde{A}_q^p)(\nabla_{\bar{k}}\tilde{A}_s^r) = \nabla_{\bar{k}}\nabla_k g_{1\bar{1}} + \tilde{\lambda}_1^{pq,rs}(\nabla_k\tilde{A}_q^p)(\nabla_{\bar{k}}\tilde{A}_s^r), \quad (3.8)$$

where we used (3.6) and the fact that $\nabla_{\bar{k}}\nabla_k B_1^1 = 0$ at the origin. To rewrite this in terms of partial derivatives, note first that

$$\begin{aligned}\nabla_k g_{1\bar{1}} &= g_{1\bar{1}k} - \Gamma_{k1}^m g_{m\bar{1}} \\ \nabla_{\bar{k}}\nabla_k g_{1\bar{1}} &= g_{1\bar{1}k\bar{k}} - (\partial_{\bar{k}}\Gamma_{k1}^m)g_{m\bar{1}} - \Gamma_{k1}^m g_{m\bar{1}\bar{k}} - \overline{\Gamma_{k1}^q} g_{1\bar{q}k} + \overline{\Gamma_{k1}^q}\Gamma_{k1}^m g_{m\bar{q}} \\ &= g_{1\bar{1}k\bar{k}} + O\left(\sum_m |g_{1\bar{m}k}| + \lambda_1\right).\end{aligned}\quad (3.9)$$

In addition, we have

$$\begin{aligned}\tilde{\lambda}_1^{pq,rs}(\nabla_k\tilde{A}_q^p)(\nabla_{\bar{k}}\tilde{A}_s^r) &= \sum_{p>1} \frac{(\nabla_k\tilde{A}_1^p)(\nabla_{\bar{k}}\tilde{A}_p^1) + (\nabla_k\tilde{A}_p^1)(\nabla_{\bar{k}}\tilde{A}_1^p)}{\lambda_1 - \tilde{\lambda}_p} \\ &= \sum_{p>1} \frac{(\nabla_k g_{1\bar{p}} + \Gamma_{k1}^q B_q^p)(\nabla_{\bar{k}} g_{p\bar{1}}) + (\nabla_k g_{p\bar{1}} - \Gamma_{kq}^1 B_p^q)(\nabla_{\bar{k}} g_{1\bar{p}})}{\lambda_1 - \tilde{\lambda}_p} \\ &\geq \frac{1}{2} \sum_{p>1} \frac{|\nabla_k g_{1\bar{p}}|^2 + |\nabla_k g_{p\bar{1}}|^2}{\lambda_1 - \tilde{\lambda}_p} - C,\end{aligned}\quad (3.10)$$

where we used (3.3) and (3.6). Recall that, due to our choice of B , we have $\sum_i \tilde{\lambda}_i > -1$, which implies $1/(\lambda_1 - \tilde{\lambda}_p) \geq 1/(n\lambda_1 + 1)$ for $p > 1$, and so

$$\tilde{\lambda}_1^{pq,rs} (\nabla_k \tilde{A}_q^p) (\nabla_{\bar{k}} \tilde{A}_s^r) \geq \frac{1}{2(n\lambda_1 + 1)} \sum_{p>1} (|\nabla_k g_{1\bar{p}}|^2 + |\nabla_k g_{p\bar{1}}|^2) - C.$$

To rewrite this in terms of partial derivatives, note that

$$\nabla_k g_{1\bar{p}} = g_{1\bar{p}k} - \Gamma_{k1}^\ell g_{\ell\bar{p}} = g_{1\bar{p}k} + O(\lambda_1),$$

where we made use of the fact that $\sum_i \lambda_i > 0$ to conclude that $|\lambda_i| \leq (n-1)\lambda_1$ for all i . It follows, since we assume $\lambda_1 > 1$, that

$$\tilde{\lambda}_1^{pq,rs} (\nabla_k \tilde{A}_q^p) (\nabla_{\bar{k}} \tilde{A}_s^r) \geq \frac{1}{4n\lambda_1} \sum_{p>1} (|g_{1\bar{p}k}|^2 + |g_{p\bar{1}k}|^2) - C\lambda_1.$$

Combining this with (3.8) and (3.9), we obtain

$$\begin{aligned} \tilde{\lambda}_{1,k\bar{k}} &\geq g_{1\bar{1}k\bar{k}} + \frac{1}{4n\lambda_1} \sum_{p>1} (|g_{1\bar{p}k}|^2 + |g_{p\bar{1}k}|^2) - C \left(\sum_m |g_{1\bar{m}k}| + \lambda_1 \right) \\ &\geq g_{1\bar{1}k\bar{k}} + \frac{1}{8n\lambda_1} \sum_{p>1} (|g_{1\bar{p}k}|^2 + |g_{p\bar{1}k}|^2) - C(|g_{1\bar{1}k}| + \lambda_1). \end{aligned} \quad (3.11)$$

Rewriting g in terms of u , we have

$$\begin{aligned} g_{1\bar{1}k\bar{k}} &= \chi_{1\bar{1}k\bar{k}} + u_{1\bar{1}k\bar{k}} + W_{1\bar{1}k\bar{k}} = \chi_{1\bar{1}k\bar{k}} + u_{k\bar{k}1\bar{1}} + W_{1\bar{1}k\bar{k}} \\ &= \chi_{1\bar{1}k\bar{k}} - \chi_{k\bar{k}1\bar{1}} + g_{k\bar{k}1\bar{1}} - W_{k\bar{k}1\bar{1}} + W_{1\bar{1}k\bar{k}}, \end{aligned} \quad (3.12)$$

and so

$$\begin{aligned} F^{kk} \tilde{\lambda}_{1,k\bar{k}} &\geq F^{kk} g_{k\bar{k}1\bar{1}} + F^{kk} (W_{1\bar{1}k\bar{k}} - W_{k\bar{k}1\bar{1}}) \\ &\quad + \frac{1}{8n\lambda_1} \sum_{p>1} F^{kk} (|g_{1\bar{p}k}|^2 + |g_{p\bar{1}k}|^2) - C(F^{kk} |g_{1\bar{1}k}| + \lambda_1 \mathcal{F}), \end{aligned} \quad (3.13)$$

where F^{pq} denotes the partial derivative of the function $F(A)$ with respect to the (p, q) -entry of the matrix A (as explained earlier), and we have set $\mathcal{F} = \sum_k F^{kk}$. Observe that, due to (2.7) and (3.6), at the origin we have that F^{pq} vanishes whenever $p \neq q$, while on the other hand $F^{kk} = f_k$, using the notation from §2. Recall from the last assertion of Proposition 2.3 that

$$\mathcal{F} \geq \varkappa > 0, \quad (3.14)$$

for a uniform $\varkappa > 0$.

• **The term $F^{kk}g_{k\bar{k}1\bar{1}}$**

We now differentiate the equation $F(A)=h$, using covariant derivatives to simplify a term that appears below. Applying ∇_i , we obtain

$$F^{pq}\nabla_i g_{q\bar{p}} = h_i,$$

namely,

$$F^{kk}g_{k\bar{k}i} + F^{kk}(\alpha^{k\bar{k}})_i g_{k\bar{k}} = h_i. \quad (3.15)$$

Applying $\nabla_{\bar{i}}$ and setting $i=1$,

$$F^{pq,r s}\nabla_1 g_{q\bar{p}}\nabla_{\bar{1}} g_{s\bar{r}} + F^{kk}\nabla_{\bar{1}}\nabla_1 g_{k\bar{k}} = h_{1\bar{1}}. \quad (3.16)$$

To rewrite this using partial derivatives, note that

$$\begin{aligned} \nabla_{\bar{1}}\nabla_1 g_{k\bar{k}} &= g_{k\bar{k}1\bar{1}} - (\partial_{\bar{1}}\Gamma_{1k}^m)g_{m\bar{k}} - \Gamma_{1k}^m g_{m\bar{k}\bar{1}} - \overline{\Gamma}_{1k}^{\bar{q}} g_{k\bar{q}1} + \overline{\Gamma}_{1k}^{\bar{q}} \Gamma_{1k}^m g_{m\bar{q}} \\ &= g_{k\bar{k}1\bar{1}} - 2\operatorname{Re}(\overline{\Gamma}_{1k}^{\bar{q}} g_{k\bar{q}1}) + O(\lambda_1). \end{aligned} \quad (3.17)$$

By rewriting g in terms of u , we have

$$g_{k\bar{q}1} = g_{1\bar{q}k} + \chi_{k\bar{q}1} - \chi_{1\bar{q}k} + W_{k\bar{q}1} - W_{1\bar{q}k},$$

and hence

$$\nabla_{\bar{1}}\nabla_1 g_{k\bar{k}} = g_{k\bar{k}1\bar{1}} - 2\operatorname{Re}(\overline{\Gamma}_{1k}^{\bar{q}}(W_{k\bar{q}1} - W_{1\bar{q}k})) + O\left(\sum_q |g_{1\bar{q}k}| + \lambda_1\right).$$

Returning to (3.16), and making use of (3.14), we obtain

$$\begin{aligned} F^{kk}g_{k\bar{k}1\bar{1}} &\geq -F^{pq,r s}\nabla_1 g_{q\bar{p}}\nabla_{\bar{1}} g_{s\bar{r}} - CF^{kk}\sum_q |g_{1\bar{q}k}| - C\mathcal{F}\lambda_1 \\ &\quad + 2\operatorname{Re}(F^{kk}\overline{\Gamma}_{1k}^{\bar{q}}(W_{k\bar{q}1} - W_{1\bar{q}k})). \end{aligned} \quad (3.18)$$

To bound the term involving $W_{1\bar{q}k}$, we observe that

$$|F^{kk}W_{1\bar{q}k}| \leq C\left(\mathcal{F}\lambda_1 + \sum_p F^{kk}|u_{pk}|\right). \quad (3.19)$$

To see (3.19), we use the fact that $\lambda_1 > K$ to bound the terms involving the gradient of u that arise from $W_{1\bar{q}k}$ when taking the $\partial/\partial z^k$ derivative of $W_{1\bar{q}} = (\operatorname{tr}_\alpha Z)\alpha_{1\bar{q}} - (n-1)Z_{1\bar{q}}$. For the term involving $W_{k\bar{q}1}$, note that

$$\begin{aligned} \nabla_{\bar{1}}\nabla_1 W_{k\bar{k}} &= \nabla_{\bar{1}}(W_{k\bar{k}1} - \Gamma_{1k}^q W_{q\bar{k}}) \\ &= W_{k\bar{k}1\bar{1}} - \overline{\Gamma}_{1k}^{\bar{q}} W_{k\bar{q}1} - \Gamma_{1k}^q W_{q\bar{k}\bar{1}} - (\Gamma_{1k}^q)_{\bar{1}} W_{q\bar{k}} + \overline{\Gamma}_{1k}^{\bar{p}} \Gamma_{1k}^q W_{q\bar{p}}. \end{aligned}$$

In particular,

$$2 \operatorname{Re}(\overline{\Gamma_{1k}^q} W_{k\bar{q}1}) = W_{k\bar{k}1\bar{1}} - \nabla_{\bar{1}} \nabla_1 W_{k\bar{k}} + O(K^{1/2}).$$

Using this (and that we may assume $\lambda_1 > K$), we have

$$2F^{kk} \operatorname{Re}(\overline{\Gamma_{1k}^q} W_{k\bar{q}1}) = F^{kk} W_{k\bar{k}1\bar{1}} - F^{kk} \nabla_{\bar{1}} \nabla_1 W_{k\bar{k}} + O(\mathcal{F}\lambda_1). \quad (3.20)$$

Combining (3.18)–(3.20) gives

$$\begin{aligned} F^{kk} g_{k\bar{k}1\bar{1}} &\geq -F^{pq,rs} \nabla_1 g_{q\bar{p}} \nabla_{\bar{1}} g_{s\bar{r}} + F^{kk} W_{k\bar{k}1\bar{1}} - F^{kk} \nabla_{\bar{1}} \nabla_1 W_{k\bar{k}} \\ &\quad - C \left(F^{kk} \sum_q |g_{1\bar{q}k}| + F^{kk} \sum_p |u_{pk}| + \mathcal{F}\lambda_1 \right). \end{aligned} \quad (3.21)$$

Going back to (3.13), using the square terms there to control the terms in (3.21) involving $|g_{1\bar{q}k}|$ for $q \neq 1$, we obtain

$$\begin{aligned} F^{kk} \tilde{\lambda}_{1,k\bar{k}} &\geq -F^{pq,rs} \nabla_1 g_{q\bar{p}} \nabla_{\bar{1}} g_{s\bar{r}} + F^{kk} (W_{1\bar{1}k\bar{k}} - \nabla_{\bar{1}} \nabla_1 W_{k\bar{k}}) \\ &\quad - C \left(F^{kk} |g_{1\bar{1}k}| + \sum_p F^{kk} |u_{pk}| + \lambda_1 \mathcal{F} \right). \end{aligned} \quad (3.22)$$

• **The term $F^{kk} \nabla_{\bar{1}} \nabla_1 W_{k\bar{k}}$**

Using (2.5) and (2.11), we have

$$\begin{aligned} F^{kk} \nabla_{\bar{1}} \nabla_1 W_{k\bar{k}} &= \frac{1}{n-1} \sum_k \sum_{i \neq k} \tilde{F}^{ii} \nabla_{\bar{1}} \nabla_1 W_{k\bar{k}} \\ &= \frac{1}{n-1} \sum_i \tilde{F}^{ii} \sum_{k \neq i} \nabla_{\bar{1}} \nabla_1 W_{k\bar{k}} = \tilde{F}^{ii} \nabla_{\bar{1}} \nabla_1 Z_{i\bar{i}}. \end{aligned} \quad (3.23)$$

Recall from (2.12) and (2.13) that $\tilde{F}^{11} = \tilde{f}_1$ is “large”, equivalent to $F^{kk} = f_k$ for any $k > 1$, while $\tilde{F}^{ii} = \tilde{f}_i$ for $i > 1$, is “small”, bounded by $F^{11} = f_1$. We also recall that, as explained earlier, the crucial assumption on $Z_{i\bar{j}}$ implies that $\nabla_{\bar{1}} \nabla_1 Z_{1\bar{1}}$ does not contain the terms $u_{11\bar{1}}$, u_{11} or their complex conjugates.

Hence, using the fact that $\sup_{i,j} |u_{i\bar{j}}| \leq C\lambda_1$ and $\lambda_1 \geq K$,

$$\begin{aligned} F^{kk} \nabla_{\bar{1}} \nabla_1 W_{k\bar{k}} &\leq C \left(\tilde{F}^{11} \sum_{k>1} (|u_{k1}| + |u_{k\bar{1}\bar{1}}|) + F^{11} \sum_k (|u_{k1}| + |u_{k\bar{1}\bar{1}}|) + \lambda_1 \mathcal{F} \right) \\ &\leq C \left(F^{11} (|u_{11}| + |u_{1\bar{1}\bar{1}}|) + \sum_{k>1} F^{kk} (|u_{k1}| + |u_{k\bar{1}\bar{1}}|) + \lambda_1 \mathcal{F} \right) \\ &\leq C \left(\sum_p F^{kk} |u_{pk}| + F^{kk} |u_{1\bar{1}k}| + \lambda_1 \mathcal{F} \right). \end{aligned} \quad (3.24)$$

We also have

$$u_{1\bar{1}k} = g_{1\bar{1}k} - \chi_{1\bar{1}k} - W_{1\bar{1}k},$$

and so

$$F^{kk}|u_{1\bar{1}k}| \leq F^{kk}|g_{1\bar{1}k}| + C \left(\sum_p F^{kk}|u_{pk}| + \lambda_1 \mathcal{F} \right).$$

From (3.24), we then obtain

$$F^{kk} \nabla_{\bar{1}} \nabla_1 W_{k\bar{k}} \leq C \left(\sum_p F^{kk}|u_{pk}| + F^{kk}|g_{1\bar{1}k}| + \lambda_1 \mathcal{F} \right). \quad (3.25)$$

• **The term $F^{kk}W_{1\bar{1}k\bar{k}}$**

Let us write $W_{1\bar{1}} = W^p u_p + W^{\bar{p}} u_{\bar{p}}$. We have

$$(W^p u_p)_{k\bar{k}} = (W^p)_{k\bar{k}} u_p + (W^p)_k u_{p\bar{k}} + (W^p)_{\bar{k}} u_{pk} + W^p u_{pk\bar{k}},$$

and so, since we may assume that $\lambda_1 \gg K$, it follows that

$$F^{kk}(W^p u_p)_{k\bar{k}} \geq W^p F^{kk} u_{k\bar{k}p} - C \left(\sum_p F^{kk}|u_{pk}| + \lambda_1 \mathcal{F} \right). \quad (3.26)$$

Using (3.15), we have

$$\begin{aligned} |F^{kk} u_{k\bar{k}p}| &= |F^{kk}(g_{k\bar{k}p} - \chi_{k\bar{k}p} - W_{k\bar{k}p})| \leq C F^{kk}|g_{k\bar{k}}| + C \mathcal{F} + |F^{kk} W_{k\bar{k}p}| \\ &\leq C F^{kk}|u_{k\bar{k}}| + C K^{1/2} \mathcal{F} + |F^{kk} W_{k\bar{k}p}|. \end{aligned}$$

To deal with this last term, note that, due to (2.5), as in (3.23) we have

$$F^{kk} W_{k\bar{k}p} = F^{kk} \nabla_p W_{k\bar{k}} + O(K^{1/2} \mathcal{F}) = \tilde{F}^{ii} \nabla_p Z_{i\bar{i}} + O(K^{1/2} \mathcal{F}),$$

and using the crucial assumption on $Z_{i\bar{j}}$, as explained earlier, we see that $\nabla_p Z_{1\bar{1}}$ is independent of u_{11} and $u_{1\bar{1}}$. It follows that these Hessian terms can appear only with the “small” coefficients \tilde{F}^{ii} with $i > 1$. We obtain

$$\begin{aligned} |F^{kk} W_{k\bar{k}p}| &\leq C \left(\tilde{F}^{11} \sum_{k>1} (|u_{kp}| + |u_{\bar{k}p}|) + F^{11} \sum_k (|u_{kp}| + |u_{\bar{k}p}|) + K^{1/2} \mathcal{F} \right) \\ &\leq C \left(\sum_p F^{kk}|u_{pk}| + \sum_p F^{kk}|u_{p\bar{k}}| + K^{1/2} \mathcal{F} \right), \end{aligned}$$

and so

$$|F^{kk} u_{k\bar{k}p}| \leq C \left(\sum_p F^{kk}|u_{pk}| + \sum_p F^{kk}|u_{p\bar{k}}| + K^{1/2} \mathcal{F} \right). \quad (3.27)$$

From (3.26), we then have (using $\lambda_1 \gg K$)

$$F^{kk}(W^p u_p)_{k\bar{k}} \geq -C \left(\sum_p F^{kk} |u_{pk}| + \sum_p F^{kk} |u_{p\bar{k}}| + \lambda_1 \mathcal{F} \right).$$

A similar argument gives the same estimate for $F^{kk}(W^{\bar{p}} u_{\bar{p}})_{k\bar{k}}$, and this completes the required estimate for $F^{kk} W_{1\bar{1}k\bar{k}}$:

$$F^{kk} W_{1\bar{1}k\bar{k}} \geq -C \left(\sum_p F^{kk} |u_{pk}| + \lambda_1 \mathcal{F} \right).$$

Putting together this last inequality and (3.25) into (3.22), we obtain

$$F^{kk} \tilde{\lambda}_{1,k\bar{k}} \geq -F^{pq,rs} \nabla_1 g_{q\bar{p}} \nabla_{\bar{1}} g_{s\bar{r}} - C \left(F^{kk} |g_{1\bar{1}k}| + \sum_p F^{kk} |u_{pk}| + \lambda_1 \mathcal{F} \right).$$

We now use this in equation (3.5) to give

$$\begin{aligned} F^{kk} \tilde{H}_{k\bar{k}} &\geq -\frac{F^{pq,rs} \nabla_1 g_{q\bar{p}} \nabla_{\bar{1}} g_{s\bar{r}}}{\lambda_1} - \frac{F^{kk} |\tilde{\lambda}_{1,k}|^2}{\lambda_1^2} + F^{kk} \phi' (u_p u_{\bar{p}k\bar{k}} + u_{\bar{p}} u_{pk\bar{k}}) \\ &\quad + \sum_p \frac{F^{kk}}{5K} (|u_{pk}|^2 + |u_{p\bar{k}}|^2) + \phi'' F^{kk} |V_k|^2 + \psi'' F^{kk} |u_k|^2 + \psi' F^{kk} u_{k\bar{k}} \\ &\quad - C \left(F^{kk} \lambda_1^{-1} |g_{1\bar{1}k}| + \sum_p F^{kk} \lambda_1^{-1} |u_{pk}| + \mathcal{F} \right). \end{aligned}$$

We can use (3.27) and the fact that $\phi' < 1/2K$ to bound the terms involving $u_{\bar{p}k\bar{k}}$ and $u_{pk\bar{k}}$:

$$\sum_p \phi' F^{kk} |u_p u_{\bar{p}k\bar{k}}| \leq C \left(\frac{1}{K^{1/2}} \sum_p F^{kk} |u_{pk}| + \frac{1}{K^{1/2}} \sum_p F^{kk} |u_{p\bar{k}}| + \mathcal{F} \right),$$

which in turn can be controlled by the good squared terms $|u_{pk}|^2 + |u_{p\bar{k}}|^2$ at the cost of an extra multiple of \mathcal{F} . In addition, since we assume $\lambda_1 \gg K$, we can control the $F^{kk} \lambda_1^{-1} |u_{pk}|$ term in the same way. We therefore have

$$\begin{aligned} 0 \geq F^{kk} \tilde{H}_{k\bar{k}} &\geq -\frac{F^{pq,rs} \nabla_1 g_{q\bar{p}} \nabla_{\bar{1}} g_{s\bar{r}}}{\lambda_1} - \frac{F^{kk} |\tilde{\lambda}_{1,k}|^2}{\lambda_1^2} \\ &\quad + \sum_p \frac{F^{kk}}{6K} (|u_{pk}|^2 + |u_{p\bar{k}}|^2) + \phi'' F^{kk} |V_k|^2 \\ &\quad + \psi'' F^{kk} |u_k|^2 + \psi' F^{kk} u_{k\bar{k}} - C(F^{kk} \lambda_1^{-1} |g_{1\bar{1}k}| + \mathcal{F}). \end{aligned} \tag{3.28}$$

We now deal with two cases separately, as was done in Hou–Ma–Wu [20], depending on a small constant $\delta = \delta_{D_1, D_2} > 0$ to be determined shortly, and which will depend on the constants D_1 and D_2 .

Case 1. Assume $\delta\lambda_1 \geq -\lambda_n$. Define the set

$$I = \{i : F^{ii} > \delta^{-1} F^{11}\}.$$

From (3.4) and the fact that $\tilde{H}_k = 0$ at the maximum, we get

$$\begin{aligned} -\sum_{k \notin I} \frac{F^{kk} |\tilde{\lambda}_{1,k}|^2}{\lambda_1^2} &= -\sum_{k \notin I} F^{kk} |\phi' V_k + \psi' u_k|^2 \\ &\geq -2(\phi')^2 \sum_{k \notin I} F^{kk} |V_k|^2 - 2(\psi')^2 \sum_{k \notin I} F^{kk} |u_k|^2 \\ &\geq -\phi'' \sum_{k \notin I} F^{kk} |V_k|^2 - 2(\psi')^2 \delta^{-1} F^{11} K. \end{aligned} \quad (3.29)$$

For $k \in I$ we have, in the same way,

$$-2\delta \sum_{k \in I} \frac{F^{kk} |\tilde{\lambda}_{1,k}|^2}{\lambda_1^2} \geq -2\delta\phi'' \sum_{k \in I} F^{kk} |V_k|^2 - 4\delta(\psi')^2 \sum_{k \in I} F^{kk} |u_k|^2. \quad (3.30)$$

We wish to use some of the good $\psi'' F^{kk} |u_k|^2$ term in (3.28) to control the last term in (3.30). For this, we assume that δ is chosen so small (depending on ψ , i.e. on D_1, D_2 and the maximum of $|u|$), such that

$$4\delta(\psi')^2 < \frac{1}{2}\psi''. \quad (3.31)$$

Since ψ'' is strictly positive, such a $\delta > 0$ exists.

Using this together with (3.29) and (3.30) in (3.28), we have

$$\begin{aligned} 0 &\geq -\frac{F^{pq,rs} \nabla_1 g_{q\bar{p}} \nabla_{\bar{1}} g_{s\bar{r}}}{\lambda_1} - (1-2\delta) \sum_{k \in I} \frac{F^{kk} |\tilde{\lambda}_{1,k}|^2}{\lambda_1^2} \\ &\quad + \sum_p \frac{F^{kk}}{6K} (|u_{pk}|^2 + |u_{p\bar{k}}|^2) + \frac{1}{2}\psi'' F^{kk} |u_k|^2 + \psi' F^{kk} u_{k\bar{k}} \\ &\quad - 2(\psi')^2 \delta^{-1} F^{11} K - C(F^{kk} \lambda_1^{-1} |g_{1\bar{1}k}| + \mathcal{F}). \end{aligned} \quad (3.32)$$

To deal with the first two terms, note that (as in [34, equation (67)]) the concavity of the operator F implies that

$$-F^{pq,rs} \nabla_1 g_{q\bar{p}} \nabla_{\bar{1}} g_{s\bar{r}} \geq \sum_{k \in I} \frac{F^{kk} - F^{11}}{\lambda_1 - \lambda_k} |\nabla_1 g_{k\bar{1}}|^2, \quad (3.33)$$

where note that the denominator involves λ_k instead of $\tilde{\lambda}_k$ because we are evaluating F at A . We also remark that the denominator on the right-hand side does not vanish, because the assumption $k \in I$ implies that $F^{kk} > F^{11}$, which in turn implies that $\lambda_k < \lambda_1$ because f is symmetric. By definition, for $k \in I$ we have $F^{11} \leq \delta F^{kk}$, and the assumption $\delta \lambda_1 \geq -\lambda_n$ implies that

$$\frac{1-\delta}{\lambda_1-\lambda_k} \geq \frac{1-2\delta}{\lambda_1}.$$

It follows that

$$\sum_{k \in I} \frac{F^{kk} - F^{11}}{\lambda_1 - \lambda_k} |\nabla_1 g_{k\bar{1}}|^2 \geq \sum_{k \in I} \frac{(1-\delta)F^{kk}}{\lambda_1 - \lambda_k} |\nabla_1 g_{k\bar{1}}|^2 \geq \frac{1-2\delta}{\lambda_1} \sum_{k \in I} F^{kk} |\nabla_1 g_{k\bar{1}}|^2. \quad (3.34)$$

Combining this with (3.32) and (3.33), we then obtain

$$\begin{aligned} 0 &\geq (1-2\delta) \sum_{k \in I} \frac{F^{kk} (|\nabla_1 g_{k\bar{1}}|^2 - |\tilde{\lambda}_{1,k}|^2)}{\lambda_1^2} \\ &\quad + \sum_p \frac{F^{kk}}{6K} (|u_{pk}|^2 + |u_{p\bar{k}}|^2) + \frac{1}{2} \psi'' F^{kk} |u_k|^2 + \psi' F^{kk} u_{k\bar{k}} \\ &\quad - 2(\psi')^2 \delta^{-1} F^{11} K - C(F^{kk} \lambda_1^{-1} |g_{1\bar{1}k}| + \mathcal{F}). \end{aligned} \quad (3.35)$$

We wish to obtain a lower bound for the first term in (3.35). We make the following claim.

CLAIM. *For any $\varepsilon > 0$ there exists a constant C_ε such that*

$$\begin{aligned} \sum_{k \in I} \frac{F^{kk} |\nabla_1 g_{k\bar{1}}|^2}{\lambda_1^2} &\geq \sum_{k \in I} \frac{F^{kk} |\tilde{\lambda}_{1,k}|^2}{\lambda_1^2} - \sum_p \frac{F^{kk}}{12K} (|u_{p\bar{k}}|^2 + |u_{pk}|^2) \\ &\quad + C_\varepsilon \psi' F^{kk} |u_k|^2 + \varepsilon C \psi' \mathcal{F} - C\mathcal{F}, \end{aligned} \quad (3.36)$$

as long as λ_1/K is sufficiently large compared to ψ' (the constants D_1 and D_2 of ψ will be chosen uniformly later).

Proof. First, we compare $\nabla_1 g_{k\bar{1}}$ to $\tilde{\lambda}_{1,k}$. We have

$$\begin{aligned} \nabla_1 g_{k\bar{1}} &= g_{k\bar{1}1} - \Gamma_{1k}^p g_{p\bar{1}} = \chi_{k\bar{1}1} + u_{k\bar{1}1} + W_{k\bar{1}1} + O(\lambda_1) = \chi_{k\bar{1}1} + u_{1\bar{1}k} + W_{k\bar{1}1} + O(\lambda_1) \\ &= g_{1\bar{1}k} - W_{1\bar{1}k} + W_{k\bar{1}1} + O(\lambda_1) = \tilde{\lambda}_{1,k} - W_{1\bar{1}k} + W_{k\bar{1}1} + O(\lambda_1), \end{aligned}$$

absorbing bounded terms into $O(\lambda_1)$ and using (3.7). It follows that for any k , without summing,

$$|\nabla_1 g_{k\bar{1}}|^2 \geq |\tilde{\lambda}_{1,k}|^2 - C(|\tilde{\lambda}_{1,k}|(|W_{1\bar{1}k}| + |W_{k\bar{1}1}|) + \lambda_1 |\tilde{\lambda}_{1,k}| + |W_{1\bar{1}k}|^2 + |W_{k\bar{1}1}|^2 + \lambda_1^2). \quad (3.37)$$

• **The terms in (3.37) involving W**

Note that if $k \in I$ then $k \neq 1$, and so from (2.5) we have

$$W_{k\bar{1}1} = (\text{tr}_\alpha Z)\alpha_{k\bar{1}1} - (n-1)Z_{k\bar{1}1}.$$

Our basic assumption for Z implies that $Z_{k\bar{1}1} = \nabla_1 Z_{k\bar{1}} + O(Z)$ does not contain the Hessian terms u_{11} and $u_{\bar{1}\bar{1}}$. It follows that $W_{k\bar{1}1}$ and its complex conjugate do not contain these Hessian terms. The term $W_{1\bar{1}k}$ and its complex conjugate also do not contain the Hessian terms u_{11} and $u_{\bar{1}\bar{1}}$, since each Hessian term must contain a k -derivative. To simplify the formulas, let us write

$$U = \sum_{\substack{p>1 \\ q \geq 1}} |u_{pq}|.$$

It follows that

$$|W_{k\bar{1}1}| + |W_{1\bar{1}k}| \leq C(\lambda_1 + U), \quad (3.38)$$

and so the terms $|W_{k\bar{1}1}|^2 + |W_{1\bar{1}k}|^2$ in (3.37) can be bounded by $C(\lambda_1^2 + U^2)$.

We now use these to estimate the negative terms in (3.37). Using $\tilde{H}_k = 0$ together with (3.38), we have

$$\begin{aligned} & |\tilde{\lambda}_{1,k}|(|W_{1\bar{1}k}| + |W_{k\bar{1}1}|) \\ &= \lambda_1 |\phi'(u_r u_{\bar{r}k} + u_{rk} u_{\bar{r}} + (\alpha^{r\bar{s}})_k u_r u_{\bar{s}}) + \psi' u_k| (|W_{1\bar{1}k}| + |W_{k\bar{1}1}|) \\ &\leq \frac{C\lambda_1}{2K^{1/2}} \left(\sum_r |u_{\bar{r}k}| + \sum_r |u_{rk}| + K^{1/2} \right) (\lambda_1 + U) + C\lambda_1 |\psi'| |u_k| (\lambda_1 + U). \end{aligned} \quad (3.39)$$

We have

$$\frac{C\lambda_1}{2K^{1/2}} \left(\sum_r |u_{\bar{r}k}| + \sum_r |u_{rk}| + K^{1/2} \right) \lambda_1 \leq \frac{C\lambda_1^2}{2K^{1/2}} \sum_r |u_{\bar{r}k}| + \frac{C\lambda_1^2}{2K^{1/2}} U + C\lambda_1^2 \quad (3.40)$$

and

$$\begin{aligned} \frac{C\lambda_1}{2K^{1/2}} \left(\sum_r |u_{\bar{r}k}| + \sum_r |u_{rk}| + K^{1/2} \right) U &\leq \frac{C\lambda_1^2}{K^{1/2}} U + \frac{C\lambda_1}{K^{1/2}} U^2 + C\lambda_1 U \\ &\leq \frac{C\lambda_1^2}{K^{1/2}} U + \frac{C\lambda_1}{K^{1/2}} U^2. \end{aligned} \quad (3.41)$$

Next, for any $\varepsilon > 0$, there exists a constant C_ε such that

$$\lambda_1 |\psi'| |u_k| \lambda_1 \leq -\varepsilon \lambda_1^2 \psi' - \lambda_1^2 C_\varepsilon \psi' |u_k|^2, \quad (3.42)$$

where we have used the fact that $\psi' < 0$. Also,

$$\lambda_1 |\psi'| |u_k| U \leq -\lambda_1 \psi' U^2 - \lambda_1 \psi' |u_k|^2. \quad (3.43)$$

Combining (3.39) with (3.40)–(3.43), we obtain

$$\begin{aligned} |\tilde{\lambda}_{1,k}| (|W_{1\bar{1}k}| + |W_{k\bar{1}1}|) &\leq C \left(\frac{\lambda_1^2}{2K^{1/2}} \sum_r |u_{\bar{r}k}| + \frac{\lambda_1^2}{K^{1/2}} U + \frac{\lambda_1}{K^{1/2}} U^2 + \lambda_1^2 \right. \\ &\quad \left. - \lambda_1^2 C_\varepsilon \psi' |u_k|^2 - \varepsilon \lambda_1^2 \psi' - \lambda_1 \psi' |u_k|^2 - \lambda_1 \psi' U^2 \right). \end{aligned} \quad (3.44)$$

Using $\tilde{H}_k = 0$ again,

$$\begin{aligned} \frac{|\tilde{\lambda}_{1,k}|}{\lambda_1} &= |\phi' (u_p u_{\bar{p}k} + u_{pk} u_{\bar{p}} + (\alpha^{p\bar{q}})_k u_p u_{\bar{q}}) + \psi' u_k| \\ &\leq \frac{1}{2K^{1/2}} \sum_p |u_{\bar{p}k}| + \frac{1}{2K^{1/2}} U - C_\varepsilon \psi' |u_k|^2 - \varepsilon \psi' + C. \end{aligned} \quad (3.45)$$

We then obtain

$$\begin{aligned} \frac{|\nabla_1 g_{k\bar{1}}|^2}{\lambda_1^2} &\geq \frac{|\tilde{\lambda}_{1,k}|^2}{\lambda_1^2} - C \left(\frac{1}{K^{1/2}} \sum_p |u_{\bar{p}k}| + \frac{1}{K^{1/2}} U \right. \\ &\quad \left. + \frac{1}{\lambda_1 K^{1/2}} U^2 + 1 - \varepsilon \psi' - \frac{1}{\lambda_1} \psi' |u_k|^2 - \frac{1}{\lambda_1} \psi' U^2 \right) + C_\varepsilon \psi' |u_k|^2. \end{aligned}$$

Summing over $k \in I$, we have

$$\begin{aligned} \sum_{k \in I} \frac{F^{kk} |\nabla_1 g_{k\bar{1}}|^2}{\lambda_1^2} &\geq \sum_{k \in I} \frac{F^{kk} |\tilde{\lambda}_{1,k}|^2}{\lambda_1^2} - C \left(\frac{1}{K^{1/2}} \sum_p F^{kk} |u_{p\bar{k}}| + \frac{1}{K^{1/2}} \mathcal{F} U \right. \\ &\quad \left. + \frac{1}{\lambda_1 K^{1/2}} \mathcal{F} U^2 + \mathcal{F} - \varepsilon \mathcal{F} \psi' - \frac{1}{\lambda_1} \psi' F^{kk} |u_k|^2 - \frac{1}{\lambda_1} \psi' \mathcal{F} U^2 \right) \\ &\quad + C_\varepsilon \psi' F^{kk} |u_k|^2. \end{aligned} \quad (3.46)$$

First, we use

$$\frac{C}{K^{1/2}} \sum_p F^{kk} |u_{p\bar{k}}| \leq \frac{1}{12K} \sum_p F^{kk} |u_{p\bar{k}}|^2 + C\mathcal{F}. \quad (3.47)$$

Note that all F^{kk} , with $k > 1$, are comparable to \mathcal{F} . It follows that

$$\frac{C}{K^{1/2}} \mathcal{F} U \leq \frac{1}{50K} \sum_p F^{kk} |u_{pk}|^2 + C\mathcal{F} \quad (3.48)$$

and

$$\frac{C}{\lambda_1 K^{1/2}} \mathcal{F} U^2 \leq \frac{C}{\lambda_1 K^{1/2}} \sum_p F^{kk} |u_{pk}|^2 \leq \frac{1}{50K} \sum_p F^{kk} |u_{pk}|^2. \quad (3.49)$$

As long as λ_1/K is sufficiently large depending on ψ' (i.e. depending on D_1 and D_2 , which will be chosen later), we have

$$\frac{C}{\lambda_1} \psi' \mathcal{F} U^2 \leq \frac{1}{50K} \sum_p F^{kk} |u_{pk}|^2. \quad (3.50)$$

and using (3.47)–(3.50) in (3.46), we finally obtain

$$\begin{aligned} \sum_{k \in I} \frac{F^{kk} |\nabla_1 g_{k\bar{1}}|^2}{\lambda_1^2} &\geq \sum_{k \in I} \frac{F^{kk} |\tilde{\lambda}_{1,k}|^2}{\lambda_1^2} - \sum_p \frac{F^{kk}}{12K} (|u_{p\bar{k}}|^2 + |u_{pk}|^2) \\ &\quad + C_\varepsilon \psi' F^{kk} |u_k|^2 + \varepsilon C \psi' \mathcal{F} - C\mathcal{F}. \end{aligned}$$

This completes the proof of the claim. \square

We now use the claim in (3.35) to obtain

$$\begin{aligned} 0 &\geq \sum_p \frac{F^{kk}}{12K} (|u_{pk}|^2 + |u_{p\bar{k}}|^2) + \frac{1}{2} \psi'' F^{kk} |u_k|^2 + \psi' F^{kk} u_{k\bar{k}} \\ &\quad - 2(\psi')^2 \delta^{-1} F^{11} K - C(F^{kk} \lambda_1^{-1} |g_{1\bar{1}k}| + \mathcal{F}) + C_\varepsilon \psi' F^{kk} |u_k|^2 + \varepsilon C \psi' \mathcal{F}. \end{aligned} \quad (3.51)$$

• **The terms involving $|g_{1\bar{1}k}|$ and $F^{kk} u_{k\bar{k}}$**

From (3.7) we know that

$$g_{1\bar{1}k} = \tilde{\lambda}_{1,k} + O(\lambda_1),$$

and so using (3.45) we get

$$F^{kk} \lambda_1^{-1} |g_{1\bar{1}k}| \leq \frac{1}{2K^{1/2}} \sum_p F^{kk} (|u_{p\bar{k}}| + |u_{pk}|) - C_\varepsilon \psi' F^{kk} |u_k|^2 - \varepsilon \psi' \mathcal{F} + C\mathcal{F}. \quad (3.52)$$

The terms involving $|u_{p\bar{k}}|$ and $|u_{pk}|$ can be absorbed by the squared terms $|u_{p\bar{k}}|^2$ and $|u_{pk}|^2$ in (3.51), and so we obtain

$$\begin{aligned} 0 &\geq \sum_p \frac{F^{kk}}{20K} (|u_{pk}|^2 + |u_{p\bar{k}}|^2) + \frac{1}{2} \psi'' F^{kk} |u_k|^2 + \psi' F^{kk} u_{k\bar{k}} \\ &\quad - 2(\psi')^2 \delta^{-1} F^{11} K - C\mathcal{F} + C_\varepsilon \psi' F^{kk} |u_k|^2 + \varepsilon C \psi' \mathcal{F}. \end{aligned} \quad (3.53)$$

As for the term involving $u_{k\bar{k}}$, we have

$$\psi' F^{kk} u_{k\bar{k}} = \psi' F^{kk} (g_{k\bar{k}} - \chi_{k\bar{k}} - W_{k\bar{k}}).$$

As in (3.23), we have

$$\sum_k F^{kk} W_{k\bar{k}} = \sum_i \tilde{F}^{ii} Z_{i\bar{i}}.$$

Recall that $Z_{1\bar{1}}$ does not contain u_1 and $u_{\bar{1}}$, and \tilde{F}^{11} is the only “large” coefficient of order F^{kk} for $k > 1$. It follows that

$$|F^{kk} W_{k\bar{k}}| \leq C F^{kk} |u_k| \leq C_\varepsilon F^{kk} |u_k|^2 + \varepsilon \mathcal{F},$$

and so

$$\psi' F^{kk} u_{k\bar{k}} \geq \psi' F^{kk} (g_{k\bar{k}} - \chi_{k\bar{k}}) + C_\varepsilon \psi' F^{kk} |u_k|^2 + \varepsilon \psi' \mathcal{F}. \quad (3.54)$$

From (3.53), we then finally obtain (if necessary replacing C_ε by another constant depending only on ε and the allowed data), that

$$\begin{aligned} 0 \geq F^{11} \left(\frac{\lambda_1^2}{40K} - 2(\psi')^2 \delta^{-1} K \right) + \left(\frac{1}{2} \psi'' + C_\varepsilon \psi' \right) F^{kk} |u_k|^2 \\ - C_0 \mathcal{F} + \varepsilon C_0 \psi' \mathcal{F} - \psi' F^{kk} (\chi_{k\bar{k}} - g_{k\bar{k}}), \end{aligned} \quad (3.55)$$

for a uniform C_0 . We have used the fact that $|u_{1\bar{1}}|^2 \geq \frac{1}{2} \lambda_1^2 - CK$.

Under the assumption that the function $\underline{u} = 0$ is a \mathcal{C} -subsolution, and that $\lambda_1 \gg 1$, we may apply Proposition 2.3 and see that there is a uniform positive number $\varkappa > 0$ such that one of two possibilities occurs:

(a) We have $F^{kk} (\chi_{k\bar{k}} - g_{k\bar{k}}) > \varkappa \mathcal{F}$. In this case we have

$$0 \geq F^{11} \left(\frac{\lambda_1^2}{40K} - 2(\psi')^2 \delta^{-1} K \right) + \left(\frac{1}{2} \psi'' + C_\varepsilon \psi' \right) F^{kk} |u_k|^2 - C_0 \mathcal{F} + \varepsilon C_0 \psi' \mathcal{F} - \psi' \varkappa \mathcal{F}.$$

We first choose $\varepsilon > 0$ such that $\varepsilon C_0 < \frac{1}{2} \varkappa$. We then choose the parameter D_2 in the definition of $\psi(t) = D_1 e^{-D_2 t}$ to be large enough so that

$$\frac{1}{2} \psi'' > C_\varepsilon |\psi'|.$$

At this point, we have

$$0 \geq F^{11} \left(\frac{\lambda_1^2}{40K} - 2(\psi')^2 \delta^{-1} K \right) - C_0 \mathcal{F} - \frac{1}{2} \psi' \varkappa \mathcal{F}.$$

We now choose D_1 so large that $-\frac{1}{2} \psi' \varkappa > C_0$, which implies

$$\frac{\lambda_1^2}{40K} \leq 2(\psi')^2 \delta^{-1} K.$$

Note that δ is determined by the choices of D_1 and D_2 , according to (3.31), so we obtain the required upper bound for λ_1/K .

(b) We have $F^{11} > \varkappa \mathcal{F}$. With the choices of constants made above, (3.55) implies that

$$0 \geq \varkappa \mathcal{F} \left(\frac{\lambda_1^2}{40K} - 2(\psi')^2 \delta^{-1} K \right) - C_0 \mathcal{F} + \varepsilon C_0 \psi' \mathcal{F} + C_1 \psi' \mathcal{F} + \psi' F^{kk} g_{k\bar{k}},$$

for another uniform constant C_1 . Since $F^{kk} g_{k\bar{k}} \leq \mathcal{F} \lambda_1$, we can divide through by $\mathcal{F} K$ and obtain

$$0 \geq \frac{\varkappa \lambda_1^2}{40K^2} - C_2(1 + K^{-1} + \lambda_1 K^{-1}),$$

for a uniform C_2 . The required upper bound for λ_1/K follows from this.

Case 2. We now assume that $\delta \lambda_1 < -\lambda_n$, with all the constants D_1 , D_2 and δ fixed as in the previous case. We first use that $F^{nn} \geq \mathcal{F}/n$, as well as $\lambda_n^2 > \delta^2 \lambda_1^2$, to bound

$$\begin{aligned} \sum_p \frac{F^{kk}}{6K} (|u_{pk}|^2 + |u_{p\bar{k}}|^2) &\geq \frac{F^{nn}}{6K} |u_{n\bar{n}}|^2 \geq \frac{\mathcal{F}}{6nK} |\lambda_n - \chi_{n\bar{n}} - W_{n\bar{n}}|^2 \\ &\geq \frac{\mathcal{F}}{10nK} |\lambda_n|^2 - \frac{C\mathcal{F}}{K} (1+K) \geq \frac{\delta^2}{10nK} \mathcal{F} \lambda_1^2 - C\mathcal{F}. \end{aligned}$$

In (3.28) we now discard the positive first term and the term involving ψ'' , and use this to obtain

$$0 \geq -\frac{F^{kk} |\tilde{\lambda}_{1,k}|^2}{\lambda_1^2} + \frac{\delta^2}{10nK} \mathcal{F} \lambda_1^2 + \phi'' F^{kk} |V_k|^2 + \psi' F^{kk} u_{k\bar{k}} - C(F^{kk} \lambda_1^{-1} |g_{1\bar{1}k}| + \mathcal{F}).$$

To deal with the terms involving $F^{kk} u_{k\bar{k}}$ and $|g_{1\bar{1}k}|$, we note that

$$F^{kk} |u_{k\bar{k}}| \leq C\mathcal{F} \lambda_1$$

and, since $g_{1\bar{1}k} = \tilde{\lambda}_{1,k} + O(\lambda_1)$,

$$CF^{kk} \lambda_1^{-1} |g_{1\bar{1}k}| \leq CF^{kk} \lambda_1^{-1} |\tilde{\lambda}_{1,k}| + C\mathcal{F} \leq \frac{1}{2} \frac{F^{kk} |\tilde{\lambda}_{1,k}|^2}{\lambda_1^2} + C\mathcal{F}.$$

Then, we obtain

$$0 \geq -\frac{3}{2} \frac{F^{kk} |\tilde{\lambda}_{1,k}|^2}{\lambda_1^2} + \frac{\delta^2}{10nK} \mathcal{F} \lambda_1^2 + F^{kk} \phi'' |V_k|^2 - C\mathcal{F} \lambda_1. \quad (3.56)$$

Using $\tilde{H}_k = 0$ we have, since ψ' is fixed now and bounded,

$$\begin{aligned} \frac{3}{2} \frac{F^{kk} |\tilde{\lambda}_{1,k}|^2}{\lambda_1^2} &= \frac{3}{2} F^{kk} |\phi' V_k + \psi' u_k|^2 \leq 2F^{kk} (\phi')^2 |V_k|^2 + CF^{kk} (\psi')^2 |u_k|^2 \\ &\leq F^{kk} \phi'' |V_k|^2 + C\mathcal{F}K. \end{aligned}$$

Returning to (3.56), we obtain, since we may assume $\lambda_1 \geq K$,

$$0 \geq \frac{\delta^2 \lambda_1^2}{10nK} \mathcal{F} - C\lambda_1 \mathcal{F}.$$

Dividing by $\lambda_1 \mathcal{F}$ gives the required bound for λ_1/K .

Then, we immediately deduce the bound (3.1), namely

$$\sup_M |\sqrt{-1} \partial \bar{\partial} u|_\alpha \leq C \left(\sup_M |\nabla u|_\alpha^2 + 1 \right). \tag{3.57}$$

A blow-up argument as in [34, §6] combined with a Liouville theorem [34, §5] (see also [8], [39], [40]), shows that $\sup_M |\nabla u|_\alpha^2 \leq C$, and so we get a uniform bound $|\Delta u| \leq C$. Here we remark that in the blow-up argument the only difference from the setup here (compared to [34]) is the presence of the term $W_{i\bar{j}}$. However, this term is linear in ∇u and so converges to zero uniformly on compact sets under the rescaling procedure of [34] (compare [40, §6]).

We can then apply the Evans–Krylov-type result in [36, Theorem 1.1] and deduce a uniform bound

$$\|u\|_{C^{2,\beta}(M,\alpha)} \leq C$$

for a uniform $0 < \beta < 1$. Differentiating the equation and applying a standard bootstrapping argument, we finally obtain uniform higher-order estimates.

4. Proof of Theorem 1.4

In this section, we explain how Theorem 1.4 follows from Theorem 2.2.

Write $*$ for the Hodge star operator with respect to α . This acts on real $(n-1, n-1)$ -forms as follows. Consider a real $(n-1, n-1)$ -form Θ given by

$$\Theta = (\sqrt{-1})^{n-1} \sum_{i,j} \text{sgn}(i, j) \Theta_{i\bar{j}} dz^1 \wedge d\bar{z}^1 \wedge \dots \wedge \widehat{dz^i} \wedge d\bar{z}^i \wedge \dots \wedge dz^j \wedge d\bar{z}^j \wedge \dots \wedge dz^n \wedge d\bar{z}^n,$$

with

$$\text{sgn}(i, j) = \begin{cases} 1, & \text{if } i \leq j, \\ -1, & \text{if } i > j. \end{cases}$$

If we are computing at a point in coordinates so that $\alpha_{i\bar{j}} = \delta_{ij}$, then

$$*\Theta = \sqrt{-1} \sum_{i,j} \Theta_{i\bar{j}} dz^j \wedge d\bar{z}^i.$$

A basic property is that for any Hermitian metric ω we have (see [39, §2], for example)

$$\left(\frac{\omega^n}{\alpha^n}\right)^{n-1} = \frac{*(\omega^{n-1})^n}{*(\alpha^{n-1})^n} = \frac{*(\omega^{n-1})^n}{((n-1)!\alpha)^n}.$$

Then, taking ω as in Theorem 1.4, we see that equation (1.7) is equivalent to

$$\log \frac{*(\omega^{n-1})^n}{((n-1)!\alpha)^n} = h, \quad (4.1)$$

with $h=(n-1)(F+b)$ being a smooth function. Recall that

$$\omega^{n-1} = \alpha_0^{n-1} + \sqrt{-1}\partial\bar{\partial}u \wedge \alpha^{n-2} + \operatorname{Re}(\sqrt{-1}\partial u \wedge \bar{\partial}(\alpha^{n-2})).$$

As in [39], we have

$$\frac{1}{(n-1)!}*(\sqrt{-1}\partial\bar{\partial}u \wedge \alpha^{n-2}) = \frac{1}{n-1}((\Delta u)\alpha - \sqrt{-1}\partial\bar{\partial}u).$$

Define

$$Z_{i\bar{j}} = \left(\frac{1}{(n-1)!} * \operatorname{Re}(\sqrt{-1}\partial u \wedge \bar{\partial}(\alpha^{n-2})) \right)_{i\bar{j}}. \quad (4.2)$$

A straightforward but long calculation gives

$$\begin{aligned} Z_{i\bar{j}} = & \frac{1}{2(n-1)} (\alpha^{p\bar{q}} \alpha^{k\bar{\ell}} u_p \bar{T}_{q\ell\bar{k}} \alpha_{i\bar{j}} - \alpha^{k\bar{\ell}} u_i \bar{T}_{j\ell\bar{k}} - \alpha^{k\bar{\ell}} u_k \bar{T}_{\ell j\bar{i}} \\ & + \alpha^{p\bar{q}} \alpha^{k\bar{\ell}} u_{\bar{q}} T_{pk\bar{\ell}} \alpha_{i\bar{j}} - \alpha^{k\bar{\ell}} u_{\bar{j}} T_{ik\bar{\ell}} - \alpha^{k\bar{\ell}} u_{\bar{\ell}} T_{ki\bar{j}}), \end{aligned} \quad (4.3)$$

where we are writing T_{ij}^k for the torsion of α , and $T_{i\bar{j}\bar{\ell}} = T_{ij}^k \alpha_{k\bar{\ell}}$. An important point to note is that, since the torsion is skew-symmetric ($T_{i\bar{j}\bar{\ell}} = -T_{j\bar{i}\bar{\ell}}$), in orthonormal coordinates for α we see that $Z_{i\bar{j}}$ is independent of $u_{\bar{i}}$ and u_j , and that $\nabla_i Z_{i\bar{i}}$ is independent of $u_{\bar{i}}$. Indeed, in local orthonormal coordinates for α , we have

$$Z_{i\bar{i}} = \frac{1}{2(n-1)} \left(\sum_{p \neq i} \sum_{k \neq i} u_p \bar{T}_{pk\bar{k}} + \sum_{p \neq i} \sum_{k \neq i} u_{\bar{p}} T_{pk\bar{k}} \right)$$

and, for $i \neq j$,

$$Z_{i\bar{j}} = \frac{1}{2(n-1)} \left(- \sum_{k \neq j} (u_i \bar{T}_{jk\bar{k}} + u_k \bar{T}_{kj\bar{i}}) - \sum_{k \neq i} (u_{\bar{j}} T_{ik\bar{k}} + u_{\bar{k}} T_{ki\bar{j}}) \right),$$

using the skew-symmetry of the torsion. Also,

$$\nabla_i Z_{i\bar{i}} = \frac{1}{2(n-1)} \left(\sum_{p \neq i} \sum_{k \neq i} (u_p \nabla_i \bar{T}_{pk\bar{k}} + \nabla_i u_p \bar{T}_{pk\bar{k}}) + \sum_{p \neq i} \sum_{k \neq i} (u_{\bar{p}} \nabla_i T_{pk\bar{k}} + \nabla_i u_{\bar{p}} T_{pk\bar{k}}) \right),$$

and the statement follows. We also define

$$\tilde{\chi}_{i\bar{j}} = \left(\frac{1}{(n-1)!} * (\alpha_0^{n-1}) \right)_{i\bar{j}}.$$

Given this, we see that (4.1) is equivalent to

$$\log(\mu_1 \dots \mu_n) = h,$$

where μ_i are the eigenvalues of $\alpha^{i\bar{p}} \tilde{g}_{j\bar{p}}$, for \tilde{g} given by

$$\tilde{g}_{i\bar{j}} = \tilde{\chi}_{i\bar{j}} + \frac{1}{n-1} ((\Delta u) \alpha_{i\bar{j}} - u_{i\bar{j}}) + Z_{i\bar{j}}.$$

Since $\tilde{\chi}_{i\bar{j}}$ is positive definite, we have that 0 is a \mathcal{C} -subsolution. From the discussion in §2, it is now immediate to see that this equation falls into the setup of Theorem 2.2, and so we obtain the uniform a-priori estimate (3.57). Therefore, Theorem 1.4 follows from [40, Theorem 1.7].

References

- [1] ANGELLA, D., *Cohomological Aspects in Complex Non-Kähler Geometry*. Lecture Notes in Mathematics, 2095. Springer, Cham, 2014.
- [2] BŁOCKI, Z., On uniform estimate in Calabi–Yau theorem. *Sci. China Ser. A*, 48 (2005), 244–247.
- [3] CAFFARELLI, L., NIRENBERG, L. & SPRUCK, J., The Dirichlet problem for nonlinear second-order elliptic equations. III. Functions of the eigenvalues of the Hessian. *Acta Math.*, 155 (1985), 261–301.
- [4] CHERRIER, P., Équations de Monge–Ampère sur les variétés hermitiennes compactes. *Bull. Sci. Math.*, 111 (1987), 343–385.
- [5] CHOU, K.-S. & WANG, X.-J., A variational theory of the Hessian equation. *Comm. Pure Appl. Math.*, 54 (2001), 1029–1064.
- [6] CHU, J., The complex Monge–Ampère equation on some compact Hermitian manifolds. *Pacific J. Math.*, 276 (2015), 369–386.
- [7] COLLINS, T. C. & SZÉKELYHIDI, G., Convergence of the J -flow on toric manifolds. *J. Differential Geom.*, 107 (2017), 47–81.
- [8] DINEW, S. & KOŁODZIEJ, S., Liouville and Calabi–Yau type theorems for complex Hessian equations. *Amer. J. Math.*, 139 (2017), 403–415.
- [9] FINO, A., GRANTCHAROV, G. & VEZZONI, L., Astheno–Kähler and balanced structures on fibrations. To appear in *Int. Math. Res. Not.*
- [10] FU, J., WANG, Z. & WU, D., Form-type Calabi–Yau equations. *Math. Res. Lett.*, 17 (2010), 887–903.
- [11] — Form-type equations on Kähler manifolds of nonnegative orthogonal bisectional curvature. *Calc. Var. Partial Differential Equations*, 52 (2015), 327–344.

- [12] GAUDUCHON, P., Le théorème de l'excentricité nulle. *C. R. Acad. Sci. Paris Sér. A-B*, 285 (1977), A387–A390.
- [13] — La 1-forme de torsion d'une variété hermitienne compacte. *Math. Ann.*, 267 (1984), 495–518.
- [14] GUAN, B., Second-order estimates and regularity for fully nonlinear elliptic equations on Riemannian manifolds. *Duke Math. J.*, 163 (2014), 1491–1524.
- [15] — The Dirichlet problem for fully nonlinear elliptic equations on Riemannian manifolds. Preprint, 2014. [arXiv:1403.2133 \[math.AP\]](https://arxiv.org/abs/1403.2133).
- [16] GUAN, B. & JIAO, H., Second order estimates for Hessian type fully nonlinear elliptic equations on Riemannian manifolds. *Calc. Var. Partial Differential Equations*, 54 (2015), 2693–2712.
- [17] GUAN, B. & LI, Q., Complex Monge–Ampère equations and totally real submanifolds. *Adv. Math.*, 225 (2010), 1185–1223.
- [18] HARVEY, F. R. & LAWSON, H. B., JR., Dirichlet duality and the nonlinear Dirichlet problem on Riemannian manifolds. *J. Differential Geom.*, 88 (2011), 395–482.
- [19] — Geometric plurisubharmonicity and convexity: an introduction. *Adv. Math.*, 230 (2012), 2428–2456.
- [20] HOU, Z., MA, X.-N. & WU, D., A second order estimate for complex Hessian equations on a compact Kähler manifold. *Math. Res. Lett.*, 17 (2010), 547–561.
- [21] JOST, J. & YAU, S.-T., A nonlinear elliptic system for maps from Hermitian to Riemannian manifolds and rigidity theorems in Hermitian geometry. *Acta Math.*, 170 (1993), 221–254. Correction in *Acta Math.*, 173 (1994), 307.
- [22] LATORRE, A. & UGARTE, L., On non-Kähler compact complex manifolds with balanced and astheno-Kähler metrics. *C. R. Math. Acad. Sci. Paris*, 355 (2017), 90–93.
- [23] LI, Y., A priori estimates for Donaldson's equation over compact Hermitian manifolds. *Calc. Var. Partial Differential Equations*, 50 (2014), 867–882.
- [24] LI, Y. Y., Some existence results for fully nonlinear elliptic equations of Monge–Ampère type. *Comm. Pure Appl. Math.*, 43 (1990), 233–271.
- [25] LI, S.-Y., On the Dirichlet problems for symmetric function equations of the eigenvalues of the complex Hessian. *Asian J. Math.*, 8 (2004), 87–106.
- [26] MICHELSON, M. L., On the existence of special metrics in complex geometry. *Acta Math.*, 149 (1982), 261–295.
- [27] NIE, X., Regularity of a complex Monge–Ampère equation on Hermitian manifolds. *Comm. Anal. Geom.*, 22 (2014), 833–856.
- [28] PHONG, D. H., SONG, J. & STURM, J., Complex Monge–Ampère equations, in *Surveys in Differential Geometry*. Vol. XVII, Surv. Differ. Geom., 17, pp. 327–410. Int. Press, Boston, MA, 2012.
- [29] POPOVICI, D., Deformation limits of projective manifolds: Hodge numbers and strongly Gauduchon metrics. *Invent. Math.*, 194 (2013), 515–534.
- [30] — Aepli cohomology classes associated with Gauduchon metrics on compact complex manifolds. *Bull. Soc. Math. France*, 143 (2015), 763–800.
- [31] SPRUCK, J., Geometric aspects of the theory of fully nonlinear elliptic equations, in *Global Theory of Minimal Surfaces*, Clay Math. Proc., 2, pp. 283–309. Amer. Math. Soc., Providence, RI, 2005.
- [32] SUN, W., On a class of fully nonlinear elliptic equations on closed Hermitian manifolds. *J. Geom. Anal.*, 26 (2016), 2459–2473.
- [33] — On a class of fully nonlinear elliptic equations on closed Hermitian manifolds II: L^∞ estimate. *Comm. Pure Appl. Math.*, 70 (2017), 172–199.

- [34] SZÉKELYHIDI, G., Fully non-linear elliptic equations on compact Hermitian manifolds. To appear in *J. Differential Geom.*
- [35] TOSATTI, V., Non-Kähler Calabi–Yau manifolds, in *Analysis, Complex Geometry, and Mathematical Physics*, Contemp. Math., 644, pp. 261–277. Amer. Math. Soc., Providence, RI, 2015.
- [36] TOSATTI, V., WANG, Y., WEINKOVE, B. & YANG, X., $C^{2,\alpha}$ estimates for nonlinear elliptic equations in complex and almost complex geometry. *Calc. Var. Partial Differential Equations*, 54 (2015), 431–453.
- [37] TOSATTI, V. & WEINKOVE, B., The complex Monge–Ampère equation on compact Hermitian manifolds. *J. Amer. Math. Soc.*, 23 (2010), 1187–1195.
- [38] — Estimates for the complex Monge–Ampère equation on Hermitian and balanced manifolds. *Asian J. Math.*, 14 (2010), 19–40.
- [39] — The Monge–Ampère equation for $(n-1)$ -plurisubharmonic functions on a compact Kähler manifold. *J. Amer. Math. Soc.*, 30 (2017), 311–346.
- [40] — Hermitian metrics, $(n-1, n-1)$ forms and Monge–Ampère equations. To appear in *J. Reine Angew. Math.*
- [41] TRUDINGER, N. S., On the Dirichlet problem for Hessian equations. *Acta Math.*, 175 (1995), 151–164.
- [42] URBAS, J., Hessian equations on compact Riemannian manifolds, in *Nonlinear Problems in Mathematical Physics and Related Topics*, II, Int. Math. Ser. (N. Y.), 2, pp. 367–377. Kluwer/Plenum, New York, 2002.
- [43] YAU, S.-T., On the Ricci curvature of a compact Kähler manifold and the complex Monge–Ampère equation. I. *Comm. Pure Appl. Math.*, 31 (1978), 339–411.
- [44] ZHANG, D., Hessian equations on closed Hermitian manifolds. *Pacific J. Math.*, 291 (2017), 485–510.
- [45] ZHANG, X., A priori estimates for complex Monge–Ampère equation on Hermitian manifolds. *Int. Math. Res. Not.*, 19 (2010), 3814–3836.
- [46] ZHANG, X. & ZHANG, X., Regularity estimates of solutions to complex Monge–Ampère equations on Hermitian manifolds. *J. Funct. Anal.*, 260 (2011), 2004–2026.

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