Acta Math., 219 (2017), 1–16
DOI: 10.4310/ACTA.2017.v219.n1.a1

Asymptotic behavior of flows by powers of the Gaussian curvature

by

Simon Brendle
Columbia University
New York, NY, U.S.A.

Kyeongsu Choi
Columbia University
New York, NY, U.S.A.

Panagiota Daskalopoulos
Columbia University
New York, NY, U.S.A.

1. Introduction

Parabolic flows for hypersurfaces play an important role in differential geometry. One fundamental example is the flow by mean curvature (see [18]). In this paper, we consider flows where the speed is given by some power of the Gaussian curvature. More precisely, given an integer $n\geqslant 2$ and a real number $\alpha>0$, a 1-parameter family of immersions $F: M^n\times[0,T)\to\mathbb{R}^{n+1}$ is a solution of the α-Gauss curvature flow if, for each $t\in[0,T)$, $F(M^n,t)=\Sigma_t$ is a complete convex hypersurface in $\mathbb{R}^{n+1}$ and $F(\,\cdot\,,t)$ satisfies

$$\frac{\partial}{\partial t}F(p,t)=-K^\alpha(p,t)\nu(p,t).$$

Here, $K(p,t)$ and $\nu(p,t)$ are the Gauss curvature and the outward-pointing unit normal vector of Σ_t at the point $F(p,t)$, respectively.

Theorem 1. *Let Σ_t be a family of closed, strictly convex hypersurfaces in $\mathbb{R}^{n+1}$ moving with speed $-K^\alpha\nu$, where $\alpha\geqslant 1/(n+2)$. Then, either the hypersurfaces Σ_t converge to a round sphere after rescaling, or we have $\alpha=1/(n+2)$ and the hypersurfaces Σ_t converge to an ellipsoid after rescaling.*

Flows by powers of the Gaussian curvature have been studied by many authors, starting with the seminal paper of Firey [14] in 1974. Tso [21] showed that the flow

The first-named author was supported in part by the National Science Foundation under grant DMS-1649174. The third-named author was supported in part by the National Science Foundation under grant DMS-1600658.

exists up to some maximal time, when the enclosed volume converges to zero. In the special case $\alpha=1/n$, Chow [9] proved convergence to a round sphere. Moreover, Chow [10] obtained interesting Harnack inequalitites for flows, by powers of the Gaussian curvature (see also [17]). In the affine invariant case $\alpha=1/(n+2)$, Andrews [1] showed that the flow converges to an ellipsoid. This result can alternatively be derived from a theorem of Calabi [7], which asserts that the only self-similar solutions for $\alpha=1/(n+2)$ are ellipsoids. The arguments in [7] and [1] rely crucially on the affine invariance of the equation, and do not generalize to other exponents. In the special case of surfaces in $\mathbb{R}^3$ ($n=2$), Andrews [2] proved that flow converges to a sphere when $\alpha=1$; this was later extended in [4] to the case $n=2$ and $\alpha\in\left[\frac{1}{2},2\right]$. The results in [2] and [4] rely on an application of the maximum principle to a suitably chosen function of the curvature eigenvalues; these techniques do not appear to work in higher dimensions. However, it is known that the flow converges to a self-similar solution for every $n\geqslant 2$ and every $\alpha\geqslant 1/(n+2)$. This was proved by Andrews [3] for $\alpha\in[1/(n+2),1/n]$; by Guan and Ni [16] for $\alpha=1$; and by Andrews, Guan, and Ni [5] for all $\alpha\in(1/(n+2),\infty)$. One of the key ingredients in these results is a monotonicity formula for an entropy functional. This monotonicity was discovered by Firey [14] in the special case $\alpha=1$.

Thus, the problem can be reduced to the classification of self-similar solutions. In the affine invariant case $\alpha=1/(n+2)$, the self-similar solutions were already classified by Calabi [7]. In the special case when $\alpha\geqslant 1$ and the hypersurfaces are invariant under antipodal reflection, it was shown in [5] that the only self-similar solutions are round spheres. Very recently, the case $1/n\leqslant\alpha<1+1/n$ was solved in [8] as part of K. Choi's Ph. D. thesis. In particular, this includes the case $\alpha=1$ conjectured by Firey [14].

Finally, we note that there is a substantial literature on other fully non-linear parabolic flows for hypersurfaces (see e.g. [13], [6], [20]) and for Riemannian metrics (cf. [12]). In particular, Gerhardt [15] studied convex hypersurfaces moving outward with speed $K^\alpha\nu$, where $\alpha<0$. Note that, for $\alpha<0$, one can show using the method of moving planes that any convex hypersurface satisfying $K^\alpha=\langle x,\nu\rangle$, where $\alpha<0$, is a round sphere. Using an a-priori estimate in [11], Gerhardt [15] proved that the flow converges to a round sphere after rescaling.

We now give an outline of the proof of Theorem 1. In view of the discussion above, it suffices to classify all closed self-similar solutions to the flow. The self-similar solutions $\Sigma=F(M^n)$ satisfy the equation

$$K^\alpha=\langle F,\nu\rangle. \qquad (*_\alpha)$$

To classify the solutions of $(*_\alpha)$, we distinguish two cases.

First, suppose that $\alpha \in \big[1/(n+2), \frac{1}{2}\big]$. In this case, we consider the quantity

$$Z = K^\alpha \operatorname{tr}(b) - \frac{n\alpha - 1}{2\alpha} |F|^2,$$

where b denotes the inverse of the second fundamental form. The motivation for the quantity Z is that Z is constant when $\alpha = 1/(n+2)$ and Σ is an ellipsoid. Indeed, if $\alpha = 1/(n+2)$ and $\Sigma = \{x \in \mathbb{R}^{n+1} : \langle Sx, x\rangle = 1\}$ for some positive definite matrix S with determinant 1, then $K^{1/(n+2)} = \langle F, \nu\rangle$ and

$$Z = K^{1/(n+2)} \operatorname{tr}(b) + |F|^2 = \operatorname{tr}(S^{-1}).$$

Hence, in this case Z is constant, and equals the sum of the squares of the semi-axes of the ellipsoid.

Suppose now that $\Sigma = F(M^n)$ is a solution of $(*_\alpha)$ for some $\alpha \in \big[1/(n+2), \frac{1}{2}\big]$. We show that Z satisfies an inequality of the form

$$\alpha K^\alpha b^{ij} \nabla_i \nabla_j Z + (2\alpha - 1) b^{ij} \nabla_i K^\alpha \nabla_j Z \geqslant 0.$$

The strong maximum principle then implies that Z is constant. By examining the case of equality, we are able to show that either $\nabla h = 0$, or $\alpha = 1/(n+2)$ and the cubic form vanishes. This shows that either Σ is a round sphere, or $\alpha = 1/(n+2)$ and Σ is an ellipsoid.

Finally, we consider the case $\alpha \in \big(\frac{1}{2}, \infty\big)$. As in [8], we consider the quantity

$$W = K^\alpha \lambda_1^{-1} - \frac{n\alpha - 1}{2n\alpha} |F|^2.$$

By applying the maximum principle, we can show that any point where W attains its maximum is umbilical. From this, we deduce that any maximum point of W is also a maximum point of Z. Applying the strong maximum principle to Z, we are able to show that Z and W are both constant. This implies that Σ is a round sphere.

2. Preliminaries

We first recall the notation:

- The metric is given by $g_{ij} = \langle F_i, F_j\rangle$, where $F_i := \nabla_i F$. Moreover, g^{ij} denotes the inverse of g_{ij}, so that $g^{ij} g_{jk} = \delta^i_k$. Also, we use the notation $F^i = g^{ij} F_j$.
- We denote by H and h_{ij} the mean curvature and second fundamental form, respectively.
- For a strictly convex hypersurface, we denote by b^{ij} the inverse of the second fundamental form h_{ij}, so that $b^{ij} h_{jk} = \delta^i_k$. Moreover, $\operatorname{tr}(b)$ will denote the trace of b, i.e. the reciprocal of the harmonic mean curvature.

- We denote by $\mathcal{L}$ the operator $\mathcal{L}=\alpha K^\alpha b^{ij}\nabla_i\nabla_j$.
- We denote by C_{ijk} the cubic form

$$C_{ijk}=\tfrac{1}{2}K^{-1(n+2)}\nabla_k h_{ij}+\tfrac{1}{2}h_{jk}\nabla_i K^{-1/(n+2)}+\tfrac{1}{2}h_{ki}\nabla_j K^{-1/(n+2)}+\tfrac{1}{2}h_{ij}\nabla_k K^{-1/(n+2)}.$$

We next derive some basic equations.

Proposition 2. *Given a strictly convex smooth solution $F\colon M^n\to\mathbb{R}^{n+1}$ of ($*_\alpha$), the following equations hold*:

$$\nabla_i b^{jk}=-b^{jl}b^{km}\nabla_i h_{lm}, \tag{1}$$

$$\mathcal{L}|F|^2=2\alpha K^\alpha b^{ij}(g_{ij}-h_{ij}K^\alpha)=2\alpha K^\alpha\,\mathrm{tr}(b)-2n\alpha K^{2\alpha}, \tag{2}$$

$$\nabla_i K^\alpha=h_{ij}\langle F,F^j\rangle, \tag{3}$$

$$\mathcal{L}K^\alpha=\langle F,F_i\rangle\nabla_i K^\alpha+n\alpha K^\alpha-\alpha K^{2\alpha}H, \tag{4}$$

$$\begin{aligned}\mathcal{L}h_{ij}=&-K^{-\alpha}\nabla_i K^\alpha\nabla_j K^\alpha+\alpha K^\alpha b^{pr}b^{qs}\nabla_i h_{rs}\nabla_j h_{pq}\\&+\langle F,F_k\rangle\nabla^k h_{ij}+h_{ij}+(n\alpha-1)h_{ik}h_j^k K^\alpha-\alpha K^\alpha H h_{ij},\end{aligned} \tag{5}$$

$$\begin{aligned}\mathcal{L}b^{pq}=&K^{-\alpha}b^{pr}b^{qs}\nabla_r K^\alpha\nabla_s K^\alpha+\alpha K^\alpha b^{pr}b^{qs}b^{ij}b^{km}\nabla_r h_{ik}\nabla_s h_{jm}\\&+\langle F,F_i\rangle\nabla^i b^{pq}-b^{pq}-(n\alpha-1)g^{pq}K^\alpha+\alpha K^\alpha H b^{pq}.\end{aligned} \tag{6}$$

Proof. The relation $\nabla_i(b^{jk}h_{kl})=\nabla_i\delta_l^j=0$ gives $h_{kl}\nabla_i b^{jk}=-b^{jk}\nabla_i h_{kl}$. This directly implies (1). We next compute

$$\nabla_i\nabla_j|F|^2=2\langle\nabla_i F,\nabla_j F\rangle+2\langle F,\nabla_i\nabla_j F\rangle=2g_{ij}-2h_{ij}\langle F,\nu\rangle=2g_{ij}-2K^\alpha h_{ij},$$

and hence

$$\mathcal{L}|F|^2=2\alpha K^\alpha\,\mathrm{tr}(b)-2n\alpha K^{2\alpha}.$$

This proves (2).

To derive equation (3), we differentiate ($*_\alpha$):

$$\nabla_i K^\alpha=h_{ik}\langle F,F^k\rangle.$$

If we differentiate this equation again, we obtain

$$\nabla_i\nabla_j K^\alpha=\langle F,F^k\rangle\nabla_i h_{jk}+h_{ij}-h_{ik}h_j^k\langle F,\nu\rangle=\langle F,F^k\rangle\nabla_k h_{ij}+h_{ij}-K^\alpha h_{ik}h_j^k,$$

and hence

$$\mathcal{L}K^\alpha=\langle F,F^k\rangle\nabla_k K^\alpha+n\alpha K^\alpha-\alpha K^{2\alpha}H.$$

On the other hand, using (1) we compute

$$\begin{aligned}\nabla_i\nabla_j K^\alpha &= \nabla_i(\alpha K^\alpha b^{pq}\nabla_j h_{pq})\\ &= \alpha K^\alpha b^{pq}\nabla_i\nabla_j h_{pq} + \alpha^2 K^\alpha b^{rs} b^{pq}\nabla_i h_{rs}\nabla_j h_{pq} - \alpha K^\alpha b^{pr} b^{qs}\nabla_i h_{rs}\nabla_j h_{pq}.\end{aligned}$$

Using the commutator identity

$$\begin{aligned}\nabla_i\nabla_j h_{pq} = \nabla_i\nabla_p h_{jq} &= \nabla_p\nabla_i h_{jq} + R_{ipjm} h_q^m + R_{ipqm} h_j^m\\ &= \nabla_p\nabla_q h_{ij} + (h_{ij}h_{pm} - h_{im}h_{jp})h_q^m + (h_{iq}h_{pm} - h_{im}h_{pq})h_j^m,\end{aligned}$$

we deduce that

$$\begin{aligned}\alpha K^\alpha b^{pq}\nabla_i\nabla_j h_{pq} &= \alpha K^\alpha b^{pq}\nabla_p\nabla_q h_{ij} + \alpha K^\alpha H h_{ij} - n\alpha K^\alpha h_{im} h_j^m\\ &= \mathcal{L} h_{ij} + \alpha K^\alpha H h_{ij} - n\alpha K^\alpha h_{im} h_j^m.\end{aligned}$$

Combining the equations above yields

$$\begin{aligned}\mathcal{L} h_{ij} = &-\alpha^2 K^\alpha b^{rs} b^{pq}\nabla_i h_{rs}\nabla_j h_{pq} + \alpha K^\alpha b^{pr} b^{qs}\nabla_i h_{rs}\nabla_j h_{pq}\\ &+ \langle F, F^k\rangle\nabla_k h_{ij} + h_{ij} + (n\alpha-1)h_{ik}h_j^k K^\alpha - \alpha K^\alpha H h_{ij}.\end{aligned}$$

This completes the proof of (5).

Finally, using (1), we obtain

$$\begin{aligned}\mathcal{L} b^{pq} &= \alpha K^\alpha b^{ij}\nabla_i(-b^{pr} b^{qs}\nabla_j h_{rs})\\ &= 2\alpha K^\alpha b^{ij} b^{pk} b^{rm} b^{qs}\nabla_i h_{km}\nabla_j h_{rs} - b^{pr} b^{qs}\mathcal{L} h_{rs}.\end{aligned}$$

Applying (5), we conclude that

$$\begin{aligned}\mathcal{L} b^{pq} = &\,\alpha^2 K^\alpha b^{pr} b^{qs} b^{ij} b^{km}\nabla_r h_{ij}\nabla_s h_{km} + \alpha K^\alpha b^{pr} b^{qs} b^{ij} b^{km}\nabla_r h_{ik}\nabla_s h_{jm}\\ &+ \langle F, F^k\rangle\nabla_k b^{pq} - b^{pq} - (n\alpha-1)g^{pq}K^\alpha + \alpha K^\alpha H b^{pq}.\end{aligned}$$

Since $\nabla K^\alpha = \alpha K^\alpha b^{ij}\nabla h_{ij}$, the identity (6) follows. □

3. Classification of self-similar solutions: the case $\alpha \in \left[1/(n+2), \frac{1}{2}\right]$

In this section, we consider the case $\alpha \in \left[1/(n+2), \frac{1}{2}\right]$. We begin with an algebraic lemma.

LEMMA 3. *Assume* $\alpha \in \big[1/(n+2), \frac{1}{2}\big]$, $\lambda_1, \dots, \lambda_n$ *are positive real numbers* (*not necessarily arranged in increasing order*), *and* $\sigma_1, \dots, \sigma_n$ *are arbitrary real numbers. Then*

$$\begin{aligned} Q := \sum_{i=1}^n \sigma_i^2 + 2 \sum_{i=1}^{n-1} \lambda_n \lambda_i^{-1} \sigma_i^2 \\ - 4\alpha\lambda_n \bigg(\sum_{i=1}^n \lambda_i^{-1} \sigma_i + \bigg((n\alpha - 1)\lambda_n^{-1} - \alpha \sum_{i=1}^n \lambda_i^{-1} \bigg) \sum_{i=1}^n \sigma_i \bigg) \bigg(\sum_{i=1}^n \sigma_i \bigg) \\ - 2\alpha^2 \lambda_n \bigg(\sum_{i=1}^n \lambda_i^{-1} \bigg) \bigg(\sum_{i=1}^n \sigma_i \bigg)^2 + (2n\alpha^2 + (n-1)\alpha - 1) \bigg(\sum_{i=1}^n \sigma_i \bigg)^2 \geqslant 0. \end{aligned}$$

Moreover, if equality holds, then we either have $\sigma_1 = \dots = \sigma_n = 0$, *or we have* $\alpha = 1/(n+2)$ *and* $\sigma_1 = \dots = \sigma_{n-1} = \frac{1}{3}\sigma_n$.

Proof. If $\sum_{i=1}^n \sigma_i = 0$, the assertion is trivial. Hence, it suffices to consider the case $\sum_{i=1}^n \sigma_i \neq 0$. By scaling, we may assume $\sum_{i=1}^n \sigma_i = 1$. Let us define

$$\tau_i = \begin{cases} \sigma_i - \alpha, & \text{for } i = 1, \dots, n-1, \\ \sigma_n - 1 + (n-1)\alpha, & \text{for } i = n. \end{cases}$$

Then

$$\sum_{i=1}^n \tau_i = \sum_{i=1}^n \sigma_i - 1 = 0$$

and

$$\sum_{i=1}^n \lambda_i^{-1} \tau_i = \sum_{i=1}^n \lambda_i^{-1} \sigma_i + (n\alpha - 1)\lambda_n^{-1} - \alpha \sum_{i=1}^n \lambda_i^{-1}.$$

Therefore, the quantity Q satisfies

$$\begin{aligned} Q &= \sum_{i=1}^{n-1} (\tau_i + \alpha)^2 + (\tau_n + 1 - (n-1)\alpha)^2 + 2 \sum_{i=1}^{n-1} \lambda_n \lambda_i^{-1} (\tau_i + \alpha)^2 \\ &\quad - 4\alpha \sum_{i=1}^n \lambda_n \lambda_i^{-1} \tau_i - 2\alpha^2 \sum_{i=1}^n \lambda_n \lambda_i^{-1} + 2n\alpha^2 + (n-1)\alpha - 1 \\ &= \sum_{i=1}^n \tau_i^2 + 2\alpha \sum_{i=1}^{n-1} \tau_i + 2(1 - (n-1)\alpha)\tau_n + (n-1)\alpha^2 + (1 - (n-1)\alpha)^2 \\ &\quad + 2 \sum_{i=1}^{n-1} \lambda_n \lambda_i^{-1} (\tau_i^2 + 2\alpha\tau_i + \alpha^2) - 4\alpha \sum_{i=1}^{n-1} \lambda_n \lambda_i^{-1} \tau_i - 4\alpha\tau_n \\ &\quad - 2\alpha^2 \sum_{i=1}^{n-1} \lambda_n \lambda_i^{-1} - 2\alpha^2 + 2n\alpha^2 + (n-1)\alpha - 1 \\ &= \sum_{i=1}^n \tau_i^2 + 2\alpha \sum_{i=1}^{n-1} \tau_i + 2(1 - (n+1)\alpha)\tau_n + 2 \sum_{i=1}^{n-1} \lambda_n \lambda_i^{-1} \tau_i^2 + (n-1)\alpha((n+2)\alpha - 1). \end{aligned}$$

Using the identity $\sum_{i=1}^n \tau_i = 0$, we obtain

$$Q = \sum_{i=1}^n \tau_i^2 + 2(1-(n+2)\alpha)\tau_n + 2\sum_{i=1}^{n-1} \lambda_n \lambda_i^{-1} \tau_i^2 + (n-1)\alpha((n+2)\alpha - 1).$$

Moreover, the identity $\sum_{i=1}^n \tau_i = 0$ gives

$$\sum_{i=1}^n \tau_i^2 = \sum_{i=1}^{n-1} \left(\tau_i + \frac{1}{n-1}\tau_n\right)^2 - \frac{2}{n-1}\tau_n \sum_{i=1}^{n-1} \tau_i + \frac{n-2}{n-1}\tau_n^2 = \sum_{i=1}^{n-1}\left(\tau_i + \frac{1}{n-1}\tau_n\right)^2 + \frac{n}{n-1}\tau_n^2.$$

Thus,

$$\begin{aligned} Q = {} & \frac{n}{n-1}\left(\tau_n + \frac{n-1}{n}(1-(n+2)\alpha)\right)^2 + \sum_{i=1}^{n-1}\left(\tau_i + \frac{1}{n-1}\tau_n\right)^2 \\ & + 2\sum_{i=1}^{n-1} \lambda_n \lambda_i^{-1}\tau_i^2 + \frac{n-1}{n}(1-2\alpha)((n+2)\alpha - 1). \end{aligned}$$

The right-hand side is clearly non-negative. Also, if equality holds, then $\tau_1 = \ldots = \tau_n = 0$ and $\alpha = 1/(n+2)$. This proves the lemma. ☐

THEOREM 4. *Assume* $\alpha \in \left[1/(n+2), \frac{1}{2}\right]$ *and* Σ *is a strictly convex closed smooth solution of* ($*_\alpha$). *Then, either* Σ *is a round sphere, or* $\alpha = 1/(n+2)$ *and* Σ *is an ellipsoid.*

Proof. Taking the trace in equation (6) gives

$$\begin{aligned} \mathcal{L}\,\mathrm{tr}(b) = {} & K^{-\alpha} b^{pr} b_p^s \nabla_r K^\alpha \nabla_s K^\alpha + \alpha K^\alpha b^{pr} b_p^s b^{ij} b^{km} \nabla_r h_{ik} \nabla_s h_{jm} \\ & + \langle F, F_i \rangle \nabla^i \,\mathrm{tr}(b) - \mathrm{tr}(b) - n(n\alpha - 1)K^\alpha + \alpha K^\alpha H\, \mathrm{tr}(b). \end{aligned}$$

Using equation (4), we obtain

$$\begin{aligned} \mathcal{L}(K^\alpha\, \mathrm{tr}(b)) & = K^\alpha \mathcal{L}\,\mathrm{tr}(b) + \mathcal{L}K^\alpha\, \mathrm{tr}(b) + 2\alpha K^\alpha b^{ij} \nabla_i K^\alpha \nabla_j\, \mathrm{tr}(b) \\ & = b^{pr} b_p^s \nabla_r K^\alpha \nabla_s K^\alpha + \alpha K^{2\alpha} b^{pr} b_p^s b^{ij} b^{km} \nabla_r h_{ik} \nabla_s h_{jm} \\ & \quad + \langle F, F_i\rangle \nabla^i (K^\alpha\, \mathrm{tr}(b)) + (n\alpha - 1) K^\alpha\, \mathrm{tr}(b) - n(n\alpha - 1) K^{2\alpha} \\ & \quad + 2\alpha K^\alpha b^{ij} \nabla_i K^\alpha \nabla_j\, \mathrm{tr}(b). \end{aligned}$$

Using (2), it follows that the function

$$Z = K^\alpha\, \mathrm{tr}(b) - \frac{n\alpha - 1}{2\alpha}|F|^2 \tag{7}$$

satisfies

$$\begin{aligned} \mathcal{L}Z = {} & b^{pr} b_p^s \nabla_r K^\alpha \nabla_s K^\alpha + \alpha K^{2\alpha} b^{pr} b_p^s b^{ij} b^{km} \nabla_r h_{ik} \nabla_s h_{jm} \\ & + \langle F, F_i \rangle \nabla^i (K^\alpha\, \mathrm{tr}(b)) + 2\alpha K^\alpha b^{ij} \nabla_i K^\alpha \nabla_j\, \mathrm{tr}(b). \end{aligned} \tag{8}$$

Using (3), we obtain

$$\tfrac{1}{2}\nabla^i |F|^2 = \langle F, F^i\rangle = b^{ij}\nabla_j K^\alpha,$$

and hence

$$\nabla_i Z = K^\alpha \nabla_i \operatorname{tr}(b) + \operatorname{tr}(b)\nabla_i K^\alpha - \frac{n\alpha-1}{\alpha} b_i^j \nabla_j K^\alpha.$$

This gives

$$\langle F, F_i\rangle \nabla^i (K^\alpha \operatorname{tr}(b)) = b^{ij}\nabla_i K^\alpha \nabla_j Z + \frac{n\alpha-1}{\alpha} b^{ik} b_k^j \nabla_i K^\alpha \nabla_j K^\alpha$$

and

$$2\alpha K^\alpha b^{ij}\nabla_i K^\alpha \nabla_j \operatorname{tr}(b) = 2\alpha b^{ij}\nabla_i K^\alpha \nabla_j Z - (2\alpha b^{ij}\operatorname{tr}(b) - 2(n\alpha-1) b^{ik} b_k^j)\nabla_i K^\alpha \nabla_j K^\alpha.$$

Substituting these identities into (8) gives

$$\begin{aligned} &\mathcal{L}Z + (2\alpha-1) b^{ij}\nabla_i K^\alpha \nabla_j Z \\ &\quad = 4\alpha b^{ij}\nabla_i K^\alpha \nabla_j Z + \alpha K^{2\alpha} b^{pr} b_p^s b^{ij} b^{km} \nabla_r h_{ik} \nabla_s h_{jm} \\ &\qquad + (-2\alpha b^{ij}\operatorname{tr}(b) + (2n\alpha + n - 1 - \alpha^{-1}) b^{ik} b_k^j)\nabla_i K^\alpha \nabla_j K^\alpha. \end{aligned} \tag{9}$$

Let us fix an arbitrary point p. We can choose an orthonormal frame so that $h_{ij}(p) = \lambda_i \delta_{ij}$. With this understood, we have

$$\nabla_i K^\alpha = \alpha K^\alpha \sum_{j=1}^n \lambda_j^{-1} \nabla_i h_{jj} \quad \text{and} \quad \nabla_i \operatorname{tr}(b) = -\sum_{j=1}^n \lambda_j^{-2} \nabla_i h_{jj}. \tag{10}$$

Let D denote the set of all triplets (i,j,k) such that i, j and k are pairwise distinct. Then, by using (9) and (10), we have

$$\begin{aligned} &\alpha^{-1} K^{-2\alpha}(\mathcal{L}Z + (2\alpha-1) b^{ij}\nabla_i K^\alpha \nabla_j Z) \\ &\quad = \sum_D \lambda_i^{-2}\lambda_j^{-1}\lambda_k^{-1} (\nabla_i h_{jk})^2 + 4\alpha \sum_k \lambda_k^{-1} (\nabla_k \log K)(K^{-\alpha}\nabla_k Z) \\ &\qquad + \sum_k \sum_i \lambda_k^{-2}\lambda_i^{-2} (\nabla_k h_{ii})^2 + 2\sum_k \sum_{i\neq k} \lambda_k^{-1}\lambda_i^{-3} (\nabla_k h_{ii})^2 \\ &\qquad + \sum_k \lambda_k^{-1} (-2\alpha^2 \operatorname{tr}(b) + (2n\alpha^2 + (n-1)\alpha - 1)\lambda_k^{-1}) (\nabla_k \log K)^2. \end{aligned} \tag{11}$$

We claim that, for each k, the following holds:

$$\begin{aligned} &\sum_i \lambda_k^{-1}\lambda_i^{-2} (\nabla_k h_{ii})^2 + 2\sum_{i\neq k} \lambda_i^{-3} (\nabla_k h_{ii})^2 + 4\alpha (\nabla_k \log K)(K^{-\alpha}\nabla_k Z) \\ &\qquad + (-2\alpha^2 \operatorname{tr}(b) + (2n\alpha^2 + (n-1)\alpha - 1)\lambda_k^{-1}) (\nabla_k \log K)^2 \geqslant 0. \end{aligned} \tag{12}$$

Notice that

$$K^{-\alpha}\nabla_k Z = -\sum_i \lambda_i^{-2}\nabla_k h_{ii} + (\alpha\,\mathrm{tr}(b) - (n\alpha-1)\lambda_k^{-1})\nabla_k \log K.$$

After relabeling indices, we may assume $k=n$. If we put $\sigma_i := \lambda_i^{-1}\nabla_n h_{ii}$, then the assertion follows from Lemma 3. This proves (12). Combining (11) and (12), we conclude that

$$\mathcal{L}Z + (2\alpha-1)b^{ij}\nabla_i K^\alpha \nabla_j Z \geqslant 0$$

at each point p. Therefore, by the strong maximum principle, Z is a constant. Hence, the left-hand side of (11) is zero. Therefore, $\nabla_i h_{jk}=0$ if i, j and k are all distinct. Moreover, since we have equality in the lemma, we either have $\lambda_i^{-1}\nabla_k h_{ii}=0$ for all i and k, or we have $\alpha=1/(n+2)$ and $\lambda_i^{-1}\nabla_k h_{ii} = \frac{1}{3}\lambda_k^{-1}\nabla_k h_{kk}$ for $i\neq k$.

In the first case, we conclude that $\nabla_i h_{jk}=0$ for all i, j and k, and thus Σ is a round sphere.

In the second case, we obtain

$$\lambda_i^{-1}\nabla_k h_{ii} = \frac{1}{n+2}\nabla_k \log K \ \text{ for } i\neq k \quad \text{and} \quad \lambda_k^{-1}\nabla_k h_{kk} = \frac{3}{n+2}\nabla_k \log K.$$

This gives

$$C_{ijk} = \tfrac{1}{2}K^{-1/(n+2)}\nabla_k h_{ij} + \tfrac{1}{2}h_{jk}\nabla_i K^{-1/(n+2)} + \tfrac{1}{2}h_{ki}\nabla_j K^{-1/(n+2)} + \tfrac{1}{2}h_{ij}\nabla_k K^{-1/(n+2)} = 0$$

for all i, j and k. Since the cubic form C_{ijk} vanishes everywhere, the surface is an ellipsoid by the Berwald–Pick theorem (see e.g. [19, Theorem 4.5]). This proves the theorem. □

4. Classification of self-similar solutions: the case $\alpha\in(\frac{1}{2},\infty)$

We now turn to the case $\alpha\in(\frac{1}{2},\infty)$. In the following, we denote by $\lambda_1\leqslant\ldots\leqslant\lambda_n$ the eigenvalues of the second fundamental form, arranged in increasing order. Each eigenvalue defines a Lipschitz continuous function on M.

Lemma 5. *Suppose that φ is a smooth function such that $\lambda_1\geqslant\varphi$ everywhere and $\lambda_1=\varphi$ at $\bar{p}$. Let μ denote the multiplicity of the smallest curvature eigenvalue at $\bar{p}$, so that $\lambda_1=\ldots=\lambda_\mu<\lambda_{\mu+1}\leqslant\ldots\leqslant\lambda_n$. Then, at $\bar{p}$, $\nabla_i h_{kl}=\nabla_i\varphi\,\delta_{kl}$ for $1\leqslant k,l\leqslant\mu$. Moreover,*

$$\nabla_i\nabla_i\varphi \leqslant \nabla_i\nabla_i h_{11} - 2\sum_{l>\mu}(\lambda_l-\lambda_1)^{-1}(\nabla_i h_{1l})^2.$$

at $\bar{p}$.

Proof. Fix an index i, and let $\gamma(s)$ be the geodesic satisfying $\gamma(0)=\bar{p}$ and $\gamma'(0)=e_i$. Moreover, let $v(s)$ be a vector field along γ such that $v(0)\in\text{span}\{e_1,\dots,e_\mu\}$ and $v'(0)\in\text{span}\{e_{\mu+1},\dots,e_n\}$. Then, the function $s\mapsto h(v(s),v(s))-\varphi(\gamma(s))|v(s)|^2$ has a local minimum at $s=0$. This gives

$$\begin{aligned}0&=\frac{d}{ds}(h(v(s),v(s))-\varphi(\gamma(s))|v(s)|^2)\Big|_{s=0}\\&=\nabla_i h(v(0),v(0))+2h(v(0),v'(0))-\nabla_i\varphi|v(0)|^2-2\langle v(0),v'(0)\rangle\\&=\nabla_i h(v(0),v(0))-\nabla_i\varphi|v(0)|^2.\end{aligned}$$

Since $v(0)\in\text{span}\{e_1,\dots,e_\mu\}$ is arbitrary, it follows that $\nabla_i h_{kl}=\nabla_i\varphi\delta_{kl}$ for $1\leqslant k,l\leqslant\mu$ at the point $\bar{p}$.

We next consider the second derivative. To that end, we choose $v(0)=e_1$,

$$v'(0)=-\sum_{l>\mu}(\lambda_l-\lambda_1)^{-1}\nabla_i h_{1l}e_l,$$

and $v''(0)=0$. Since the function $s\mapsto h(v(s),v(s))-\varphi(\gamma(s))|v(s)|^2$ has a local minimum at $s=0$, we obtain

$$\begin{aligned}0&\leqslant\frac{d^2}{ds^2}(h(v(s),v(s))-\varphi(\gamma(s))|v(s)|^2)\Big|_{s=0}\\&=\nabla_i\nabla_i h(v(0),v(0))+4\nabla_i h(v(0),v'(0))+2h(v'(0),v'(0))\\&\qquad-\nabla_i\nabla_i\varphi|v(0)|^2-4\nabla_i\varphi\langle v(0),v'(0)\rangle-2\varphi|v'(0)|^2\\&=\nabla_i\nabla_i h_{11}-4\sum_{l>\mu}(\lambda_l-\lambda_1)^{-1}(\nabla_i h_{1l})^2+2\sum_{l>\mu}\lambda_l(\lambda_l-\lambda_1)^{-2}(\nabla_i h_{1l})^2\\&\qquad-\nabla_i\nabla_i\varphi-2\sum_{l>\mu}\lambda_1(\lambda_l-\lambda_1)^{-2}(\nabla_i h_{1l})^2\\&=\nabla_i\nabla_i h_{11}-2\sum_{l>\mu}(\lambda_l-\lambda_1)^{-2}(\nabla_i h_{1l})^2-\nabla_i\nabla_i\varphi.\end{aligned}$$

This proves the assertion. ∎

THEOREM 6. *Assume $\alpha\in\left(\frac{1}{2},\infty\right)$ and Σ is a strictly convex closed smooth solution of ($*_\alpha$). Then Σ is a round sphere.*

Proof. Let us consider the function

$$W=K^\alpha\lambda_1^{-1}-\frac{n\alpha-1}{2n\alpha}|F|^2. \tag{13}$$

Let us consider an arbitrary point $\bar{p}$ where W attains its maximum. As above, we denote by μ the multiplicity of the smallest eigenvalue of the second fundamental form. Let us define a smooth function φ such that

$$W(\bar{p}) = K^\alpha \varphi^{-1} - \frac{n\alpha - 1}{2n\alpha} |F|^2.$$

Since W attains its maximum at $\bar{p}$, we have $\lambda_1 \geqslant \varphi$ everywhere and $\lambda_1 = \varphi$ at $\bar{p}$. Therefore, we may apply the previous lemma. Hence, at the point $\bar{p}$, we have $\nabla_i h_{kl} = \nabla_i \varphi \delta_{kl}$ for $1 \leqslant k, l \leqslant \mu$. Moreover,

$$\nabla_k \nabla_k \varphi \leqslant \nabla_k \nabla_k h_{11} - 2 \sum_{l>\mu} (\lambda_l - \lambda_1)^{-1} (\nabla_k h_{1l})^2$$

at $\bar{p}$. We multiply both sides by $\alpha K^\alpha \lambda_k^{-1}$ and sum over k. This gives

$$\mathcal{L}\varphi \leqslant \mathcal{L} h_{11} - 2\alpha K^\alpha \sum_k \sum_{l>\mu} \lambda_k^{-1} (\lambda_l - \lambda_1)^{-1} (\nabla_k h_{1l})^2.$$

By (5), we have

$$\begin{aligned}\mathcal{L} h_{ij} = & -K^{-\alpha} \nabla_i K^\alpha \nabla_j K^\alpha + \alpha K^\alpha b^{pr} b^{qs} \nabla_i h_{rs} \nabla_j h_{pq} \\ & + \langle F, F_k \rangle \nabla^k h_{ij} + h_{ij} + (n\alpha - 1) h_{ik} h_j^k K^\alpha - \alpha K^\alpha H h_{ij}.\end{aligned}$$

Thus,

$$\begin{aligned}\mathcal{L}\varphi \leqslant & -2\alpha K^\alpha \sum_k \sum_{l>\mu} \lambda_k^{-1} (\lambda_l - \lambda_1)^{-1} (\nabla_k h_{1l})^2 - K^{-\alpha} (\nabla_1 K^\alpha)^2 + \alpha K^\alpha \sum_{k,l} \lambda_k^{-1} \lambda_l^{-1} (\nabla_k h_{1l})^2 \\ & + \sum_k \langle F, F_k \rangle \nabla_k h_{11} + \lambda_1 + (n\alpha - 1) \lambda_1^2 K^\alpha - \alpha K^\alpha H \lambda_1.\end{aligned}$$

Using the estimate

$$\begin{aligned}& -2\alpha K^\alpha \sum_k \sum_{l>\mu} \lambda_k^{-1} (\lambda_l - \lambda_1)^{-1} (\nabla_k h_{1l})^2 + \alpha K^\alpha \sum_{k,l} \lambda_k^{-1} \lambda_l^{-1} (\nabla_k h_{1l})^2 \\ & = -\alpha K^\alpha \sum_k \sum_{l>\mu} \lambda_k^{-1} (2(\lambda_l - \lambda_1)^{-1} - \lambda_l^{-1}) (\nabla_k h_{1l})^2 + \alpha K^\alpha \sum_k \lambda_k^{-1} \lambda_1^{-1} (\nabla_k h_{11})^2 \\ & \leqslant -\alpha K^\alpha \sum_{l>\mu} \lambda_1^{-1} (2(\lambda_l - \lambda_1)^{-1} - \lambda_l^{-1}) (\nabla_1 h_{1l})^2 + \alpha K^\alpha \sum_k \lambda_k^{-1} \lambda_1^{-1} (\nabla_k h_{11})^2 \\ & = -2\alpha K^\alpha \sum_{k>\mu} \lambda_1^{-1} ((\lambda_k - \lambda_1)^{-1} - \lambda_k^{-1}) (\nabla_k h_{11})^2 + \alpha K^\alpha \lambda_1^{-2} (\nabla_1 h_{11})^2,\end{aligned}$$

we obtain

$$\begin{aligned}\mathcal{L}\varphi \leqslant& -2\alpha K^\alpha \lambda_1^{-1} \sum_{k>\mu} ((\lambda_k-\lambda_1)^{-1}-\lambda_k^{-1})(\nabla_k h_{11})^2 + \alpha K^\alpha \lambda_1^{-2}(\nabla_1 h_{11})^2 - K^{-\alpha}(\nabla_1 K^\alpha)^2 \\ &+ \sum_k \langle F, F_k\rangle \nabla_k h_{11} + \lambda_1 + (n\alpha-1)\lambda_1^2 K^\alpha - \alpha K^\alpha H \lambda_1.\end{aligned}$$

Since $\nabla_k\varphi = \nabla_k h_{11}$, it follows that

$$\begin{aligned}\mathcal{L}(\varphi^{-1}) \geqslant& \, 2\alpha K^\alpha \lambda_1^{-3} \sum_k \lambda_k^{-1}(\nabla_k h_{11})^2 \\ &+ 2\alpha K^\alpha \lambda_1^{-3} \sum_{k>\mu} ((\lambda_k-\lambda_1)^{-1}-\lambda_k^{-1})(\nabla_k h_{11})^2 \\ &- \alpha K^\alpha \lambda_1^{-4}(\nabla_1 h_{11})^2 + K^{-\alpha}\lambda_1^{-2}(\nabla_1 K^\alpha)^2 \\ &+ \sum_k \langle F, F_k\rangle \nabla_k(\lambda_1^{-1}) - \lambda_1^{-1} - (n\alpha-1)K^\alpha + \alpha K^\alpha H \lambda_1^{-1} \\ =& \, 2\alpha K^\alpha \lambda_1^{-3} \sum_{k>\mu} (\lambda_k-\lambda_1)^{-1}(\nabla_k h_{11})^2 \\ &+ \alpha K^\alpha \lambda_1^{-4}(\nabla_1 h_{11})^2 + K^{-\alpha}\lambda_1^{-2}(\nabla_1 K^\alpha)^2 \\ &+ \sum_k \langle F, F_k\rangle \nabla_k(\lambda_1^{-1}) - \lambda_1^{-1} - (n\alpha-1)K^\alpha + \alpha K^\alpha H \lambda_1^{-1}.\end{aligned}$$

This gives

$$\begin{aligned}\mathcal{L}(K^\alpha\varphi^{-1}) =& \, K^\alpha \mathcal{L}(\varphi^{-1}) + \varphi^{-1}\mathcal{L}(K^\alpha) + 2\alpha K^\alpha \sum_k \lambda_k^{-1} \nabla_k K^\alpha \nabla_k(\varphi^{-1}) \\ \geqslant& \, 2\alpha \sum_k \lambda_k^{-1} \nabla_k K^\alpha \nabla_k (K^\alpha \varphi^{-1}) - 2\alpha\lambda_1^{-1} \sum_k \lambda_k^{-1}(\nabla_k K^\alpha)^2 \\ &+ 2\alpha K^{2\alpha} \lambda_1^{-3} \sum_{k>\mu} (\lambda_k-\lambda_1)^{-1}(\nabla_k h_{11})^2 \\ &+ \alpha K^{2\alpha}\lambda_1^{-4}(\nabla_1 h_{11})^2 + \lambda_1^{-2}(\nabla_1 K^\alpha)^2 \\ &+ \sum_k \langle F, F_k\rangle \nabla_k(K^\alpha\varphi^{-1}) + (n\alpha-1)K^\alpha\lambda_1^{-1} - (n\alpha-1)K^{2\alpha}.\end{aligned}$$

By assumption, the function

$$K^\alpha\varphi^{-1} - \frac{n\alpha-1}{2n\alpha}|F|^2$$

is constant. Consequently,

$$\nabla_k(K^\alpha\varphi^{-1}) = \frac{n\alpha-1}{2n\alpha}\nabla_k|F|^2,$$

and

$$\begin{aligned}0 &= \mathcal{L}\left(K^\alpha\varphi^{-1} - \frac{n\alpha-1}{2n\alpha}|F|^2\right)\\ &\geqslant \frac{n\alpha-1}{n}\sum_k \lambda_k^{-1}\nabla_k K^\alpha \nabla_k |F|^2 - 2\alpha\lambda_1^{-1}\sum_k \lambda_k^{-1}(\nabla_k K^\alpha)^2\\ &\quad + 2\alpha K^{2\alpha}\lambda_1^{-3}\sum_{k>\mu}(\lambda_k-\lambda_1)^{-1}(\nabla_k h_{11})^2\\ &\quad + \alpha K^{2\alpha}\lambda_1^{-4}(\nabla_1 h_{11})^2 + \lambda_1^{-2}(\nabla_1 K^\alpha)^2\\ &\quad + \frac{n\alpha-1}{2n\alpha}\sum_k \langle F, F_k\rangle \nabla_k |F|^2 + (n\alpha-1)K^\alpha\left(\lambda_1^{-1} - \frac{1}{n}\operatorname{tr}(b)\right).\end{aligned}$$

Recall that

$$\tfrac{1}{2}\nabla_k |F|^2 = \langle F, F_k\rangle = \lambda_k^{-1}\nabla_k K^\alpha$$

by (3). Moreover, using the identity $\nabla_k\varphi = \nabla_k h_{11}$, we obtain

$$0 = \nabla_k(K^\alpha\varphi^{-1}) - \frac{n\alpha-1}{2n\alpha}\nabla_k|F|^2 = \left(\lambda_1^{-1} - \frac{n\alpha-1}{n\alpha}\lambda_k^{-1}\right)\nabla_k K^\alpha - K^\alpha\lambda_1^{-2}\nabla_k h_{11}$$

at $\bar{p}$. Note that, if $2\leqslant l\leqslant\mu$, then $\nabla_k h_{1l}=0$ for all k. Putting $k=1$ gives $\nabla_l h_{11}=0$ and $\nabla_l K=0$ for $2\leqslant l\leqslant\mu$. Putting these facts together, we obtain

$$\begin{aligned}0 &\geqslant \frac{2(n\alpha-1)}{n}\sum_k \lambda_k^{-2}(\nabla_k K^\alpha)^2 - 2\alpha\lambda_1^{-1}\sum_k \lambda_k^{-1}(\nabla_k K^\alpha)^2\\ &\quad + 2\alpha\lambda_1\sum_{k>\mu}(\lambda_k-\lambda_1)^{-1}\left(\lambda_1^{-1} - \frac{n\alpha-1}{n\alpha}\lambda_k^{-1}\right)^2(\nabla_k K^\alpha)^2\\ &\quad + \frac{1}{n^2\alpha}\lambda_1^{-2}(\nabla_1 K^\alpha)^2 + \lambda_1^{-2}(\nabla_1 K^\alpha)^2\\ &\quad + \frac{n\alpha-1}{n\alpha}\sum_k \lambda_k^{-2}(\nabla_k K^\alpha)^2 + (n\alpha-1)K^\alpha\left(\lambda_1^{-1} - \frac{1}{n}\operatorname{tr}(b)\right).\end{aligned}$$

Using the identities

$$\begin{aligned}\frac{2(n\alpha-1)}{n}\lambda_k^{-2} - 2\alpha\lambda_1^{-1}\lambda_k^{-1} + 2\alpha\lambda_1(\lambda_k-\lambda_1)^{-1}\left(\lambda_1^{-1} - \frac{n\alpha-1}{n\alpha}\lambda_k^{-1}\right)^2 + \frac{n\alpha-1}{n\alpha}\lambda_k^{-2}\\ = \left(\frac{n\alpha-1}{n\alpha} + \frac{2}{n} + \frac{2}{n^2\alpha}\lambda_1(\lambda_k-\lambda_1)^{-1}\right)\lambda_k^{-2}\end{aligned}$$

and

$$\frac{2(n\alpha-1)}{n}\lambda_1^{-2} - 2\alpha\lambda_1^{-2} + \frac{1}{n^2\alpha}\lambda_1^{-2} + \lambda_1^{-2} + \frac{n\alpha-1}{n\alpha}\lambda_1^{-2} = \frac{n-1}{n^2\alpha}(2n\alpha-1)\lambda_1^{-2},$$

the previous inequality can be rewritten as follows:

$$0 \geqslant \sum_{k>\mu} \bigg(\frac{n\alpha - 1}{n\alpha} + \frac{2}{n} + \frac{2}{n^2\alpha} \lambda_1 (\lambda_k - \lambda_1)^{-1} \bigg) \lambda_k^{-2} (\nabla_k K^\alpha)^2 + \frac{n-1}{n^2\alpha} (2n\alpha - 1) \lambda_1^{-2} (\nabla_1 K^\alpha)^2 + (n\alpha - 1) K^\alpha \bigg(\lambda_1^{-1} - \frac{1}{n} \operatorname{tr}(b) \bigg).$$

Since $\alpha > 1/n$, it follows that $\bar{p}$ is an umbilical point. As $\bar{p}$ is an umbilical point and $\alpha > \frac{1}{2}$, there exists a neighborhood U of $\bar{p}$ with the property that

$$\begin{aligned} &\alpha^{-1} K^{-2\alpha} (\mathcal{L}Z - (2\alpha+1) b^{ij} \nabla_i K^\alpha \nabla_j Z) \\ &\quad = \sum_D \lambda_i^{-2} \lambda_j^{-1} \lambda_k^{-1} (\nabla_i h_{jk})^2 \\ &\qquad + \sum_k \sum_i \lambda_k^{-2} \lambda_i^{-2} (\nabla_k h_{ii})^2 + 2 \sum_k \sum_{i \neq k} \lambda_k^{-1} \lambda_i^{-3} (\nabla_k h_{ii})^2 \\ &\qquad + \sum_k \lambda_k^{-1} (-2\alpha^2 \operatorname{tr}(b) + (2n\alpha^2 + (n-1)\alpha - 1) \lambda_k^{-1}) (\nabla_k \log K)^2 \geqslant 0 \end{aligned}$$

at each point in U. (Indeed, if $n \geqslant 3$, the last inequality follows immediately from the fact that $(n-1)\alpha - 1 \geqslant 0$. For $n = 2$ the last inequality follows from a straightforward calculation.)

Now, since $\bar{p}$ is an umbilical point, we have $Z(p) \leqslant nW(p) \leqslant nW(\bar{p}) = Z(\bar{p})$ for each point $p \in U$. Thus, Z attains a local maximum at $\bar{p}$. By the strong maximum principle, $Z(p) = Z(\bar{p})$ for all points $p \in U$. This implies that $W(p) = W(\bar{p})$ for all points $p \in U$. Thus, the set of all points where W attains its maximum is open. Consequently, W is constant. This implies that Σ is umbilical, and hence a round sphere. □

5. Proof of Theorem 1

Suppose that we have any strictly convex solution to the flow with speed $-K^\alpha \nu$, where $\alpha \in [1/(n+2), \infty)$. It is known that the flow converges to a soliton after rescaling; for $\alpha > 1/(n+2)$, this follows from [5, Theorem 6.2], while for $\alpha = 1/(n+2)$ this follows from results in [3, §9]. By Theorems 4 and 6, either the limit is a round sphere, or $\alpha = 1/(n+2)$ and the limit is an ellipsoid.

References

[1] Andrews, B., Contraction of convex hypersurfaces by their affine normal. *J. Differential Geom.*, 43 (1996), 207–230.

[2] — Gauss curvature flow: the fate of the rolling stones. *Invent. Math.*, 138 (1999), 151–161.

[3] — Motion of hypersurfaces by Gauss curvature. *Pacific J. Math.*, 195 (2000), 1–34.

[4] Andrews, B. & Chen, X., Surfaces moving by powers of Gauss curvature. *Pure Appl. Math. Q.*, 8 (2012), 825–834.

[5] Andrews, B., Guan, P. & Ni, L., Flow by powers of the Gauss curvature. *Adv. Math.*, 299 (2016), 174–201.

[6] Bethuel, F., Huisken, G., Müller, S. & Steffen, K., *Calculus of Variations and Geometric Evolution Problems*. Lecture Notes in Mathematics, 1713. Springer, Berlin–Heidelberg, 1999.

[7] Calabi, E., Complete affine hyperspheres. I, in *Symposia Mathematica*, Vol. X (Convegno di Geometria Differenziale, INdAM, Rome, 1971), pp. 19–38. Academic Press, London, 1972.

[8] Choi, K., *The Gauss Curvature Flow*: *Regularity and Asymptotic Behavior*. Ph.D. Thesis, Columbia University, New York, NY, 2017.

[9] Chow, B., Deforming convex hypersurfaces by the nth root of the Gaussian curvature. *J. Differential Geom.*, 22 (1985), 117–138.

[10] — On Harnack's inequality and entropy for the Gaussian curvature flow. *Comm. Pure Appl. Math.*, 44 (1991), 469–483.

[11] Chow, B. & Gulliver, R., Aleksandrov reflection and nonlinear evolution equations. I. The n-sphere and n-ball. *Calc. Var. Partial Differential Equations*, 4 (1996), 249–264.

[12] Chow, B. & Hamilton, R. S., The cross curvature flow of 3-manifolds with negative sectional curvature. *Turkish J. Math.*, 28 (2004), 1–10.

[13] Chow, B. & Tsai, D. H., Nonhomogeneous Gauss curvature flows. *Indiana Univ. Math. J.*, 47 (1998), 965–994.

[14] Firey, W. J., Shapes of worn stones. *Mathematika*, 21 (1974), 1–11.

[15] Gerhardt, C., Non-scale-invariant inverse curvature flows in Euclidean space. *Calc. Var. Partial Differential Equations*, 49 (2014), 471–489.

[16] Guan, P. & Ni, L., Entropy and a convergence theorem for Gauss curvature flow in high dimension. *J. Eur. Math. Soc.* (*JEMS*), 19 (2017), 3735–3761.

[17] Hamilton, R. S., Remarks on the entropy and Harnack estimates for the Gauss curvature flow. *Comm. Anal. Geom.*, 2 (1994), 155–165.

[18] Huisken, G., Flow by mean curvature of convex surfaces into spheres. *J. Differential Geom.*, 20 (1984), 237–266.

[19] Nomizu, K. & Sasaki, T., *Affine Differential Geometry*. Cambridge Tracts in Mathematics, 111. Cambridge University Press, Cambridge, 1994.

[20] Schnürer, O. C., Surfaces expanding by the inverse Gauß curvature flow. *J. Reine Angew. Math.*, 600 (2006), 117–134.

[21] Tso, K., Deforming a hypersurface by its Gauss–Kronecker curvature. *Comm. Pure Appl. Math.*, 38 (1985), 867–882.

Simon Brendle
Department of Mathematics
Columbia University
2990 Broadway
New York, NY 10027
U.S.A.
simon.brendle@columbia.edu

Kyeongsu Choi
Department of Mathematics
Columbia University
2990 Broadway
New York, NY 10027
U.S.A.
kschoi@math.columbia.edu

Panagiota Daskalopoulos
Department of Mathematics
Columbia University
2990 Broadway
New York, NY 10027
U.S.A.
pdaskalo@math.columbia.edu

Received December 6, 2016